Gluon saturation and the Color Glass Condensate

*Part III*

Edmond Iancu
SPhT Saclay & CNRS
Reminder: Dipole factorization for DIS

\[ \sigma_{\gamma^* p}(x, Q^2) = \int_0^1 dz \int d^2r \left| \Psi_{\gamma}(z, r; Q^2) \right|^2 \sigma_{\text{dipole}}(x, r) \]

\[ \sigma_{\text{dipole}}(x, r) = 2 \int d^2b \ T(x, r, b) \]

- Unitarity bound on the dipole amplitude: \[ T(x, r, b) \leq 1 \]
Reminder: Balitsky–Kovchegov equation

\[ \frac{\partial}{\partial Y} \langle T(x, y) \rangle_Y = \frac{\bar{\alpha}_s}{2\pi} \int_z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \left\langle -T(x, y) + T(x, z) + T(z, y) - T(x, z)T(z, y) \right\rangle_Y \]

BFKL (linear) \hspace{2cm} \text{non–linear}

- Mean field approximation assuming factorization:

\[ \langle T(x, z)T(z, y) \rangle_Y \approx \langle T(x, z) \rangle_Y \langle T(z, y) \rangle_Y \]

- Neglects correlations in the target wavefunction.
Reminder: Saturation & Geometric scaling

Saturation line: \[ \ln Q_s^2(Y) = \lambda Y \] with \( \lambda \approx 0.3 \) (NLO)

Geometric scaling above saturation (in the ‘dilute regime’)

\[ T(r, Y) \approx e^{-\gamma_s (\rho - \rho_s)} = (r^2 Q_s^2) \gamma_s \]
Gluon production in $pp$ or $pA$ collisions

- ‘Dense–dilute’ scattering
  - $pA$ collisions (RHIC, LHC)
  - $pp$ collisions at ‘forward rapidity’ (LHC)

- Only one parton from the dilute projectile gets involved

- A probe of the gluon distribution inside the dense target! 
Forward gluon production (cf. lecture by F. Gelis)

- Effective gluon–gluon dipole in the cross–section

\[
\frac{d\sigma^{pp\to JX}}{d\eta d^2k_\perp} \sim \frac{1}{k_\perp^2} \ xG_p(x_1, k_\perp^2) \int d^2r \ e^{-ik \cdot r} \ \nabla_r^2 \sigma_{(gg)}(r, x_2)
\]

\[
x_{1,2} = (k_\perp/\sqrt{s}) e^{\pm \eta}
\]
What about nucleus–nucleus collisions?

- ‘Dense–dense scattering’: multi–particle interactions probe higher gluon correlations in the nuclear wavefunctions
  ➞ a more complete description of the gluon distribution

- The CGC formalism
What about nucleus–nucleus collisions?

- complicated evolution of the wavefunction (JIMWLK)
- additional ‘complications’ due to final state interactions
- factorization is far from being obvious
- Factorization recently proven for the inclusive parton production (see lectures by F. Gelis and R. Venugopalan)
The Color Glass Condensate

*(McLerran, Venugopalan, 1994; E.I., Leonidov, McLerran, 2000)*

- Effective theory for the small–\(x\) gluons at/near saturation

- Small–\(x\) gluons: Classical color fields radiated by fast color sources \((x' \gg x)\) ‘frozen’ in some random configuration

- Quantum modes with \(x' > x\) (or \(k^+ > \Lambda^+ = xP^+\)) have been ‘integrated out’ and replaced with a random color charge distribution with density \(\rho^a\) and probability distribution \(W_x[\rho]\)
The **Color Glass Condensate**

*(McLerran, Venugopalan, 1994; E.I., Leonidov, McLerran, 2000)*

- Effective theory for the small–$x$ gluons at/near saturation
- Small–$x$ gluons: Classical color fields radiated by fast color sources ($x' \gg x$) ‘frozen’ in some random configuration

Classical field equations (Yang–Mills) for the field $A^\mu_\alpha [\rho]$

Probability distribution for the charge density at $x$ : $W_x [\rho]$

Renormalization group equation for $W_x [\rho]$ : JIMWLK
**Light Cone notations & Kinematics**

- **The hadron moves in the positive \( z \) direction, with \( v \approx c = 1 \)**

- **Longitudinal momentum** \( P \gg M \Rightarrow P^\mu = (E \approx P, 0, 0, P) \)

  \[
P^+ = \frac{1}{\sqrt{2}}(E + P) \approx \sqrt{2}P, \quad P^- = \frac{1}{\sqrt{2}}(E - P) \approx 0
  \]

- **A classical particle:** \( z \approx t \) or \( x^- \approx 0 \) with

  \[
x^+ = \frac{1}{\sqrt{2}}(t + z) \approx \sqrt{2}t, \quad x^- = \frac{1}{\sqrt{2}}(t - z) \approx 0
  \]

  - **LC time**
  - **LC longitudinal coord.**
Light Cone notations & Kinematics

- The hadron moves in the positive $z$ direction, with $v \approx c = 1$

\[ t \quad x^- \quad x^+ \]

\[ z \]

- Longitudinal momentum $P \gg M \implies P^\mu = (E \approx P, 0, 0, P)$

\[
P^+ \equiv \frac{1}{\sqrt{2}}(E + P) \approx \sqrt{2}P, \quad P^- \equiv \frac{1}{\sqrt{2}}(E - P) \approx 0
\]

- Even for the quantum system, the wavefunction is strongly localized near $x^- = 0$ (“pancake”)

\[
\Delta x^- \sim \frac{1}{P^+} \sim \frac{1}{\gamma M} \ll \frac{1}{M}
\]
The hadron moves in the positive $z$ direction, with $v \simeq c = 1$

**Longitudinal momentum** $P \gg M \implies P^\mu = (E \approx P, 0, 0, P)$

$$P^+ \equiv \frac{1}{\sqrt{2}}(E + P) \simeq \sqrt{2}P, \quad P^- \equiv \frac{1}{\sqrt{2}}(E - P) \simeq 0$$

**Small–$x$ gluons** are, however, more delocalized

$$\Delta x^- \sim \frac{1}{xP^+} \ll \frac{1}{P^+}$$
The Yang–Mills equations

- The ‘color source’ : a current in the ‘plus’ direction

\[
(D_\nu F^{\nu\mu})_a(x) = \delta^{\mu+} \rho_a(x^- \cdot x_\perp)
\]

- The source \(\rho_a\) is
  - independent of the LC time \(x^+\) (Lorentz time dilation)
  - localized near \(x^- = 0\) (i.e., \(z = t\)) within a distance

\[
\Delta x^- \sim 1/\Lambda^+ \quad \text{with} \quad \Lambda^+ = xP^+
\]
Classical solution (cf. lecture by F. Gelis)

- Only one independent field d.o.f. (independent of $x^+$)

- Coulomb gauge: $\nabla^i A^i_a = 0 \implies A^i_a = 0, \ i = 1, 2$

  $$-\nabla^2_{\perp} A_a^+ (x^-, x_{\perp}) = \rho_a(x^-, x_{\perp})$$

- Exercise: Show that the solution is of the form

  $$A^+_a(x^-, x) = \int d^2 y \frac{1}{4\pi} \ln \frac{1}{(x - y)^2 \mu^2} \rho_a(x^-, y)$$

  NB: Localized near $x^- = 0$, so like the color charge itself.
  ‘Weiszäcker–Williams color field’

- One can easily trade $\rho^a$ for $A_a^+$ (e.g., $W_x[\rho] \rightarrow W_x[A^+]$)

- Where are the non–linear effects ?!
Eikonal Approximation

- Right moving target (CGC) + Left moving projectile \((q, \bar{q}, g\ldots)\)

- Field equations in the background field:

\[
\gamma_\mu D^\mu \psi(x) = 0 \quad (D^\mu = \partial^\mu - igT^a A^\mu_a)
\]

-\(D^+ \sim 1/\Delta x^-\) is much larger than \(D_\perp\)

\[
\Rightarrow D^+ S(x^-, x_\perp) \approx 0
\]

\[
\Rightarrow \text{straight line trajectory: } x_\perp = \text{const.}
\]
Wilson lines

\[ D^+ S(x^-, x_\perp) \equiv \left( \frac{\partial}{\partial x^-} - igT^a A^+_a \right) S(x^-, x_\perp) = 0 \]

\[ \implies S(x^-, x_\perp) = P \exp\left( ig \int_{-\infty}^{x^-} dy^- A^+_a(x^-, x_\perp) T^a \right) \equiv V(x^-, x_\perp) \]

- Path–ordered exponential: color rotation, non–linear in \( A^+ \)

\[ V(x^-) = e^{ig\epsilon A^+(x^-)} e^{ig\epsilon A^+(x^-_{N-1})} \ldots e^{ig\epsilon A^+(x^-_0)} \]
Dipole scattering off the CGC

\[ S(x, y)[A^+] = \frac{1}{N_c} \text{tr}(V(x)V^\dagger(y)) \]

\[ V(x) \equiv \text{Pexp}\left(ig \int dx^- A_a^+(x^-, x)t^a\right) \in SU(N_c) \]

- Color trace: the dipole is color neutral
- Color transparency: when \( x \to y, \) \( S \to 1 \)
- Unitarity manifest: \( |S| \leq 1 \) (multiple scattering)
The weight function $W_Y[\rho]$

- $S(x, y)[A^+] = \text{the ‘event-by-event’ } S\text–matrix:$ valid for a given configuration of the color sources (fields) in the target

- The physical amplitude: average over all configurations average over $\rho$ with the weight function $W_Y[A^+]$:

  $$\langle S(x, y) \rangle_Y = \int D[A^+] \ W_Y[A^+] \ S(x, y)[A^+]$$

- Computing $W_Y[A^+]$: the main issue in the CGC formalism
The (unintegrated) gluon distribution

\[ n(Y, k_{\perp}, b_{\perp}) \equiv \frac{(2\pi)^3}{2(N_c^2 - 1)} \frac{dN}{dY \, d^2k_{\perp} \, d^2b_{\perp}} \]

■ What is ‘gluon number’?! (a priori, gauge–dependent)

■ A Fourier transform of the gauge–invariant 2–point function

\[ \left\langle \text{Tr} \left\{ E^i(x) W_\gamma(x, y) E^i(y) W_\gamma(y, x) \right\} \right\rangle_Y, \quad W_\gamma(x, y) = V(x)V^\dagger(y) \]
Rapidity $Y = \ln \frac{1}{x}$: The relevant information about the gluon distribution has been included in the weight function $W_Y[\rho]$

Alternatively: $W_Y[A^+]$ (a simple change of variables)
Renormalization group at small $x$

- Rapidity $Y + dY$: One addition ‘color source’ (gluon) is being radiated, from one of the color sources at $Y$.

- Low density/energy: The new gluon is incoherently produced from any of the previous sources: $\delta \rho \propto \rho$.

\[ \rho_a(x) \]
\[ W_Y [\rho] \]
\[ Y < y < Y + dY \]
\[ A^+ + \delta A^+ \]
Renormalization group at small $x$

High density/energy: The new gluon can rescatter off the color field produced by other sources: $\delta \rho = \text{non-linear in } \rho$

Strategy: Absorb the change in $\rho$ and in the correlations into a change of the weight function: $W_Y [\rho] \rightarrow W_{Y+dY} [\rho]$
Renormalization group at small $x$

Evolution equation for $W_Y[\rho]$ (‘JIMWLK’)

$$\frac{\partial W_Y[\rho]}{\partial Y} = H_{\text{JIMWLK}} \left( \rho, \frac{\delta}{\delta \rho} \right) W_Y[\rho]$$

(Jalilian-Marian, E.I., McLerran, Weigert, Leonidov, and Kovner, 97–00)
Low density: BFKL Hamiltonian

- \( W_Y[\rho] = \) a probability distribution \( \implies \) ‘cut’ diagrams:
  (amplitude \( \times \) complex conjugate amplitude)

- Weak fields/low density: one must recover BFKL evolution

\[
H_{\text{JIMWLK}} \approx \frac{1}{2} (K_{\text{BFKL}} A A) \frac{\delta}{\delta A} \frac{\delta}{\delta A} \equiv H_{\text{BFKL}}
\]
BFKL as color glass evolution

- The weak–field (BFKL) limit of the JIMWLK Hamiltonian

\[
\frac{\partial W_Y[A]}{\partial Y} \approx \frac{1}{2} \mathcal{K}_{\text{BFKL}} \left( \frac{\delta}{\delta A} A A \frac{\delta}{\delta A} \right) W_Y \equiv H_{\text{BFKL}} W_Y[A]
\]

- Each step in the evolution: 2 → 2 gluon vertex
  Insert a ‘BFKL exchange’ in between each pair of fields

- All gluon correlations rise exponentially with \(Y\)
The general case: JIMWLK Hamiltonian

Strong fields $gA^+ \sim \mathcal{O}(1)$: The quantum gluon rescatters of the background field in the eikonal approximation

$$H_{JIMWLK} = \frac{1}{2} \int_{\mathbf{x}_\perp, \mathbf{y}_\perp} \frac{\delta}{\delta A^a(\mathbf{x}_\perp)} \chi_{ab}(\mathbf{x}_\perp, \mathbf{y}_\perp)[V, V^\dagger] \frac{\delta}{\delta A^b(\mathbf{y}_\perp)}$$

$V$ and $V^\dagger$: Wilson lines in the adjoint representation.
Coherent emission and saturation

\[ \chi_{ab}(x, y) = \int_z \frac{z^i - x^i}{(z - x)^2} \frac{z^i - y^i}{(z - y)^2} \left( 1 - V^\dagger(x) V(z) \right)_{ac} \left( 1 - V^\dagger(z) V(y) \right)_{cb} \]

\[ \rightarrow (A(x) - A(z))(A(z) - A(y)) \]

- \( \chi[A] \): the gluon emission rate in the background field
- \( n \rightarrow 2 \) gluon vertices with arbitrary \( n \) (non–linear evolution)

\[ \text{‘Gluon recombination’ — more correctly, coherent emission} \]
Coherent emission and saturation

\[ \chi_{ab}(x, y) = \int \frac{z^i - x^i}{(z - x)^2} \frac{z^i - y^i}{(z - y)^2} (1 - V^\dagger(x)V(z))_{ac} (1 - V^\dagger(z)V(y))_{cb} \]

Lipatov vertex

\[ \to (A(x) - A(z))(A(z) - A(y)) \]

- The new gluon is coherently produced out of the preexisting color sources
- Strong fields \( A \sim 1/g \)
  - the Wilson lines \( V \sim \exp(i g A^+) \) rapidly oscillate
  - the bilinear \( V^\dagger(z)V(x) \) self–averages to zero
  - the emission rate saturates at a field–independent value
- However strong the field is, there will be only one additional gluon emitted per unit rapidity
  \[ \Rightarrow \text{the gluon density rises linearly in } Y \sim \ln s, \text{ rather than exponentially (as in the low density/weak field regime)} \]
Coherent emission and saturation

\[ \chi_{ab}(x, y) = \int_z \frac{z^i - x^i}{(z - x)^2} \frac{z^i - y^i}{(z - y)^2} \left( 1 - V^\dagger(x) V(z) \right)_{ac} \left( 1 - V^\dagger(z) V(y) \right)_{cb} \]

- This is the mechanism for gluon saturation!

- This only happens on sufficiently large distances, i.e., for sufficiently small transverse momenta for the emitted gluons.

- Remember: \( V^\dagger(z) V(x) = \) the \( S\)-matrix for a color dipole \( V^\dagger(z) V(x) \to 0 \) over distances \( |z - x| \gtrsim 1/Q_s(Y) \)

- For a given \( Y \), saturation occurs only for gluons with transverse momenta \( k_\perp \lesssim Q_s(Y) \)

\[ \chi(x, y) \approx \ln \left( (x - y)^2 Q_s^2(Y) \right) \implies n(k_\perp, Y) \sim \ln \frac{Q_s^2(Y)}{k_\perp^2} \propto Y \]

- The connection saturation \( \leftrightarrow \) dipole unitarity is now manifest.
The gluon occupation number

\[ n(Y, k_\perp, b_\perp) \equiv \frac{(2\pi)^3}{2(N_c^2 - 1)} \frac{dN}{dY \, d^2k_\perp \, d^2b_\perp} \]

\[ Y = 5 \text{ red} \]
\[ Y = 10 \text{ green} \]
\[ Y = 15 \text{ blue} \]
Gluon occupation number

For \( k_\perp \gg Q_s(Y) \): \( n(k, Y) \propto 1/k_\perp^2 \) (bremsstrahlung)
For $k_\perp \gg Q_s(Y)$: $n(k, Y) \propto (Q_s^2/k_\perp^2)^{\gamma_s}$ (geometric scaling)
Gluon occupation number

For $k_{\perp} \lesssim Q_s(Y)$:

$$n(Y, k_{\perp}) \approx \frac{1}{\alpha_s N_c} \ln \frac{Q_s^2(Y)}{k_{\perp}^2} \sim \frac{1}{\bar{\alpha}_s}$$
Gluon occupation number

\[ xG(x, Q^2) = \int d^2 b \int^Q dk \ n(k, Y) \]

\[ n(k) \]

\[ k \ n(k) \]

- \( Q_s(Y) \): the typical transverse momentum of the gluons
- High energy \( (Y) \) \( \implies \) large \( Q_s^2 \) \( \implies \) weak coupling!
- At sufficiently high energy, gluon saturation allows for a meaningful perturbation theory!
Color neutrality at saturation

- In a low energy hadron, color is screened by confinement:
  \[ r \sim \Lambda_{\text{QCD}}^{-1} \sim 1 \text{ fm} \]

- At high energy, the densely packed gluons screen each other, in such a way that color neutralization occurs already at the perturbative scale \( Q_s^{-1} \ll \Lambda_{\text{QCD}}^{-1} \)

\( Q_s(Y) \) is the “infrared cutoff” at high energy, and is hard!
Evolution of observables

- **Recall:** Observables are obtained by averaging over $\rho$, or $A$:

\[
\langle \mathcal{O}[A]\rangle_Y = \int D[A] \ W_Y[A] \ \mathcal{O}[A]
\]

- **Example:** The dipole $S$–matrix:

\[
S(x, y) = \frac{1}{N_c} \text{tr} \left( V(x) V^\dagger(y) \right)
\]

- Differentiate w.r.t. $Y$, use JIMWLK, and integrate by parts:

\[
\partial_Y \langle \mathcal{O}[A]\rangle_Y = \int D[A] \ (\partial_Y W_Y) \ \mathcal{O}[A]
\]

\[
= \int D[A] \ 1 \ 2 \left( \frac{\delta}{\delta A_a} \chi_{ab} \frac{\delta W_Y}{\delta A_b} \right) \ \mathcal{O}[A]
\]

\[
= \int D[A] \ W_Y[A] \ 1 \ 2 \frac{\delta}{\delta A_a} \chi_{ab} \frac{\delta}{\delta A_b} \ \mathcal{O}[A]
\]

- ... or, simply,

\[
\partial_Y \langle \mathcal{O}[A]\rangle_Y = \langle H_{\text{JIMWLK}} \ \mathcal{O}[A]\rangle_Y
\]
Recovering Balitsky equations

■ **Exercise:** Use JIMWLK to deduce the first Balitsky equation:

\[
\frac{\partial}{\partial Y} \langle S(x, y) \rangle_Y = \frac{\bar{\alpha}_s}{2\pi} \int_z \frac{(x - y)^2}{(x - z)^2(y - z)^2} \left\{ -\langle S(x, y) \rangle_Y + \langle S(x, z) S(z, y) \rangle_Y \right\}
\]

■ The action of the functional derivatives on Wilson lines

\[
V(x) \equiv \text{Pexp} \left( ig \int_{-\infty}^{\infty} dx^- A_+^a(x^-, x) T^a \right)
\]

\[
\frac{\delta}{\delta A_+^a(x)} V(x) \equiv \frac{\delta}{\delta A_+^a(x^- \to \infty, x)} V(x) = igT^a V(x)
\]

■ The derivative w.r.t. the field at \( x^- = \infty \) ! ("Lie derivative")

■ The **CGC** color source/field gets built in layers of \( x^- \)

■ The higher equations in the Balitsky hierarchy can be similarly obtained.
Gluon number fluctuations

- What about the reversed, missing, diagrams?
- $2 \rightarrow n$ gluon vertices with $n > 2$

Clearly, irrelevant at high density! (higher powers of $g$ without compensating factors of the strong field $A \sim 1/g$)

- Generate gluon number fluctuations at low density
- And so what?! ... Higher–order effects anyway!
The front is driven by the linear (BFKL) growth in the tail of the amplitude at large $\rho \gg \rho_s(Y)$, where $T \ll 1$ \quad \Rightarrow \quad \text{potentially strong sensitivity to fluctuations!}

High–energy evolution in pQCD: a stochastic process in the same universality as the reaction–diffusion process $A \leftrightarrow 2A$

(E.I., Mueller, Munier; E.I., Triantafyllopoulos, 2004)
The Pomeron loop equations

- Additional terms in the evolution equations (large $N_c$)

\[
\frac{\partial \langle T \rangle}{\partial t} \approx \langle T \rangle - \langle TT \rangle
\]

\[
\frac{\partial \langle TT \rangle}{\partial t} \approx 2\langle TT \rangle - 2\langle TTT \rangle + \alpha_s^2 \langle T \rangle
\]

- The fluctuation term dominates when $\langle T \rangle \lesssim \alpha_s^2$ (dilute tail)
New phenomena due to fluctuations

- Front diffusion: geometric scaling is progressively washed out and replaced by diffusive scaling

\[ \log\left(\frac{r_0^2}{r^2}\right) \]
New phenomena due to fluctuations

- Highly inhomogeneous gluon distribution in impact parameter space: black spots

... All that mostly coming from the correspondence with statistical physics.
Running coupling effects

- Unfortunately, the fluctuations appear to be ‘washed out’ by the effects of the running of the coupling

‘washed out’ = delayed up to very high energies

\[ Y \gtrsim 100 \implies \sqrt{s} \gg M_{Planck} \]

(Dumitru, E. I., Portugal, Soyez, Triantafyllopoulos, 07)

- To be followed!
Prediction: **Next week end**
Initial condition: The MV model

- The gluon distribution of a large nucleus \( A \gg 1 \) at not so large energy: \( \alpha_s Y \ll 1 \) (say, \( x = 10^{-1} \div 10^{-2} \))

- Gluon (BFKL) evolution is negligible
  \[ W_A[\rho] = \mathcal{N} \exp \left\{ -\frac{1}{2} \int d^2x \frac{\rho_a(x)\rho_a(x)}{\mu_A} \right\} , \quad \mu_A \propto A^{1/3} \]

- Reasonable initial condition \( Y = 0 \) for the JIMWLK equation