Gluon saturation and the Color Glass Condensate

Part II

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Reminder : Yesterday morning

Reminder : This morning

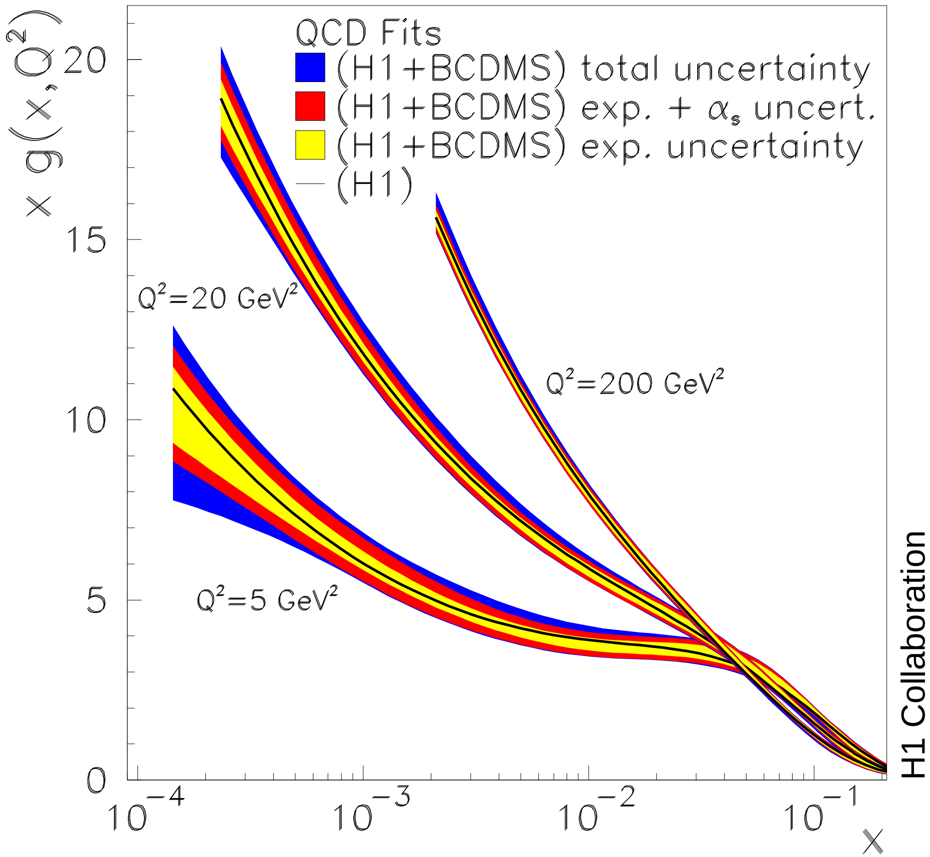
Reminder : Last coffee break

New students (and/or professors)

Reminder: Gluon distribution at HERA

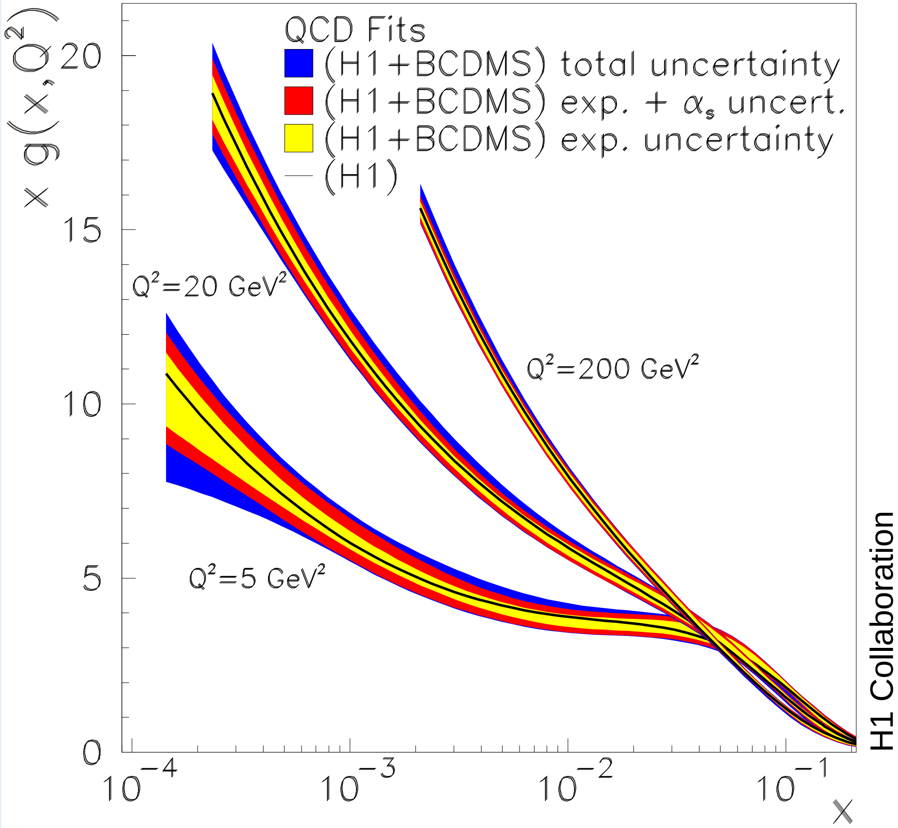
A brief reminder
Dipole picture for DIS
BK equation
Saturation line

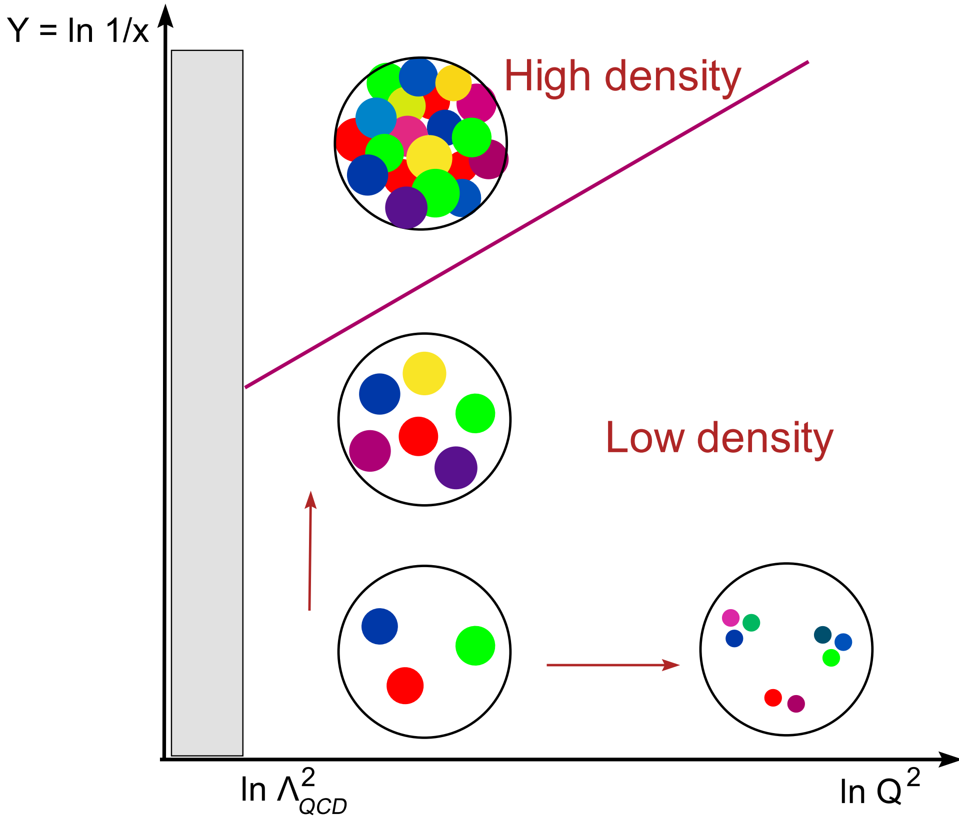
▷ The gluon distribution rises very fast at small x ! $(\sim 1/x^{0.3})$



$xG(x,Q^2) \approx$ # of gluons with transverse size $\Delta x_\perp \sim 1/Q$ and $k_z = xP$

Reminder: Gluon distribution at HERA

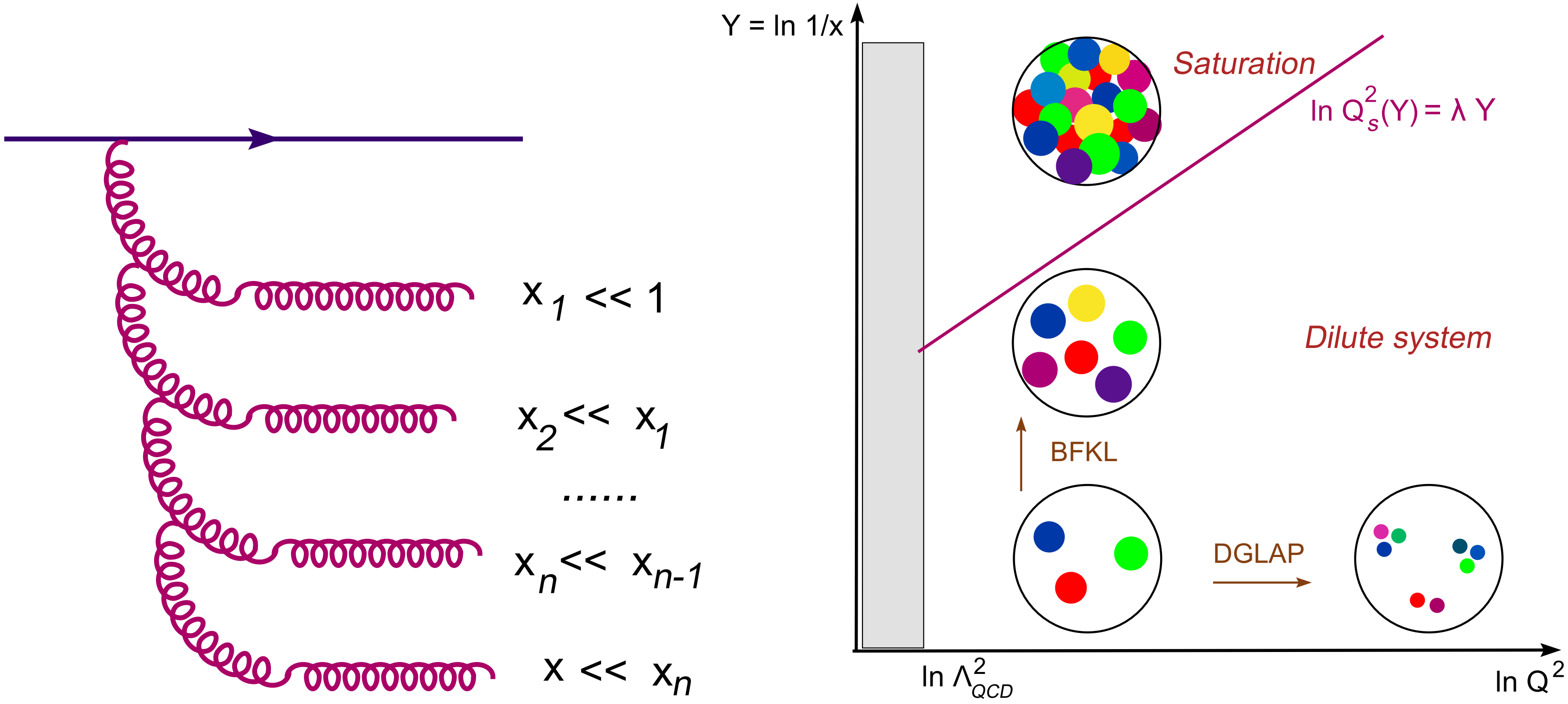




▷ Increasing Q^2 (DGLAP) : An evolution towards diluteness

▷ Increasing rapidity $Y \sim \ln s$: An evolution towards higher density

Reminder: BFKL evolution

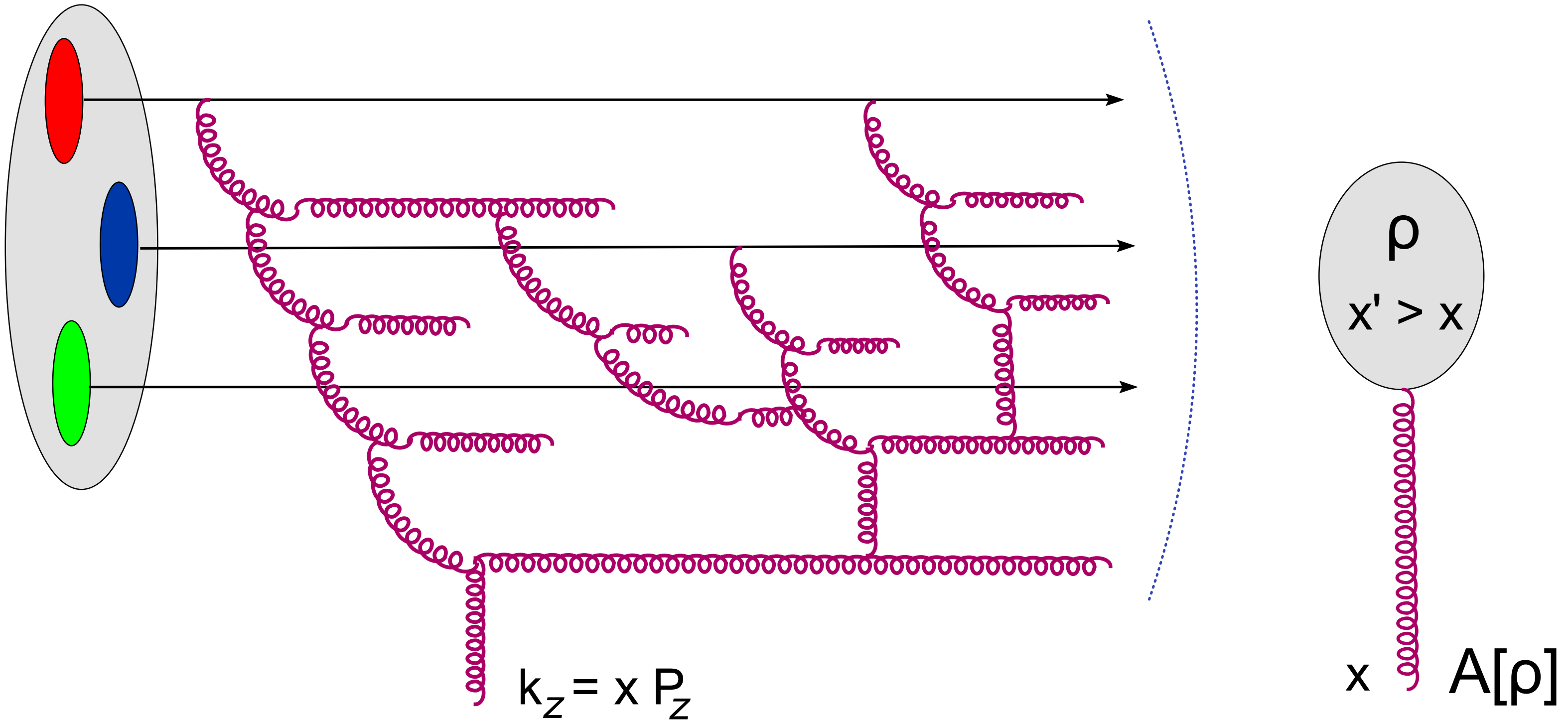


$$n(Y,k_\perp) \approx \frac{\alpha_s C_F}{\pi} \left(\frac{k_0^2}{k_\perp^2}\right)^{\gamma} \mathrm{e}^{\omega\alpha_s Y}$$

- Valid so long as the gluon occupation number remains small

$$n \ll 1/\alpha_s \iff k_\perp \gg Q_s(Y,A) \quad \text{with} \quad Q_s^2(Y,A) \propto A^{1/3}\,\mathrm{e}^{\lambda Y}$$

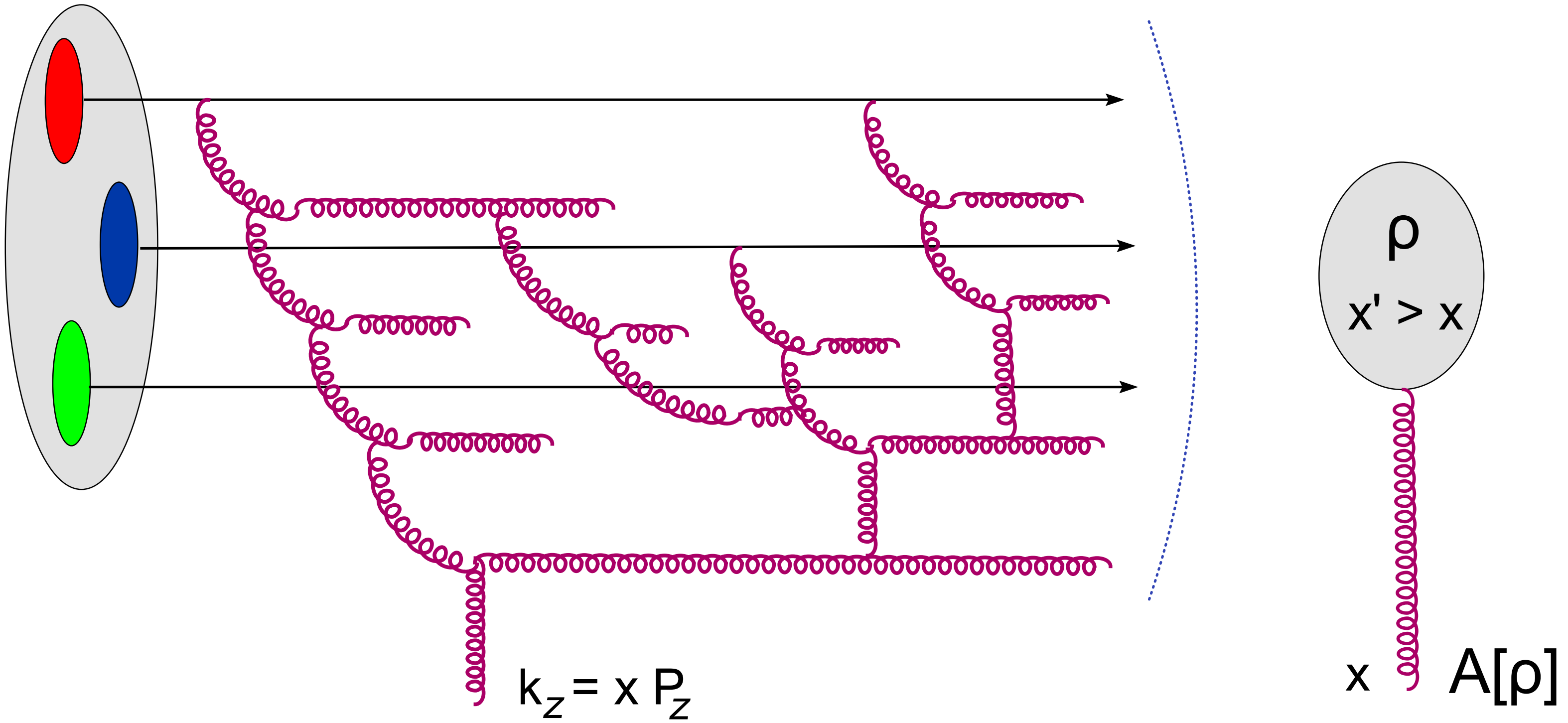
Reminder: Color Glass Condensate



- Small–x gluons: Classical color fields radiated by fast color sources ($x' \gg x$) 'frozen' in some random configuration
- Probability distribution for the charge density at Y : $W_Y[\rho]$

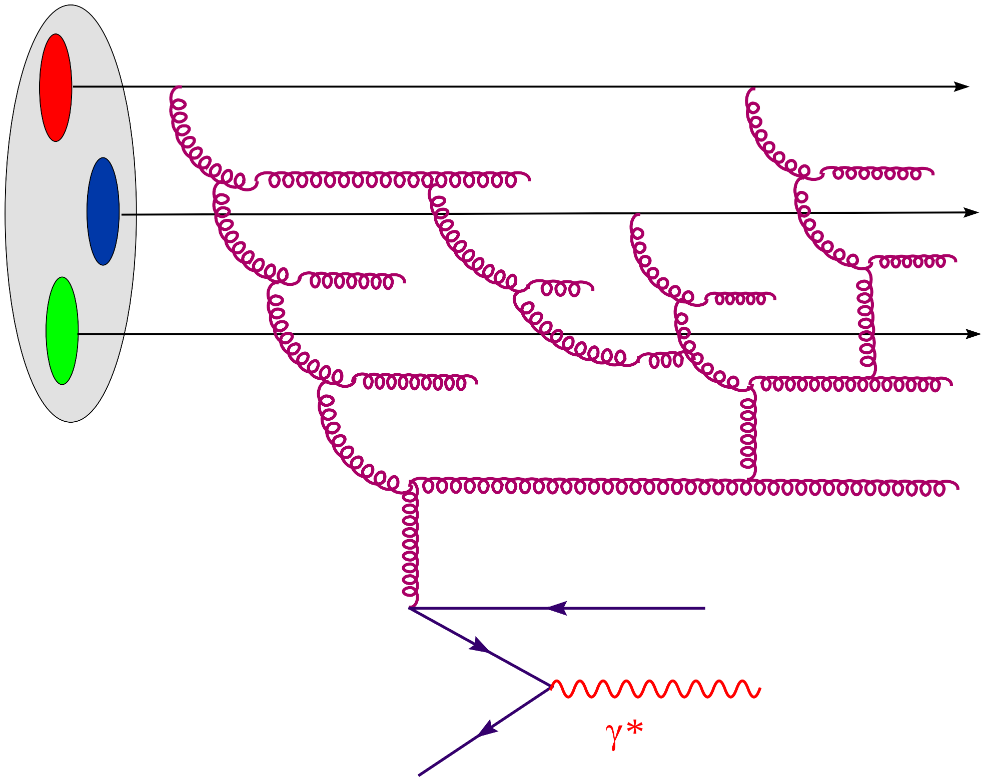
  A kind of 'super' gluon distribution (many–body correlations)

Reminder: Color Glass Condensate



- Renormalization group equation for $W_Y[\rho]$: JIMWLK
  Complicated since it keeps trace of all the gluon correlations
- However, a simple (= dilute) projectile probes only a particular subset of these correlations.

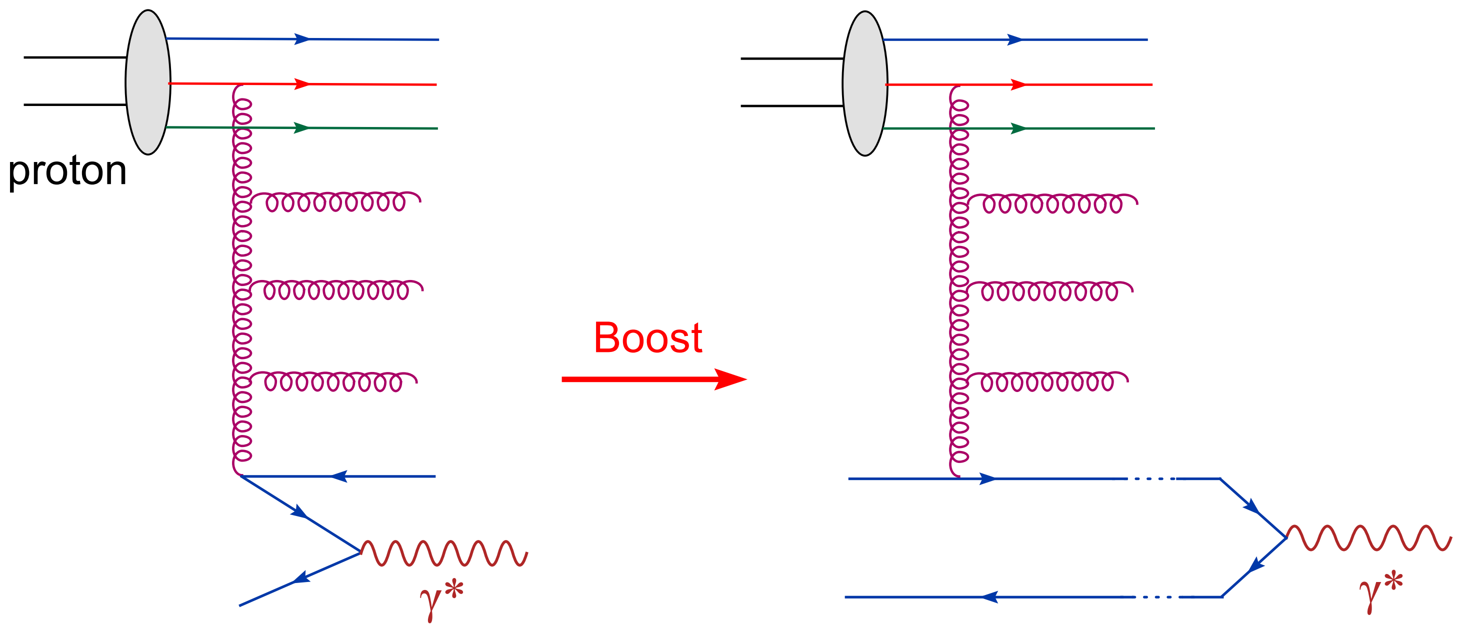
Reminder: DIS off the CGC



- ‘Dense–dilute scattering’ (also proton–nucleus)
- Specialize to observables related to the scattering of the dilute projectile (here, to DIS)
- Use boost invariance to associate one step in the high energy evolution with the wavefunction of the projectile (γ^*)

Dipole frame

- Lorentz boost: from IMF to the 'dipole frame' :

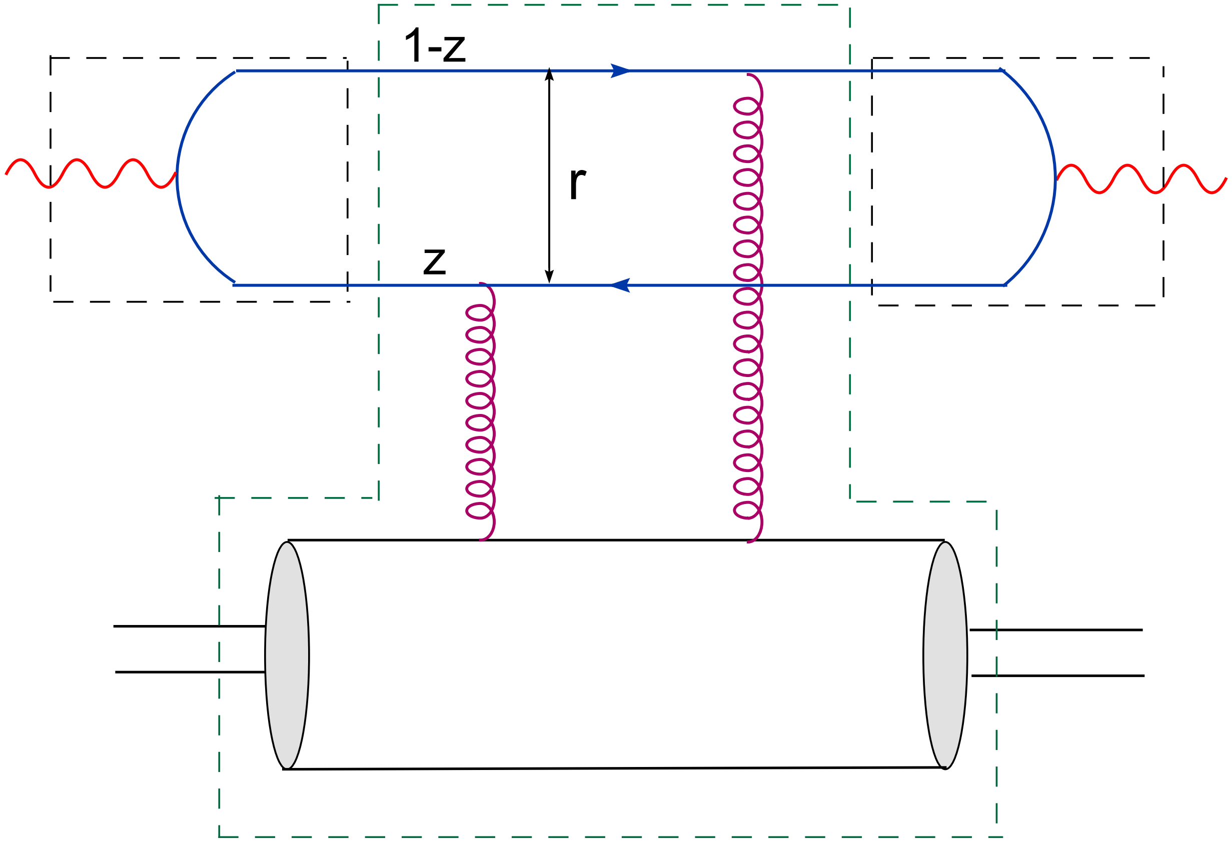


- γ^* has a relatively high longitudinal momentum $q \gg Q$

$$\Delta t_{\text{pair}} \sim \frac{q}{Q^2} \qquad \text{(Lorentz time dilation)}$$

$\Longrightarrow \gamma^*$ fluctuates into a colorless $q\bar{q}$ pair ('color dipole') with transverse size $r \sim 1/Q$ which then scatters off the proton

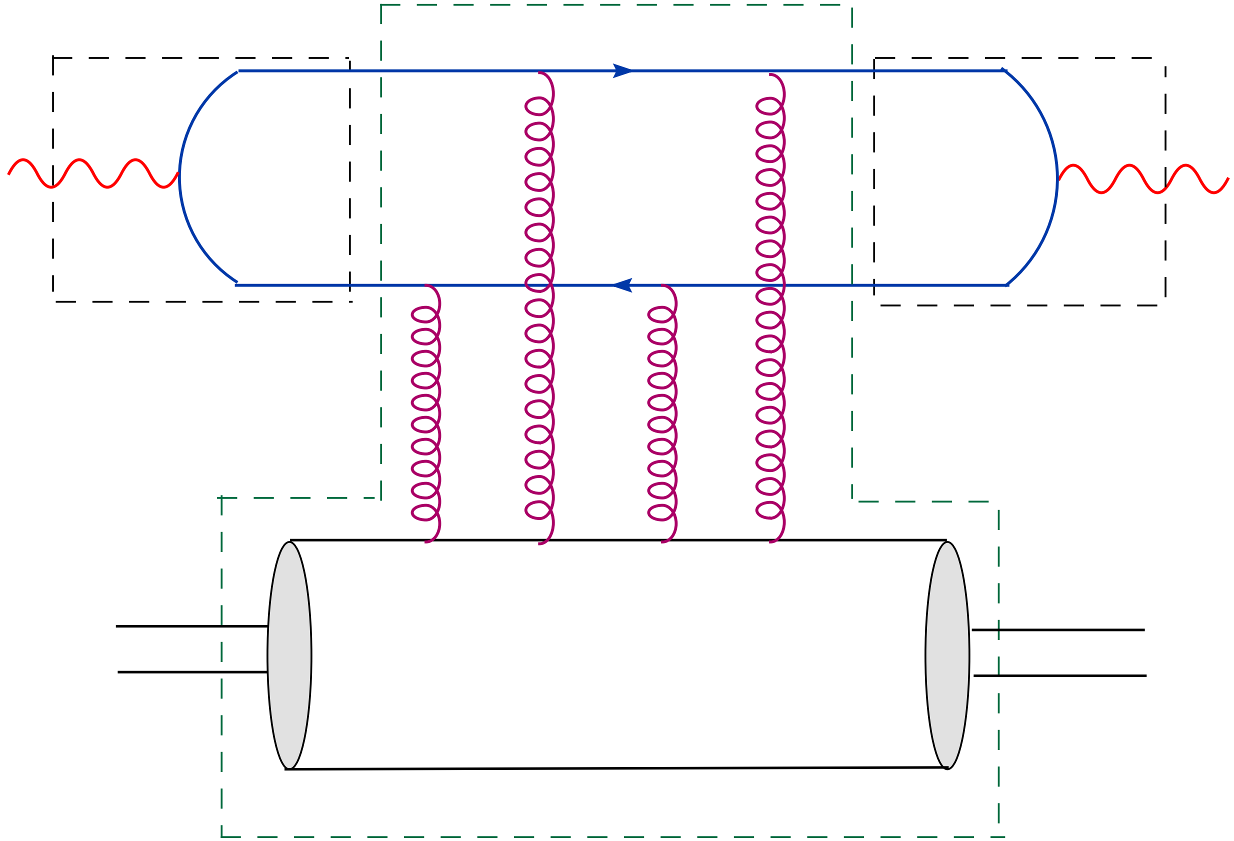
$$\sigma_{\gamma^* p}(x, Q^2) = \int_0^1 \mathrm{d}z \int \mathrm{d}^2\boldsymbol{r}\, |\Psi_\gamma(z, \boldsymbol{r}; Q^2)|^2\, \sigma_{\mathrm{dipole}}(x, \boldsymbol{r})$$



▷ Two gluon exchange: single scattering approximation

Dipole factorization for DIS

$$\sigma_{\gamma^* p}(x, Q^2) = \int_0^1 \mathrm{d}z \int \mathrm{d}^2\boldsymbol{r}\, |\Psi_\gamma(z, \boldsymbol{r}; Q^2)|^2\, \sigma_{\mathrm{dipole}}(x, \boldsymbol{r})$$

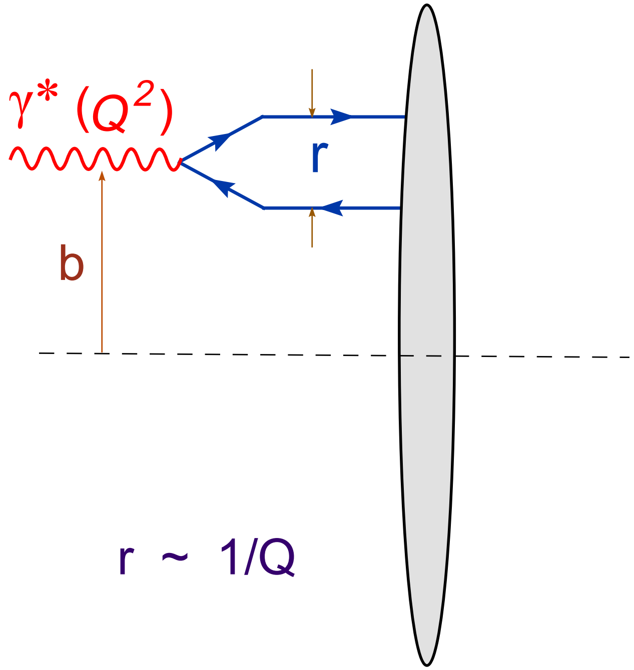


▷ Not restricted to single scattering !

Dipole cross–section: Unitarity bound

- Dipole S–matrix : $S = \mathbb{1} + \mathrm{i}\mathcal{A}$

$$\sigma_{\rm dipole}(x,r) \;=\; 2\,\Im\mathrm{m}\,\mathcal{A}(r,s,t=0) \;=\; 2\int \mathrm{d}^2\boldsymbol{b}\;\Im\mathrm{m}\,\mathcal{A}(x,\boldsymbol{r},\boldsymbol{b})$$



High energy: $\mathcal{A} \approx \mathrm{i}T$ with real T

$T = 1 - S$: 'scattering amplitude'

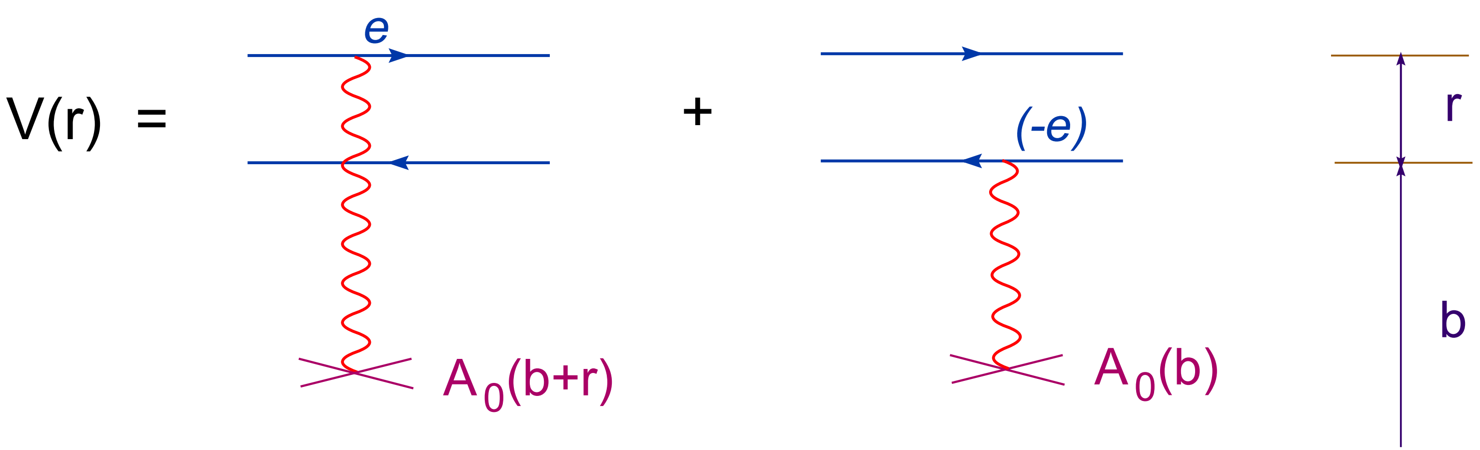
$$\sigma_{\rm dipole}(x,r) \;=\; 2\int \mathrm{d}^2\boldsymbol{b}\;T(x,\boldsymbol{r},\boldsymbol{b})$$

- Unitarity bound : $SS^\dagger = 1 \Longrightarrow T(x,\boldsymbol{r},\boldsymbol{b}) \le 1$
  - $T \ll 1$: weak scattering
  - $T = 1$: 'black disk limit' (multiple scattering)

- Reminder (classical electrodynamics) :

A small dipole couples to the electric field :

$$V(\boldsymbol{r}) = e\big[A_0(\boldsymbol{b}+\boldsymbol{r}) - A_0(\boldsymbol{b})\big] \simeq e\,r^i\partial_i A_0(\boldsymbol{b}) = -e\,\boldsymbol{r}\cdot\boldsymbol{E}(\boldsymbol{b})$$



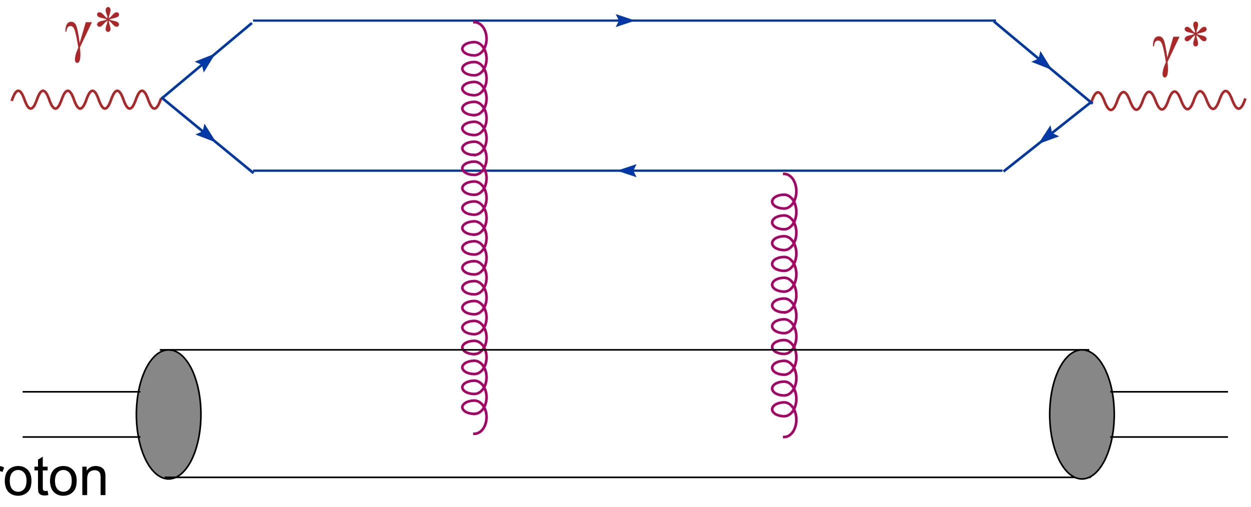
- QCD : ‘Color dipole’ = $q\bar{q}$ pair in a color singlet state

$e\,\boldsymbol{r}\cdot\boldsymbol{E} \longrightarrow g t^a \boldsymbol{r}\cdot\boldsymbol{E}_a$ + average over color: $\frac{1}{N_c}\mathrm{tr}\{\ldots\}$

Color dipole: single scattering

- A small color dipole scatters off the gluon field in the target

$$V(\boldsymbol{r}) \simeq g t^a \boldsymbol{r} \cdot \boldsymbol{E}_a \implies T(x, \boldsymbol{r}, \boldsymbol{b}) \propto \frac{g^2}{N_c} r^2 \langle \boldsymbol{E}_a(\boldsymbol{b}) \cdot \boldsymbol{E}_a(\boldsymbol{b}) \rangle_x$$



$$T(x, r, b) \simeq \frac{\alpha_s}{N_c} r^2 \frac{xG(x, 1/r^2)}{\pi R^2}$$

▷ A direct measure of the gluon distribution $xG(x, Q^2)$ at $Q^2 \sim 1/r^2$

Color dipole: single scattering

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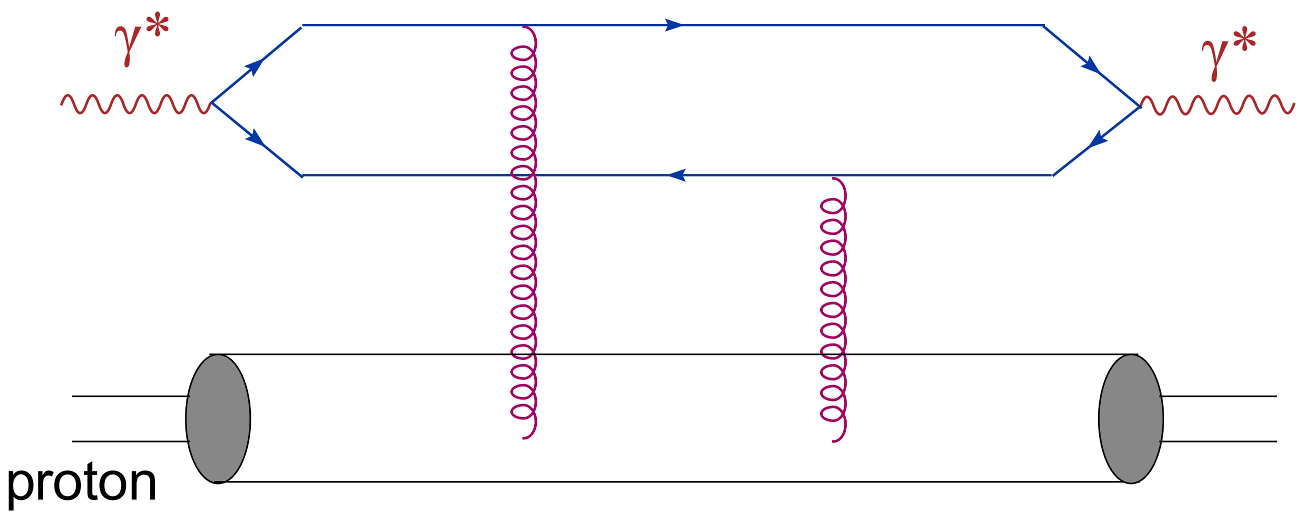


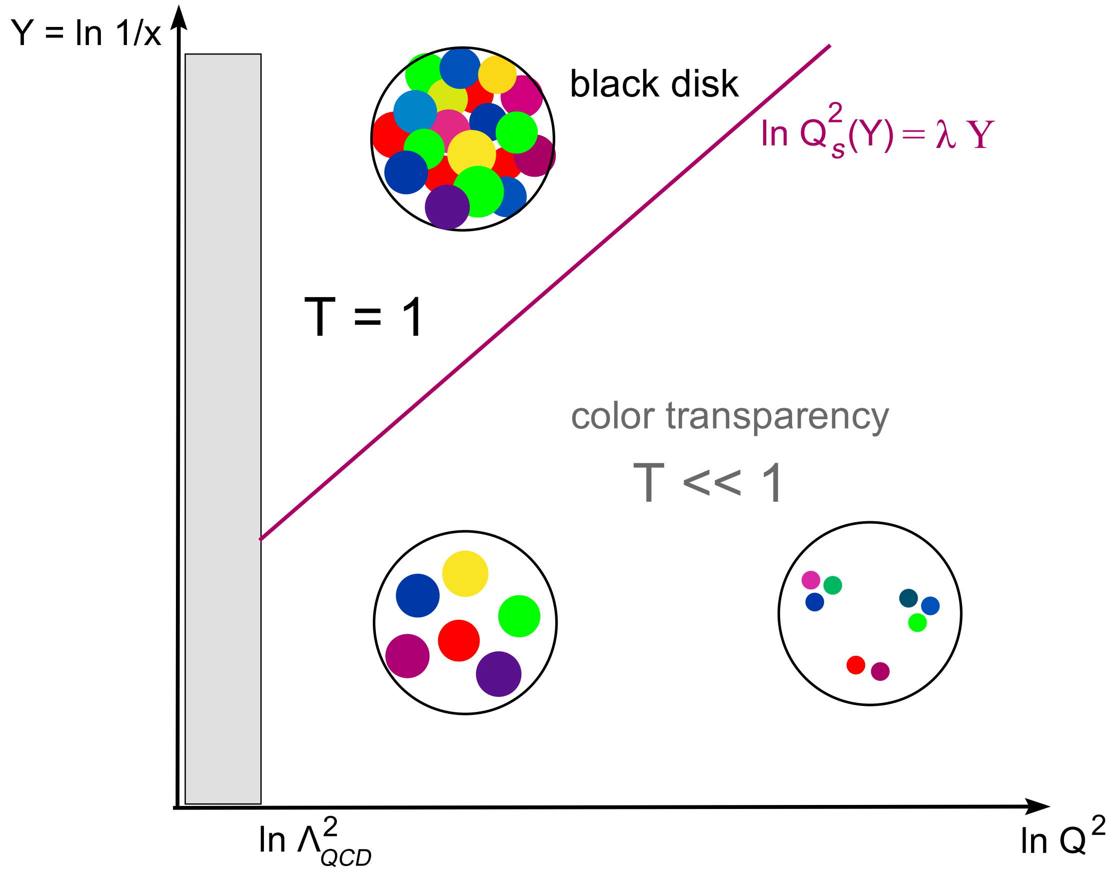
$$T(x, r, b) \propto \alpha_s^2 \frac{r^2}{R^2} \frac{1}{x^\lambda}$$

- $T(x, r, b) \to 0$ as $r \to 0$: ‘color transparency’
- When decreasing x and/or increasing r : $T(x, r) \sim \mathcal{O}(1)$

Unitarity = Saturation

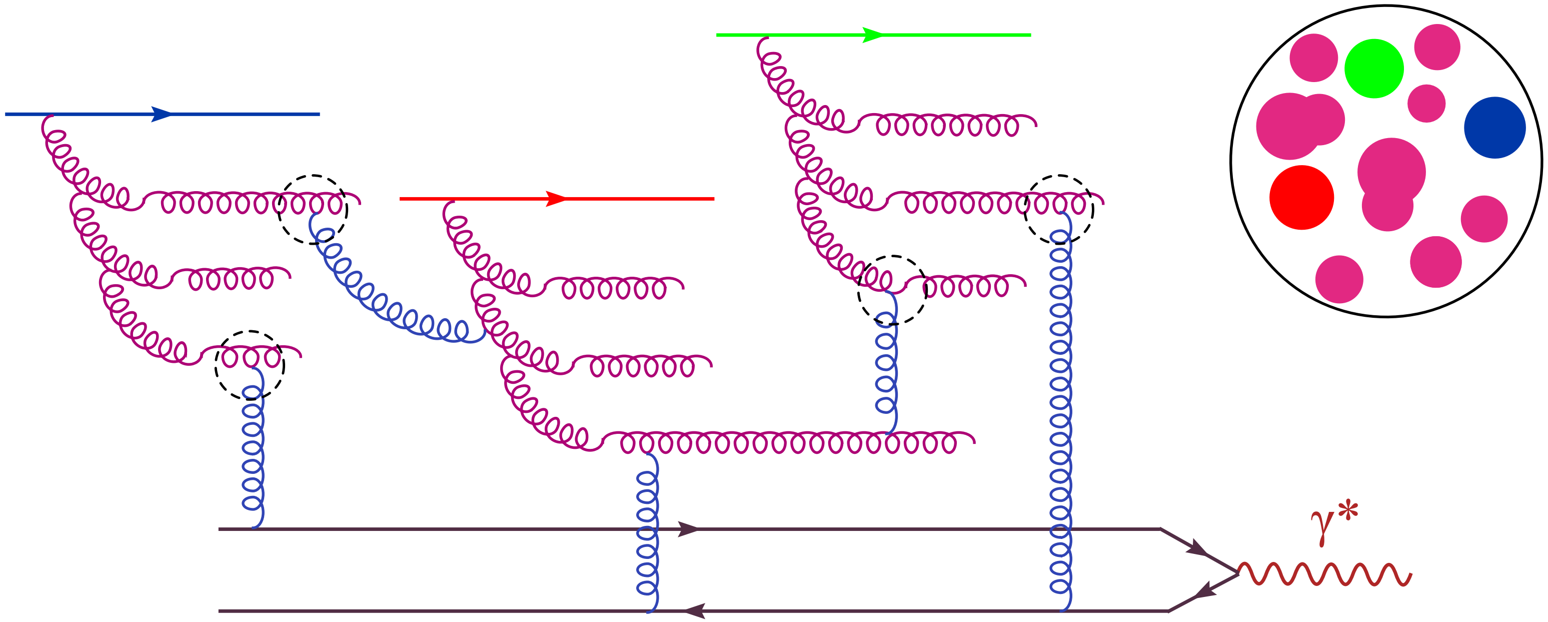
- Onset of ‘unitarity corrections’: $T(x,r) \sim 1 \Longleftrightarrow r \sim 1/Q_s(x)$

$$Q_s^2(x) \simeq \frac{\alpha_s}{N_c} \frac{xG(x,Q_s^2)}{\pi R^2} \sim e^{\lambda Y}$$



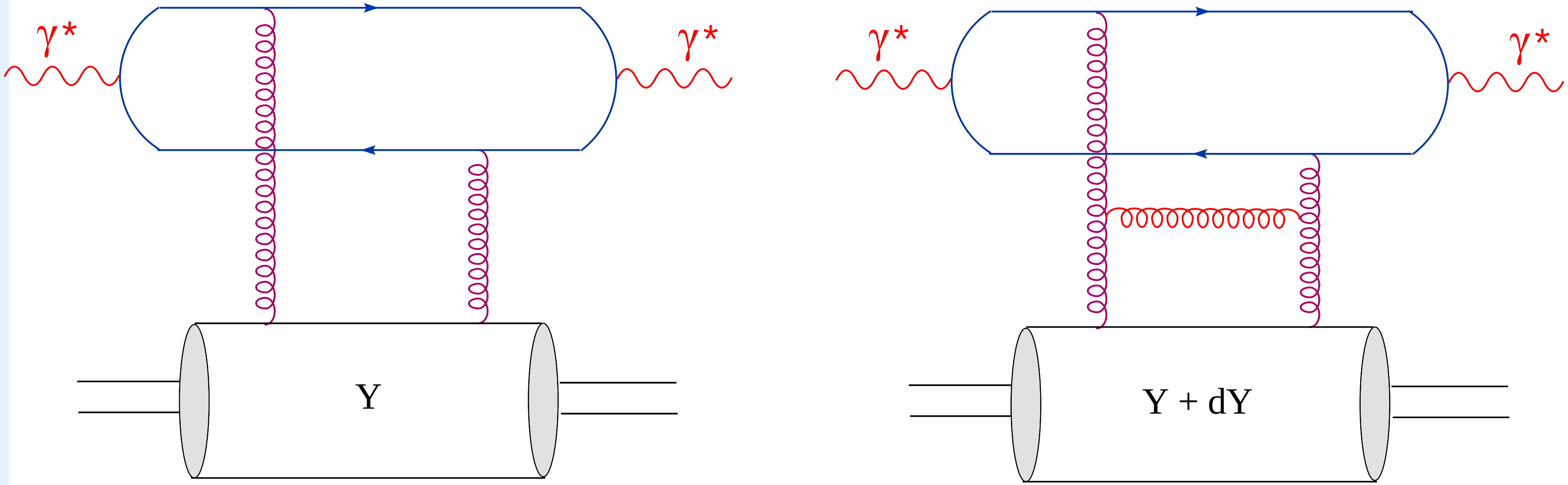
Unitarity corrections

- Multiple scattering & Gluon saturation



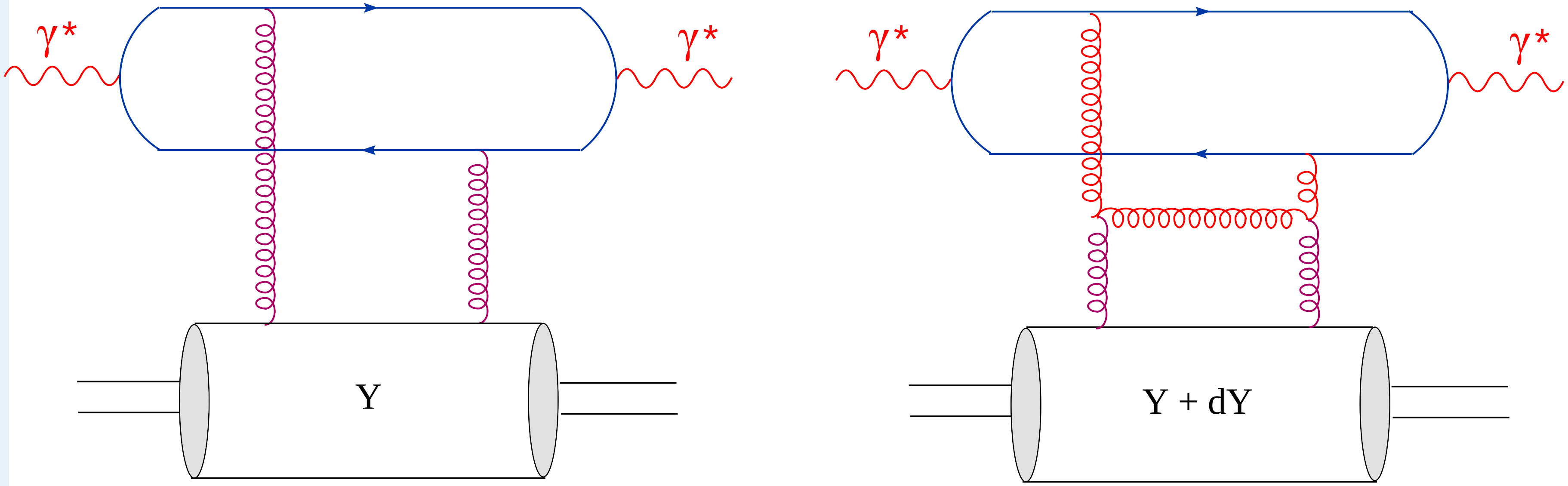
- Unitarity : a property of scattering (frame–independent)
- Saturation : a property of the wavefunction, manifest in the target infinite momentum frame (and also in the dipole frame)
- One can study saturation by following dipole evolution

One–step (BFKL) evolution in DIS

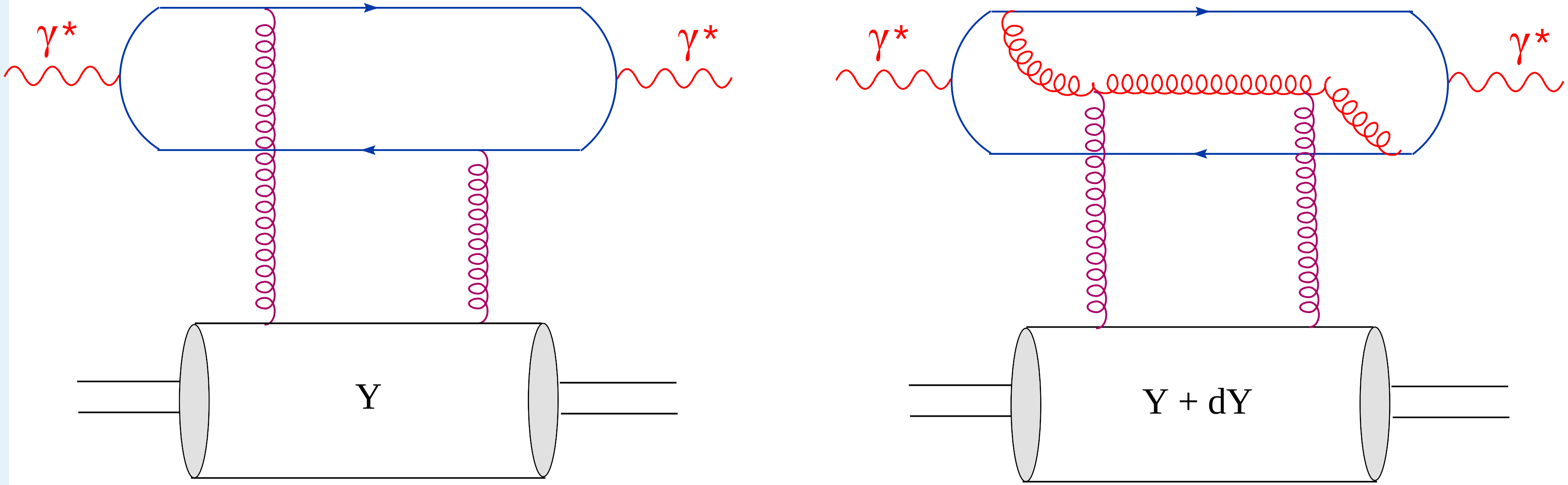


- Increase the proton rapidity by $\mathrm{d}Y$
- A probability of $\mathcal{O}(\alpha_s \mathrm{d}Y)$ to emit an extra gluon, which is softer $(Y < y < Y + \mathrm{d}Y)$

One–step (BFKL) evolution in DIS

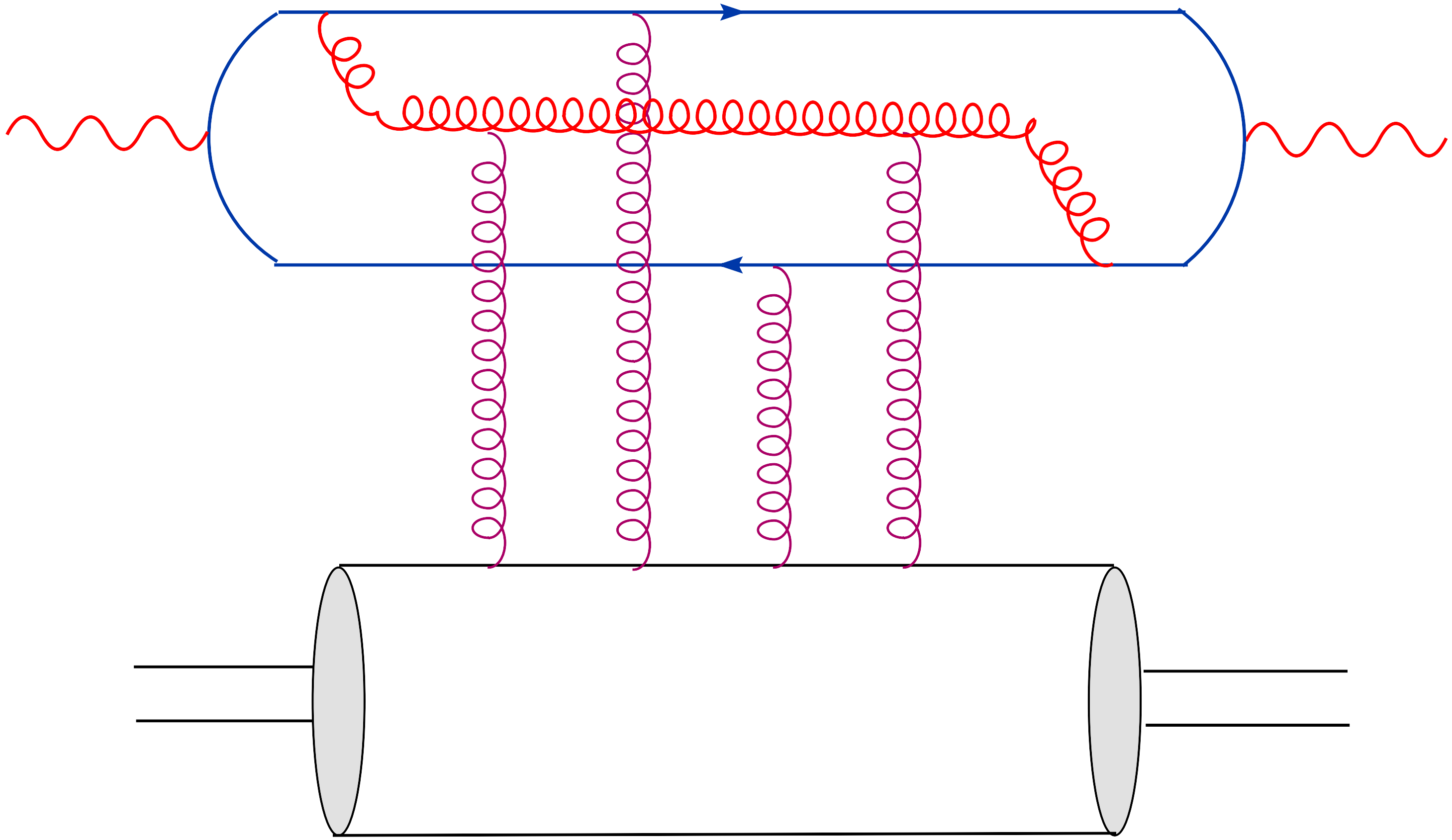


- One can instead give the rapidity increment $\mathrm{d}Y$ to the dipole (merely a change of frame)
- An extra gluon in the dipole wavefunction



- One can instead give the rapidity increment $\mathrm{d}Y$ to the dipole (merely a change of frame)
- $Y + \mathrm{d}Y$: A quark–antiquark–gluon system which scatters off the target

One–step (BFKL) evolution in DIS



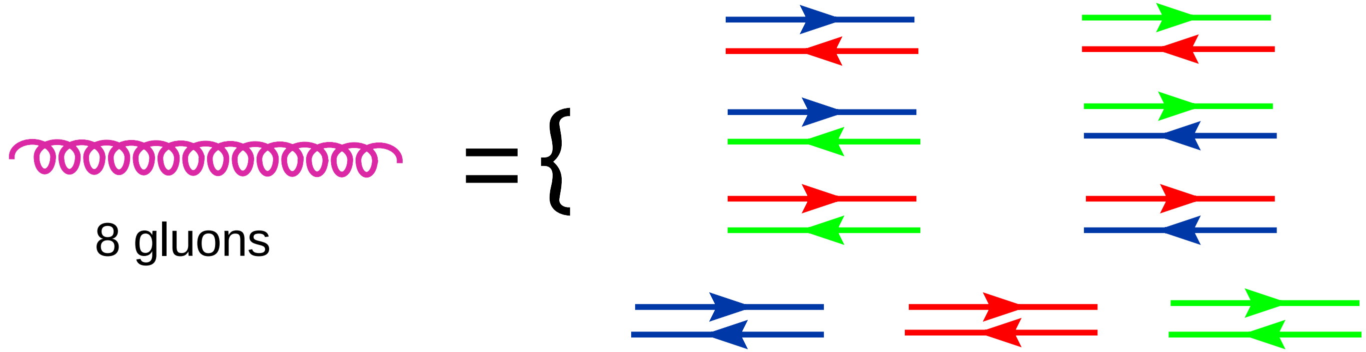
- Not restricted to single scattering !
- Evolution is simple, but scattering can be complicated.

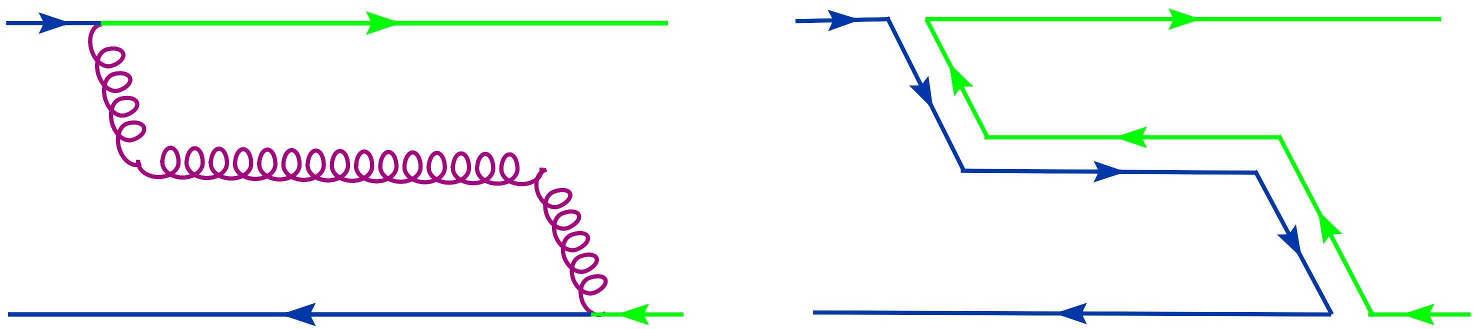
Dipole splitting (large N_c)

A brief reminder
Dipole picture for DIS
BK equation
- Target evolution
- Dipole evolution
- Balitsky equation
- Gluon evolution
- BK equation
Saturation line

- A quark–antiquark pair = a ‘gluon’ + a color singlet

$$3 \otimes \bar{3} = 8 \oplus 1\,, \qquad N_c \otimes \bar{N}_c = (N_c^2 - 1) \oplus 1$$





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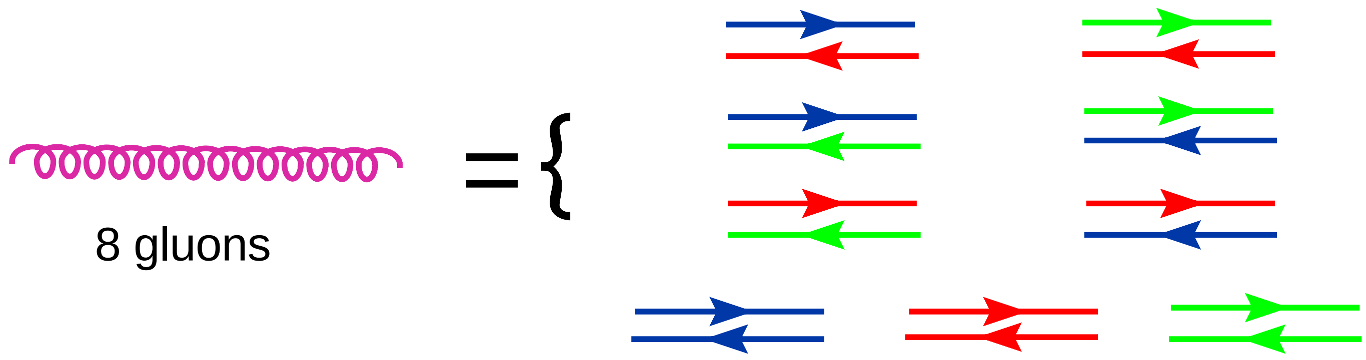
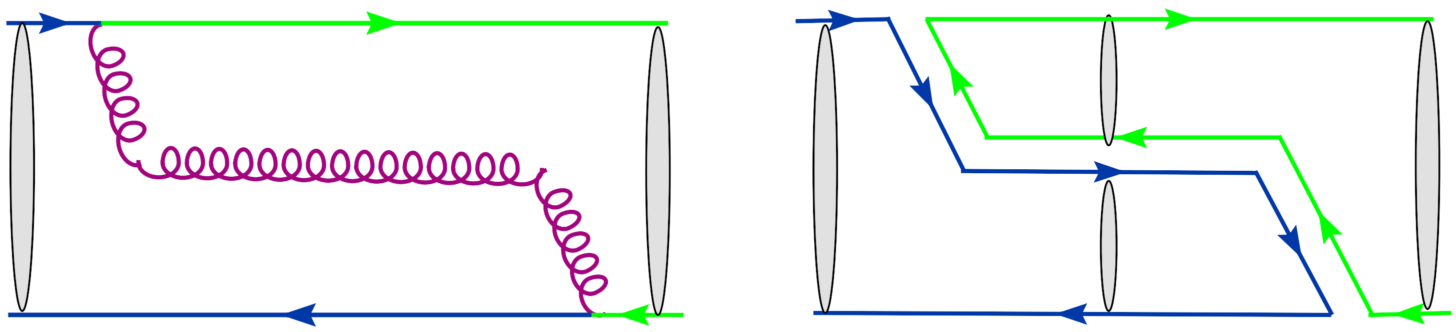
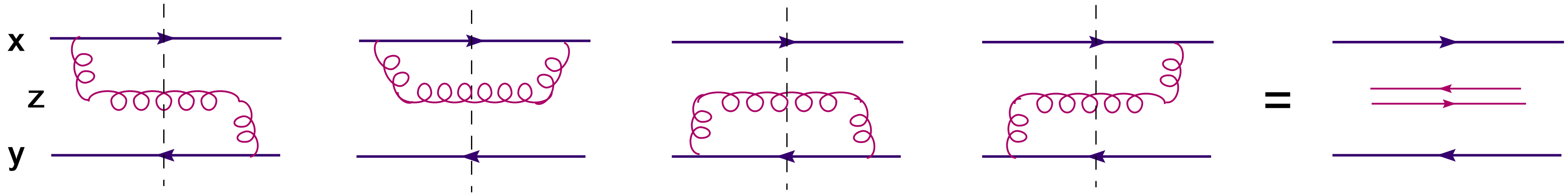




Dipole kernel

- Dipole splitting at large N_c : four diagrams ...



... most conveniently evaluated in coordinate space ($x_\perp$).

(Eikonal approx: soft gluons are emitted without recoil.)

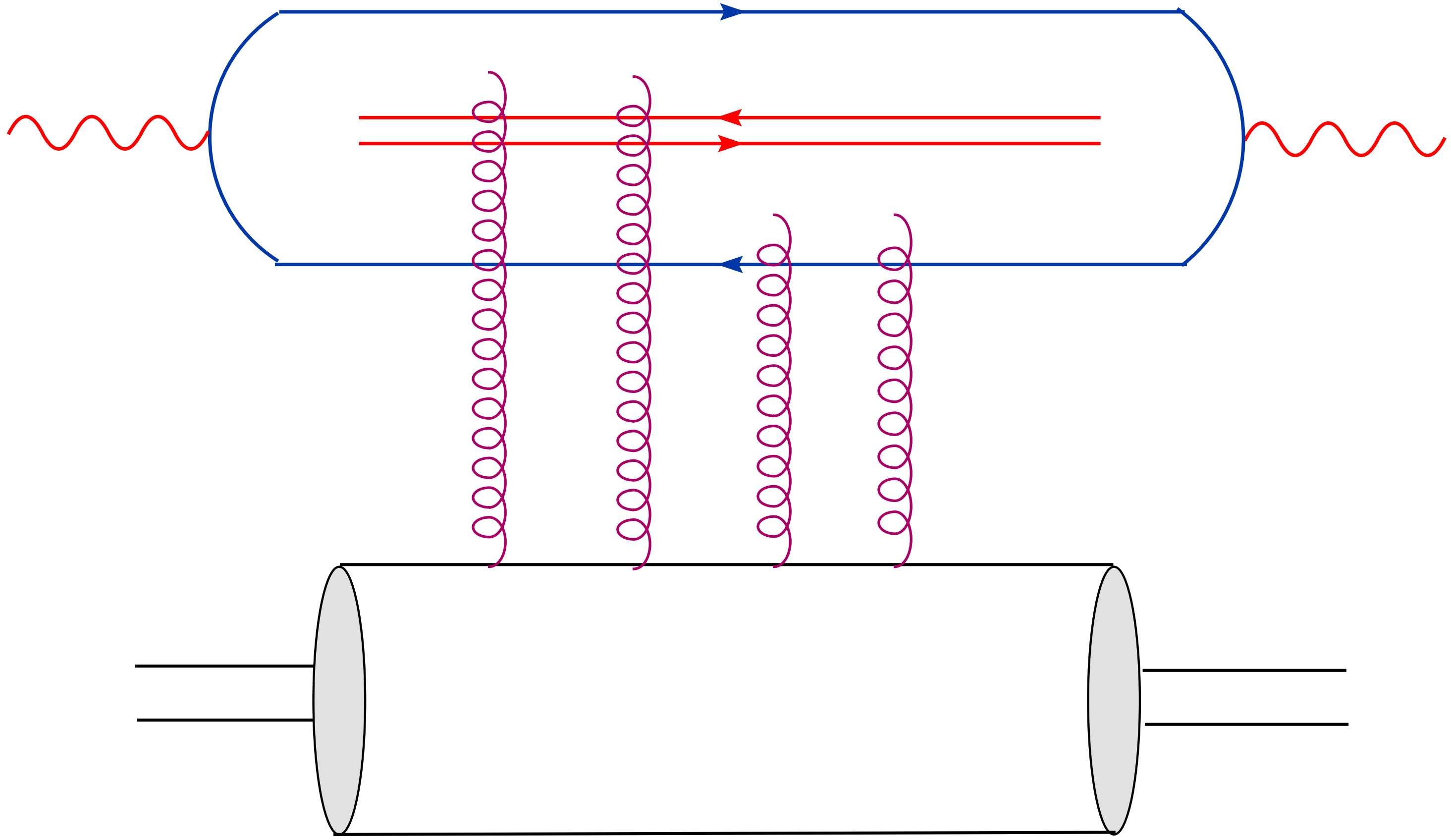
- One dipole $(\boldsymbol{x}, \boldsymbol{y})$ splits into two dipoles $(\boldsymbol{x}, \boldsymbol{z})$ and $(\boldsymbol{z}, \boldsymbol{y})$

(“dipole picture of BFKL evolution”, Al Mueller, 94)

- Differential probability (or ‘dipole kernel’)

$$\mathrm{d}P_{\rm split} = \frac{\bar{\alpha}_s}{2\pi} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, \mathrm{d}^2\boldsymbol{z}\, \mathrm{d}Y, \quad \text{where} \quad \bar{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}$$

Dipole evolution in DIS



▷ The original dipole splits into two new dipoles which then can scatter off the gluon fields in the target.

The first Balitsky equation

- $\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y$: S–matrix element for the scattering between the dipole $(\boldsymbol{x},\boldsymbol{y})$ and the target for a rapidity separation Y
  - $|\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y|^2 =$ the dipole survival probability
- $\langle S(\boldsymbol{x}_1,\boldsymbol{y}_1)S(\boldsymbol{x}_2,\boldsymbol{y}_2)\rangle_Y$: the S–matrix for a projectile made with 2 dipoles: $(\boldsymbol{x}_1,\boldsymbol{y}_1)$ and $(\boldsymbol{x}_2,\boldsymbol{y}_2)$
- One evolution step : $Y \to Y+\mathrm{d}Y$ with $\alpha_s \mathrm{d}Y \ll 1$

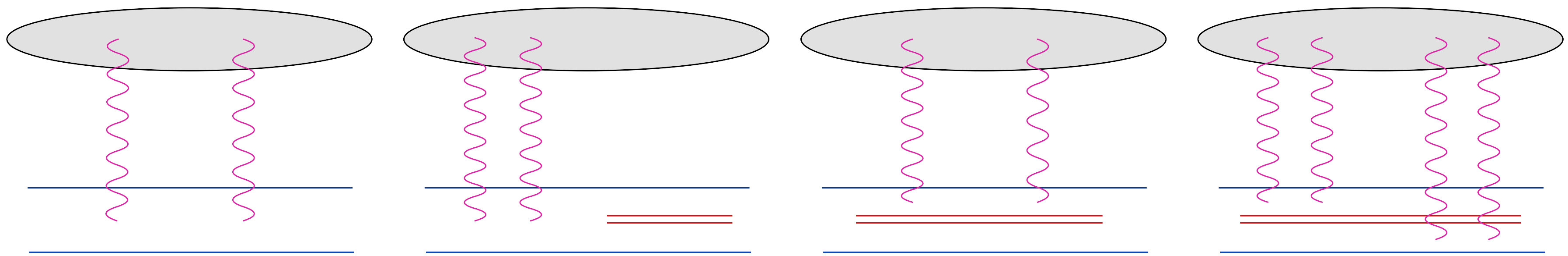
$$
\begin{aligned}
\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_{Y+\mathrm{d}Y} &= \mathrm{d}Y \int \mathrm{d}^2\boldsymbol{z}\, \frac{\mathrm{d}P_{\mathrm{split}}}{\mathrm{d}Y\,\mathrm{d}^2\boldsymbol{z}}\, \langle S(\boldsymbol{x},\boldsymbol{z})S(\boldsymbol{z},\boldsymbol{y})\rangle_Y \\
&+ \left(1-\mathrm{d}Y\int \mathrm{d}^2\boldsymbol{z}\,\frac{\mathrm{d}P_{\mathrm{split}}}{\mathrm{d}Y\,\mathrm{d}^2\boldsymbol{z}}\right)\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y
\end{aligned}
$$

The first Balitsky equation

$$\frac{\partial}{\partial Y}\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\Big\{-\langle S(\boldsymbol{x},\boldsymbol{y})\rangle_Y + \langle S(\boldsymbol{x},\boldsymbol{z})S(\boldsymbol{z},\boldsymbol{y})\rangle_Y\Big\}$$

- Not a closed equation ! (one dipole $\longrightarrow$ two dipoles)
  First equation from the infinite Balitsky hierarchy
- Rewritten for the dipole scattering amplitude $T \equiv 1 - S$:

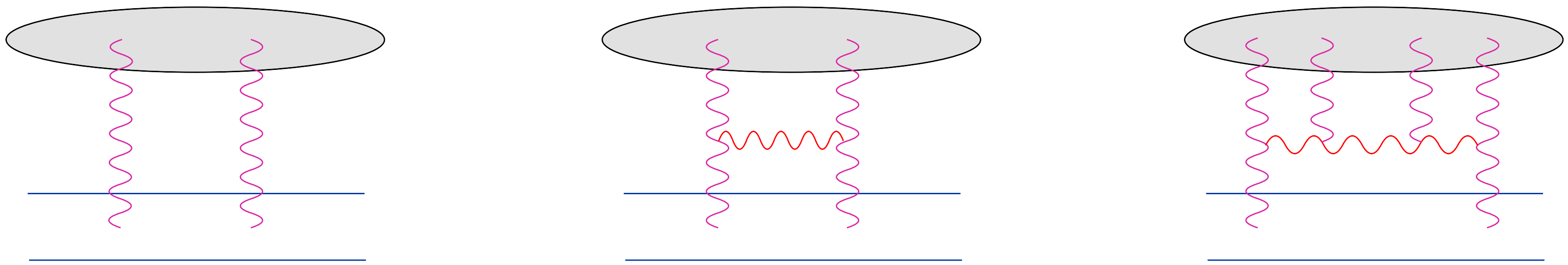
$$\frac{\partial}{\partial Y}\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\langle \underbrace{-T(\boldsymbol{x},\boldsymbol{y}) + T(\boldsymbol{x},\boldsymbol{z}) + T(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})}_{\text{non–linear}}\Big\rangle_Y$$



Non–linear gluon evolution

- Alternatively: Use the rapidity increment $\mathrm{d}Y$ to accelerate the target hadron $\Longrightarrow$ the same equation ...

$$\frac{\partial}{\partial Y}\langle T(\boldsymbol{x},\boldsymbol{y})\rangle_Y = \frac{\bar{\alpha}_s}{2\pi}\int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big\langle \underbrace{-T(\boldsymbol{x},\boldsymbol{y})+T(\boldsymbol{x},\boldsymbol{z})+T(\boldsymbol{z},\boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})}_{\text{non–linear}} \Big\rangle_Y$$



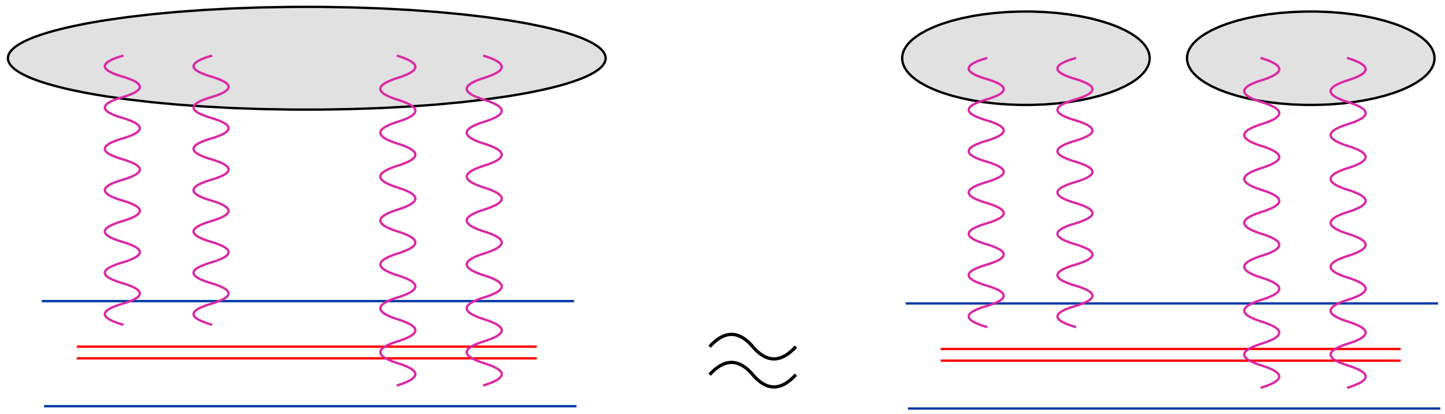
- ... but with a different physical interpretation !

  Saturation effects in the target wavefunction

The Balitsky–Kovchegov equation

- Mean field approximation assuming factorization:

$$\langle T(\boldsymbol{x},\boldsymbol{z})T(\boldsymbol{z},\boldsymbol{y})\rangle_Y \approx \langle T(\boldsymbol{x},\boldsymbol{z})\rangle_Y \, \langle T(\boldsymbol{z},\boldsymbol{y})\rangle_Y$$



- Neglects correlations in the target wavefunction.
- Some usual justifications in the literature:
  - large nucleus $A \gg 1$
  - large N_c
  - leads to a relatively simple (closed) equation ✓

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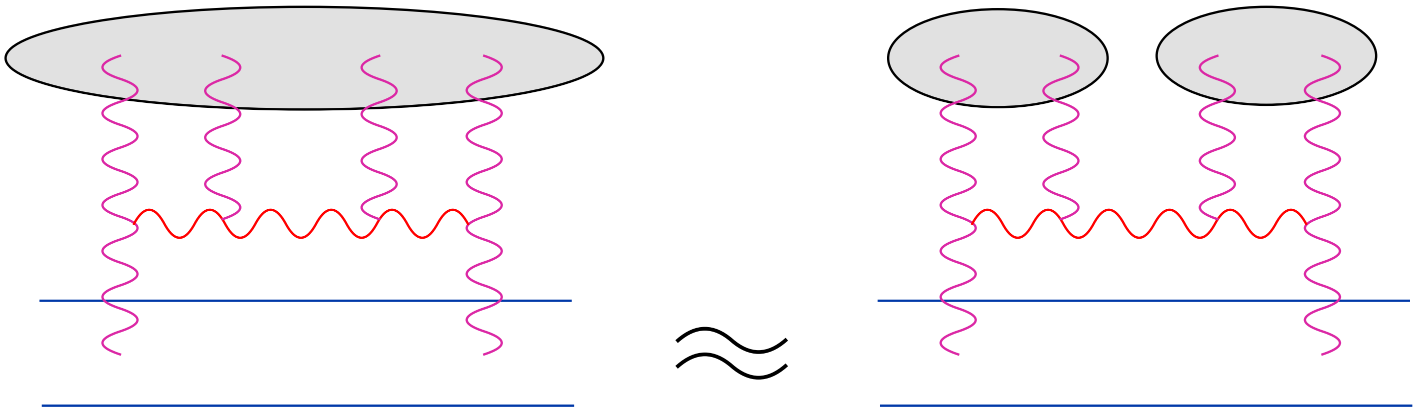

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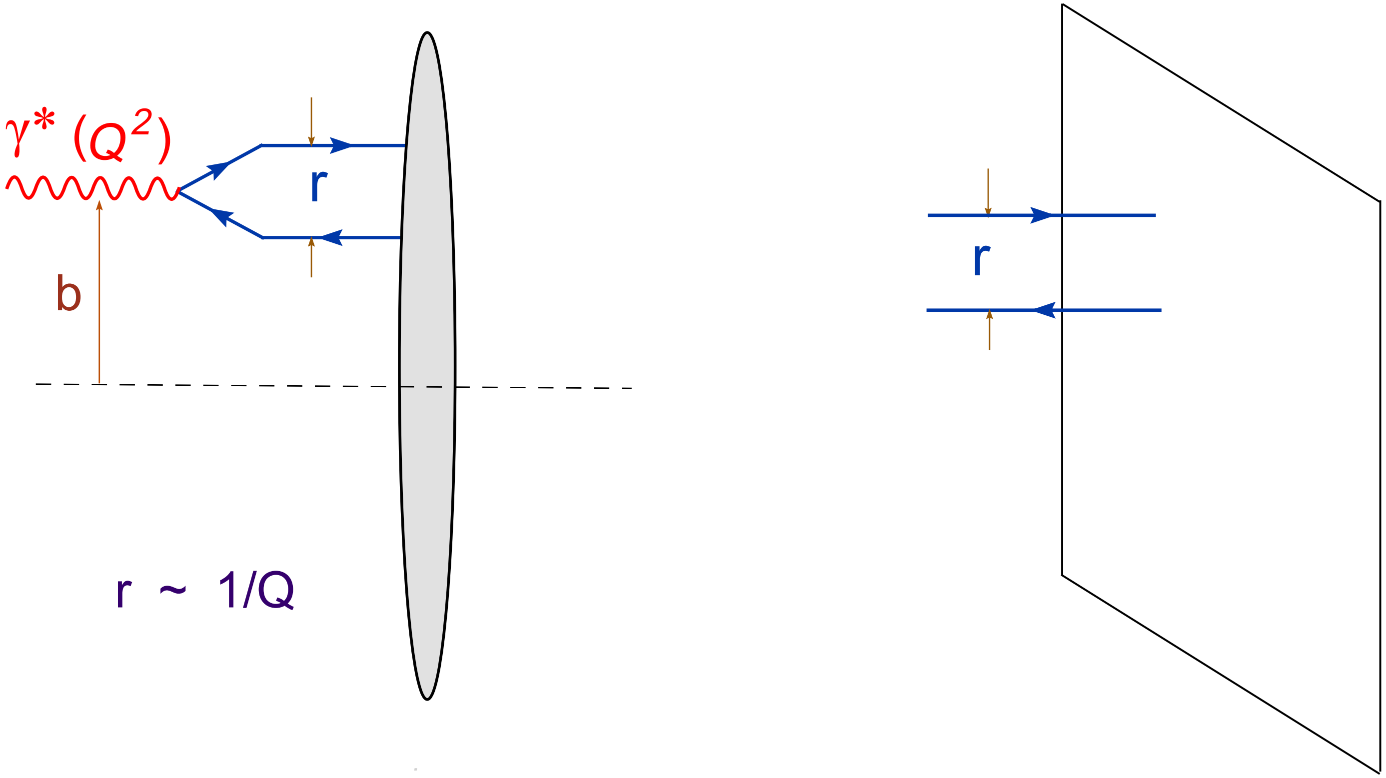
The Balitsky–Kovchegov equation

$$\frac{\partial}{\partial Y} T(\boldsymbol{x}, \boldsymbol{y}) = \frac{\bar{\alpha}_s}{2\pi} \int_{\boldsymbol{z}} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2 (\boldsymbol{y}-\boldsymbol{z})^2} \Big\{ \underbrace{-T(\boldsymbol{x}, \boldsymbol{y}) + T(\boldsymbol{x}, \boldsymbol{z}) + T(\boldsymbol{z}, \boldsymbol{y})}_{\text{BFKL (linear)}} - \underbrace{T(\boldsymbol{x}, \boldsymbol{z}) T(\boldsymbol{z}, \boldsymbol{y})}_{\text{non–linear}} \Big\}$$

- Notations: $\bar{\alpha}_s \equiv \alpha_s N_c / \pi, \; T(\boldsymbol{x}, \boldsymbol{y}) \equiv \langle T(\boldsymbol{x}, \boldsymbol{y}) \rangle_Y$
- Weak scattering $T \ll 1$ (low energy/small dipole) $\Longrightarrow$ BFKL
  - $T = 0$: unstable fixed point of BK equation
  - unitarity violations, infrared diffusion
- Strong scattering $T \sim 1 \Longrightarrow$ Non–linear effects
  - $T = 1$: stable fixed point of BK equation
- BK equation: a simple framework to study unitarization

BK solution : target = ‘large nucleus’



- $T(\boldsymbol{x},\boldsymbol{y}|Y) = T(r,Y)$ with $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ (dipole size)

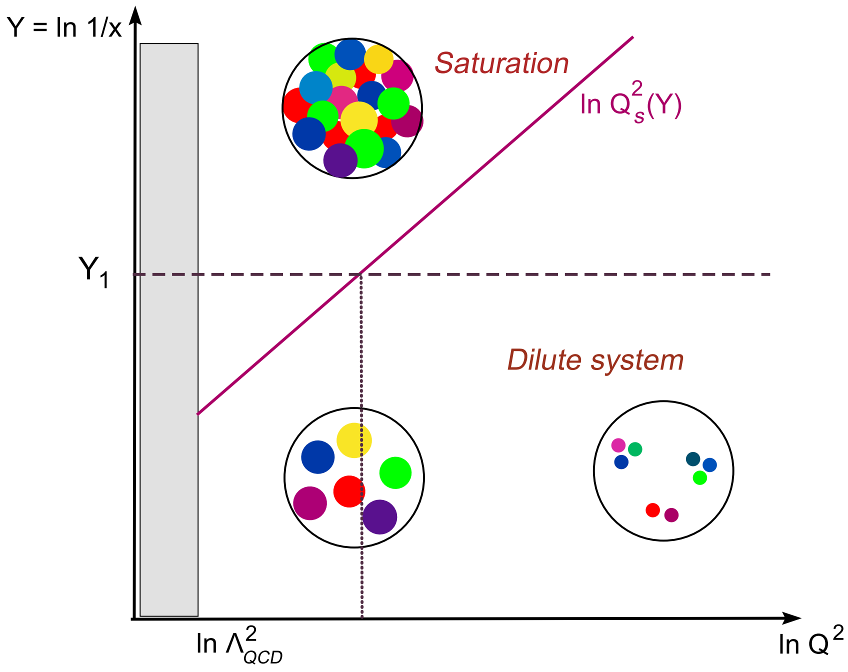
$$T(r, Y=0) \approx \begin{cases} r^2 Q_0^2 & \text{for } r \ll 1/Q_0 \\ 1 & \text{for } r \gtrsim 1/Q_0 \end{cases}$$

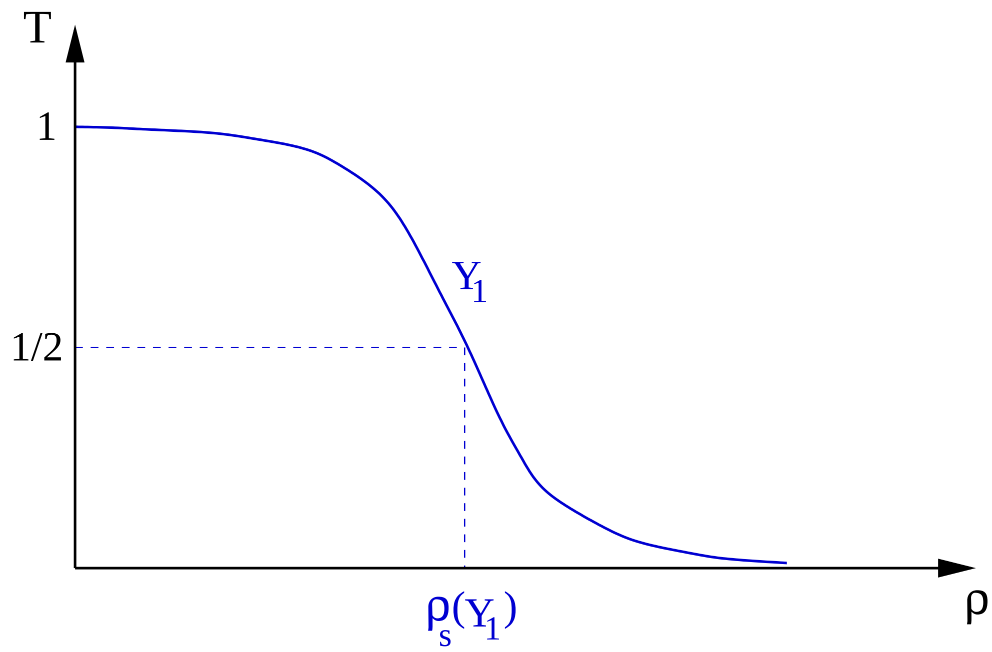
- Q_0 : saturation momentum in the initial conditions

BK solution : the saturation front

- $T(r,Y) \equiv T(\rho,Y)$ with $\rho \equiv \ln(1/Q_0^2 r^2) \equiv \ln(Q^2/Q_0^2)$

  a front interpolating between $T=0$ and $T=1$

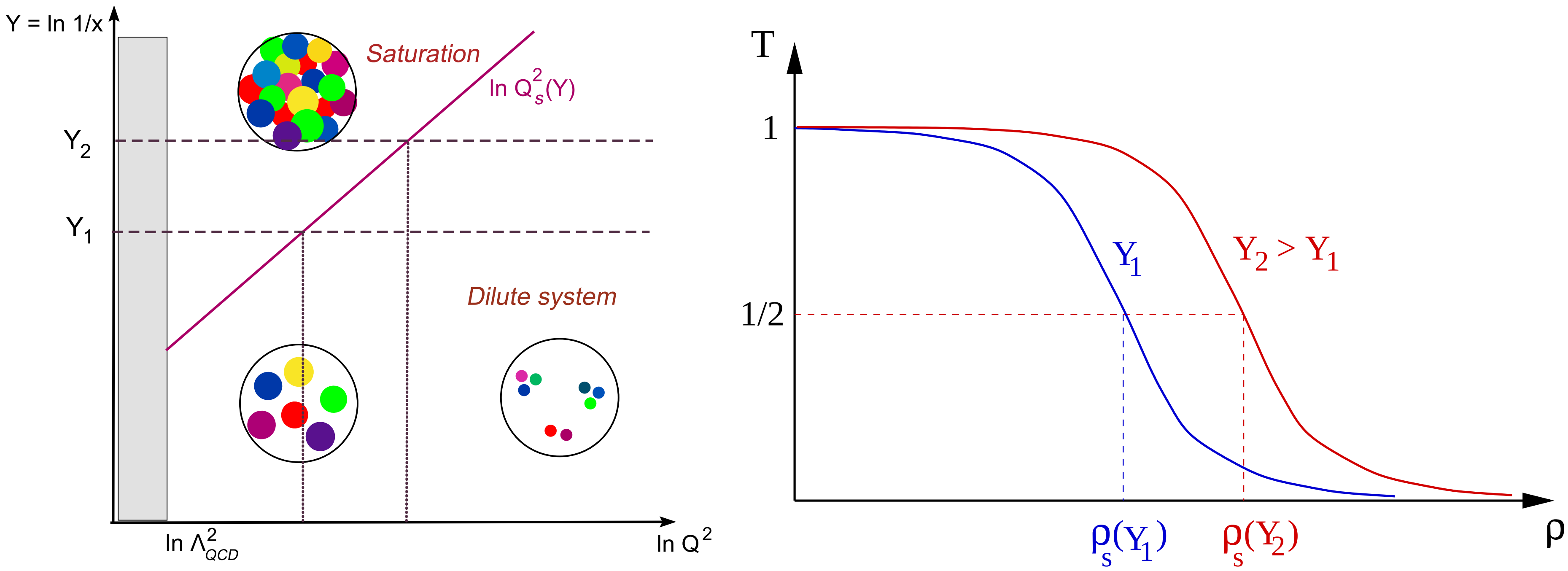




- $T=1$ for $\rho \lesssim \rho_s(Y)$ and $T \propto e^{-\rho} = r^2$ for $\rho \gg \rho_s(Y)$
- The front position: the saturation scale $\rho_s \equiv \ln(Q_s^2/Q_0^2)$

BK solution : the saturation front

- Increase Y : the front propagates towards larger values of ρ

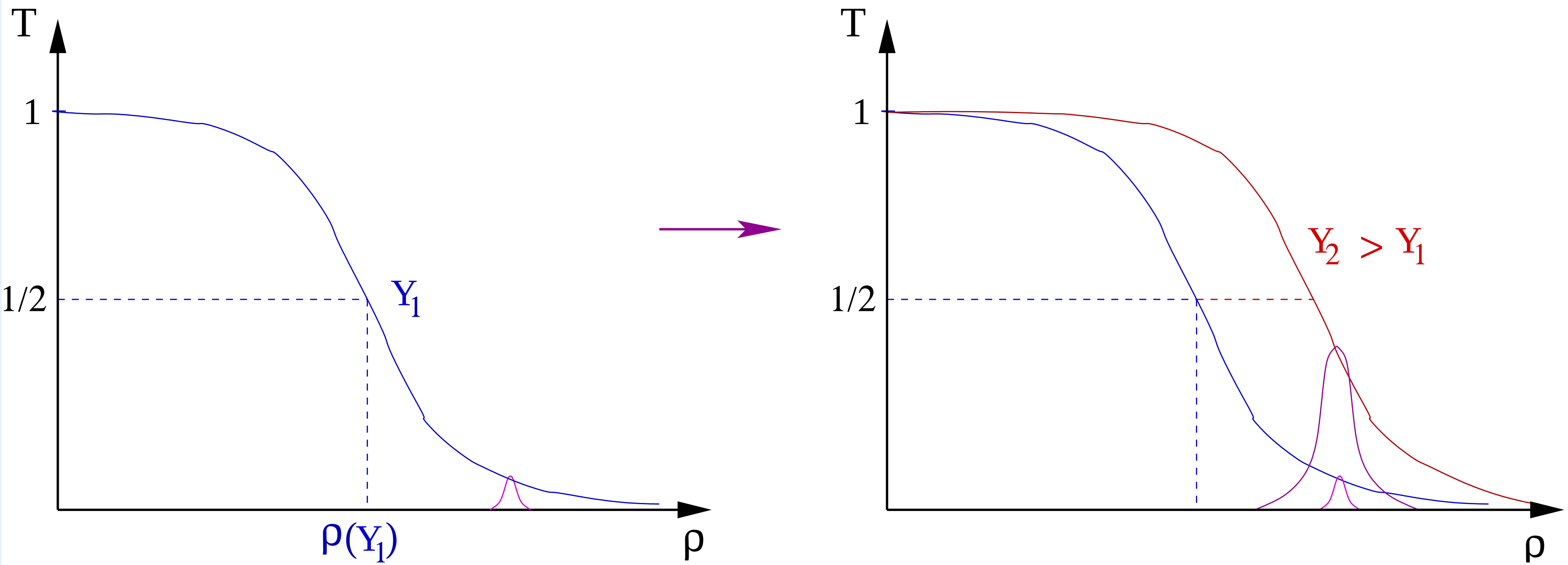


- Main questions :

  What are the front position (ρ_s) and its shape around ρ_s ?

- A priori, a very complicated problem: non–linear equation

BK solution : pulled front



- The front is pulled by the rapid growth in the tail of the amplitude at large $\rho \gg \rho_s(Y)$, where $T \ll 1$
- The evolution is controlled by the linear (BFKL) equation with saturation boundary condition: $T(\rho_s(Y), Y) \sim 1$

E.I., Itakura, McLerran (02); Mueller, Triantafyllopoulos (02); Munier, Peschanski (03) $\longrightarrow$ Fischer–Kolmogorov equation

BK solution : FKPP equation

- Gradient expansion of the non–locality $\Longrightarrow$ FKPP equation:

$$\partial_Y T(\rho,Y) = \underbrace{\partial_\rho^2 T(\rho,Y)}_{\text{diffusion}} + \underbrace{T(\rho,Y)}_{\text{growth}} \underbrace{- T^2(\rho,Y)}_{\text{saturation}}$$

- Linearized equation ($T \ll 1$) : diffusion + exponential growth
- Boundary condition : $T(\rho,Y) \sim 1$ when $\rho \sim \rho_s(Y)$

$$T(\rho,Y) \simeq \mathrm{e}^{-\gamma_s(\rho-\rho_s)} \exp\left\{-\frac{(\rho-\rho_s)^2}{2D\bar{\alpha}_s Y}\right\}$$

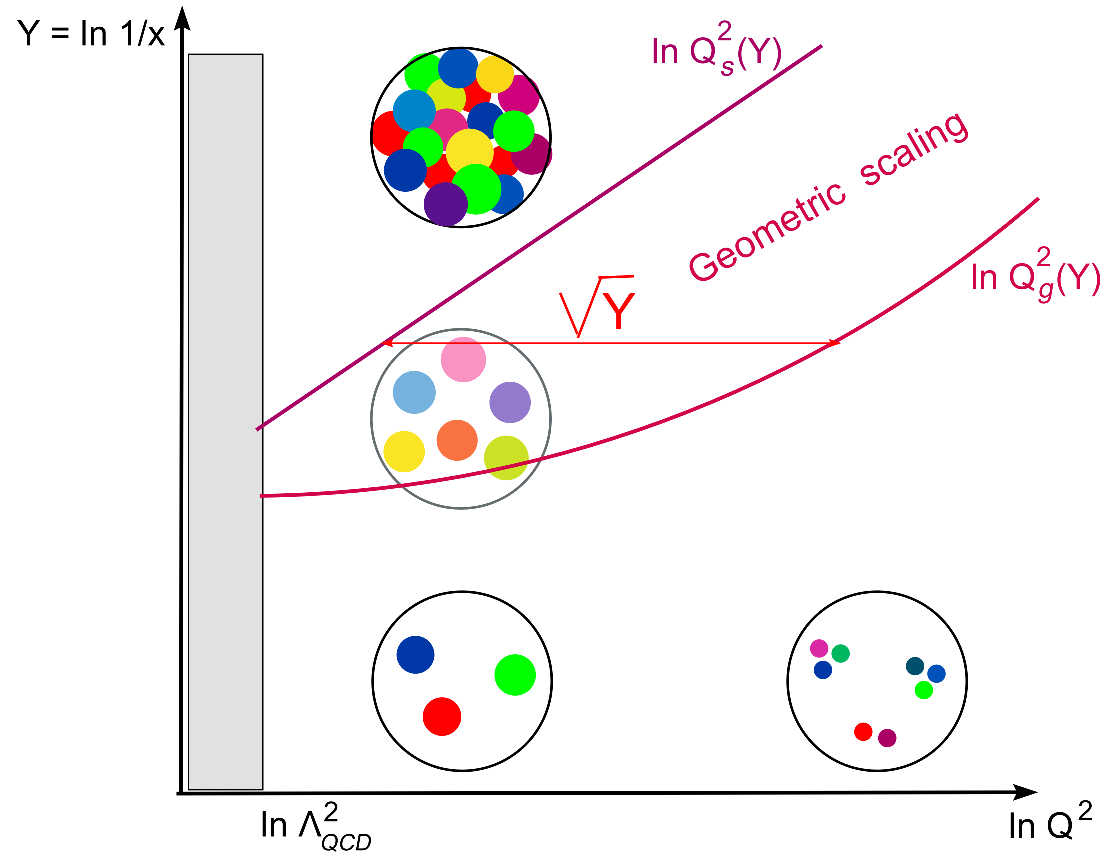
with the anomalous dimension $\gamma_s = 0.63...$

and the saturation scale $\rho_s(Y) = \lambda_s \bar{\alpha}_s Y$ with $\lambda_s = 4.88...$

- NB: $\lambda_s \bar{\alpha}_s = \mathcal{O}(1)$, whereas HERA data require $\lambda \sim 0.2 \div 0.3$
- NLO BFKL + saturation $\Longrightarrow$ $\lambda \approx 0.3$ *(Triantafyllopoulos, 02)*

BK solution : Geometric scaling

- Diffusion is negligible for $\rho - \rho_s \ll \sqrt{2D\bar{\alpha}_s Y}$



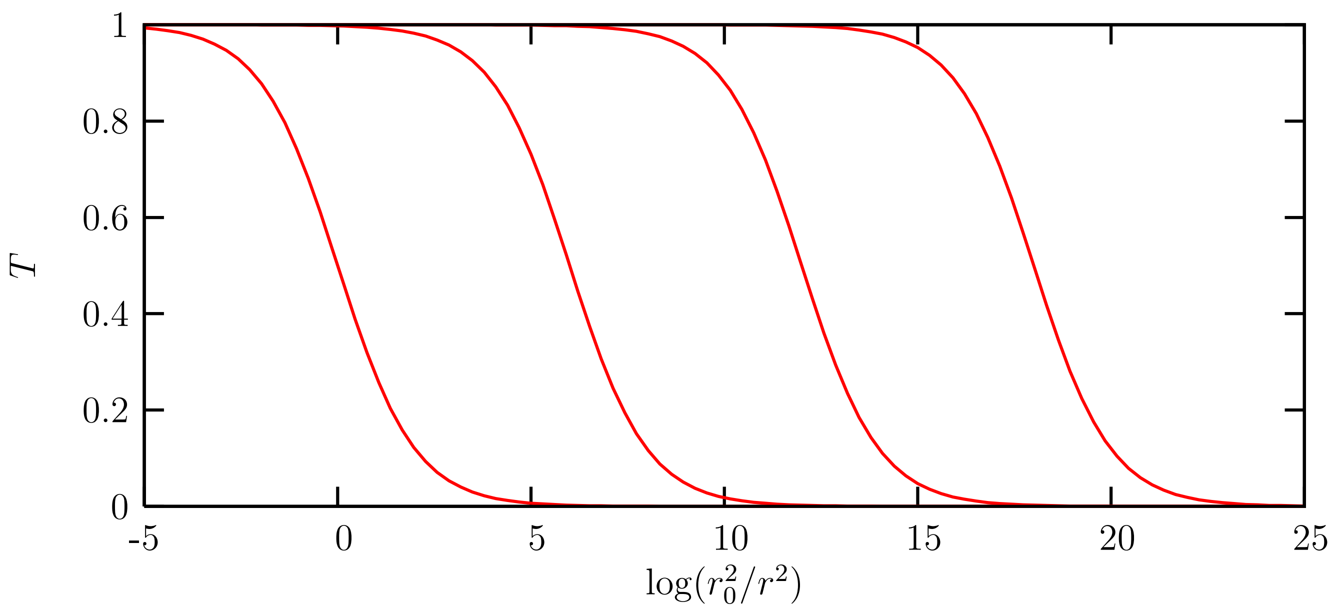
$$T(\rho, Y) \simeq e^{-\gamma_s(\rho-\rho_s)} \iff T(r, Y) \simeq (r^2 Q_s^2)^{\gamma_s}$$

- Saturation makes itself felt in the dilute regime $(Q^2 > Q_s^2)$

The traveling wave

- The amplitude 'scales' as a function of the variable $r^2 Q_s^2(Y)$

$$T(\rho, Y) \simeq T(\rho - \rho_s(Y)) = T(r^2 Q_s^2(Y))$$



- The shape of the front is not altered by the evolution
- The front propagates like a 'traveling wave'
  (Munier, Peschanski,03)

Geometric Scaling at HERA

(Staśto, Golec-Biernat and Kwieciński, 2000)

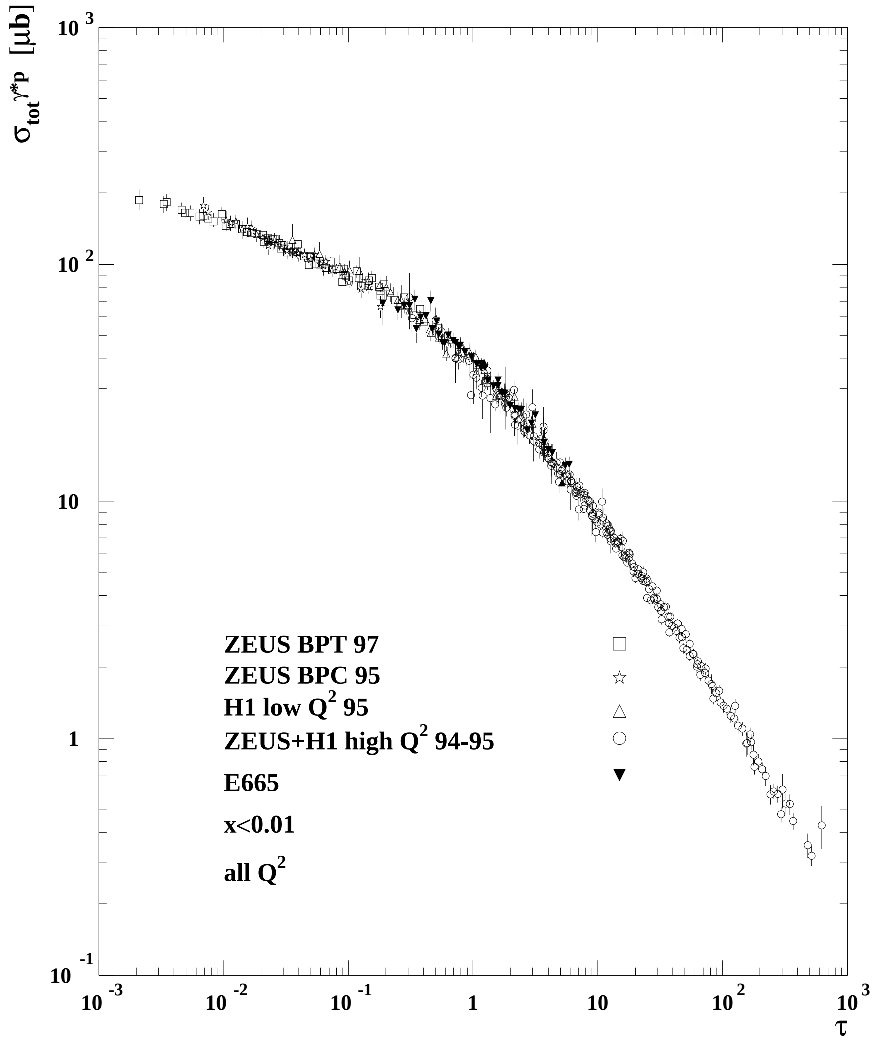
$$\sigma(x, Q^2) \approx \sigma(\tau) \quad \text{with} \quad \tau \equiv Q^2/Q_s^2(x), \quad Q_s^2(x) = (x_0/x)^\lambda \, \text{GeV}^2 \,, \quad \lambda \simeq 0.3$$

$x \leq 0.01$

$Q^2 \leq 450$ GeV^2

$Q_s^2 \sim 1$ GeV^2

for $x \sim 10^{-4}$



Geometric Scaling at HERA (2)

(Marquet and Schoeffel 2006)