Gluon saturation and the Color Glass Condensate

Part II

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Reminder: Yesterday morning
Reminder: This morning
Reminder : Last coffee break
New students (and/or professors)
Reminder: Gluon distribution at HERA

- The gluon distribution rises very fast at small $x$! ($\sim 1/x^{0.3}$)

$$xG(x, Q^2) \approx \text{# of gluons with transverse size } \Delta x_\perp \sim 1/Q \text{ and } k_z = xP$$
Reminder: Gluon distribution at HERA

Increasing $Q^2$ (DGLAP): An evolution towards diluteness

Increasing rapidity $Y \sim \ln s$: An evolution towards higher density
Reminder: BFKL evolution

\[ n(Y, k_\perp) \approx \frac{\alpha_s C_F}{\pi} \left( \frac{k_0^2}{k_\perp^2} \right)^\gamma e^{\omega \alpha_s Y} \]

- Valid so long as the gluon occupation number remains small

\[ n \ll 1/\alpha_s \iff k_\perp \gg Q_s(Y, A) \quad \text{with} \quad Q_s^2(Y, A) \propto A^{1/3} e^{\lambda Y} \]
Reminder: **Color Glass Condensate**

- **Small–**\(x\) gluons: Classical color fields radiated by fast color sources (\(x' \gg x\)) ‘frozen’ in some random configuration

- Probability distribution for the charge density at \(Y\) : \(W_Y[\rho]\)

  A kind of ‘super’ gluon distribution (many–body correlations)
Reminder: Color Glass Condensate

- Renormalization group equation for $W_Y[\rho]:$ JIMWLK
  Complicated since it keeps trace of all the gluon correlations
- However, a simple (= dilute) projectile probes only a particular subset of these correlations.
Reminder: DIS off the CGC

- ‘Dense–dilute scattering’ (also proton–nucleus)
- Specialize to observables related to the scattering of the dilute projectile (here, to DIS)
- Use boost invariance to associate one step in the high energy evolution with the wavefunction of the projectile ($\gamma^*$)
Dipole frame

- Lorentz boost: from IMF to the ‘dipole frame’ :

\[ \Delta t_{\text{pair}} \sim \frac{q}{Q^2} \] (Lorentz time dilation)

\[ \Rightarrow \gamma^* \text{ fluctuates into a colorless } q\bar{q} \text{ pair ('color dipole') with transverse size } r \sim \frac{1}{Q} \text{ which then scatters off the proton} \]
Dipole factorization for DIS

\[ \sigma_{\gamma^* p}(x, Q^2) = \int_0^1 dz \int d^2r \left| \Psi_{\gamma}(z, r; Q^2) \right|^2 \sigma_{\text{dipole}}(x, r) \]

- Two gluon exchange: single scattering approximation
Dipole factorization for DIS

\[ \sigma_{\gamma^* p}(x, Q^2) = \int_0^1 dz \int d^2r \ |\Psi_{\gamma}(z, r; Q^2)|^2 \sigma_{\text{dipole}}(x, r) \]

▷ Not restricted to single scattering!
Dipole cross–section: **Unitarity bound**

- Dipole $S$–matrix: $S = 1 + iA$

  $$\sigma_{\text{dipole}}(x, r) = 2 \Im m A(r, s, t = 0) = 2 \int d^2b \Im m A(x, r, b)$$

- Unitarity bound: $SS^\dagger = 1 \implies T(x, r, b) \leq 1$
  - $T \ll 1$: weak scattering
  - $T = 1$: ‘black disk limit’ (multiple scattering)

High energy: $A \approx iT$ with real $T$

$$T = 1 - S: \text{‘scattering amplitude’}$$

$$\sigma_{\text{dipole}}(x, r) = 2 \int d^2b \ T(x, r, b)$$

$r \sim 1/Q$
Dipole interactions

- **Reminder (classical electrodynamics):**
  A small dipole couples to the electric field:

  \[ V(r) = e \left[ A_0(b + r) - A_0(b) \right] \approx e r^i \partial_i A_0(b) = -e r \cdot E(b) \]

- **QCD:** ‘Color dipole’ = \( q \bar{q} \) pair in a color singlet state

  \( e r \cdot E \rightarrow g t^a r \cdot E_a + \text{average over color:} \quad \frac{1}{N_c} \text{tr}\{\ldots\} \)
A small color dipole scatters off the gluon field in the target

\[ V(r) \simeq g t^a r \cdot E_a \implies T(x, r, b) \propto \frac{g^2}{N_c} r^2 \langle E_a(b) \cdot E_a(b) \rangle_x \]

\[ T(x, r, b) \simeq \frac{\alpha_s}{N_c} r^2 \frac{xG(x, 1/r^2)}{\pi R^2} \]

A direct measure of the gluon distribution \( xG(x, Q^2) \) at \( Q^2 \sim 1/r^2 \)
A brief reminder

Dipole picture for DIS
- Dipole frame
- Dipole factorization
- Dipole cross-section
- Single scattering
- Unitarity

BK equation

Saturation line

**Color dipole: single scattering**

- A small color dipole scatters off the gluon field in the target

\[ V(r) \sim g t^a r \cdot E_a \quad \Rightarrow \quad T(x, r, b) \propto \frac{g^2}{N_c} r^2 \langle E_a(b) \cdot E_a(b) \rangle_x \]

\[ T(x, r, b) \propto \alpha_s^2 \frac{r^2}{R^2} \frac{1}{x^\lambda} \]

- \( T(x, r, b) \to 0 \) as \( r \to 0 \) : ‘color transparency’
- When decreasing \( x \) and/or increasing \( r \) : \( T(x, r) \sim \mathcal{O}(1) \)
Onset of ‘unitarity corrections’: \( T(x, r) \sim 1 \iff r \sim 1/Q_s(x) \)

\[
Q_s^2(x) \sim \frac{\alpha_s}{N_c} \frac{xG(x, Q_s^2)}{\pi R^2} \sim e^{\lambda Y}
\]
Unitarity corrections

- **Multiple scattering & Gluon saturation**

- **Unitarity** : a property of scattering (frame–independent)

- **Saturation** : a property of the wavefunction, manifest in the target infinite momentum frame (and also in the dipole frame)

- One can study saturation by following dipole evolution
One–step (BFKL) evolution in DIS

- Increase the proton rapidity by $dY$
- A probability of $\mathcal{O}(\alpha_s dY)$ to emit an extra gluon, which is softer ($Y < y < Y + dY$)
One–step (BFKL) evolution in DIS

- One can instead give the rapidity increment $dY$ to the dipole (merely a change of frame)
- An extra gluon in the dipole wavefunction
One–step (BFKL) evolution in DIS

- One can instead give the rapidity increment $dY$ to the dipole (merely a change of frame)

- $Y + dY$: A quark–antiquark–gluon system which scatters off the target
One–step (BFKL) evolution in DIS

- Not restricted to single scattering!
- Evolution is simple, but scattering can be complicated.
Dipole splitting (large $N_c$)

A quark–antiquark pair = a ‘gluon’ + a color singlet

\[
\begin{align*}
3 \otimes 3 &= 8 \oplus 1, \\
N_c \otimes \bar{N}_c &= (N_c^2 - 1) \oplus 1
\end{align*}
\]

8 gluons
A quark–antiquark pair = a ‘gluon’ + a color singlet

\[ 3 \otimes \bar{3} = 8 \oplus 1, \quad N_c \otimes \bar{N}_c = (N_c^2 - 1) \oplus 1 \]

8 gluons
Dipole kernel

- Dipole splitting at large $N_c$: four diagrams ...

- One dipole $(x, y)$ splits into two dipoles $(x, z)$ and $(z, y)$
  (“dipole picture of BFKL evolution”, Al Mueller, 94)

- Differential probability (or ‘dipole kernel’)

$$dP_{\text{split}} = \frac{\tilde{\alpha}_s}{2\pi} \frac{(x - y)^2}{(x - z)^2(y - z)^2} \, d^2z \, dY,$$

where

$$\tilde{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}$$
The original dipole splits into two new dipoles which then can scatter off the gluon fields in the target.
The first Balitsky equation

- $\langle S(x, y) \rangle_Y$ : $S$–matrix element for the scattering between the dipole $(x, y)$ and the target for a rapidity separation $Y$
  - $|\langle S(x, y) \rangle_Y|^2$ = the dipole survival probability

- $\langle S(x_1, y_1)S(x_2, y_2) \rangle_Y$ : the $S$–matrix for a projectile made with 2 dipoles: $(x_1, y_1)$ and $(x_2, y_2)$

- One evolution step : $Y \rightarrow Y + dY$ with $\alpha_s dY \ll 1$

$$
\langle S(x, y) \rangle_{Y+dY} = dY \int d^2z \frac{dP_{\text{split}}}{dY d^2z} \langle S(x, z)S(z, y) \rangle_Y \\
+ \left(1 - dY \int d^2z \frac{dP_{\text{split}}}{dY d^2z}\right) \langle S(x, y) \rangle_Y
$$
The first Balitsky equation

\[
\frac{\partial}{\partial Y} \langle S(x, y) \rangle_Y = \frac{\bar{\alpha}_s}{2\pi} \int_z \frac{(x-y)^2}{(x-z)^2(y-z)^2} \left\{ -\langle S(x, y) \rangle_Y + \langle S(x, z)S(z, y) \rangle_Y \right\}
\]

- Not a closed equation! (one dipole → two dipoles)
  First equation from the infinite Balitsky hierarchy

- Rewritten for the dipole scattering amplitude \( T \equiv 1 - S \):

\[
\frac{\partial}{\partial Y} \langle T(x, y) \rangle_Y = \frac{\bar{\alpha}_s}{2\pi} \int_z \frac{(x-y)^2}{(x-z)^2(y-z)^2} \langle -T(x, y) + T(x, z) + T(z, y) - T(x, z)T(z, y) \rangle_Y
\]

BFKL (linear)  non-linear
Non–linear gluon evolution

- Alternatively: Use the rapidity increment $dY$ to accelerate the target hadron $\implies$ the same equation ...

$$\frac{\partial}{\partial Y} \left\langle T(x, y) \right\rangle_Y = \frac{\bar{\alpha}_s}{2\pi} \int_z \frac{(x-y)^2}{(x-z)^2(y-z)^2}$$

$$\left\langle -T(x, y) + T(x, z) + T(z, y) - T(x, z)T(z, y) \right\rangle_Y$$

BFKL (linear) non–linear

... but with a different physical interpretation!

Saturation effects in the target wavefunction
The Balitsky–Kovchegov equation

- Mean field approximation assuming factorization:

\[
\langle T(x, z)T(z, y) \rangle_Y \approx \langle T(x, z) \rangle_Y \langle T(z, y) \rangle_Y
\]

- Neglects correlations in the target wavefunction.

- Some usual justifications in the literature:
  - large nucleus \( A \gg 1 \)
  - large \( N_c \)
  - leads to a relatively simple (closed) equation \( \checkmark \)
The Balitsky–Kovchegov equation

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The Balitsky–Kovchegov equation

\[
\frac{\partial}{\partial Y} T(x, y) = \frac{\bar{\alpha}_s}{2\pi} \int_z \frac{(x-y)^2}{(x-z)^2(y-z)^2} \left\{ -T(x, y) + T(x, z) + T(z, y) - T(x, z)T(z, y) \right\}
\]

\[\text{BFKL (linear)} \quad \nonumber \]
\[\text{non–linear} \quad \nonumber \]

- **Notations:** \( \bar{\alpha}_s \equiv \alpha_s N_c / \pi \), \( T(x, y) \equiv \langle T(x, y) \rangle_Y \)

- **Weak scattering** \( T \ll 1 \) (low energy/small dipole) \( \Rightarrow \) BFKL
  - \( T = 0 \) : unstable fixed point of BK equation
  - unitarity violations, infrared diffusion

- **Strong scattering** \( T \sim 1 \) \( \Rightarrow \) Non–linear effects
  - \( T = 1 \) : stable fixed point of BK equation

- **BK equation:** a simple framework to study unitarization
BK solution: target = ‘large nucleus’

$T(x, y|Y) = T(r, Y)$ with $r = x - y$ (dipole size)

$$T(r, Y = 0) \approx \begin{cases} 
  r^2 Q_0^2 & \text{for } r \ll 1/Q_0 \\
  1 & \text{for } r \gtrsim 1/Q_0
\end{cases}$$

$Q_0$: saturation momentum in the initial conditions
BK solution: the saturation front

\[ T(r, Y) \equiv T(\rho, Y) \text{ with } \rho \equiv \ln(1/Q_0^2 r^2) \equiv \ln(Q^2/Q_0^2) \]

A front interpolating between \( T = 0 \) and \( T = 1 \)

\[ T = 1 \text{ for } \rho \lesssim \rho_s(Y) \quad \text{and} \quad T \propto e^{-\rho} = r^2 \text{ for } \rho \gg \rho_s(Y) \]

The front position: the saturation scale \( \rho_s \equiv \ln(Q_s^2/Q_0^2) \)
BK solution: the saturation front

- Increase $Y$: the front propagates towards larger values of $\rho$

Main questions:
- What are the front position ($\rho_s$) and its shape around $\rho_s$?
- A priori, a very complicated problem: non-linear equation
The front is **pulled** by the **rapid growth** in the tail of the amplitude at large $\rho \gg \rho_s(Y)$, where $T \ll 1$.

The evolution is controlled by the **linear (BFKL)** equation with saturation boundary condition: $T(\rho_s(Y), Y) \sim 1$.

E.I., Itakura, McLerran (02); Mueller, Triantafyllopoulos (02); Munier, Peschanski (03) $\rightarrow$ Fischer–Kolmogorov equation.
BK solution : FKPP equation

- Gradient expansion of the non–locality $\implies$ FKPP equation:

$$\partial_Y T(\rho, Y) = \partial_\rho^2 T(\rho, Y) + T(\rho, Y) - T^2(\rho, Y)$$

- Linearized equation ($T \ll 1$) : diffusion + exponential growth

- Boundary condition : $T(\rho, Y) \sim 1$ when $\rho \sim \rho_s(Y)$

$$T(\rho, Y) \simeq e^{-\gamma_s(\rho-\rho_s)} \exp\left\{ -\frac{(\rho - \rho_s)^2}{2D\bar{\alpha}_s Y} \right\}$$

with the anomalous dimension $\gamma_s = 0.63$...

and the saturation scale $\rho_s(Y) = \lambda_s \bar{\alpha}_s Y$ with $\lambda_s = 4.88$...

- NB: $\lambda_s \bar{\alpha}_s = \mathcal{O}(1)$, whereas HERA data require $\lambda \sim 0.2 \div 0.3$

- NLO BFKL + saturation $\implies \lambda \approx 0.3$ (Triantafyllopoulos, 02)
BK solution: Geometric scaling

- Diffusion is negligible for $\rho - \rho_s \ll \sqrt{2D\bar{\alpha}_s Y}$

$$T(\rho, Y) \approx e^{-\gamma_s (\rho - \rho_s)} \iff T(r, Y) \approx (r^2 Q_s^2)^{\gamma_s}$$

- Saturation makes itself felt in the dilute regime ($Q^2 > Q_s^2$)
The traveling wave

- The amplitude ‘scales’ as a function of the variable $r^2 Q_s^2(Y)$

$$T(\rho, Y) \sim T(\rho - \rho_s(Y)) = T(r^2 Q_s^2(Y))$$

- The shape of the front is not altered by the evolution

- The front propagates like a ‘traveling wave’

  (Munier, Peschanski, 03)
Geometric Scaling at HERA

(Staśto, Golec-Biernat and Kwieciński, 2000)

\[ \sigma(x, Q^2) \approx \sigma(\tau) \quad \text{with} \quad \tau \equiv \frac{Q^2}{Q_s^2(x)}, \quad Q_s^2(x) = \left(\frac{x_0}{x}\right)^\lambda \text{GeV}^2, \quad \lambda \simeq 0.3 \]

\[ x \leq 0.01 \]
\[ Q^2 \leq 450 \text{ GeV}^2 \]
\[ Q_s^2 \sim 1 \text{ GeV}^2 \]

for \( x \sim 10^{-4} \)
A brief reminder

- Dipole picture for DIS
- BK equation
- Saturation line
  - Saturation front
  - Pulled front
  - FKPP
  - Geometric scaling
  - Traveling wave
  - Geometric scaling at HERA

Geometric Scaling at HERA (2)

(Marquet and Schoeffel 2006)

Hadronic collisions at the LHC and QCD at high density, Centre de Physique des Houches, France, Mar 25 - Apr 4, 2008

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