Gluon saturation from DIS to AA collisions

I – Gluon saturation in DIS

François Gelis

CERN and CEA/Saclay

General outline

- Lecture I : Gluon saturation in DIS
- Lecture II : Proton-nucleus collisions
- Lecture III : AA collisions : gluon production
- Lecture IV : AA collisions : glasma instabilities

Lecture I : Gluon saturation in DIS

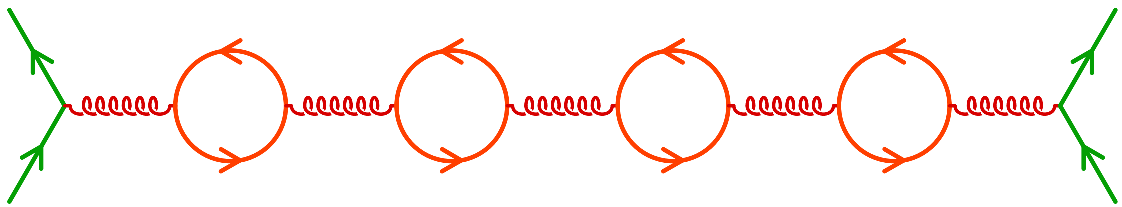
- QCD and Factorization
- Color Glass Condensate
- Eikonal scattering
- Solution of YM equations
- DIS cross-section
- Fits of DIS data

QCD and factorization

Asymptotic freedom

- Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(1/r\Lambda_{QCD})}$$

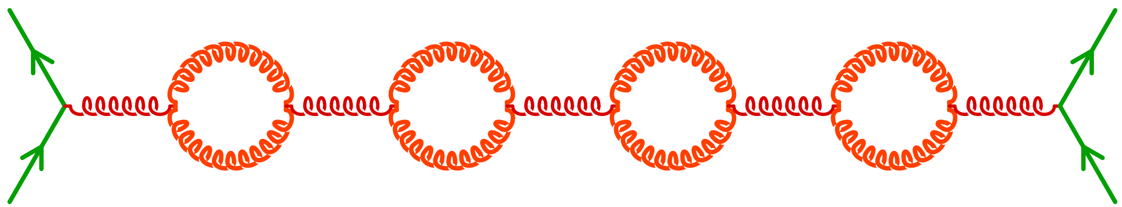


- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)

Asymptotic freedom

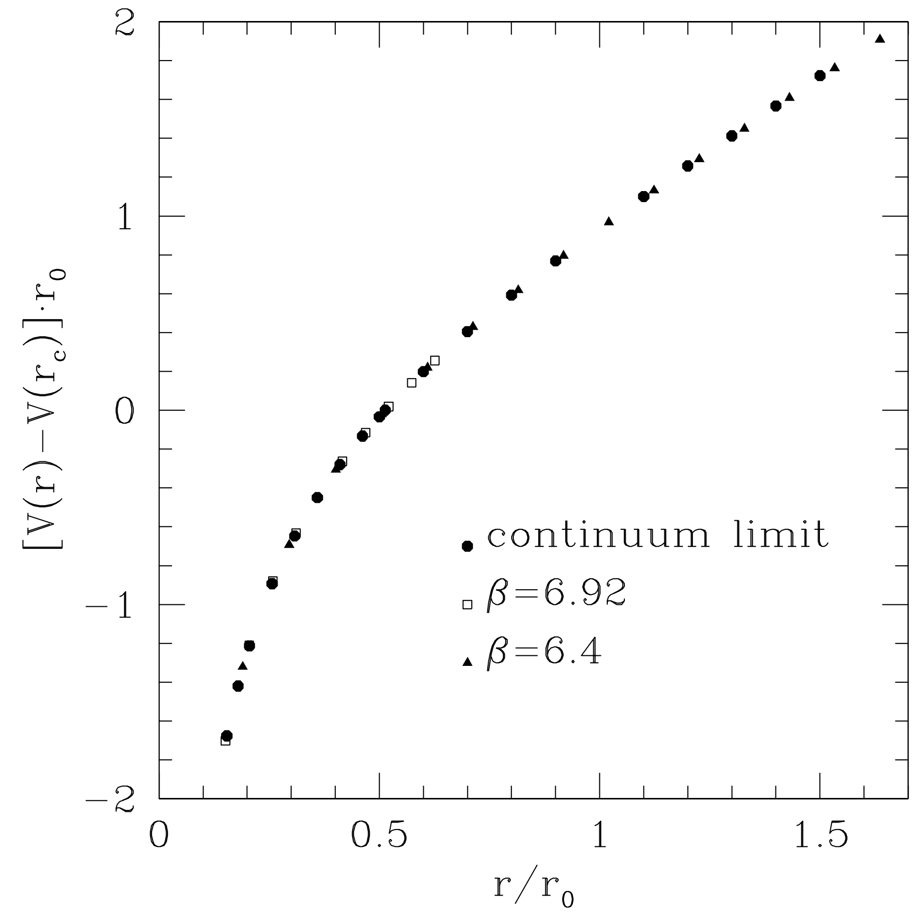
- Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(1/r\Lambda_{QCD})}$$



- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)
- But gluonic vacuum fluctuations produce an anti-screening (because of the non-abelian nature of their interactions)
- As long as $N_f < 11N_c/2 = 16.5$, the gluons win...

Quark confinement



- The quark potential increases linearly with distance
- Color singlet hadrons : no free quarks and gluons in nature

How to test QCD?

- QCD is the fundamental theory of strong interactions among quarks and gluons
- Experiments involve hadrons in their initial and final states, not quarks and gluons
- Hadrons cannot be described perturbatively in QCD
- Scattering amplitudes with time-like on-shell momenta cannot be computed on the lattice

▷ How can we compare theory and experiments?

▷ **Factorization :** separation of short distances (perturbative) and long distance (non perturbative)

Factorization

- At a superficial level, factorization means that :

$$\mathcal{O}_{\text{hadrons}} = F \otimes \mathcal{O}_{\text{partons}}$$

  - F = parton distribution
  - $\mathcal{O}_{\text{partons}}$ = observable at the partonic level (calculable in perturbation theory)

- For this to be useful, F must be universal (i.e. independent of the observable $\mathcal{O}$)

- In order to test QCD experimentally, measure as many observables as possible, and try to find common F's that fit all the data
Note : at this stage, by looking at only one observable, it is impossible to perform any meaningful test, since it is always possible to adjust F so that it works

Factorization

- Some loop corrections in $\mathcal{O}_{\rm partons}$ are enhanced by large logarithms, e.g.

$$\alpha_s \ln\left(\frac{M^2}{m_H^2}\right) \quad , \qquad \alpha_s \ln\left(\frac{s}{M^2}\right) \sim \alpha_s \ln\left(\frac{1}{x}\right)$$

  Note : the log that occurs depends on the details of the kinematics
  - Bjorken limit: $s, M^2 \to +\infty$ with s/M^2 fixed
  - Regge limit: $s \to +\infty$, M^2 fixed

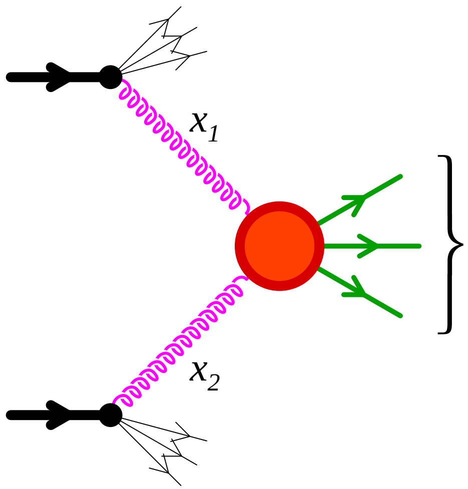
- These logs upset a naive application of perturbation theory when $\alpha_s \ln(\cdot) \sim 1$ $\triangleright$ they must be resummed

- This resummation can be performed analytically
  - the result of the resummation is universal
  - all the leading logs can be absorbed in F
    - $\triangleright$ the factorization formula remains true
    - $\triangleright$ this summation dictates how F evolves with M^2 or x

Factorization

- These logarithms tell us that the relevant parton distributions depend on the resolution scales (in time and in transverse momentum) associated to a given process

- Calculation of some process at LO :

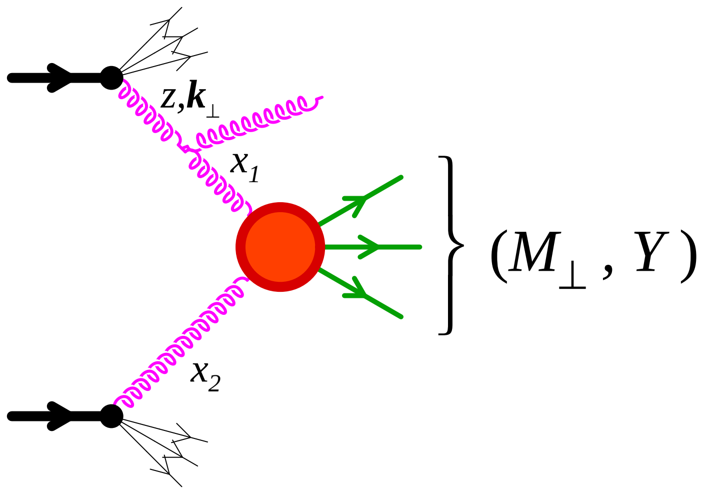


$$\left.\right\} (M_\perp\,,\,Y\,) \qquad \left\{ \begin{array}{l} x_1 = M_\perp \, e^{+Y} / \sqrt{s} \\ x_2 = M_\perp \, e^{-Y} / \sqrt{s} \end{array} \right.$$

Factorization

- These logarithms tell us that the relevant parton distributions depend on the resolution scales (in time and in transverse momentum) associated to a given process

- Radiation of an extra gluon :



$$\Longrightarrow \quad \alpha_s \int_{x_1} \frac{dz}{z} \int^{M_\perp} \frac{d^2\vec{\boldsymbol{k}}_\perp}{\boldsymbol{k}_\perp^2}$$

- Practical consequence : pQCD predicts not only $\mathcal{O}_{\rm partons}$ but also the evolution $\partial_{_M} F$ (or $\partial_x F$)
  - ▷ the only required non-perturbative input is $F(x, M_0)$ or $F(x_0, M)$

Collinear factorization

- Logs of $M_\perp \Longrightarrow$ DGLAP. Important when :
  - $M_\perp \gg \Lambda_{QCD}$, while x_1, x_2 are rather large
- Cross-sections read :

$$\frac{d\sigma}{dY\, d^2\vec{\boldsymbol{P}}_\perp} \propto F(x_1, M_\perp^2)\; F(x_2, M_\perp^2)\; |\mathcal{M}|^2$$

with $x_{1,2} = M_\perp \exp(\pm Y)/\sqrt{s}$

- Note : there are convolutions in x_1 and x_2 if some particles are integrated out in the final state
- The factorization of logarithms has been proven to all orders for sufficiently inclusive quantities (see Collins, Soper, Sterman, 1984–1985)

Kt-factorization

QCD and factorization
- Confinement
- How to test QCD?
- Factorization

Color Glass Condensate

Eikonal scattering

Solution of YM equations

DIS cross-section

Fits of DIS data

Collins, Ellis (1991), Catani, Ciafaloni, Hautmann (1991)

- Logs of $1/x \Longrightarrow$ BFKL. Important when :
  - $M_\perp$ remains moderate, while x_1 or x_2 (or both) are small
- The BFKL equation is non-local in transverse momentum
  - ▷ it applies to non-integrated gluon distributions $\varphi(x, \vec{\boldsymbol{k}}_\perp)$

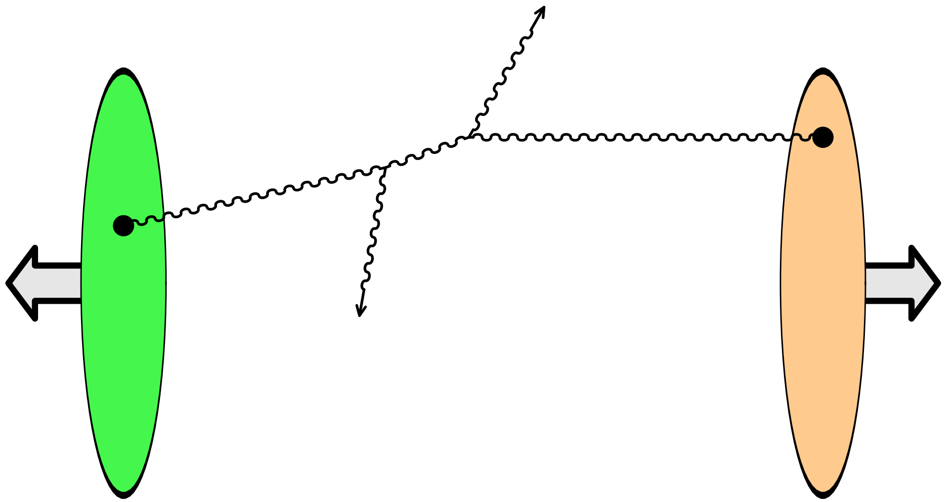
$$xG(x,Q^2) = \int^{Q^2} \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2}\;\varphi(x,\vec{\boldsymbol{k}}_\perp)$$

  - ▷ the matrix element is calculated for (off-shell) gluons with $\vec{\boldsymbol{k}}_\perp \neq \vec{0}$
- In this framework, cross-sections read :

$$\frac{d\sigma}{dY\,d^2\vec{\boldsymbol{P}}_\perp} \propto \int_{\vec{\boldsymbol{k}}_{1\perp},\vec{\boldsymbol{k}}_{2\perp}} \delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}-\vec{\boldsymbol{P}}_\perp)\;\varphi_1(x_1,k_{1\perp})\;\varphi_2(x_2,k_{2\perp})\;\frac{|\mathcal{M}|^2}{k_{1\perp}^2 k_{2\perp}^2}$$

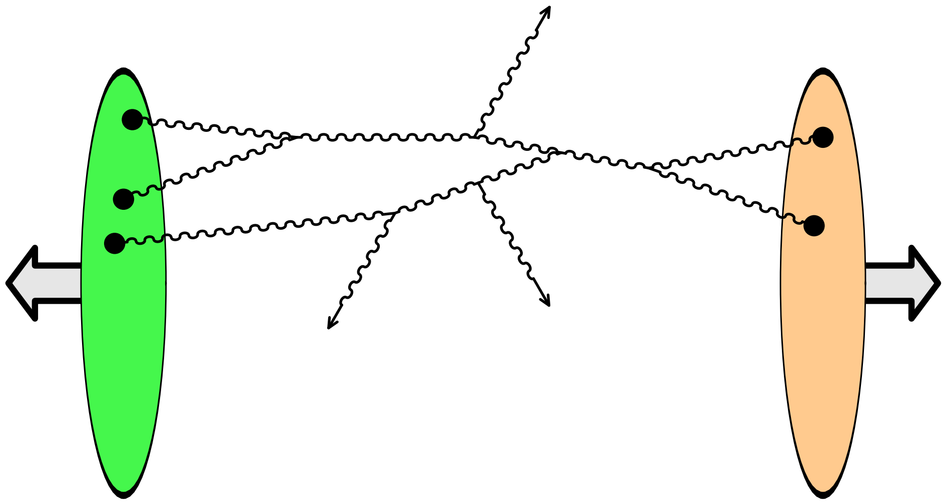
$$(x_{1,2} = M_\perp\, e^{\pm Y}\;/\;\sqrt{s})$$

Multi-parton interactions?



- Collinear or kt-factorization : only one parton in each projectile take part in the process of interest

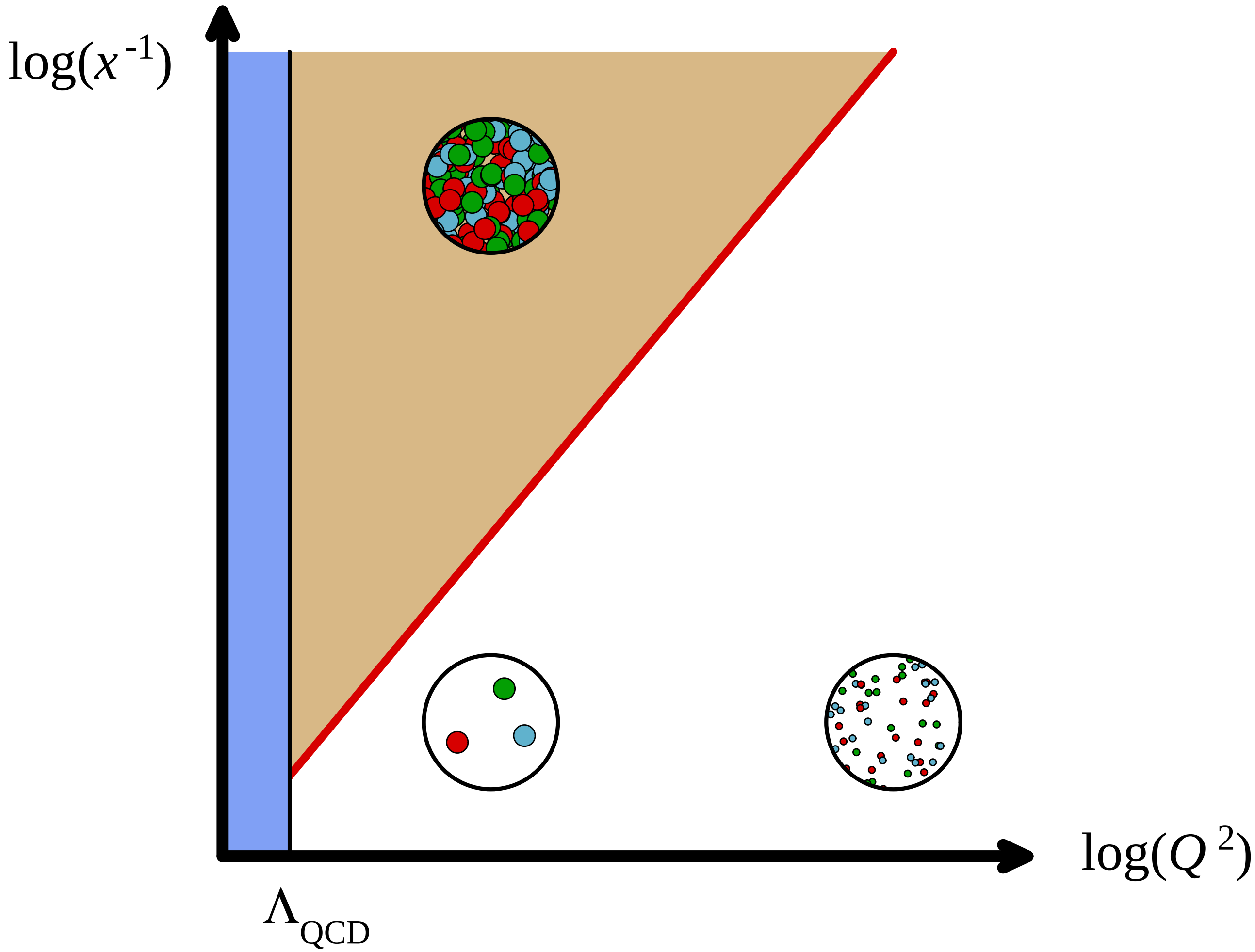
Multi-parton interactions?



- Collinear or kt-factorization : only one parton in each projectile take part in the process of interest
- If multiparton interactions are important : the above forms of factorization cannot work anymore, because the only information they retain about the distribution of partons is their 2-point correlations (i.e. the number of partons)

Color Glass Condensate

Saturation domain



CGC degrees of freedom

- The fast partons (large x) are frozen by time dilation
  ▷ described as static color sources on the light-cone :

$$J_a^\mu = \delta^{\mu +}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp) \qquad\qquad (x^- \equiv (t-z)/\sqrt{2})$$

- Slow partons (small x) cannot be considered static over the time-scales of the collision process ▷ they must be treated as the usual gauge fields

  Since they are radiated by the fast partons, they must be coupled to the current J_a^μ by a term : $A_\mu J^\mu$

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with Y the rapidity that separates "soft" and "hard"

CGC evolution

- Evolution equation (JIMWLK) :

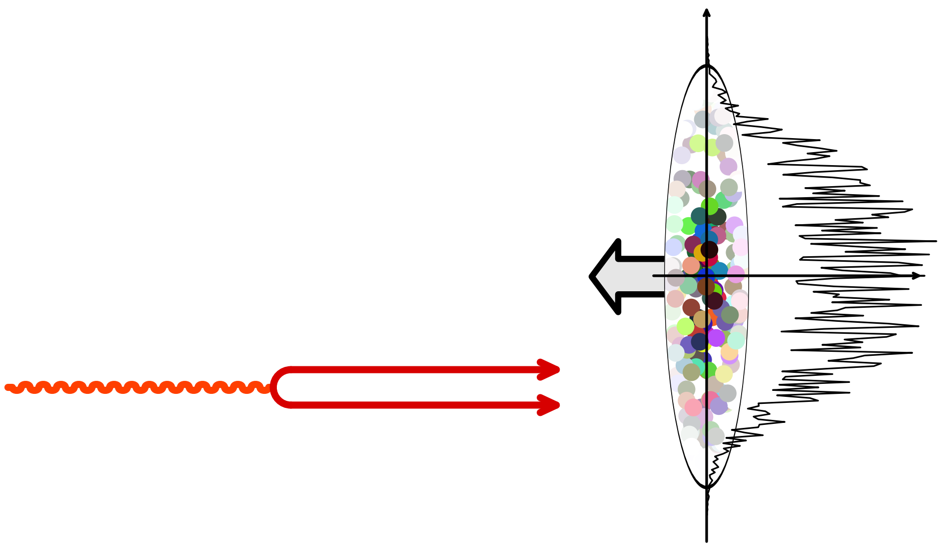
$$\frac{\partial W_Y}{\partial Y} = \mathcal{H} \; W_Y$$

$$\mathcal{H} = \frac{1}{2} \int_{\vec{\boldsymbol{y}}_\perp} \frac{\delta}{\delta \widetilde{\mathcal{A}}_b^+(\epsilon, \vec{\boldsymbol{y}}_\perp)} \eta_{ab}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \frac{\delta}{\delta \widetilde{\mathcal{A}}_a^+(\epsilon, \vec{\boldsymbol{x}}_\perp)}$$

where $-\boldsymbol{\partial}_\perp^2 \; \widetilde{\mathcal{A}}^+(\epsilon, \vec{\boldsymbol{x}}_\perp) = \rho(\epsilon, \vec{\boldsymbol{x}}_\perp)$

- η_{ab} is a non-linear functional of ρ
- This evolution equation resums the powers of $\alpha_s \ln(1/x)$ and of $Q_s/p_\perp$ that arise in loop corrections
- This equation simplifies into the BFKL equation when the color density ρ is small (one can expand η_{ab} in ρ)

Deep Inelastic Scattering



Light-cone coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$$a^{1,2} \quad \text{unchanged.} \qquad \text{Notation :} \quad \vec{a}_\perp \equiv (a^1, a^2)$$

- Under a Lorentz boost in the z direction :

$$a^+ \to \Lambda\, a^+ \quad , \quad a^- \to \Lambda^{-1}\, a^- \quad , \quad a^{1,2} \to a^{1,2}$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{x}_\perp \cdot \vec{y}_\perp$$

$$d^4x = dx^+ dx^- d^2\vec{x}_\perp$$

$$\square = 2\partial^+\partial^- - \vec{\nabla}_\perp^2 \qquad \text{Notation :} \quad \partial^+ \equiv \frac{\partial}{\partial x^-} \; , \; \partial^- \equiv \frac{\partial}{\partial x^+}$$

Parton-nucleus cross-section

- Reactions involving elementary probes can be reduced to that of individual partons with the saturated target :

$$d\sigma = \underbrace{d\Phi_1 \cdots d\Phi_n}_{} \; \frac{1}{2p^-} 2\pi\delta(p^- - \sum_i q_i^-) \; |\mathcal{M}|^2$$

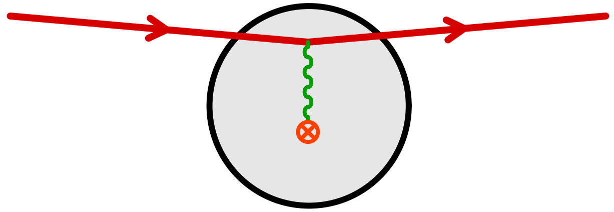
invariant phase-space
for the final state

- Invariant phase-space : $d\Phi \equiv \dfrac{d^3\vec{q}}{(2\pi)^3 2\omega_{\boldsymbol{q}}}$
- $\mathcal{M} \equiv$ transition amplitude $\left\langle \vec{\boldsymbol{q}}_1 \cdots \vec{\boldsymbol{q}}_{n\,\mathrm{out}} \middle| \vec{\boldsymbol{p}}_{\mathrm{in}} \right\rangle$ in the presence of the color field of the target
- The delta function comes from the fact that a highly boosted target field (in the $+z$ direction) is x^+-independent

Eikonal scattering

Goal

- Consider the scattering amplitude off an external potential :



$$S_{\beta\alpha} \equiv \langle\beta_{\rm out}|\alpha_{\rm in}\rangle = \langle\beta_{\rm in}|U(+\infty,-\infty)|\alpha_{\rm in}\rangle$$

where $U(+\infty,-\infty)$ is the evolution operator from $t=-\infty$ to $t=+\infty$

$$U(+\infty,-\infty) = T\,\exp\left[i\int d^4x\,\mathcal{L}_{\rm int}(\phi_{\rm in}(x))\right]$$

Note : $\mathcal{L}_{\rm int}$ contains the self-interactions of the fields and their interactions with the external potential

- We want to calculate its high energy limit :

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega\to+\infty} \langle\beta_{\rm in}|e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3}|\alpha_{\rm in}\rangle$$

where K^3 is the generator of boosts in the $+z$ direction

Eikonal scattering in a nutshell

- In a scattering at high energy, the collision time goes to zero as $s^{-1/2}$
- With scalar interactions, this implies a decrease of the scattering amplitude as $s^{-1/2}$
- With vectorial interactions, this decrease is compensated by the growth of the component J^{+} of the vector current

$\triangleright$ the eikonal approximation gives the finite limit of the scattering amplitude in the case of vectorial interactions when $s \to +\infty$

Eikonal limit

- Consider an external vector potential, that couples via $e\,\mathcal{A}_\mu(x)J^\mu(x)$ (J^μ is the current associated to some conserved charge)
- We will assume that the external potential is non-zero only in a finite range in x^+, $x^+ \in [-L,+L]$
- The action of K^3 on states and (scalar) fields is :

$$e^{-i\omega K^3}\big|\vec{\boldsymbol{p}}\cdots{}_{\rm in}\big\rangle = \big|(e^{\omega}p^+,\vec{\boldsymbol{p}}_\perp)\cdots{}_{\rm in}\big\rangle$$

$$e^{i\omega K^3}\phi_{\rm in}(x)e^{-i\omega K^3} = \phi_{\rm in}(e^{-\omega}x^+, e^{\omega}x^-, \vec{\boldsymbol{x}}_\perp)$$

- K^3 does not change the ordering in x^+. Hence,

$$e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3} = T\exp i\int d^4x\;\mathcal{L}_{\rm int}(e^{i\omega K^3}\phi_{\rm in}(x)e^{-i\omega K^3})$$

where $\mathcal{L}_{\rm int} = \mathcal{L}_{\rm self}(\phi) + e\,\mathcal{A}_\mu J^\mu$

Eikonal limit

- Split the evolution operator $U(+\infty,-\infty)$ into three factors :

$$U(+\infty,-\infty) = U(+\infty,+L)U(+L,-L)U(-L,-\infty)$$

Upon application of K^3, this becomes :

$$e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3} = e^{i\omega K^3}U(+\infty,+L)e^{-i\omega K^3} \times e^{i\omega K^3}U(+L,-L)e^{-i\omega K^3}e^{i\omega K^3}U(-L,-\infty)e^{-i\omega K^3}$$

- The external potential $\mathcal{A}_\mu(x)$ is unaffected by K^3
- The components of $J^\mu(x)$ are changed as follows :

$$e^{i\omega K^3}J^i(x)e^{-i\omega K^3} = J^i(e^{-\omega}x^+, e^{\omega}x^-, \vec{\boldsymbol{x}}_\perp)$$

$$e^{i\omega K^3}J^-(x)e^{-i\omega K^3} = e^{-\omega}\, J^-(e^{-\omega}x^+, e^{\omega}x^-, \vec{\boldsymbol{x}}_\perp)$$

$$e^{i\omega K^3}J^+(x)e^{-i\omega K^3} = e^{\omega}\, J^+(e^{-\omega}x^+, e^{\omega}x^-, \vec{\boldsymbol{x}}_\perp)$$

Eikonal limit

- The factors $U(+\infty,+L)$ and $U(-L,-\infty)$ do not contain the external potential. In order to deal with these factors, it is sufficient to change variables : $e^{-\omega}x^+ \to x^+$, $e^{\omega}x^- \to x^-$. This leads to :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+\infty,+L) e^{-i\omega K^3} = U_{\rm self}(+\infty,0)$$

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(-L,-\infty) e^{-i\omega K^3} = U_{\rm self}(0,-\infty)$$

where $U_{\rm self}$ is the same as U, but with the self-interactions only

- For the factor $U(L,-L)$, the change $e^{\omega}x^- \to x^-$ leads to :

$$e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} = \\ = T\exp i\int_{-L}^{+L} d^4x\; e^{-\omega}\Big[e\,\mathcal{A}^-(x^+,e^{-\omega}x^-,\vec{\boldsymbol{x}}_\perp) \\ \times e^{\omega} J^+(e^{-\omega}x^+,x^-,\vec{\boldsymbol{x}}_\perp) + \mathcal{O}(1)\Big]$$

Eikonal limit

- Therefore, in the limit $\omega \to +\infty$, we have :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} = \exp\left[ie \int d^2\vec{\boldsymbol{x}}_\perp \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right]$$

with

$$\begin{cases} \chi(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^+ \; \mathcal{A}^-(x^+, 0, \vec{\boldsymbol{x}}_\perp) \\ \rho(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^- \; J^+(0, x^-, \vec{\boldsymbol{x}}_\perp) \end{cases}$$

- The high-energy limit of the scattering amplitude is :

$$S^{(\infty)}_{\beta\alpha} = \left\langle \beta_{\rm in} \middle| U_{\rm self}(+\infty, 0) \; \exp\left[ie \int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right] U_{\rm self}(0,-\infty) \middle| \alpha_{\rm in} \right\rangle$$

  - Only the $-$ component of the vector potential matters
  - The self-interactions and the interactions with the external potential are factorized ▷ parton model
  - This is an exact result when $s \to +\infty$

Eikonal limit

- The previous formula still contains all the self-interactions of the fields. In order to perform the perturbative expansion, it is convenient to write first :

$$S^{(\infty)}_{\beta\alpha} = \sum_{\gamma,\delta} \big\langle \beta_{\rm in} \big| U_{\rm self}(+\infty,0) \big| \gamma_{\rm in} \big\rangle \times \big\langle \gamma_{\rm in} \big| \exp\Big[ie\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\Big] \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U_{\rm self}(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

- The factor

$$\sum_{\delta} \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U_{\rm self}(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

is the Fock expansion of the initial state: the state prepared at $x^+ = -\infty$ may have fluctuated into another state before it interacts with the external potential

Eikonal limit

- We need to calculate matrix elements such as $\left\langle\gamma_{\rm in}\middle|\boldsymbol{F}\middle|\delta_{\rm in}\right\rangle$, with :

$$\boldsymbol{F} \equiv \exp\, ie \int \chi_a(\vec{\boldsymbol{x}}_\perp)\rho^a(\vec{\boldsymbol{x}}_\perp)$$

  - having QCD in mind, we have reinstated the color indices
  - the contribution of quarks and antiquarks to $\rho^a(\vec{\boldsymbol{x}}_\perp)$ is :

$$\rho^a(\vec{\boldsymbol{x}}_\perp) = t^a_{ij} \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\frac{d^2\vec{\boldsymbol{q}}_\perp}{(2\pi)^2}\Big\{ b^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)b_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp} - d^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)d_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{-i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\Big\}$$

  - Note : one should keep the ordering of the exponential in x^+
  - the contribution of gluons is similar, with a color matrix in the adjoint representation

- The action of $\boldsymbol{F}$ on a state $\left|\delta_{\rm in}\right\rangle$ gives a state with the same particle content, the same $+$ components for the momenta, but modified transverse momenta and colors

Light-cone wavefunction

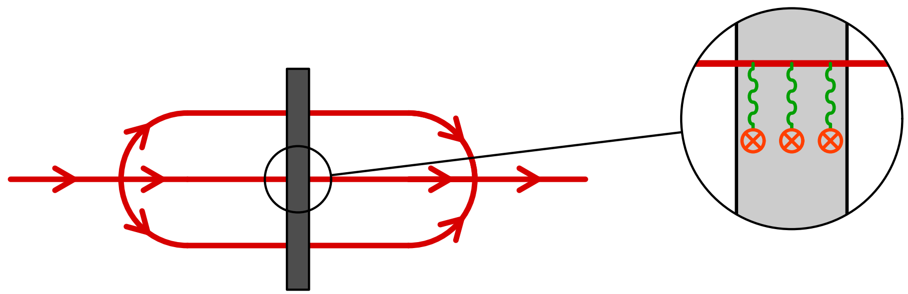
- For each intermediate state $\langle\delta_{\rm in}| \equiv \langle\{k_i^+, \vec{\boldsymbol{k}}_{i\perp}\}|$, define the corresponding light-cone wave function by :

$$\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \equiv \prod_i \int \frac{d^2\vec{\boldsymbol{k}}_{i\perp}}{(2\pi)^2} e^{-i\vec{\boldsymbol{k}}_{i\perp}\cdot\vec{\boldsymbol{x}}_{i\perp}} \langle\delta_{\rm in}|U_{\rm self}(0,-\infty)|\alpha_{\rm in}\rangle$$

- Each charged particle going through the external field acquires a phase proportional to its charge (antiparticles get an opposite phase) :

$$\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \longrightarrow \Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \prod_i U_i(\vec{\boldsymbol{x}}_\perp)$$

$$U_i(\vec{\boldsymbol{x}}_\perp) \equiv T\exp\left[ig_i \int dx^+ \, \mathcal{A}_a^-(x^+,0,\vec{\boldsymbol{x}}_\perp)t^a\right]$$



Light-cone wavefunction

- We have seen that the number and the nature of the particles is unchanged under the action of the operator $\boldsymbol{F}$. Moreover, in terms of the transverse coordinates, we simply have

$$\big\langle\gamma_{\rm in}\big|\boldsymbol{F}\big|\delta_{\rm in}\big\rangle = \delta_{NN'}\prod_i\Big[4\pi k_i^+\delta(k_i^+-k_i^{+\prime})\delta(\vec{\boldsymbol{x}}_{i\perp}-\vec{\boldsymbol{x}}'_{i\perp})U_{R_i}(\vec{\boldsymbol{x}}_{i\perp})\Big]$$

where $U_R(\vec{\boldsymbol{x}}_\perp)$ is a Wilson line operator, in the representation R appropriate for the particle going through the target

- Therefore, the high energy scattering amplitude can be written as :

$$S^{(\infty)}_{\beta\alpha} = \sum_\delta\int\Big[\prod_{i\in\delta}d\Phi_i\Big]\Psi^\dagger_{\delta\beta}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\})\Big[\prod_{i\in\delta}U_{R_i}(\vec{\boldsymbol{x}}_{i\perp})\Big]\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\})$$

- As we shall see shortly, some loop corrections are enhanced by logs of the energy. They must be resummed and drive the energy evolution of the amplitude

QCD and factorization

Color Glass Condensate

Eikonal scattering

- Eikonal limit
- Light-cone wavefunction

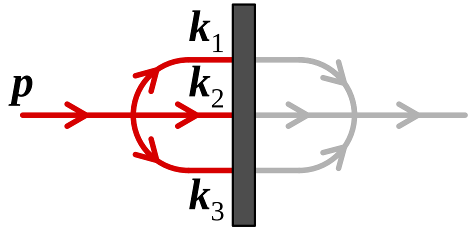
Solution of YM equations

DIS cross-section

Fits of DIS data

Light-cone wave function

- The calculation of $\langle\delta_{\rm in}|U_{\rm self}(0,-\infty)|\alpha_{\rm in}\rangle$ is similar to that of scattering amplitudes in the vacuum. The only difference is that the integration over x^+ at each vertex runs only over half of the real axis $[-\infty, 0]$
  - In Fourier space, this means that the $-$ component of the momentum is not conserved at the vertices
  - Instead of a δ function, one gets an energy denominator
- Example with a single interaction :



$$\langle\vec{\boldsymbol{k}}_1\vec{\boldsymbol{k}}_2\vec{\boldsymbol{k}}_{3\,\rm in}|U_{\rm self}(0,-\infty)|\vec{\boldsymbol{p}}_{\rm in}\rangle = -ig\int_{-\infty}^{0} d^4x\; e^{i(k_1+k_2+k_3-p)\cdot x}$$

$$= -g\,\frac{(2\pi)^3\delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}+\vec{\boldsymbol{k}}_{3\perp}-\vec{\boldsymbol{p}}_\perp)\delta(k_1^++k_2^++k_3^+-p^+)}{k_1^-+k_2^-+k_3^--p^--i\epsilon}$$

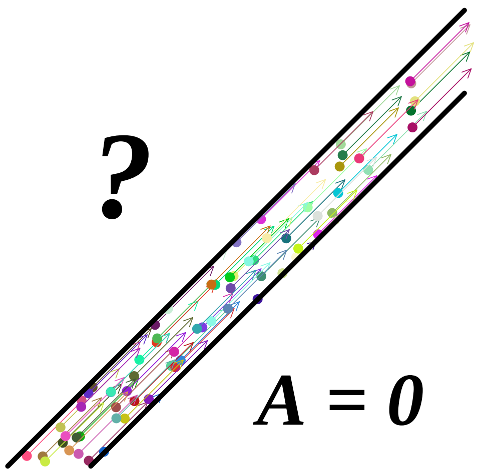
Solution of Yang-Mills equations

YM equations in covariant gauge

- Gauge condition : $\partial_\mu \mathcal{A}^\mu = 0$
- We must solve the Yang-Mills equations with the current :

$$J_a^\mu(x) \equiv \delta^{\mu +} \rho_a(x^-, \vec{\boldsymbol{x}}_\perp)$$

(in practice, the x^- dependence is close to a $\delta(x^-)$, but the solution is valid for any x^- dependence)



- The source density does not depend on x^+
- The gauge field vanishes at $x^0 \to -\infty$

YM equations in covariant gauge

- In covariant gauge, the YM equations can be rewritten as :

$$\square A^\nu = J^\nu + ig[A_\mu, F^{\mu\nu} + \partial^\mu A^\nu]$$

- One must also enforce current conservation :

$$[D_\mu, J^\mu] = 0$$

Note : this relation is satisfied trivially at order ρ^1 by our ansatz for J^μ, but it may induce higher order corrections in $\rho^2, \rho^3, \cdots$ to J^μ

- Order ρ^1 : the equation simplifies into $\square A^\mu_{(1)} = J^\mu_{(1)}$

$$A^+_{(1)} = -\frac{1}{\boldsymbol{\partial}_\perp^2}\,\rho(x^-, \boldsymbol{x}_\perp) \quad , \qquad A^-_{(1)} = A^i_{(1)} = 0$$

- Higher orders in ρ :
  - since $A^-_{(1)} = 0$, it cannot induce a change in J^+
  - the commutator in the YM equation is zero at order ρ^2
  - these properties remain true at all the following orders
    - ▷ the solution at order ρ^1 is in fact the exact solution

Light-cone gauge

- Consider a gauge transformation :

$$\widetilde{A}^\mu \equiv \Omega^\dagger \, A^\mu \, \Omega + \frac{i}{g} \, \Omega^\dagger \, \partial^\mu \, \Omega$$

- We look for Ω in the SU(N) group such that $\widetilde{A}^+ = 0$:

$$\partial^+ \, \Omega = ig \; A^+ \; \Omega$$

i.e. $$\Omega(x) = \underbrace{\mathrm{T}\, \exp \left[ig \int_{-\infty}^{x^-} dz^- \, A^+(z^-, \boldsymbol{\vec{x}}_\perp) \right]}_{U} \; \Omega_0(x^+, \boldsymbol{\vec{x}}_\perp)$$

$\Omega_0 =$ arbitrary function of $x^+, \boldsymbol{\vec{x}}_\perp$

- Residual gauge freedom fixing : if we impose that $\widetilde{A}^\mu = 0$ when $x^- \to -\infty$, we must chose $\Omega_0 \equiv 1$. This leads to :

$$\widetilde{A}^\pm = 0 \quad , \qquad \widetilde{A}^i = \frac{i}{g} \, U^\dagger \, \partial^i \, U$$

DIS cross-section

DIS amplitude

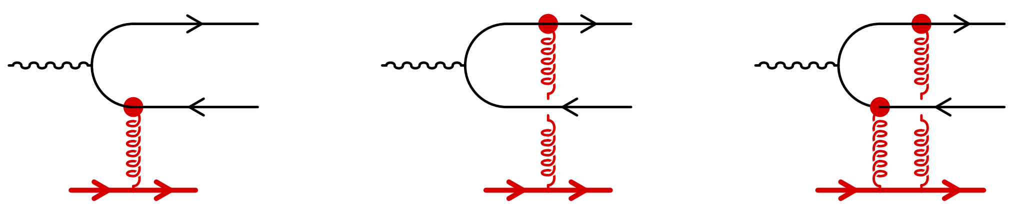
- Differential photon-target cross-section :

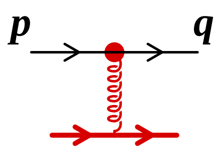
$$d\sigma_{\gamma^* T} = \frac{d^3\boldsymbol{k}}{(2\pi)^2 2E_{\boldsymbol{k}}} \frac{d^3\boldsymbol{p}}{(2\pi)^3 2E_{\boldsymbol{p}}} \; \frac{1}{2q^-} 2\pi\delta(q^- - k^- - p^-)$$
$$\times \left\langle \mathcal{M}^\mu(\boldsymbol{q}|\boldsymbol{k},\boldsymbol{p})\mathcal{M}^{\nu *}(\boldsymbol{q}|\boldsymbol{k},\boldsymbol{p})\right\rangle \epsilon_\mu(Q)\epsilon^*_\nu(Q) \;,$$

  - $\boldsymbol{k},\boldsymbol{p}$: momenta of the quark and antiquark
  - $\boldsymbol{q}$: momentum of the virtual photon

- Scattering amplitude :

$$\mathcal{M}^\mu(\boldsymbol{q}|\boldsymbol{k},\boldsymbol{p}) =$$





$$= 2\pi\,\delta(p^- - q^-)\,\gamma^- \int d^2\vec{\boldsymbol{x}}_\perp \; e^{i(\vec{\boldsymbol{q}}_\perp - \vec{\boldsymbol{p}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp} \left[U(\vec{\boldsymbol{x}}_\perp) - 1\right]$$

DIS amplitude

- The sum of the three terms simplifies considerably :

$$\mathcal{M}^\mu(\boldsymbol{k}|\boldsymbol{q},\boldsymbol{p}) = \frac{i}{2}\int \frac{d^2\vec{\boldsymbol{l}}_\perp}{(2\pi)^2}\int d^2\vec{\boldsymbol{x}}_{1\perp} d^2\vec{\boldsymbol{x}}_{2\perp}\,\left[\overline{u}(\vec{\boldsymbol{q}})\;\Gamma^\mu\; v(\vec{\boldsymbol{p}})\right]$$
$$\times\, e^{i\vec{\boldsymbol{l}}_\perp\cdot\vec{\boldsymbol{x}}_{1\perp}} e^{i(\vec{\boldsymbol{p}}_\perp+\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{q}}_\perp-\vec{\boldsymbol{l}}_\perp)\cdot\vec{\boldsymbol{x}}_{2\perp}}\left[U(\vec{\boldsymbol{x}}_{1\perp})U^\dagger(\vec{\boldsymbol{x}}_{2\perp}) - 1\right]$$

with

$$\Gamma^\mu \equiv \frac{\gamma^-(\not{K}-\not{L}+m)\gamma^\mu(\not{K}-\not{Q}-\not{L}+m)\gamma^-}{p^-[(\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{l}}_\perp)^2+m^2-2k^-q^+]+k^-[(\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{q}}_\perp-\vec{\boldsymbol{l}}_\perp)^2+m^2]}$$

- By inserting this into the DIS cross-section, we see that the differential cross-section (with two resolved quark jets in the final state) depends on the correlator of four Wilson lines

Total cross-section

- If we integrate out the final quark and antiquark, two of the Wilson lines cancel and we get :

$$\sigma_{\gamma^* T} = \int_0^1 dz \int d^2\vec{\boldsymbol{r}}_\perp \left|\psi(\boldsymbol{q}|z,\vec{\boldsymbol{r}}_\perp)\right|^2 \sigma_{\rm dipole}(\vec{\boldsymbol{r}}_\perp)$$

with

$$\sigma_{\rm dipole}(\vec{\boldsymbol{r}}_\perp) \equiv \frac{2}{N_c}\int d^2\vec{\boldsymbol{X}}_\perp \; {\rm Tr}\left\langle 1 - U(\vec{\boldsymbol{X}}_\perp + \frac{\vec{\boldsymbol{r}}_\perp}{2})U^\dagger(\vec{\boldsymbol{X}}_\perp - \frac{\vec{\boldsymbol{r}}_\perp}{2})\right\rangle$$

and

$$\left|\psi(\boldsymbol{q}|z,\vec{\boldsymbol{r}}_\perp)\right|^2 \equiv \frac{N_c\,\epsilon_\mu(Q)\epsilon^*_\nu(Q)}{64\pi(q^-)^2 z(1-z)}\int \frac{d^2\vec{\boldsymbol{l}}_\perp}{(2\pi)^2}\frac{d^2\vec{\boldsymbol{l}}'_\perp}{(2\pi)^2}\, e^{i(\vec{\boldsymbol{l}}_\perp - \vec{\boldsymbol{l}}'_\perp)\cdot\vec{\boldsymbol{r}}_\perp}$$
$$\times\, {\rm Tr}\left((\not{k}+m)\Gamma^\mu(\not{p}-m)\Gamma^{\nu\prime}\right)$$

Note : $\left|\psi\right|^2$ can be computed in closed form (in terms of the Bessel functions $K_{0,1}$)

Fits of DIS data

Dipole cross-section

- Computing F_2 requires to know $\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp) \rangle_Y$ as a function of dipole size and energy
- This object is often presented in the form of the "dipole cross-section" :

$$\sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) \equiv 2 \int d^2\vec{\boldsymbol{b}} \, \left\langle \boldsymbol{T}(\vec{\boldsymbol{b}} - \frac{\vec{\boldsymbol{r}}_\perp}{2}, \vec{\boldsymbol{b}} + \frac{\vec{\boldsymbol{r}}_\perp}{2}) \right\rangle_Y$$

Note : this formula assumes that the scattering amplitude is real

- In principle, the BK equation prescribes the energy dependence of the dipole cross-section once it is known at a certain energy
- Alternatively, one can model this cross-section (including its energy dependence)

Golec-Biernat–Wusthoff model

- GBW modeled the dipole cross-section as a Gaussian, with an energy dependence entirely contained in Q_s

$$\begin{cases} \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \sigma_0 \left[1 - e^{-Q_s(Y)^2 r_\perp^2/4} \right] \\ Q_s^2(Y) = Q_0^2 \, e^{\lambda(Y-Y_0)} \end{cases}$$

- The exponential form in $\sigma_{\rm dip}$ is inspired of Glauber scattering
- The fit parameters are σ_0, Q_0, λ and possibly an effective quark mass in the photon wave-function
- Quite good for all small-x HERA data, with some problems at large Q^2

Bartels–Golec-Biernat–Kowalski model

- This model aims at improving the agreement at large Q^2, by having a more realistic cross-section at small dipole sizes :

$$\sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \sigma_0 \left[1 - e^{-\pi^2 r_\perp^2 \alpha_s(\mu^2) xG(x,\mu^2)/3\sigma_0}\right]$$

- The scale μ^2 is chosen of the form $\mu_0^2 + C/r_\perp^2$
- The gluon distribution $xG(x, \mu^2)$ obeys the DGLAP equation. Thus, the dipole cross-section has the correct behavior at small transverse distance
- This form improves the fit quality at large Q^2
- A saturation scale is also hidden in this dipole cross-section, if one recalls the formula

$$Q_s^2 \sim \frac{\alpha_s xG(x, Q_s^2)}{\pi R^2}$$

Iancu-Itakura-Munier model

- This model of the dipole cross-section is derived from LO BFKL :

$$\begin{cases} Q_s r_\perp \le 2 \;:\quad \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \dfrac{\sigma_0}{2}\left(\dfrac{Q_s(Y) r_\perp}{2}\right)^{2(\gamma_s + \ln(2/Q_s r_\perp)/\kappa\lambda Y)} \\ Q_s r_\perp \ge 2 \;:\quad \sigma_{\rm dip}(\vec{\boldsymbol{r}}_\perp, Y) = \sigma_0 \left[1 - e^{a \ln^2(b Q_s r_\perp)}\right] \\ \\ Q_s^2(Y) = Q_0^2\, e^{\lambda(Y - Y_0)} \end{cases}$$

  - Some parameters are fixed from LO BFKL : $\gamma_s = 0.63, \kappa = 9.9$
  - σ_0, Q_0 and λ must be fitted
  - a and b are adjusted for a smooth transition at $Q_s r_\perp = 2$

Exclusive processes

Kowalski, Motyka, Watt (2006)

- So far, we have only considered the total DIS cross-section, obtained from the forward dipole amplitude via the optical theorem
- In order to study more exclusive processes, one needs non-forward amplitudes. From our general eikonal formula, they read :

$$\left\langle \Omega_{\rm out} \middle| \gamma^*{}_{\rm in} \right\rangle = \int d^2\vec{\boldsymbol{r}}_\perp \int_0^1 dz \; \Psi^*_\Omega \psi \; \underbrace{\int d^2\vec{\boldsymbol{b}}\; e^{i\vec{\boldsymbol{q}}_\perp\cdot\vec{\boldsymbol{b}}} \left\langle \boldsymbol{T}(\vec{\boldsymbol{b}} - \frac{\vec{\boldsymbol{r}}_\perp}{2}, \vec{\boldsymbol{b}} + \frac{\vec{\boldsymbol{r}}_\perp}{2}) \right\rangle_Y}_{\text{non-forward dipole cross-section with momentum transfer } \vec{\boldsymbol{q}}_\perp}$$

Note : this formula assumes that the relevant dipole sizes $r_\perp$ are small compared to the target radius (i.e. the typical $\boldsymbol{b}$)

Exclusive processes

- By squaring this amplitude, one gets the diffractive cross-section for the production of the state Ω with momentum transfer $\boldsymbol{q}_\perp$

$$\frac{d\sigma^{\rm diff}_{\gamma^* p\to\Omega p}}{d^2\vec{\boldsymbol{q}}_\perp} = \left|\left<\Omega_{\rm out}\middle|\gamma^*{}_{\rm in}\right>\right|^2$$

- The relationship to the inclusive DIS cross-section is

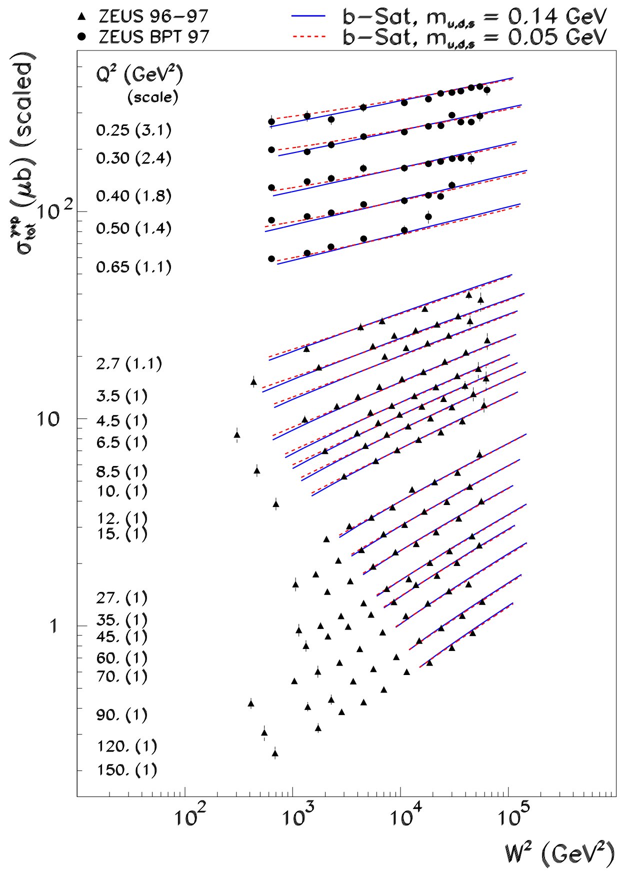
$$\sigma^{\rm tot}_{\gamma^* p}(Y, Q^2) = 2\,{\rm Re}\left<\gamma^*{}_{\rm out}\middle|\gamma^*{}_{\rm in}\right>_{\vec{\boldsymbol{q}}_\perp=0}$$

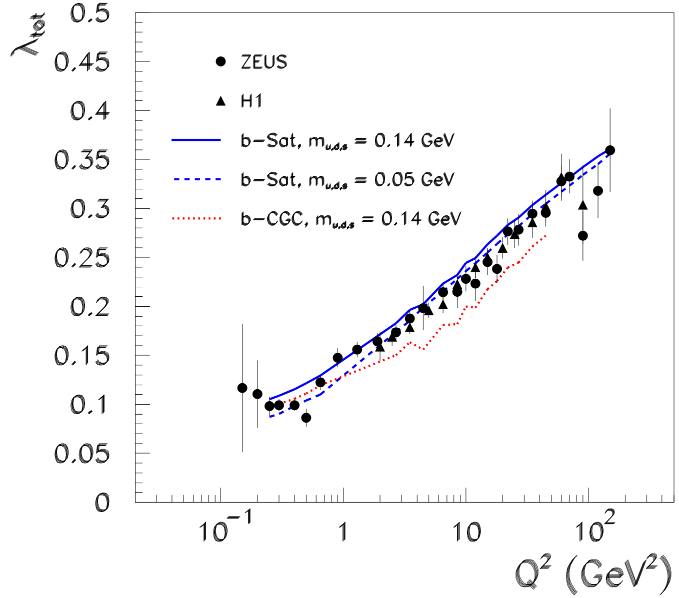
  Note : inclusive DIS only constrains the dipole amplitude averaged over impact parameter. However, if one measures the $\boldsymbol{q}_\perp$ dependence in exclusive reactions, one obtains informations about the $\boldsymbol{b}$ dependence of the dipole amplitude

- General strategy : extend the previous models in order to give them a $\boldsymbol{b}$-dependence, in a way that preserves F_2

Exclusive reactions

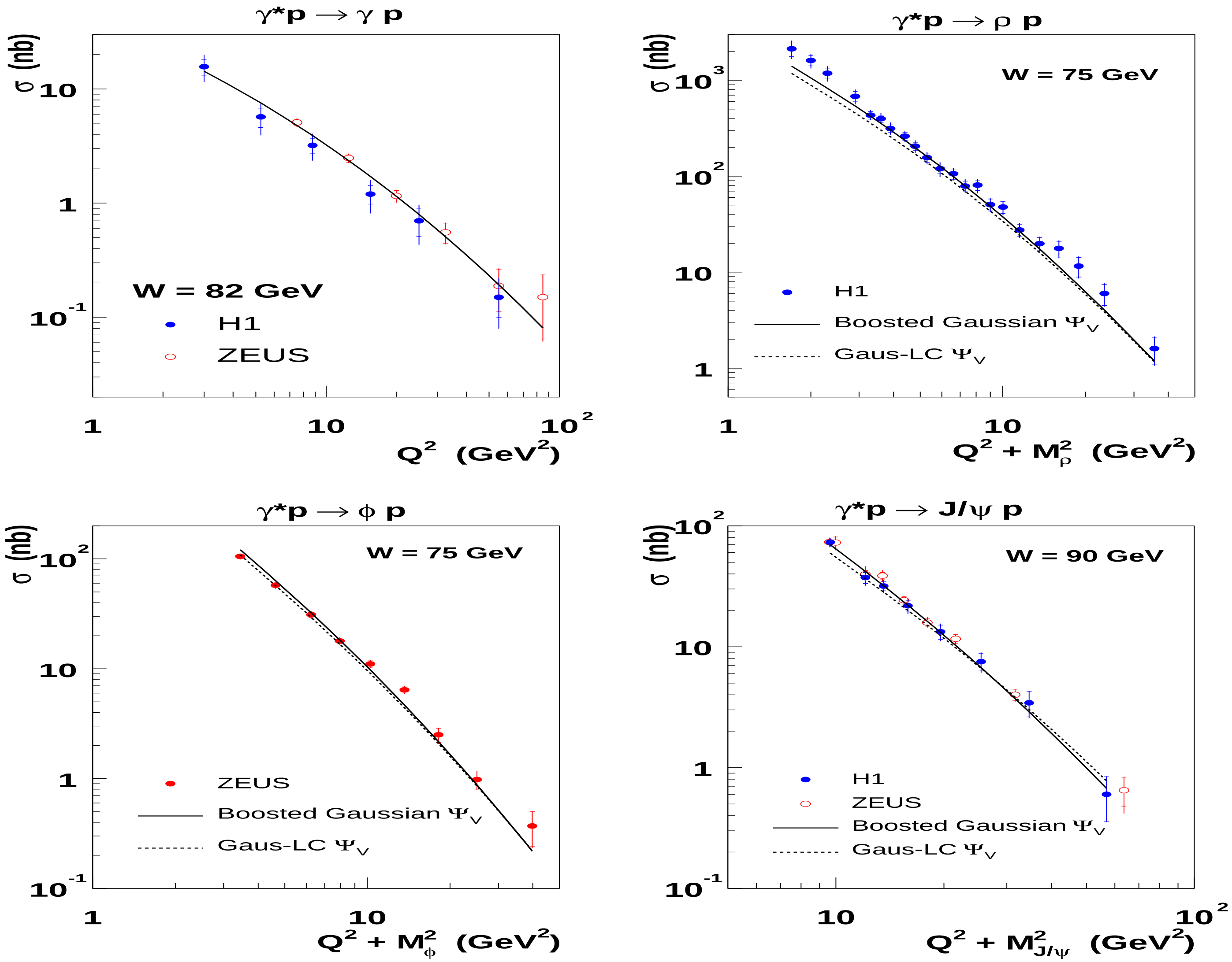
- For the total DIS cross-section, the fit is as good as before :





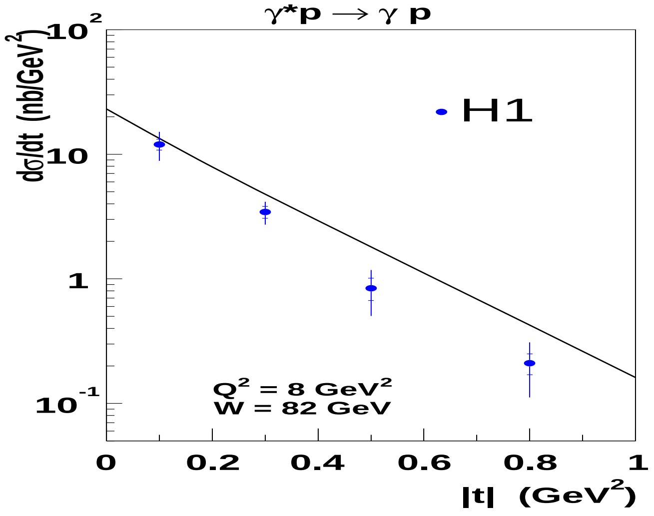
Exclusive reactions

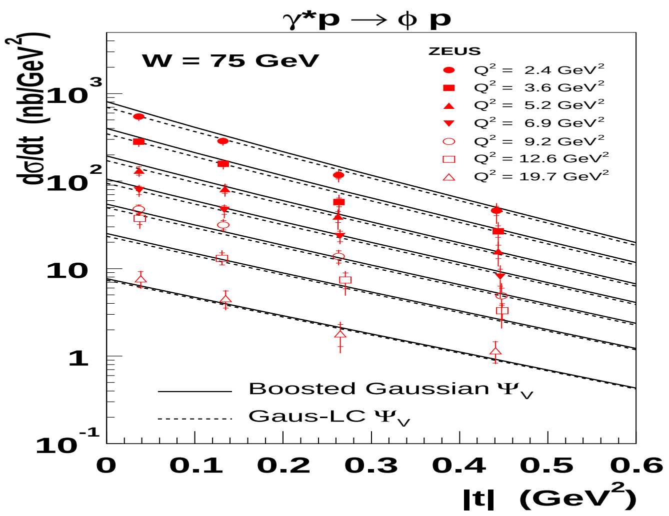
- Exclusive photon and vector meson production :

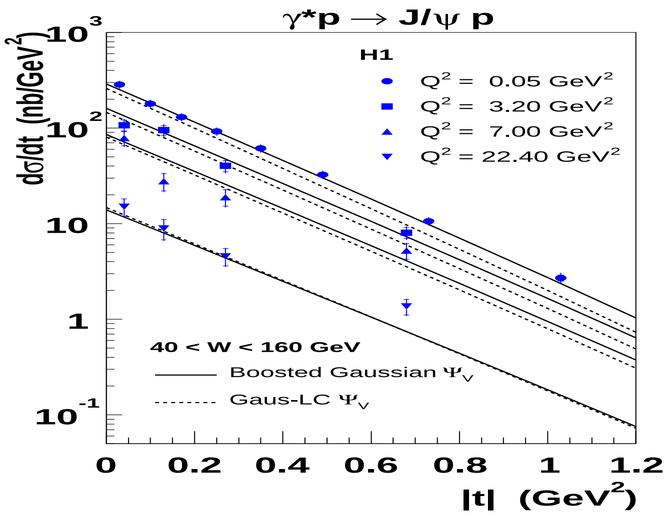


Exclusive reactions

- Exclusive photon and vector meson production :



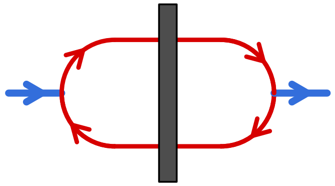




Energy dependence

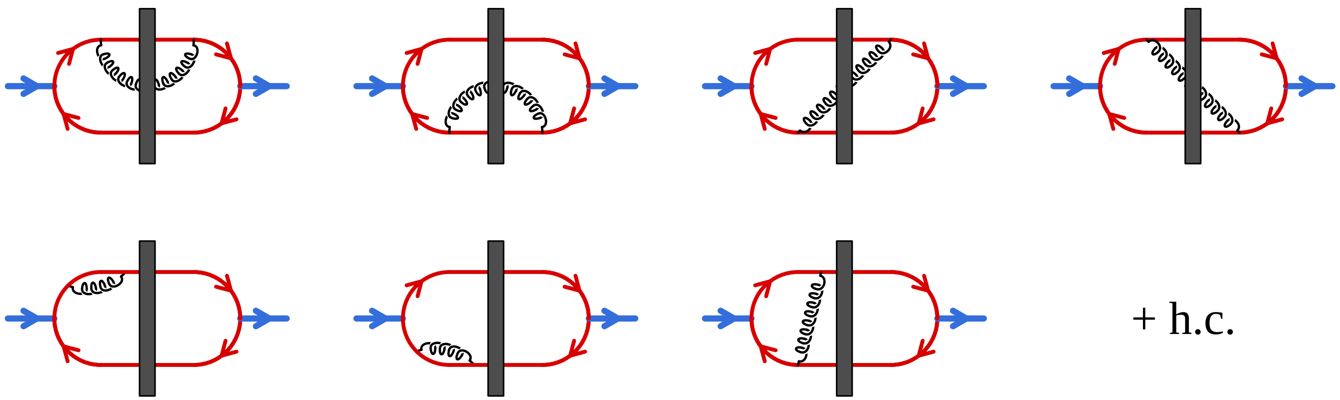
Scattering of a dipole

- Assume that the initial and final states α and β are a color singlet $Q\overline{Q}$ dipole. The bare scattering amplitude can be written as :



$$\propto \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

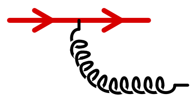
- At one loop, the following diagrams must be evaluated :



+ h.c.

Scattering of a dipole

- In the gauge $A^+ = 0$, the emission of a gluon of momentum k by a quark can be written as :



$$= \; 2gt^a \, \frac{\vec{\epsilon}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2}$$

- In coordinate space, this reads :

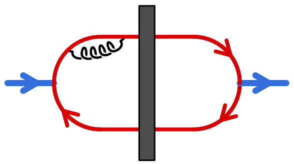
$$\int \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2} \, e^{i\vec{\boldsymbol{k}}_\perp \cdot (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)} \, 2gt^a \, \frac{\vec{\epsilon}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2} = \frac{2ig}{2\pi} t^a \, \frac{\vec{\epsilon}_\lambda \cdot (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- When connecting two gluons, one must use :

$$\sum_\lambda \vec{\epsilon}^{\,i}_\lambda \vec{\epsilon}^{\,j}_\lambda = -g^{ij}$$

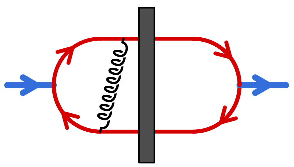
Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole
- Examples :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times -2\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp) t^a\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- Reminder : $t^a t^a = (N_c^2 - 1)/2N_c \equiv C_F$

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F\alpha_s}{\pi^2}\int\frac{dk^+}{k^+}\int d^2\vec{z}_\perp\;\frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2\mathrm{tr}\left[U(\vec{x}_\perp)U^\dagger(\vec{y}_\perp)\right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

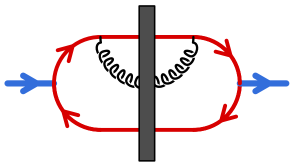
$$\int^{p^+}\frac{dk^+}{k^+}=\ln(p^+)=Y$$

▷ When Y is large, $\alpha_s Y$ may not be small. By differentiating with respect to Y, we will get an evolution equation in Y whose solution resums all the powers $(\alpha_s Y)^n$

- Note : the integral over $\vec{z}_\perp$ is divergent when $\vec{z}_\perp=\vec{x}_\perp$ or $\vec{y}_\perp$

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon
- Example :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

  - $\widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp)$ is a Wilson line in the adjoint representation
- In order to simplify the color structure, first recall that :

$$t^a \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = U(\vec{\boldsymbol{z}}_\perp) t^b U^\dagger(\vec{\boldsymbol{z}}_\perp)$$

- Then use the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Real corrections

- The Wilson lines can be rearranged into :

$$\mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = \frac{1}{2}\mathrm{tr}\left[U^\dagger(\vec{\boldsymbol{z}}_\perp)U(\vec{\boldsymbol{x}}_\perp)\right]\mathrm{tr}\left[U(\vec{\boldsymbol{z}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] - \frac{1}{2N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

  - The term in $1/2N_c$ cancels against a similar term in the virtual contribution
  - All the real terms have the same color structure

- When we sum all the real terms, we generate the same kernel as in the virtual terms :

$$\frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- In order to simplify the notations, let us denote :

$$\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

Evolution equation

- The 1-loop scattering amplitude reads :

$$-\frac{\alpha_s N_c^2 Y}{2\pi^2}\left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2} \times\left\{\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{S}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\right\}$$

- Reminder: the bare scattering amplitude was :

$$\left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 N_c\ \boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)$$

- Hence, we have :

$$\frac{\partial \boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)}{\partial Y} = -\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2} \times\left\{\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{S}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\right\}$$

  - since $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{x}}_\perp)=1$, the integral over $\vec{\boldsymbol{z}}_\perp$ is now regular

BFKL equation

Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

- The BFKL equation can be obtained by linearizing the previous equation

- Write $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv 1 - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude $\boldsymbol{T}$ is small. Drop the terms that are non-linear in $\boldsymbol{T}$:

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2} \times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \Big\}$$

- The solution of this equation grows exponentially when $Y \to +\infty$ $\quad \triangleright$ serious unitarity problem...

Non-linear evolution equation

- In fact, the first evolution equation we derived has a bounded solution. The unbounded solutions of BFKL are due to dropping the non-linear term. The full equation reads :

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

$$\times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \underline{- \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp)} \Big\}$$

(Balitsky-Kovchegov equation)

- The r.h.s. vanishes when $\boldsymbol{T}$ reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $\boldsymbol{T} = 0$ and $\boldsymbol{T} = 1$ are fixed points of this equation

$\boldsymbol{T} = \epsilon$:	r.h.s. > 0	$\Rightarrow$	$\boldsymbol{T} = 0$ is unstable
$\boldsymbol{T} = 1 - \epsilon$:	r.h.s. > 0	$\Rightarrow$	$\boldsymbol{T} = 1$ is stable

Caveats and improvements

- So far, we have studied the scattering amplitude between a color dipole and a “god given” patch of color field. This is too naive to describe any realistic situation
- We need to improve the treatment of the target
- An experimentally measured cross-section is an average over many collisions, and there is no reason why these fields should be the same in different collisions :

$$\boldsymbol{T} \quad \to \quad \langle \boldsymbol{T} \rangle$$

$\langle \cdots \rangle$ denotes the average over the target configurations, i.e.

$$\langle \cdots \rangle = \int [D\rho]\; W_Y[\rho] \cdots$$

Balitsky hierarchy

- Because of this average over the target configurations, the evolution equation we have derived should be written as :

$$\frac{\partial\,\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle}{\partial Y}=\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp\,\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\rangle+\langle \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle-\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle\underline{-\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\rangle}\Big\}$$

- As one can see, the equation is no longer a closed equation, since the equation for $\langle \boldsymbol{T}\rangle$ depends on a new object, $\langle \boldsymbol{T}\,\boldsymbol{T}\rangle$
- One can derive an evolution equation for $\langle \boldsymbol{T}\,\boldsymbol{T}\rangle$. Its right hand side contains objects with six Wilson lines
  - There is in fact an infinite hierarchy of nested evolution equations, whose generic structure is

$$\frac{\partial\,\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\rangle}{\partial Y}=\int\cdots\Big\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\Big\rangle\oplus\Big\langle(\boldsymbol{U}\boldsymbol{U}^\dagger)^{n+1}\Big\rangle$$

Balitsky-Kovchegov equation

- If one performs the large N_c approximation on all the equations of the Balitsky hierarchy, they can be rewritten in terms of the dipole operator $\boldsymbol{T} \equiv 1 - \frac{1}{N_c}\mathrm{tr}(UU^\dagger)$ only. But they still contain averages like $\langle \boldsymbol{T}^n \rangle$
- In order to truncate the hierarchy of equations, one may assume that

$$\langle \boldsymbol{T}\,\boldsymbol{T} \rangle \approx \langle \boldsymbol{T} \rangle \, \langle \boldsymbol{T} \rangle$$

- This approximation gives for $\langle \boldsymbol{T} \rangle$ the same evolution equation as the one we had for a fixed configuration of the target

Geometrical scaling

Analogy with reaction-diffusion

Munier, Peschanski (2003,2004)

- Assume translation and rotation invariance, and define :

$$N(Y,k_\perp) \equiv 2\pi \int d^2\vec{\boldsymbol{x}}_\perp\; e^{i\vec{\boldsymbol{k}}_\perp\cdot\vec{\boldsymbol{x}}_\perp}\;\frac{\left\langle \boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right\rangle_Y}{x_\perp^2}$$

- From the Balitsky-Kovchegov equation for $\langle \boldsymbol{T}\rangle$, we obtain the following equation for N :

$$\frac{\partial N(Y,k_\perp)}{\partial Y} = \frac{\alpha_s N_c}{\pi}\Big[\chi(-\partial_L)N(Y,k_\perp) - N^2(Y,k_\perp)\Big]$$

with

$$L \equiv \ln(k_\perp^2/k_0^2)$$

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

Analogy with reaction-diffusion

- Expand the function $\chi(\gamma)$ to second order around its minimum $\gamma = 1/2$
- Introduce new variables :

$$t \sim Y$$
$$z \sim L + \frac{\alpha_s N_c}{2\pi} \chi''(1/2)\, Y$$

- The equation for N becomes :

$$\partial_t N = \partial_z^2 N + N - N^2$$

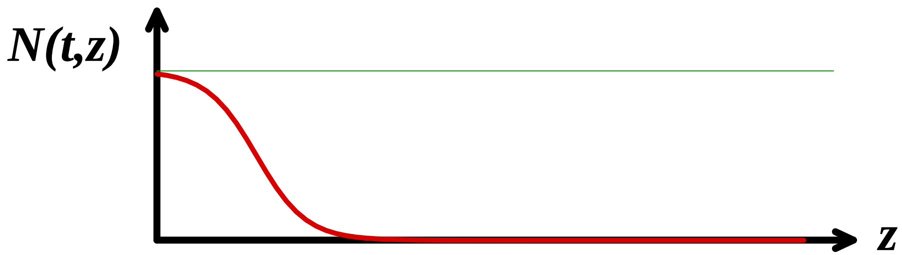
(known as the Fisher-Kolmogorov-Petrov-Piscounov equation)

Analogy with reaction-diffusion

- **Interpretation :** this equation is typical for all the diffusive systems in which a reaction $A \longleftrightarrow A + A$ takes place
  - $\partial_z^2 N$: diffusion term (the quantity under consideration can hop from a site to the neighboring sites)
  - $+N$: gain term corresponding to $A \to A + A$
  - $-N^2$: loss term corresponding to $A + A \to A$
- Note : this equation has two fixed points :
  - $N = 0$: unstable
  - $N = 1$: stable
- The stable fixed point at $N = 1$ exists only if one keeps the loss term. In other words, one would not have it from the BFKL equation

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

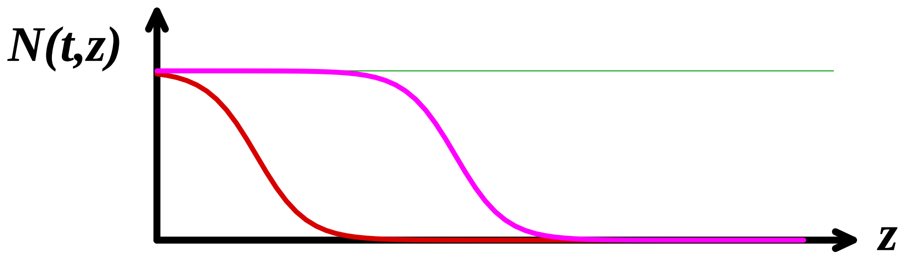
$$N(t, z) \underset{t\to+\infty}{\sim} N(z - m_\beta(t))$$

with $\quad m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1) \quad$ if $\quad \beta > 1$

▷ universal front velocity for a large class of initial conditions

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

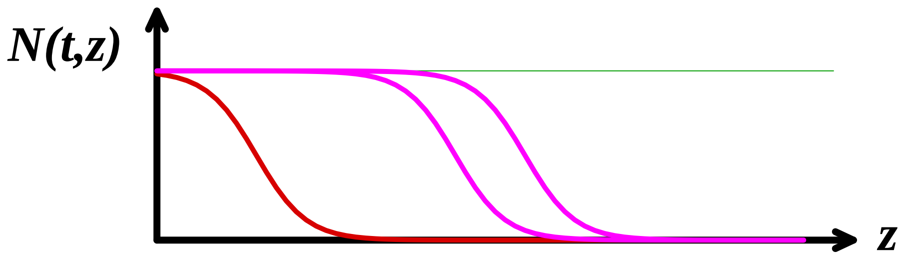
$$N(t, z) \underset{t \to +\infty}{\sim} N(z - m_\beta(t))$$

with $\quad m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1) \quad$ if $\quad \beta > 1$

▷ universal front velocity for a large class of initial conditions

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

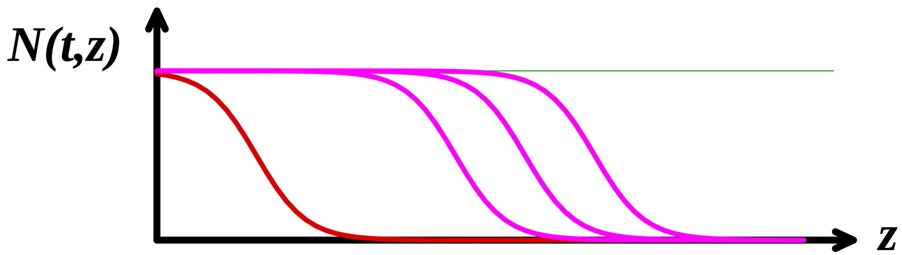
$$N(t,z) \underset{t\to+\infty}{\sim} N(z - m_\beta(t))$$

with $\quad m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1) \quad$ if $\quad \beta > 1$

▷ universal front velocity for a large class of initial conditions

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

$$N(t, z) \underset{t \to +\infty}{\sim} N(z - m_\beta(t))$$

with $\quad m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1) \quad$ if $\quad \beta > 1$

▷ universal front velocity for a large class of initial conditions

Geometrical scaling in DIS

Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)
Munier, Peschanski (2003)

- In QCD, the initial condition is of the required form, with $\beta > 1$
  $\triangleright$ front velocity independent of the initial condition

- Going back to the original variables, one gets :

$$N(Y, k_\perp) = N\left(k_\perp / Q_s(Y)\right)$$

with

$$Q_s^2(Y) = k_0^2 \, Y^{-\delta} \, e^{\lambda Y}$$

- Going from $N(Y, k_\perp)$ to $\left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp) \right\rangle_Y$, we obtain :

$$\left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp) \right\rangle_Y = T(Q_s(Y) x_\perp)$$

Geometrical scaling in DIS

- The total $\gamma^* p$ cross-section, measured in Deep Inelastic Scattering, can be written in terms of N:

$$\sigma^{\rm tot}_{\gamma^* p}(Y, Q^2) = 2\pi R^2 \int d^2\vec{\boldsymbol{x}}_\perp \int_0^1 dz \left|\psi(z, x_\perp, Q^2)\right|^2 \left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right\rangle_Y$$

  - The photon wavefunction ψ is calculable in QED. It depends on the dipole size $x_\perp$ only via

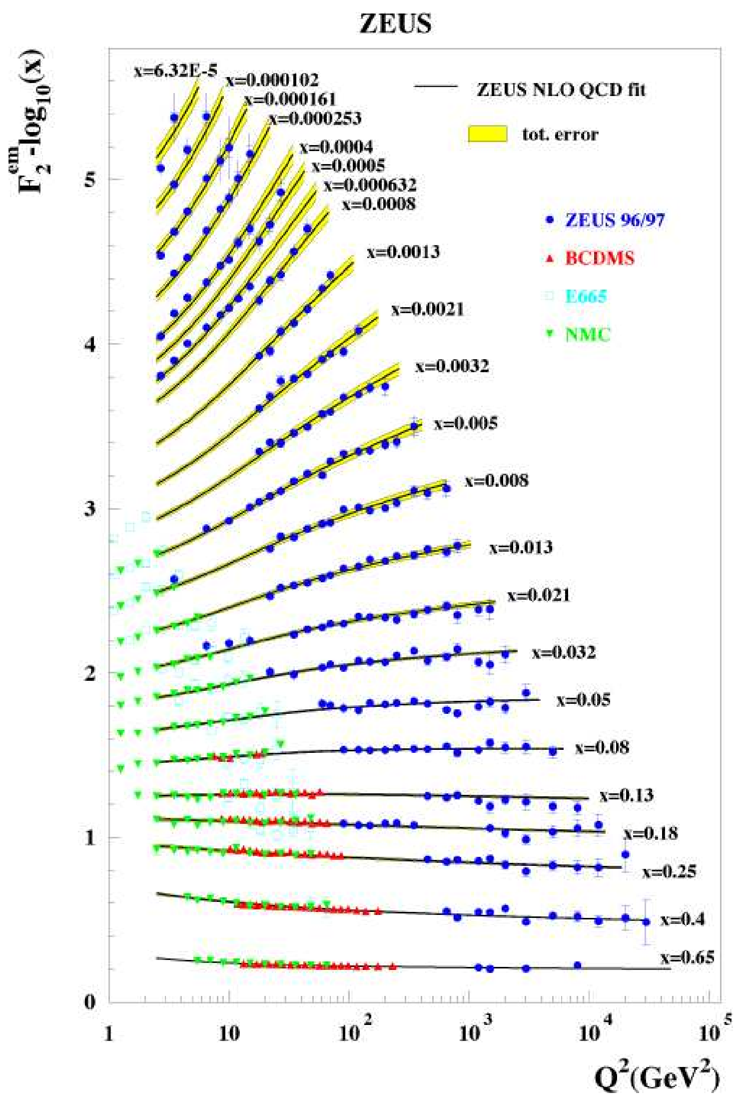
$$\left|\psi(z, x_\perp, Q^2)\right|^2 = f(\overline{Q}_f x_\perp)$$

with $\overline{Q}_f^2 \equiv m_f^2 + Q^2 z^2 (1 - z^2)$

- If one neglects the quark masses, the scaling properties of $\langle \boldsymbol{T} \rangle_Y$ imply that $\sigma_{\gamma^* p}$ depends only on the ratio $Q^2/Q_s^2(Y)$, rather than on Q^2 and Y separately

Geometrical scaling in DIS

- HERA data as a function of Q^2 and x :



Geometrical scaling in DIS

Stasto, Golec-Biernat, Kwiecinski (2000)