Some Theoretical Aspects of the Schwinger Mechanism

François Gelis

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Institut de
Physique Théorique
CEA/DRF Saclay

Outline

QED in external fields

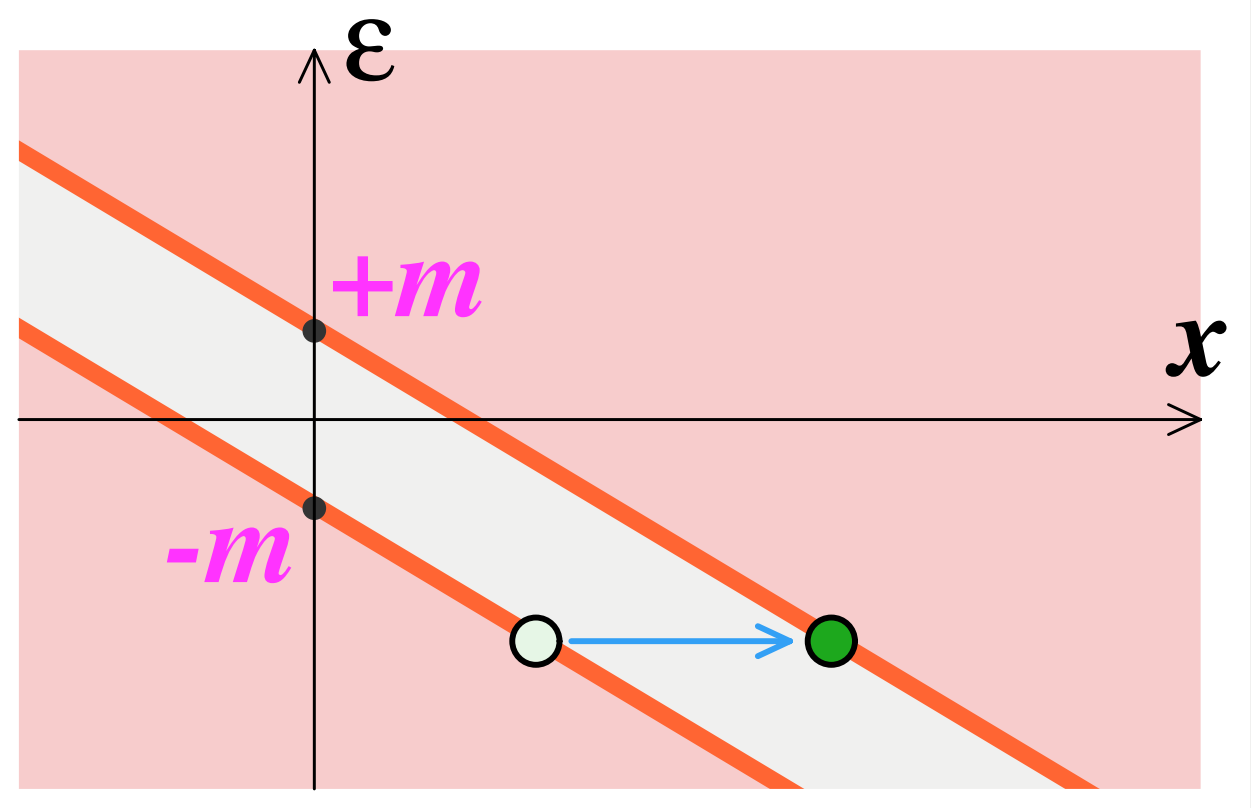
$$\mathcal{L} = \overline{\psi}\big(i\not{\partial} - m + \not{A} + \not{A}_{\mathrm{ext}}\big)\psi$$

- A^{μ}: dynamical photon field (responsible for radiation, arises in loops)
- A^{μ}_{ext}: external photon field (controlled by experimental setup, not quantized)

Momentum conservation

- The Fourier components of a time independent external field all have zero frequency
- By momentum conservation, it seems that this field cannot produce anything that has an energy > 0 (e.g., an $e^- e^+$ pair has a minimal energy $2m$)
- This is definitely correct in perturbation theory, i.e., for contributions that one may obtain from an expansion in powers of the external field
- There are also contributions in $\exp(-m^2/eE_{\text{ext}})$, non-analytic in the external field (at least for a constant E_{ext}), for which perturbation theory has nothing to say $\rightarrow$ Schwinger effect [Schwinger, 1953]

Schwinger effect ∼ tunneling



Generalities

$$\mathcal{P}_{\text{pair prod}} \sim \exp(-\pi m^2/eE)$$

- All Taylor coefficients about $e = 0$ are zero (when E is not constant but has only Fourier modes of very low frequency, the dominant Taylor coefficients are at a very large order)
- Critical field: $E_c \equiv m^2/e$ (extremely large even for the lightest known charged particle, the electron)
- Numerically more important in QCD, since the strong coupling is much larger

Outline

- Part I:
  - Quantum Fields coupled to (strong) external sources
  - Correlations in the Schwinger effect
  - Bogoliubov transformation
- Part II:
  - Numerical evaluation on the lattice
  - Worldline formalism
  - Dynamically assisted Schwinger effect

Quantum Fields coupled to external sources

[FG, Lappi, Venugopalan, 2006-2008]

Toy model: scalar fields with an external source

$$\mathcal{L} \equiv \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{g^2}{4!}\phi^4 + j\phi$$

g is dimensionless in 4 dimensions, like the QCD coupling

Power counting

$$\mathrm{Order}(\text{graph}) = g^{-2} g^{N_{\text{final particles}}} g^{2N_{\text{loops}}} (gj)^{N_{\text{sources}}}$$

- Independent of N_{sources} if $gj \sim 1 \quad \rightarrow \quad$ strong source regime
- Still perturbative in g^2, but not in j
- Vacuum graphs ($N_{\text{final particles}} = 0$) $\sim g^{-2}$ with strong sources

Vacuum diagrams

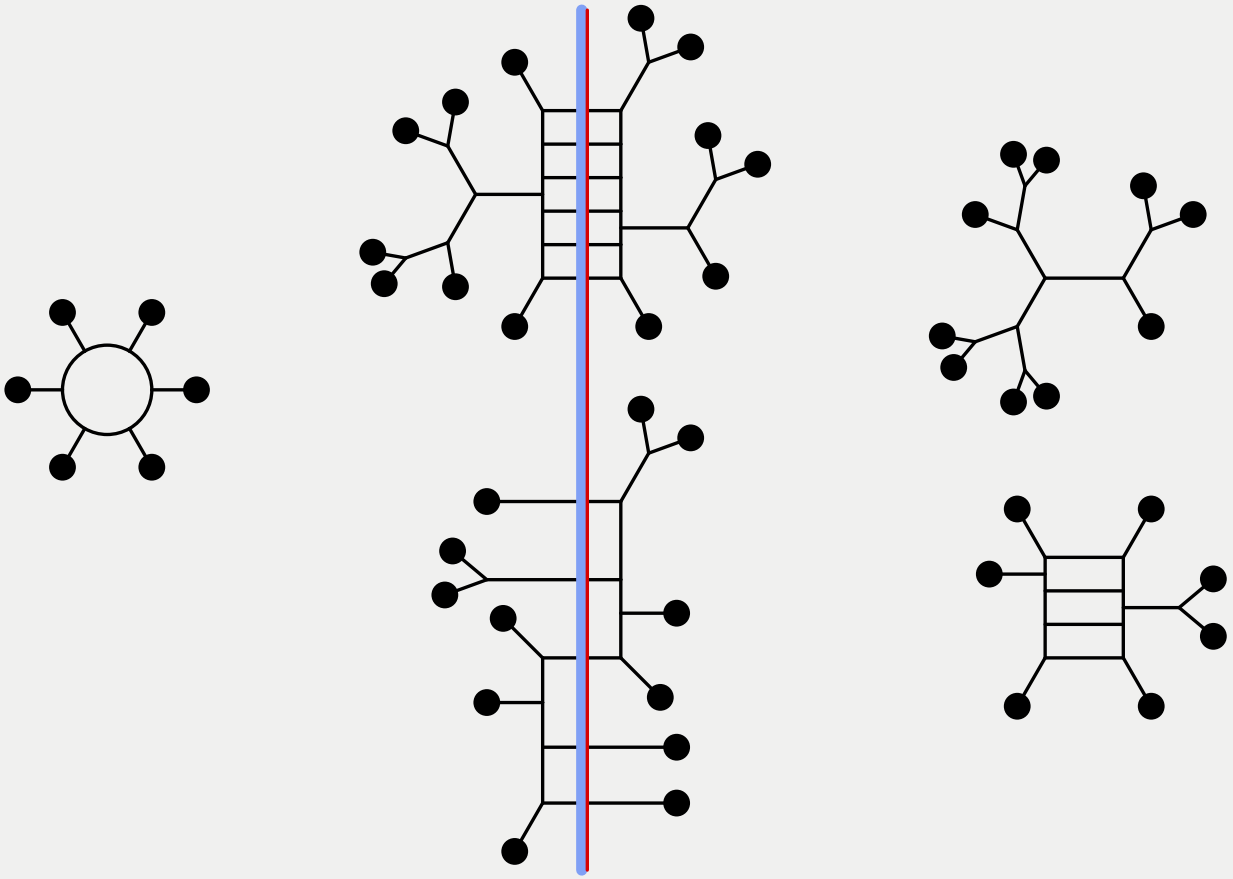
- When $j \equiv 0$: $\left|0_{\text{out}}\right\rangle = U(+\infty, -\infty)\left|0_{\text{in}}\right\rangle$ with U unitary,
  The vacuum evolves into the vacuum with probability one,
  $\left\langle 0_{\text{out}}\middle|0_{\text{in}}\right\rangle = e^{iV}$ with $V \in \mathbb{R}$,
  Vacuum graphs are purely real; their sum is a pure phase.

- When $j \neq 0$, unitarity tells this is no longer true,

$$1 = \sum_{\alpha} P(\alpha) = P(\text{vacuum}) + \sum_{\alpha \neq \text{vacuum}} P(\alpha),$$

  If $P(\alpha) \neq 0$ for some non-empty final state, then $P(\text{vacuum}) < 1$,
  $P(\text{vacuum}) = e^{-2\,\mathrm{Im}\,V}$, therefore vacuum graphs are complex,
  Their sum is not a phase, and one cannot disregard them.

Example: contribution to $P(11$ particles$)$



- Right: amplitude; Left: conjugate amplitude (dots: source j)
- Vacuum graph $\sim g^{-2}$ in the strong source regime $\rightarrow$ hopeless?

Exclusive vs Inclusive

- Probability to reach a specific final state:

$$\frac{dP_n}{d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_n} = \frac{1}{n!}\frac{1}{(2\pi)^3 2E_{\mathbf{p}_1}} \cdots \frac{1}{(2\pi)^3 2E_{\mathbf{p}_n}} \left|\langle \mathbf{p}_1 \cdots \mathbf{p}_n{}_{\text{out}} | 0_{\text{in}} \rangle\right|^2$$

  - Vacuum graphs do not cancel
  - Each of them is exponentially suppressed (tiny probability to reach a given final state)

- Inclusive particle spectrum:

$$\frac{dN_1}{d^3\mathbf{p}} \equiv \sum_{n=0}^{\infty} (n+1) \int d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_n \; \frac{dP_{n+1}}{d^3\mathbf{p}\, d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_n}$$

  - Vacuum graphs DO cancel
  - The moments have a well defined series expansion in g
  - At LO in g, can be expressed in terms of retarded classical fields

Combinatorics of multiparticle production

The probability of producing n particles can always be parameterized as

$$P_n = e^{-a} \sum_{p=1}^{n} \frac{1}{p!} \sum_{r_1+\cdots+r_p=n} b_{r_1} \cdots b_{r_p}$$

- $p =$ number of clusters (sets of correlated particles)
- $a =$ mean number of clusters
- $b_r =$ mean number of clusters with r particles
- Unitarity: $a = b_1 + b_2 + \cdots$
- Vacuum persistence probability: $P_0 = e^{-a}$
- Moments: $\langle n \rangle = \sum_r r b_r$, $\langle n^2 \rangle - \langle n \rangle^2 = \sum_r r^2 b_r$, etc...
- In general, $P_0 \neq e^{-\langle n \rangle}$ (only equal if all particles are produced uncorrelated, i.e., if there are no clusters of size > 1)
- $a, b_r \sim g^{-2}$ in the strong source regime

Schwinger-Keldysh formalism

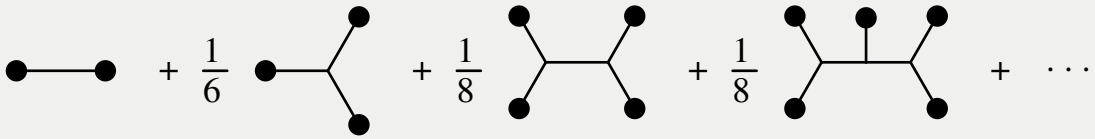
$$\langle 0_{\text{in}} | \mathcal{O} | 0_{\text{in}} \rangle = \underbrace{\sum_{\text{states } \alpha} \langle \boldsymbol{\alpha}_{\text{out}} | \mathcal{O} | \boldsymbol{\alpha}_{\text{out}} \rangle}_{\text{mixed } -+} \underbrace{\langle 0_{\text{in}} | \boldsymbol{\alpha}_{\text{out}} \rangle}_{\substack{- \text{ sector} \\ \text{(conjugate rules)}}} \underbrace{\langle \boldsymbol{\alpha}_{\text{out}} | 0_{\text{in}} \rangle}_{\substack{+ \text{ sector} \\ \text{(normal rules)}}}$$

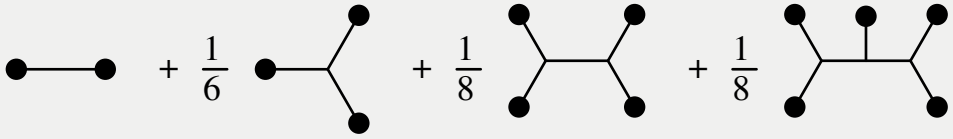
$$G^0_{++}(p) = \frac{i}{p^2 - m^2 + i\epsilon}, \qquad G^0_{--}(p) = \frac{-i}{p^2 - m^2 - i\epsilon}$$

$$G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2), \qquad G^0_{-+}(p) = 2\pi\theta(+p^0)\delta(p^2 - m^2)$$

Connection to Feynman graphs

$$\langle 0_{\text{out}} | 0_{\text{in}} \rangle = e^{i\,V}$$

$i\,V \equiv$ 

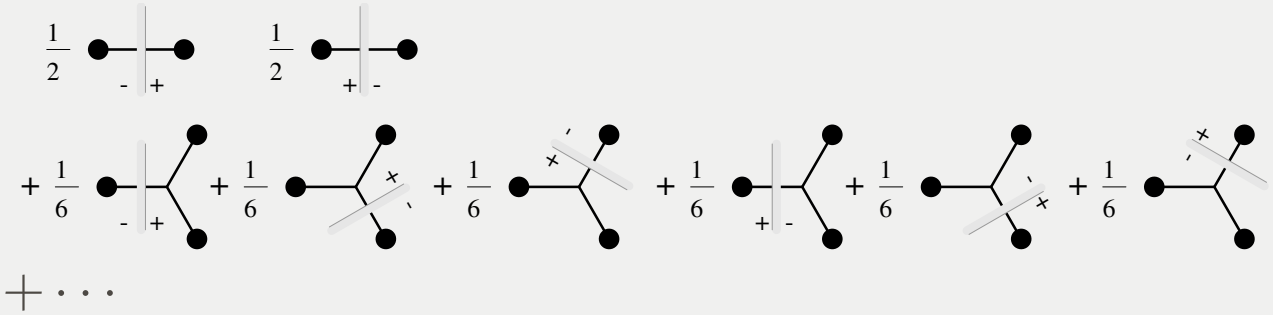
$a = 2\,\mathrm{Im}\,(V) = \sum_{\text{cuts}}$ 

$=$ sum of connected vacuum diagrams in SK formalism

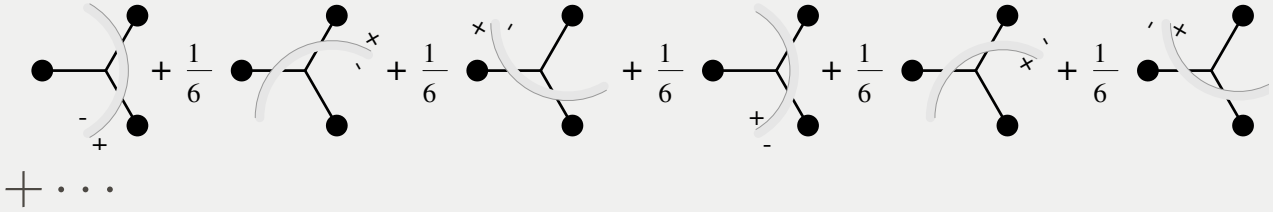
Note: vacuum graphs are all zero in SK formalism without external sources, but non-zero if $j \neq 0$

Connection to Feynman graphs (cont.)

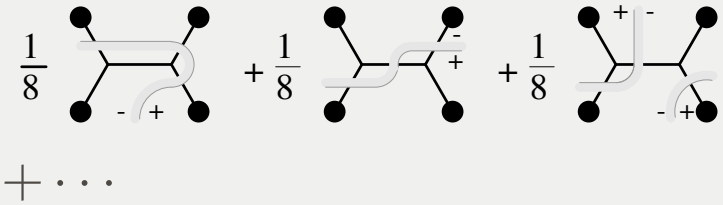
$b_1 =$



$b_2 =$

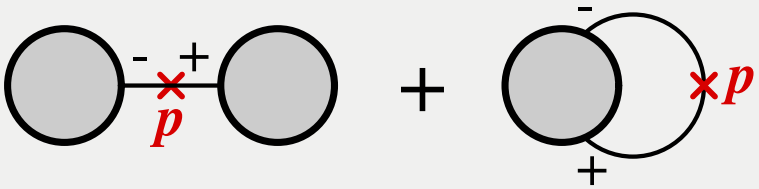


$b_3 =$



Inclusive spectrum

$$\frac{dN_1}{d^3\mathbf{p}} \propto$$



- The gray blobs are the SK 1-point ($\varphi_\pm(x)$) and 2-point ($\mathcal{G}_{-+}(x,y)$) connected correlation functions
- This formula is exact (to all orders in g and j)
- Strong source regime: $\varphi_\pm \sim g^{-1}$, $\mathcal{G}_{-+}(x,y) \sim 1$,
  $\longrightarrow$ The first term dominates.

Inclusive spectrum at Leading Order

At tree level:

$$(\Box + m^2)\varphi_\pm + \frac{g^2}{6}\varphi_\pm^3 = j,$$

$$\lim_{x^0 \to -\infty} \varphi_\pm(x) = 0$$

- Classical EOM with retarded boundary conditions
  $\longrightarrow$ numerically straightforward
- $\varphi_\pm$ are equal, and real valued
- Given the Fourier decomposition of these classical fields

$$\varphi(y) \equiv \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \left[f(y^0, \mathbf{p})\, e^{-ip\cdot y} + \text{c.c.} \right],$$

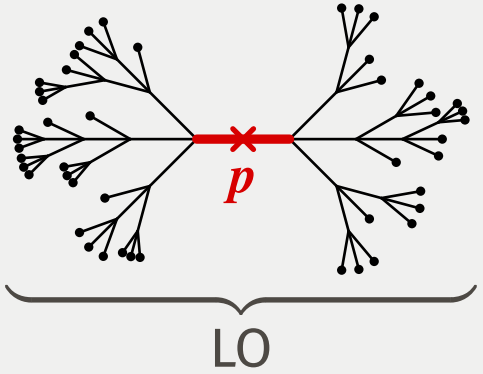
  the LO spectrum reads

$$\left.\frac{dN_1}{d^3\mathbf{p}}\right|_{\text{LO}} = \frac{1}{(2\pi)^3 2E_\mathbf{p}} \; |f(+\infty, \mathbf{p})|^2$$

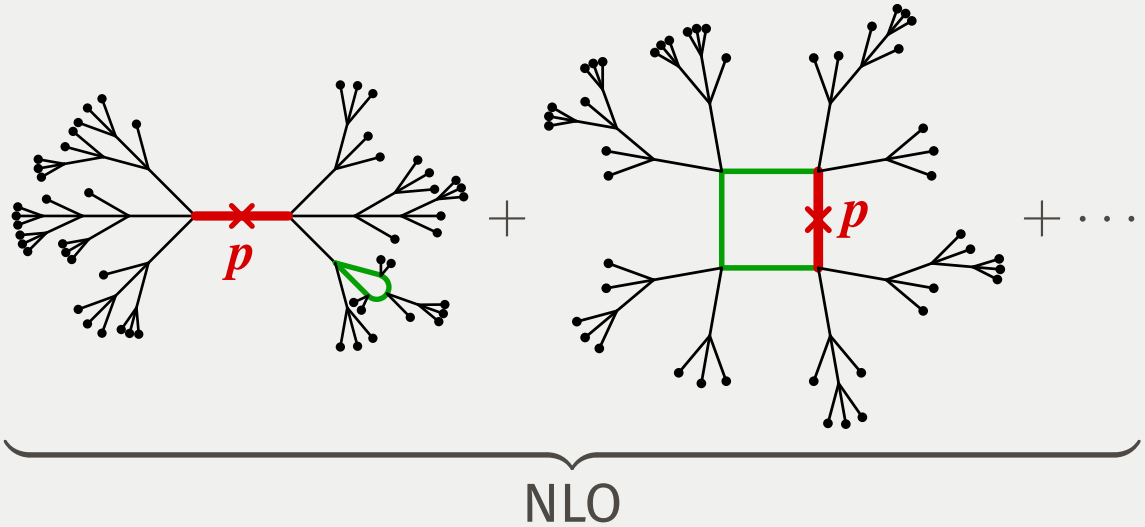
- Note: at this order, only particles that couple directly to the sources can be produced (e.g., gluons in QCD)

LO + NLO

$$\frac{dN_1}{d^3\mathbf{p}} =$$



$+$



LO + NLO (additional remarks)

- LO: can only produce particles that couple directly to the sources
- Static sources: the LO gives zero
  (same for the first of the NLO terms)
- NLO: a different particle may run in the loop $\rightarrow$ can produce particles that do not couple directly to the source
  (e.g., electrons, quarks)
- Second NLO graph: contains a non-analytic contribution when the source is static

NOTE: EXCLUSIVE QUANTITIES ARE (MUCH!) HARDER!

Simplest example:

$$\frac{dP_1}{d^3\mathbf{p}} = e^{-a}\, b_1(\mathbf{p})$$

- At LO: $a, b_1(\mathbf{p})$ given by classical fields with **non retarded** boundary conditions
- Although not impossible in principle, very hard in practice

Correlations in the Schwinger effect

[Fukushima, FG, Lappi, 2009]

Setup

- Consider scalar QED for simplicity:

$$\mathcal{L} \equiv (D_\mu \phi)(D^\mu \phi)^* - m^2 \phi \phi^* \ ,$$

- Generating functional:

$$\mathcal{F}[z, \bar{z}] \equiv \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \int \prod_{i=1}^{m} \frac{d^3 \mathbf{p}_i}{(2\pi)^3 2E_{\mathbf{p}_i}} \, z(\mathbf{p}_i) \prod_{j=1}^{n} \frac{d^3 \mathbf{q}_j}{(2\pi)^3 2E_{\mathbf{q}_j}} \, \bar{z}(\mathbf{q}_j)$$
$$\times \left| \left\langle \underbrace{\mathbf{p}_1 \cdots \mathbf{p}_m}_{\text{particles}} \underbrace{\mathbf{q}_1 \cdots \mathbf{q}_n}_{\text{antiparticles}} \, \text{out} \middle| 0_{\text{in}} \right\rangle \right|^2$$

Inclusive spectra

$$\frac{dN_1^+}{d^3\mathbf{p}} = \left.\frac{\delta\mathcal{F}[z,\bar{z}]}{\delta z(\mathbf{p})}\right|_{z=\bar{z}=1}, \qquad \frac{dN_1^-}{d^3\mathbf{q}} = \left.\frac{\delta\mathcal{F}[z,\bar{z}]}{\delta \bar{z}(\mathbf{q})}\right|_{z=\bar{z}=1}$$

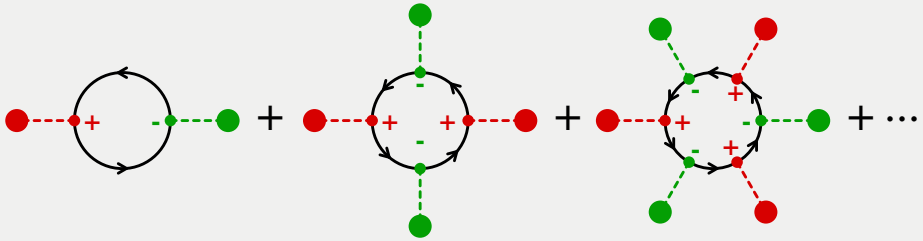
$$\frac{dN_1^{++}}{d^3\mathbf{p}_1 d^3\mathbf{p}_2} = \left.\frac{\delta^2\mathcal{F}[z,\bar{z}]}{\delta z(\mathbf{p}_1)\delta z(\mathbf{p}_2))}\right|_{z=\bar{z}=1}$$

$$\frac{dN_1^{--}}{d^3\mathbf{q}_1 d^3\mathbf{q}_2} = \left.\frac{\delta^2\mathcal{F}[z,\bar{z}]}{\delta \bar{z}(\mathbf{q}_1)\delta \bar{z}(\mathbf{q}_2))}\right|_{z=\bar{z}=1}$$

$$\frac{dN_1^{+-}}{d^3\mathbf{p} d^3\mathbf{q}} = \left.\frac{\delta^2\mathcal{F}[z,\bar{z}]}{\delta z(\mathbf{p})\delta \bar{z}(\mathbf{q}))}\right|_{z=\bar{z}=1}$$

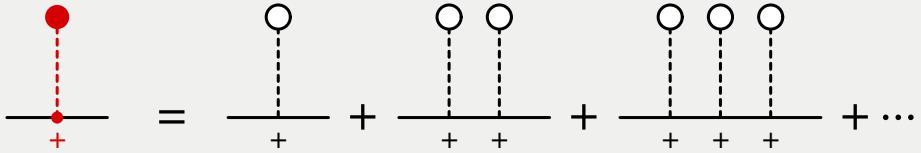
One-Loop generating functional

$$\ln \mathcal{F}[z,\bar{z}] = \text{constant} +$$

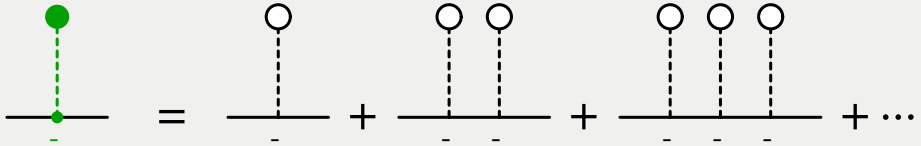


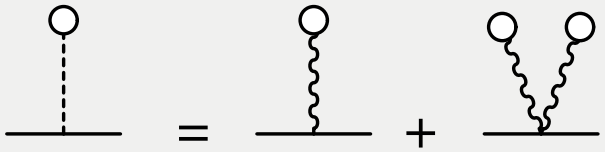
$$= \text{constant} - \mathrm{tr}\,\ln\left(1 - \mathcal{T}_+\left(z\mathrm{G}^0_{+-}\right)\mathcal{T}_-\left(\bar{z}\mathrm{G}^0_{-+}\right)\right)$$

$\mathcal{T}_+ \equiv$



$\mathcal{T}_- \equiv$





After some massaging...

- For a real external field, the time-ordered scattering matrices $\mathcal{T}_\pm$ can be related to the retarded one, $\mathcal{T}_R$
- For a spatially homogeneous external field:

$$\ln \mathcal{F}[z,\bar{z}] = \text{constant} - V \int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln\left[1 - (z(\mathbf{p})\bar{z}(-\mathbf{p}) - 1)\, f_{\mathbf{p}}\right]$$

$$i\,\mathcal{T}_R(p,-k) = 2E_{\mathbf{p}}\,(2\pi)^3\delta(\mathbf{p}+\mathbf{k})\,\beta_{\mathbf{p}}, \quad f_{\mathbf{p}} \equiv \left|\beta_{\mathbf{p}}\right|^2$$

- In practice: $\mathcal{T}_R(p,-k)$ is obtained by solving the classical EOM for ϕ, starting in the past with a negative frequency plane wave of momentum $-k$ and projecting it in the future on a positive frequency plane wave of momentum p

Spectra

$$\frac{dN_1^+}{d^3\mathbf{p}} = \frac{dN_1^-}{d^3\mathbf{p}} = \underbrace{\frac{V}{(2\pi)^3}\, f_{\mathbf{p}}}_{\equiv\, n_{\mathbf{p}}}$$

$$\frac{dN_2^{++}}{d^3\mathbf{p}\, d^3\mathbf{p}'} - \frac{dN_1^+}{d^3\mathbf{p}}\frac{dN_1^+}{d^3\mathbf{p}'} = \delta(\mathbf{p}-\mathbf{p}')\, n_{\mathbf{p}}\, f_{\mathbf{p}}$$

$$\frac{dN_2^{+-}}{d^3\mathbf{p}\, d^3\mathbf{q}} - \frac{dN_1^+}{d^3\mathbf{p}}\frac{dN_1^-}{d^3\mathbf{q}} = \delta(\mathbf{p}+\mathbf{q})\, n_{\mathbf{p}}\, (1+f_{\mathbf{p}})$$

- No correlations at different momenta for particles
- Particles and antiparticles of opposite momenta are correlated
- Consistent with Poisson distribution if $f_{\mathbf{p}} \ll 1$
- In general: Bose enhancement

Probability distribution

- For m particles and n antiparticles of momentum $\mathbf{k}$:

$$P_{\mathbf{k}}(m,n) = \delta_{m,n}\,\frac{1}{1+f_{\mathbf{k}}}\,\left(\frac{f_{\mathbf{k}}}{1+f_{\mathbf{k}}}\right)^{n}$$

  Note: longer tails than Poisson (Bose enhancement)

- Vacuum persistence probability:

$$P_0 = \exp\left\{-V\underbrace{\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\ln(1+f_{\mathbf{k}})}_{\neq\int_{\mathbf{k}} f_{\mathbf{k}}}\right\}$$

Bogoliubov Transformation

Mode functions ($A^0 = 0$ gauge)

$$\phi(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{a_{\mathbf{k},\mathrm{in}}}{\sqrt{2E_{\mathbf{k}}^{\mathrm{in}}}} \phi^+_{\mathbf{k},\mathrm{in}}(x) + \frac{b^\dagger_{\mathbf{k},\mathrm{in}}}{\sqrt{2E_{-\mathbf{k}}^{\mathrm{in}}}} \phi^-_{-\mathbf{k},\mathrm{in}}(x) \right]$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{a_{\mathbf{k},\mathrm{out}}}{\sqrt{2E_{\mathbf{k}}^{\mathrm{out}}}} \phi^+_{\mathbf{k},\mathrm{out}}(x) + \frac{b^\dagger_{\mathbf{k},\mathrm{out}}}{\sqrt{2E_{-\mathbf{k}}^{\mathrm{out}}}} \phi^-_{-\mathbf{k},\mathrm{out}}(x) \right]$$

$$E_{\mathbf{k}}^{\mathrm{in,out}} = \sqrt{m^2 + (\mathbf{k} + e\mathbf{A}^{\mathrm{in,out}})^2}$$

$$\phi^+_{\mathbf{k},\mathrm{in}}(x) = e^{-iE_{\mathbf{k}}^{\mathrm{in}} x^0 + i\mathbf{k}\cdot\mathbf{x}} \qquad \text{for } x^0 \to -\infty$$

$$\phi^-_{\mathbf{k},\mathrm{in}}(x) = e^{iE_{\mathbf{k}}^{\mathrm{in}} x^0 + i\mathbf{k}\cdot\mathbf{x}} \qquad \text{for } x^0 \to -\infty$$

$$\phi^+_{\mathbf{k},\mathrm{out}}(x) = e^{-iE_{\mathbf{k}}^{\mathrm{out}} x^0 + i\mathbf{k}\cdot\mathbf{x}} \qquad \text{for } x^0 \to +\infty$$

$$\phi^-_{\mathbf{k},\mathrm{out}}(x) = e^{iE_{\mathbf{k}}^{\mathrm{out}} x^0 + i\mathbf{k}\cdot\mathbf{x}} \qquad \text{for } x^0 \to +\infty$$

Bogoliubov transformation

- EOM linear: there is a linear mapping between the coefficients in the in and out representations
- Spatially homogeneous background: no mixing between the modes $\mathbf{k}$

$$a_{\mathbf{k},\mathrm{out}} = \alpha_{\mathbf{k}}\, a_{\mathbf{k},\mathrm{in}} + \beta_{\mathbf{k}}\, b^{\dagger}_{-\mathbf{k},\mathrm{in}}, \quad b^{\dagger}_{\mathbf{k},\mathrm{out}} = \alpha^{*}_{-\mathbf{k}}\, b^{\dagger}_{\mathbf{k},\mathrm{in}} + \beta^{*}_{-\mathbf{k}}\, a_{-\mathbf{k},\mathrm{in}}$$

- Consistency with canonical commutation relations: $|\alpha_{\mathbf{k}}|^2 - |\beta_{\mathbf{k}}|^2 = 1$
- Inverse mapping:

$$a_{\mathbf{k},\mathrm{in}} = \alpha^{*}_{\mathbf{k}}\, a_{\mathbf{k},\mathrm{out}} - \beta_{\mathbf{k}}\, b^{\dagger}_{-\mathbf{k},\mathrm{out}}, \quad b_{\mathbf{k},\mathrm{in}} = \alpha^{*}_{-\mathbf{k}}\, b_{\mathbf{k},\mathrm{out}} - \beta_{-\mathbf{k}}\, a^{\dagger}_{-\mathbf{k},\mathrm{out}}$$

In and Out vacua

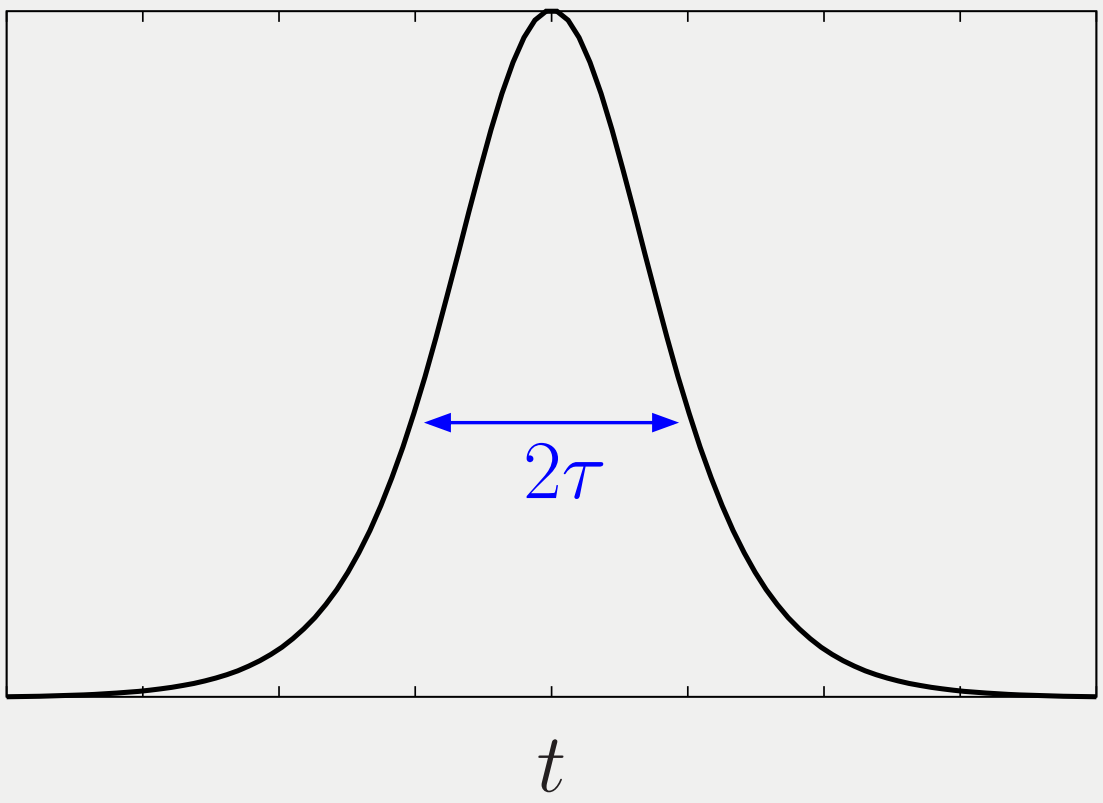
- In and Out vacua are related by:

$$\left|0_{\text{in}}\right\rangle = e^{-\frac{V}{2}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\ln(1+|\beta_{\mathbf{p}}|^2)}\, e^{\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\beta_{\mathbf{p}}}{\alpha^*_{\mathbf{p}}}\,a^\dagger_{\mathbf{p},\text{out}}\,b^\dagger_{-\mathbf{p},\text{out}}}\left|0_{\text{out}}\right\rangle$$

- This formula contains all the information about final state particle content when the initial state is empty
- Example:

$$\frac{dN_1^+}{d^3\mathbf{p}} = \frac{1}{(2\pi)^3}\left\langle 0_{\text{in}}\right| a^\dagger_{\mathbf{p},\text{out}}\, a_{\mathbf{p},\text{out}}\left|0_{\text{in}}\right\rangle = \frac{V}{(2\pi)^3}\,|\beta_{\mathbf{p}}|^2$$

Example: Sauter potential



$$E_z(t) = \frac{E}{\cosh^2(t/\tau)} \quad (\tau \sim \text{pulse duration})$$

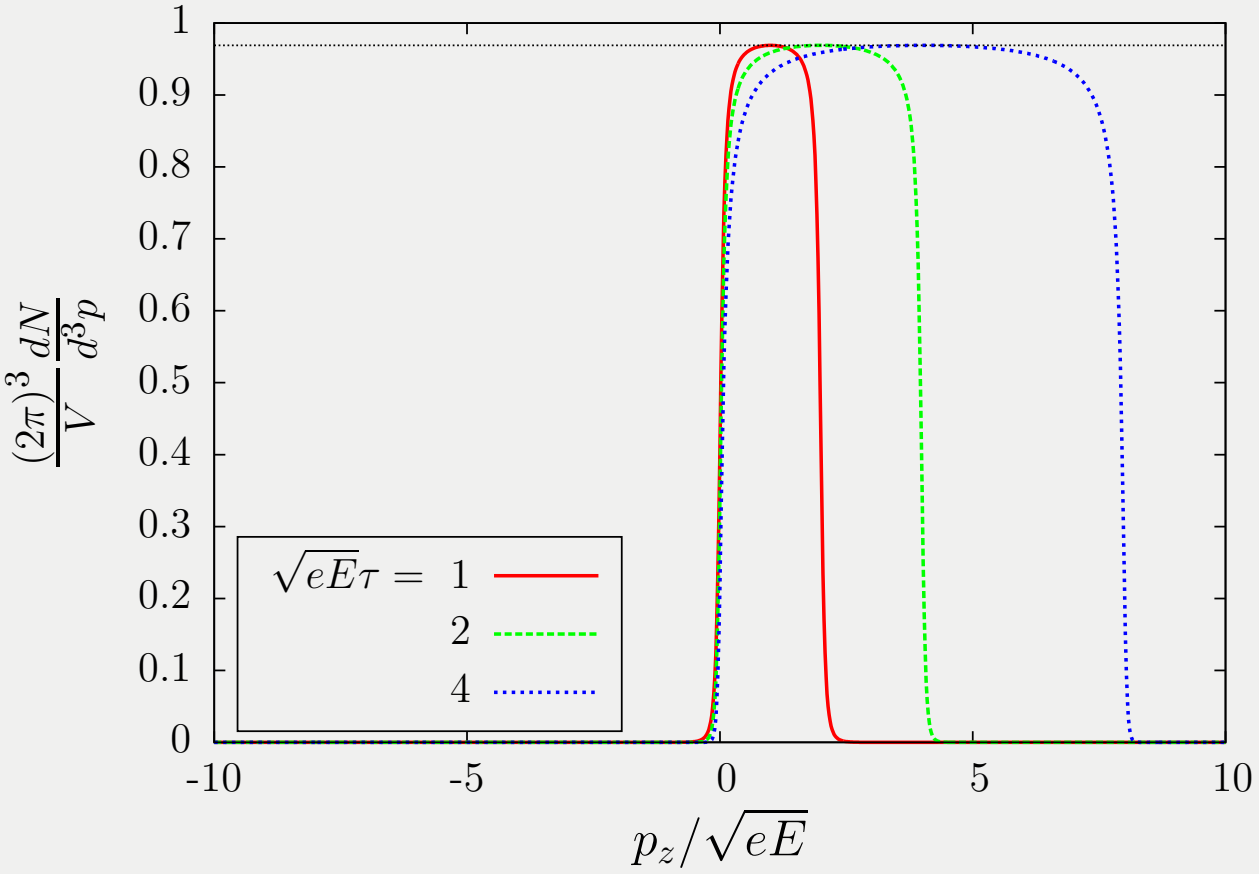
Sauter potential: inclusive spectrum at $t = +\infty$

$$\frac{dN_1}{d^3\mathbf{p}} = \frac{V}{(2\pi)^3}\left(\frac{\sinh\left[\pi(\lambda+\mu-\nu)\right]\sinh\left[\pi(\lambda-\mu+\nu)\right]}{\sinh\left(2\pi\mu\right)\sinh\left(2\pi\nu\right)}\right)$$

$$\mu \equiv \frac{\tau}{2}\sqrt{m^2+p_\perp^2+(p_z-2eE\tau)^2}$$
$$\nu \equiv \frac{\tau}{2}\sqrt{m^2+p_\perp^2+p_z^2}$$
$$\lambda \equiv eE\tau^2$$

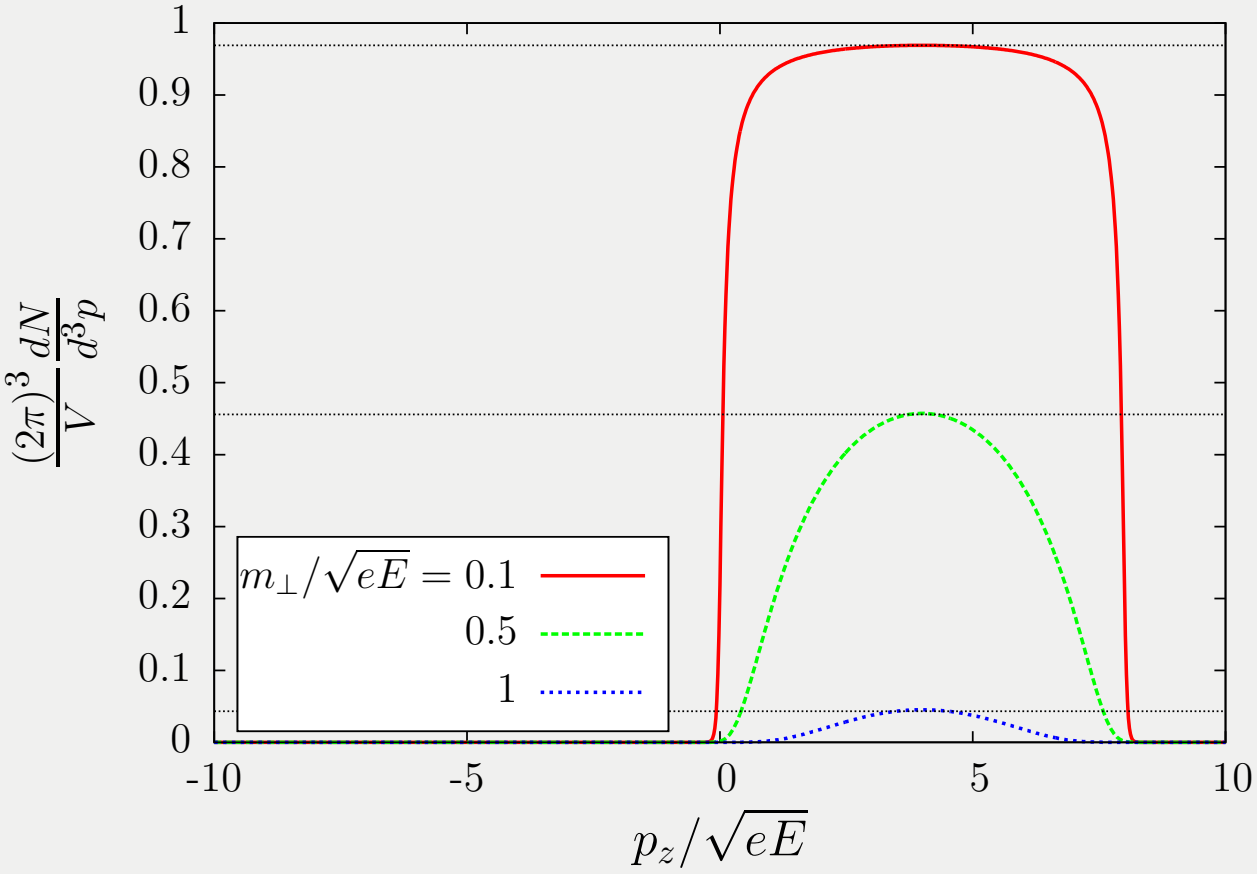
Note: analytic in eE as long as $\tau < \infty$

Sauter potential: p_z spectrum at fixed $p_\perp$



Thin horizontal dotted line: $\exp(-\pi(p_\perp^2 + m^2)/(eE))$

Sauter potential: p_z spectrum at various $p_\perp$ (fixed $\sqrt{eE}\tau = 4$)



Thin horizontal dotted line: $\exp(-\pi(p_\perp^2 + m^2)/(eE))$

Numerical evaluation on the lattice

Spectrum for a general external field

$$\frac{dN_1}{d^3\mathbf{p}} = \frac{1}{(2\pi)^3 2E_{\mathbf{p}}} \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} \left| \int d^3\mathbf{x}\, \phi^{+\dagger}_{\mathbf{p},\mathrm{out}}(x) \left(\partial_t - iE_{\mathbf{p}}\right) \phi^{-}_{\mathbf{k},\mathrm{in}}(x) \right|^2_{x_0=+\infty}$$

- Note: in general, time evolution non-diagonal in $\mathbf{p}$
- This observable belongs to a generic class of objects that can be written as

$$\left\langle \phi^\dagger \mathbf{M} \phi \right\rangle \equiv \int \frac{d^3\mathbf{k}}{(2\pi)^3 2E_{\mathbf{k}}} \left[\phi^{-\dagger}_{\mathbf{k},\mathrm{in}} \mathbf{M} \phi^{-}_{\mathbf{k},\mathrm{in}} \right]_{x^0=y^0=+\infty}$$

($\mathbf{M}_{xy}$ =Hermitean operator)

"Brute force" lattice approach

- Discretize space as a $N \times N \times N$ lattice
- Use link variables instead of A^i to ensure gauge invariance
- N^3 conjugate momenta
- Solve the EOM for N^3 incoming mode functions
- Numerical cost $\sim N_t \times N^3 \times N^3$ ($N_t =$ number of time steps)
  $\rightarrow$ quite unfavorable scaling

- Note: if the field is independent of some of the coordinates, this cost can be reduced since the evolution conserves the corresponding momentum

Statistical sampling [FG, Tanji, 2013]

- Goal: avoid summing over all the mode functions to reduce cost
- Strategy: exploit the linearity of the EOM and solve it for a random linear superposition of all the mode functions

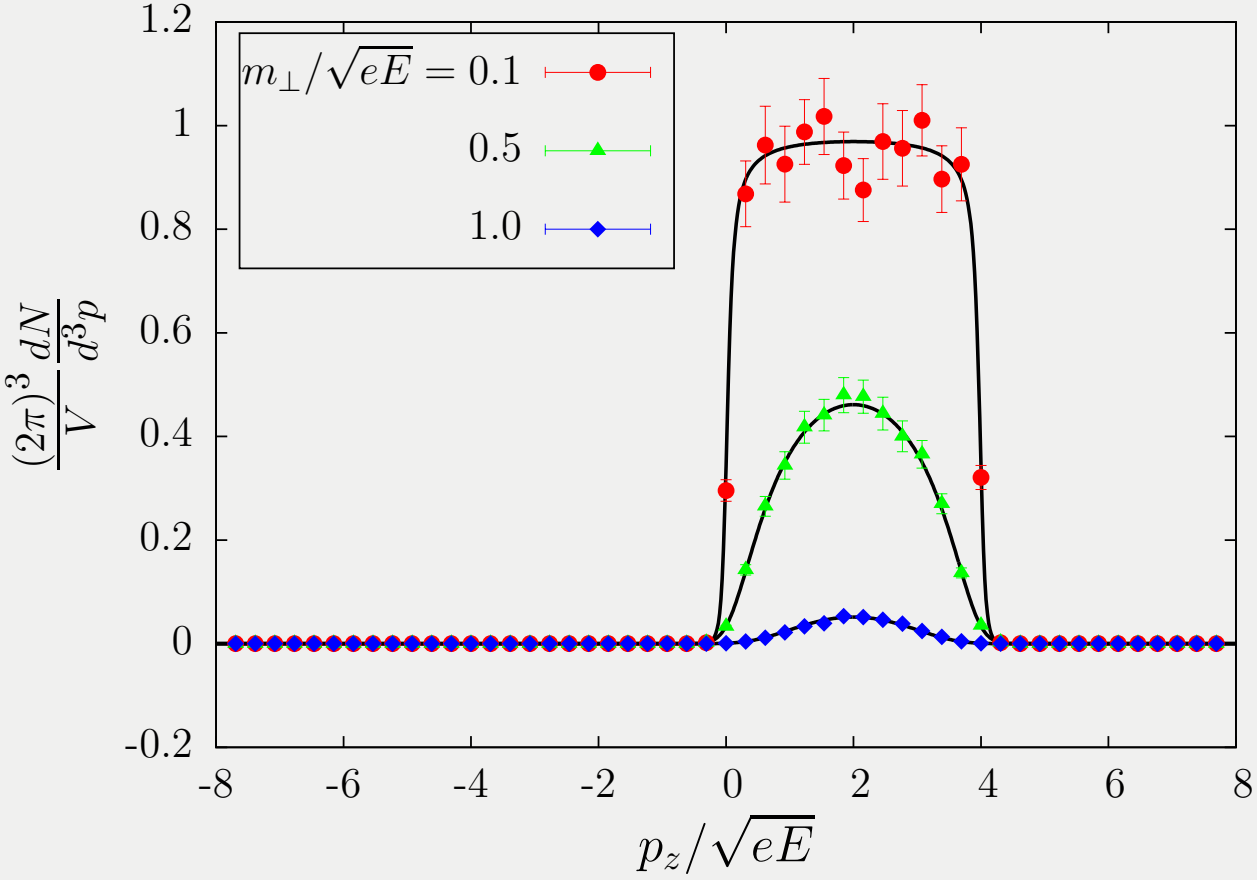
$$\phi_c^-(x) \equiv \frac{1}{\sqrt{V}} \sum_k \frac{1}{\sqrt{2E_k}}\, c_k\, \phi_{k,\mathrm{in}}^-(x)$$

$c_k =$ Gaussian random numbers with $\langle c_k \rangle = 0$ and $\langle c_k c_{k'}^* \rangle_c = \delta_{kk'}$

$$\langle \phi^\dagger M \phi \rangle = \left\langle \left[\phi_c^{-\dagger} M \phi_c^- \right]_{x^0 = y^0 = +\infty} \right\rangle_c$$

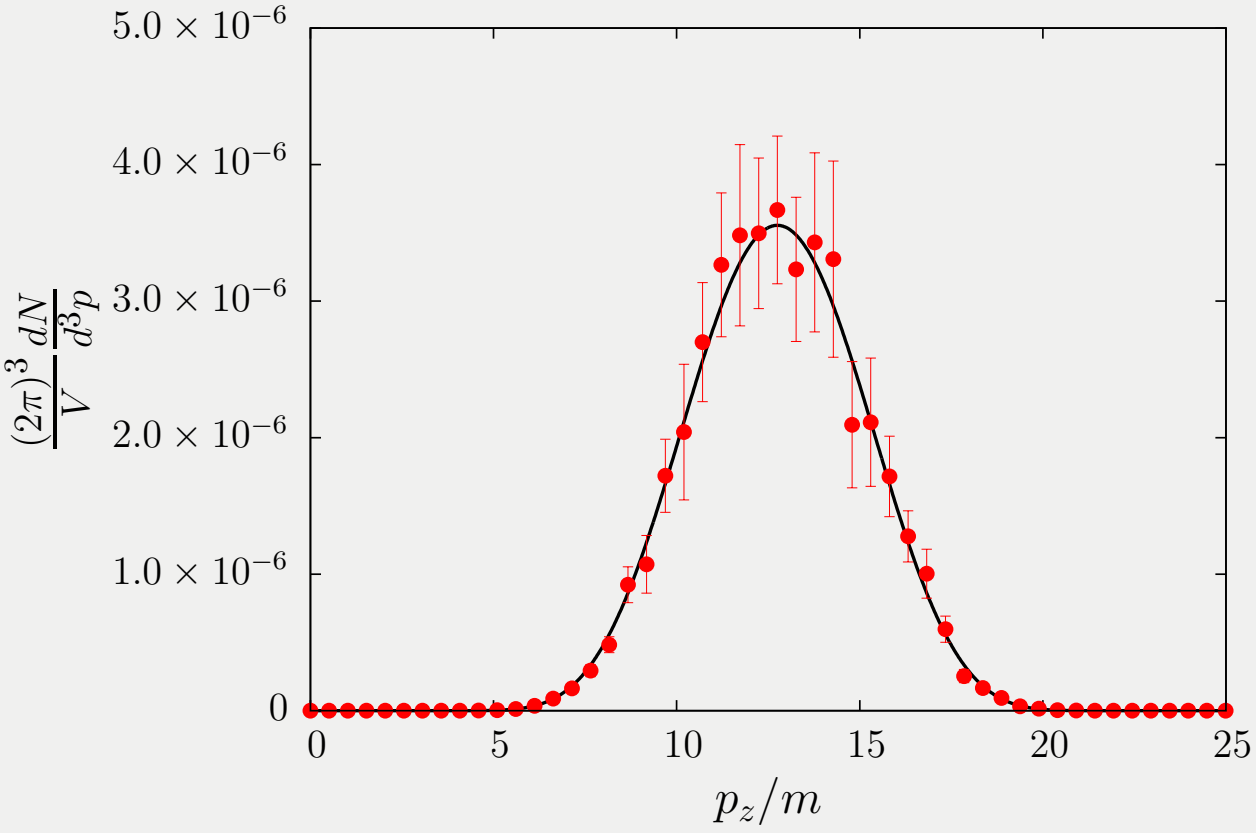
- Numerical cost: $N_t \times N^3 \times N_c$ (plus $N_c \times N^6$ for preparing the initial conditions) $\rightarrow$ favorable if $N_c \ll N^3, N_t$
- Statistical error $\sim N_c^{-1/2}$
- Related to *low cost fermions* [Borsanyi, Hindmarsh; Saffin, Tranberg; Berges, Gelfand, Sexty, Kasper, Hebenstreit, 2009-2014]

Example: Sauter field



- $N_x = N_y = 48$, $N_z = 128$
- $\sqrt{eE}\, a_x = \sqrt{eE}\, a_y = 0.42$, $\sqrt{eE}\, a_z = 0.16$,
- $N_c = 256$ ($\ll 48^2 \times 128 = 294912$)

Works also for weak fields



- $eE = 0.25\,m^2$, $\sqrt{eE}\,\tau = 25.5$
- $N_x = N_y = 48$, $N_z = 256$, $ma_z = 0.048$
- $N_c = 48$

Back Reaction

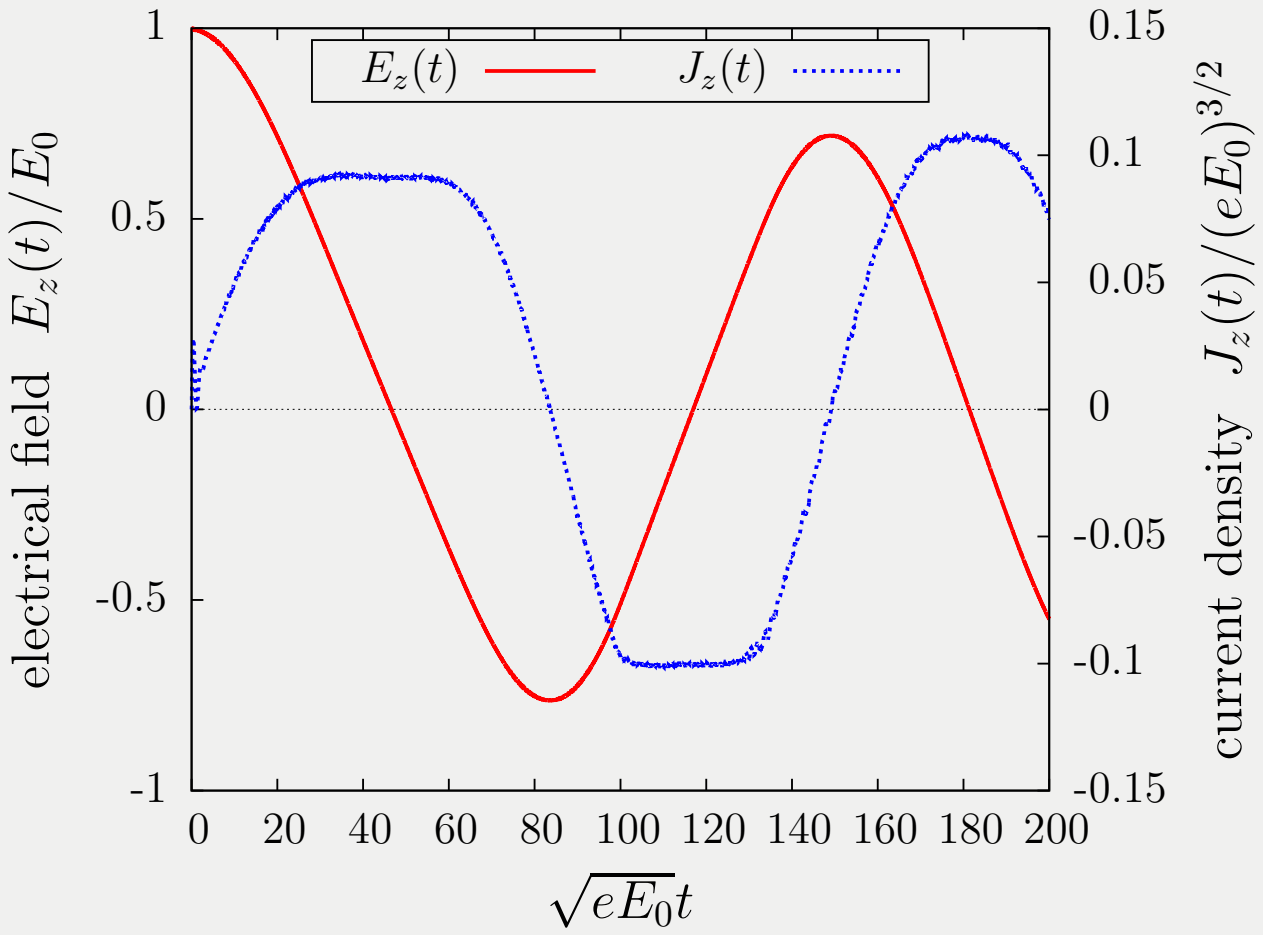
- So far, assume that the external field is unmodified by produced charged particles
- Energy is not conserved in this approximation (roughly ok if the field energy dominates)

- The produced charges screen the external field, and weaken it
- Feedback can be included by simultaneously solving Maxwell's equation:

$$\partial_\mu F^{\mu\nu}(x) = \langle \hat{J}^\nu(x) \rangle$$

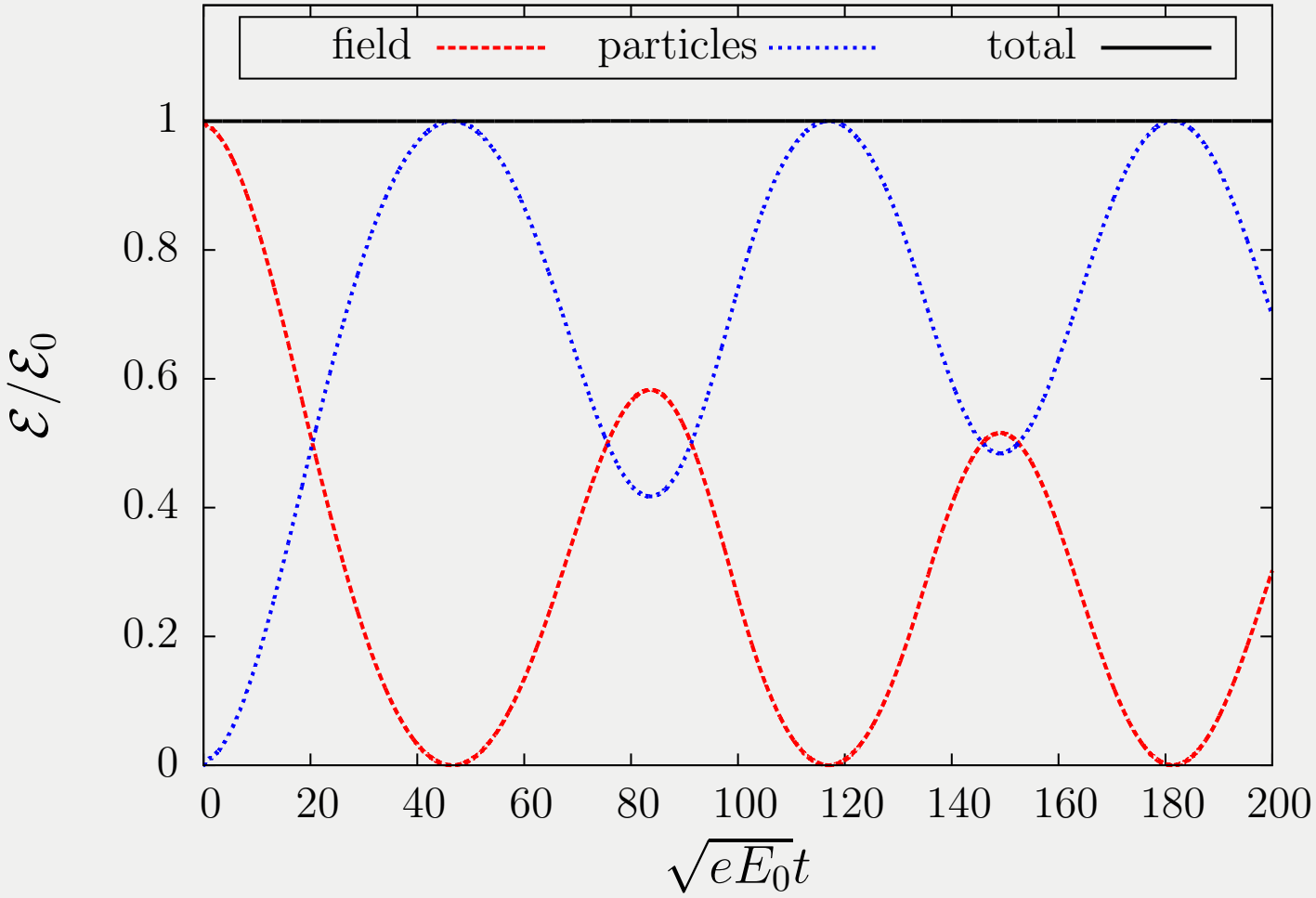
$\langle \hat{J}^\nu(x) \rangle =$ quantum expectation value of the current operator

Example: initially constant E_0



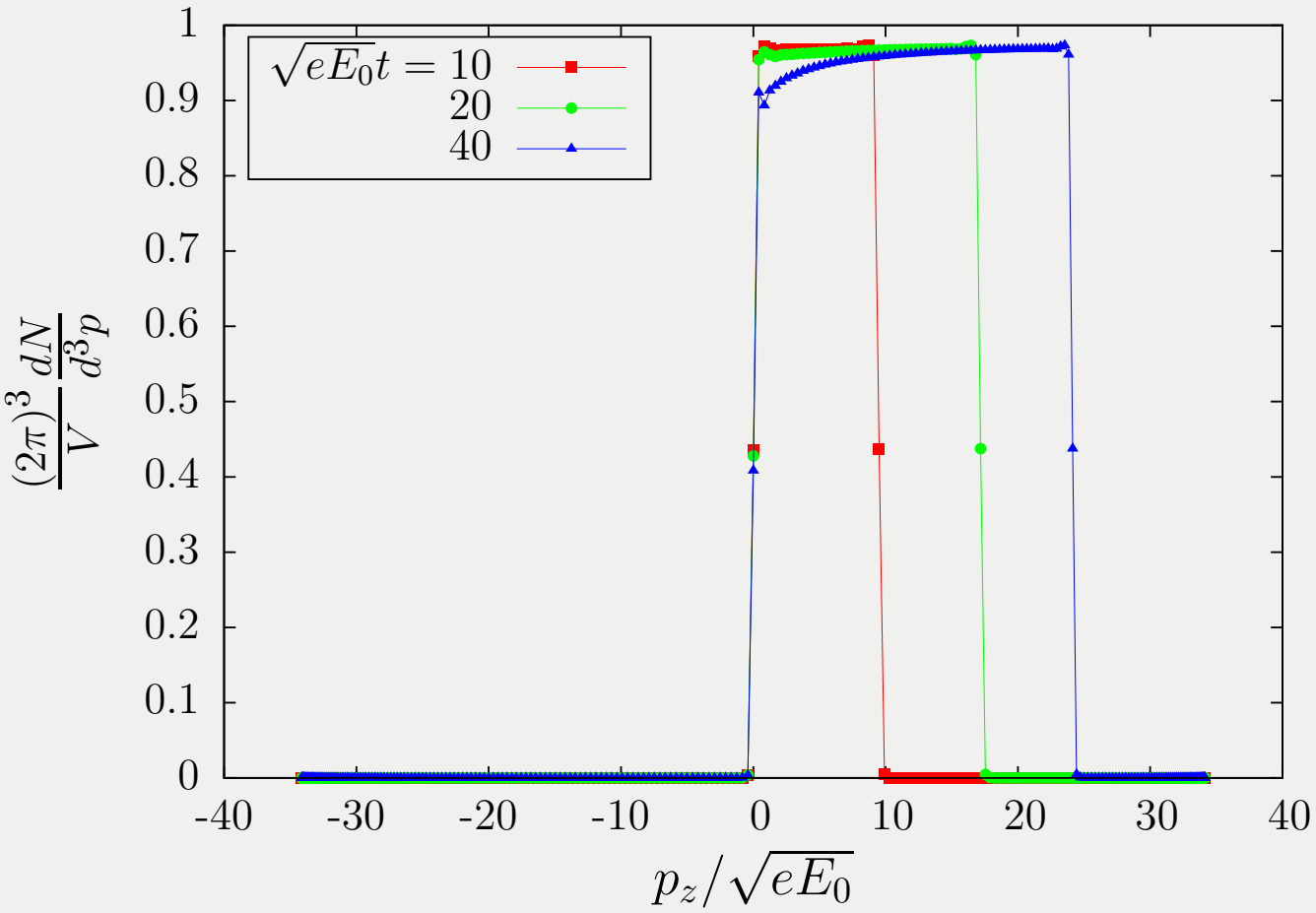
- $e = 0.3$, $m/\sqrt{eE_0} = 0.1$
- $N_x = N_y = 48$, $N_z = 512$
- $\sqrt{eE_0}\, a_x = \sqrt{eE_0}\, a_y = 0.62$, $\sqrt{eE_0}\, a_z = 0.029$

Energy conservation



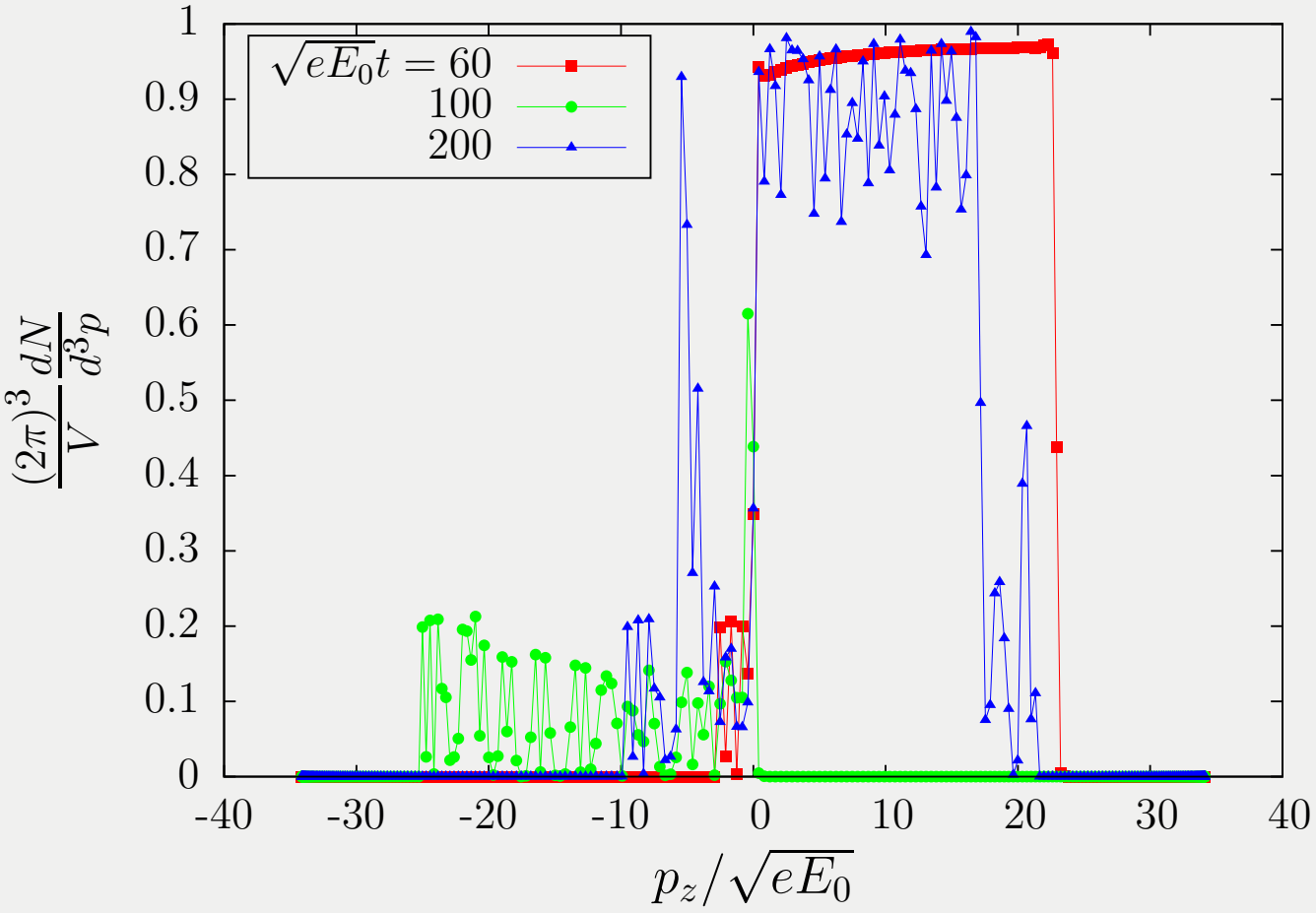
- Energy carried by the field and particles, normalized by $\mathcal{E}_0 \equiv \frac{1}{2}E_0^2$

p_z SPECTRUM (EARLY TIMES)



- Very similar to the Sauter potential (charges produced with $p_z \approx 0$ and accelerated in the $+z$ direction)

p_z SPECTRUM (LATER TIMES)



- The field direction oscillates, and the acceleration changes sign
- Existing particles encounter newly created ones, and Pauli blocking leads to interferences

Worldline Formalism

[Bern, Kosower, 1988; Strassler, 1992]
[Schubert, 1996, 2001]
[Schmidt, Schubert, 1993]
[Dunne, Schubert, 2005]

Setup

- Total particle production probability (at one loop):

$$\langle 0_{\rm out} | 0_{\rm in} \rangle = e^{i\,V}, \quad \sum_{n=1}^{\infty} P_n = 1 - P_0 = 1 - e^{-2\,{\rm Im}\,V}$$

$$i\,V = \sum \Big(\text{connected vacuum diagrams} \Big)$$

$$\text{Scalar QED}: \quad V_{1\ \rm loop} = \ln\,\det\,\big(g_{\mu\nu} D^\mu D^\nu + m^2\big)$$

- Worldline formalism is Euclidean, so consider instead:

$$V_{E,1\ \rm loop} \equiv \ln\,\det\,\big(-D^i D^i + m^2\big) = {\rm tr}\,\ln\big(-D^i D^i + m^2\big)$$

- Schwinger proper time representation:

$$\big(-D^i D^i + m^2\big)^{-1} = \int_0^\infty dT\ \exp\big(-T\big(-D^i D^i + m^2\big)\big)$$

$$\ln\big(-D^i D^i + m^2\big) = -\int_0^\infty \frac{dT}{T}\ \exp\big(-T\big(-D^i D^i + m^2\big)\big)$$

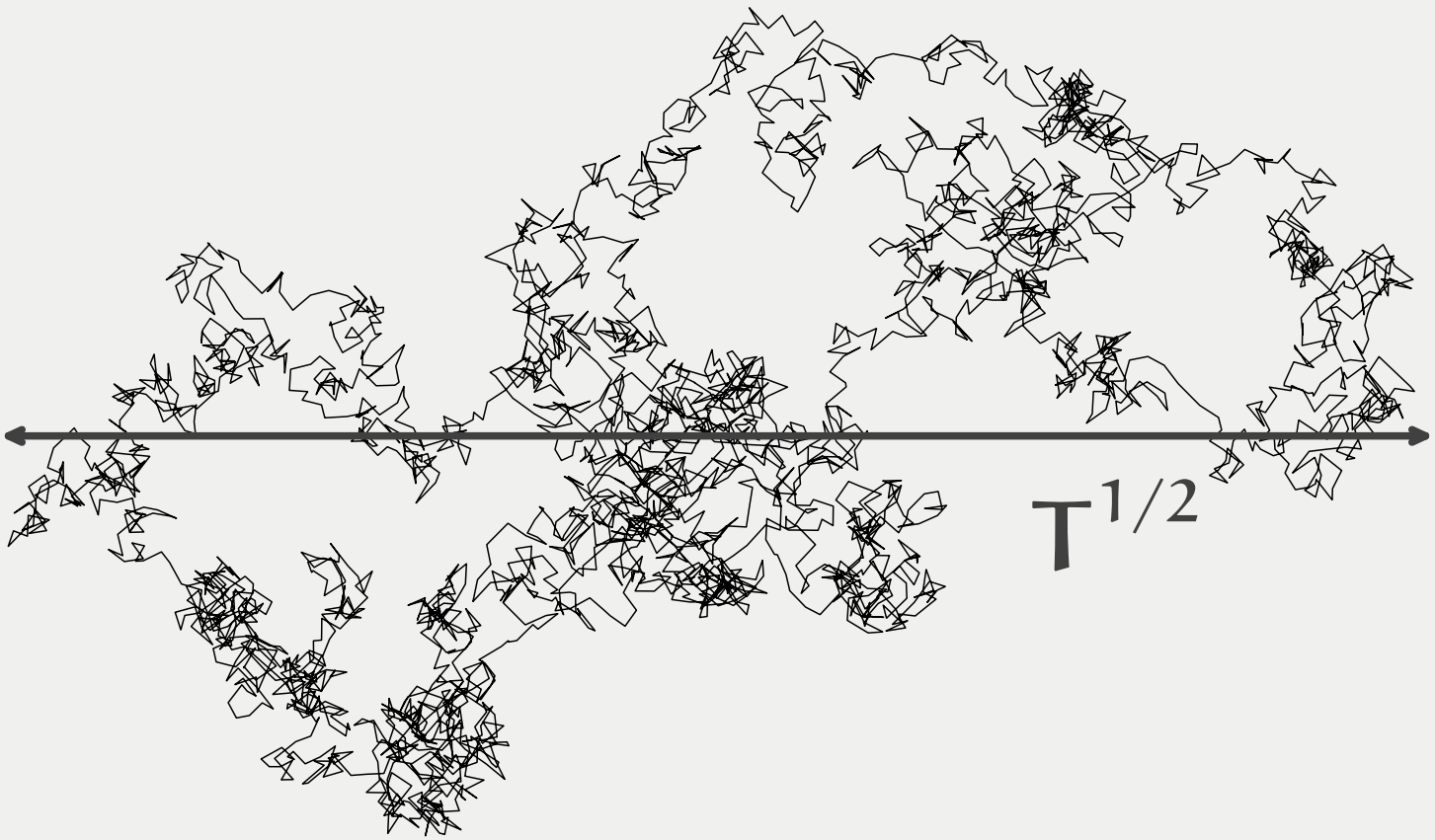
Worldline representation of $V_{E,1\text{ loop}}$

$$V_{E,1\text{ loop}} = -\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \times \int_{x^i(0)=x^i(T)} \left[Dx^i(\tau)\right] \exp\left(-\int_0^T d\tau \left(\frac{\dot{x}^i\dot{x}^i}{4} + ie\,\dot{x}^i A^i(x)\right)\right)$$

- $x^i(\tau) =$ trajectory of length T in Euclidean spacetime of a fictitious point-like particle
- Closed paths because of the trace: $x^i(0) = x^i(T)$
- The mass suppresses the long paths (longer than the Compton wavelength). $T \approx 0$ controls the UV
- Euclidean metric ensures convergence
- In vacuum, one has

$$\int_{x^i(0)=x^i(T)} \left[Dx^i(\tau)\right] \exp\left(-\int_0^T d\tau\, \frac{\dot{x}^i\dot{x}^i}{4}\right) = \frac{1}{(4\pi T)^{d/2}} \underset{d=4}{=} \frac{1}{(4\pi T)^2}$$

Scales



- Path length $= T$
- Size of explored region $\sim \sqrt{T}$
- Area $\sim T$

Barycentric coordinates

- Split x^i into barycenter of the loop and deviation:

$$x^i(\tau) \equiv X^i + r^i(\tau), \quad \int_0^T d\tau\, r^i(\tau) = 0$$

- Background field $\rightarrow$ Wilson loop centered at X^i, averaged over all paths of length T:

$$W_x[r] \equiv \exp\Big(-ie\int_0^T d\tau\, \dot{r}^i(\tau) A^i(X + r(\tau))\Big)$$

$$\langle W_x \rangle_T \equiv (4\pi T)^2 \int_{r^i(0)=r^i(T)} [Dr^i(\tau)]\, W_x[r]\, \exp\Big(-\int_0^T d\tau\, \frac{\dot{r}^i \dot{r}^i}{4}\Big)$$

  - Average is dominated by an ensemble of loops localized around the barycenter X^i (up to a distance of order $T^{1/2}$)
  - $\langle W_x \rangle_\tau$ encapsulates the local properties of the quantum field theory in the vicinity of X^i

- One-loop Euclidean vacuum diagrams:

$$\mathcal{V}_{E,1\text{ loop}} = -\frac{1}{(4\pi)^2}\int d^4X \int_0^\infty \frac{dT}{T^3}\, e^{-m^2 T}\ \langle W_X\rangle_T$$

- For a constant E, choose a gauge where A^i is linear in coordinates $\rightarrow$ $\langle W_X\rangle_T$ given by a Gaussian integral:

$$\langle W_X\rangle_T = \frac{eET}{\sin(eET)}$$

- The imaginary part of $\mathcal{V}_{E,1\text{ loop}}$ comes from poles located at $T_n = n\pi/(eE)$:

$$\mathrm{Im}\ (\mathcal{V}_{E,1\text{ loop}}) = \frac{V_4}{16\pi^3}(eE)^2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}\, e^{-n\pi m^2/(eE)}$$

- Note: the terms $n > 1$ encode Bose-Einstein correlations

Numerical worldline approach [Gies, Langfeld, 2001]

- $E \parallel z \;\rightarrow\; A^i = (0,0,0,-iEx^3)$

$$W_x[r] = e^{-eE\mathcal{A}}, \quad \text{with } \mathcal{A} \equiv \int_0^T d\tau\, \dot{r}_4(\tau) r_3(\tau)$$

($\mathcal{A} =$ projected area of the loop on the plane 34)

- Note: probability distribution for $\mathcal{A}$:

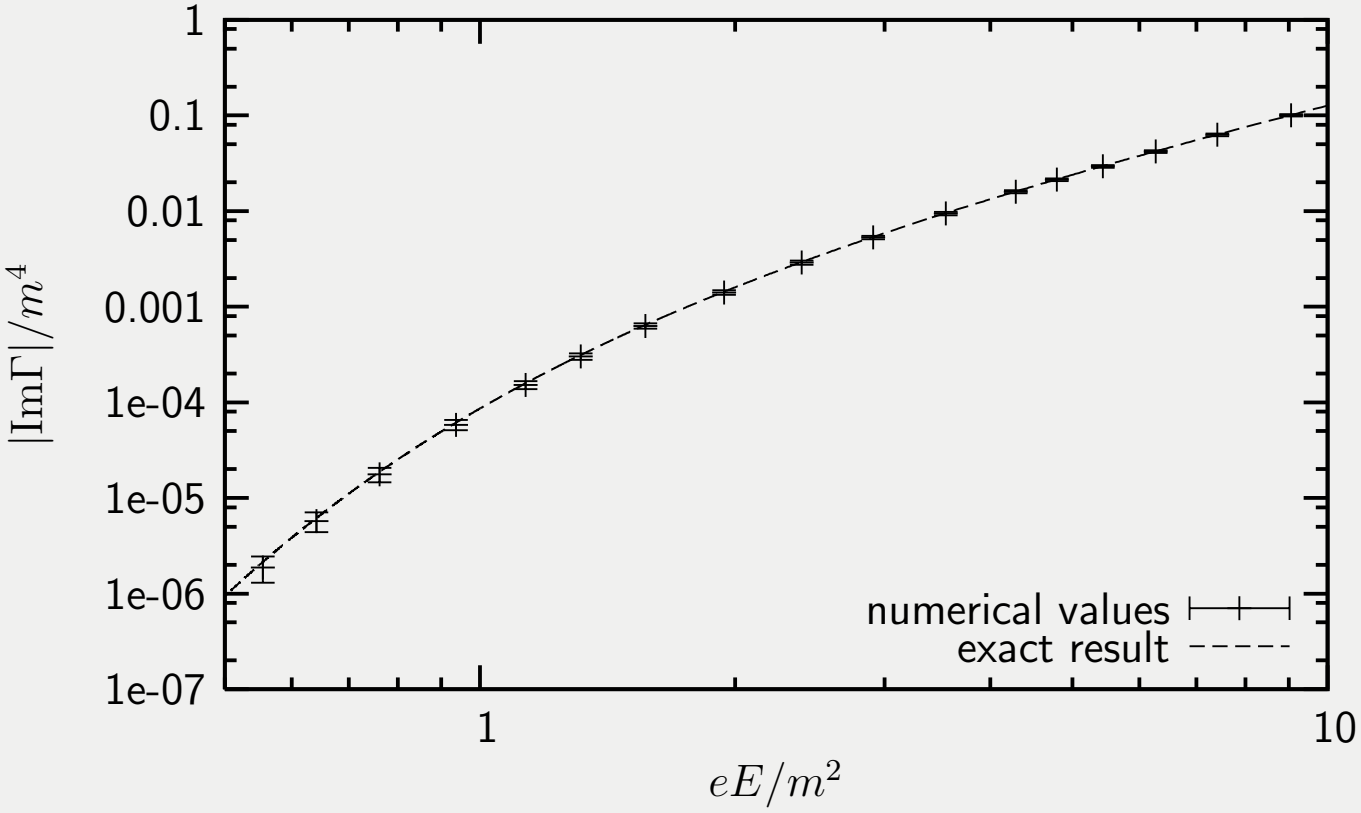
$$\mathcal{P}_T(\mathcal{A}) = \frac{\pi}{4T} \frac{1}{\cosh^2\left(\frac{\pi\mathcal{A}}{2T}\right)} \;\rightarrow\; \text{typical worldlines: } \mathcal{A} \lesssim T$$

- After a rescaling $\mathcal{J} \equiv \tau\mathcal{A}$, $s \equiv -i\tau/eE$:

$$\mathcal{V}_{E,1\text{ loop}} = \left(\frac{eE}{4\pi}\right)^2 \int d^4X \int_0^\infty \frac{ds}{s^3}\, e^{-i(m^2/(eE))s} \left\langle e^{-is\mathcal{J}} \right\rangle$$

  - Evaluate $\left\langle e^{-is\mathcal{J}} \right\rangle$ once for all (from an ensemble of loops)
  - Difficulty: small $eE/m^2 \rightarrow$ small $s \rightarrow$ large areas needed

Constant field [Gies, Klingmuller, 2005]



Algorithm for non-constant fields

- External field and ensemble of loops are entangled
- Other difficulty: integral over areas converges only for $|T| \leq \pi/(eE)$ (location of the first pole, T_1)
- Analytical ansatz for the distribution of areas:

$$W_X[r] = e^{-eE(X)T\mathcal{I}}, \quad \mathcal{I} \equiv \frac{i}{\tau E(X)} \int_0^T d\tau\, \dot{r}^i(\tau)\, A^i(X + r(\tau))$$

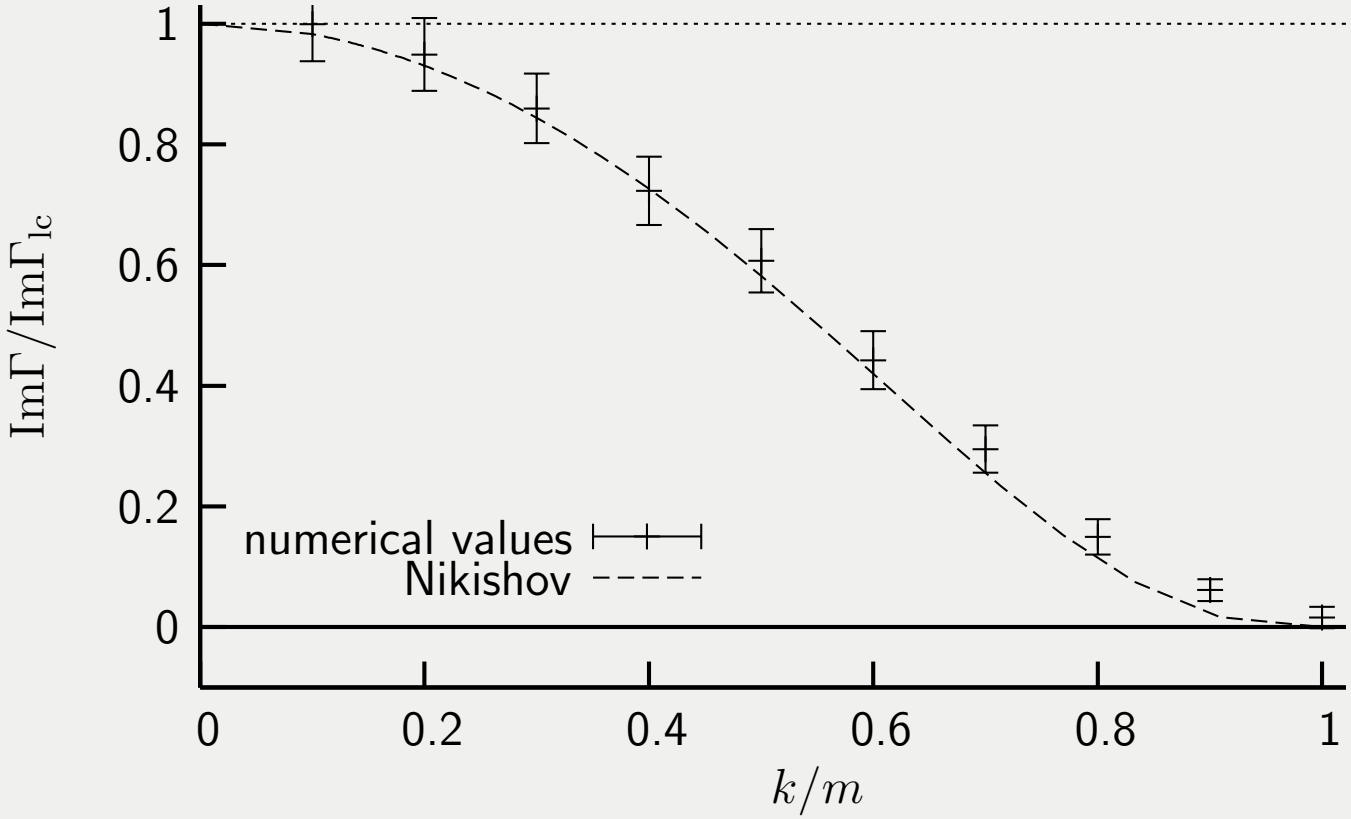
$$\text{Ansatz:} \quad \mathcal{P}_X(\mathcal{I}) = N \frac{1}{\cosh^{2\nu}\left(\frac{\pi\,\alpha\mathcal{I}}{2}\right)}$$

Fit α and ν from an ensemble of loops generated by Monte-Carlo. Then, do the $\mathcal{I}$ integral analytically

$$\int_{-\infty}^{+\infty} d\mathcal{I}\, \mathcal{P}_X(\mathcal{I})\, e^{-eE(X)\tau\mathcal{I}} = N\, \frac{4^\nu}{\pi\alpha}\, \frac{\Gamma(\nu + \frac{eE(X)\tau}{\pi\alpha})\Gamma(\nu - \frac{eE(X)\tau}{\pi\alpha})}{\Gamma(2\nu)}$$

- Integrate over T and X^i numerically

Sauter field $E^1 = E/\cosh^2(kx^1)$ [Gies, Klingmuller, 2005]

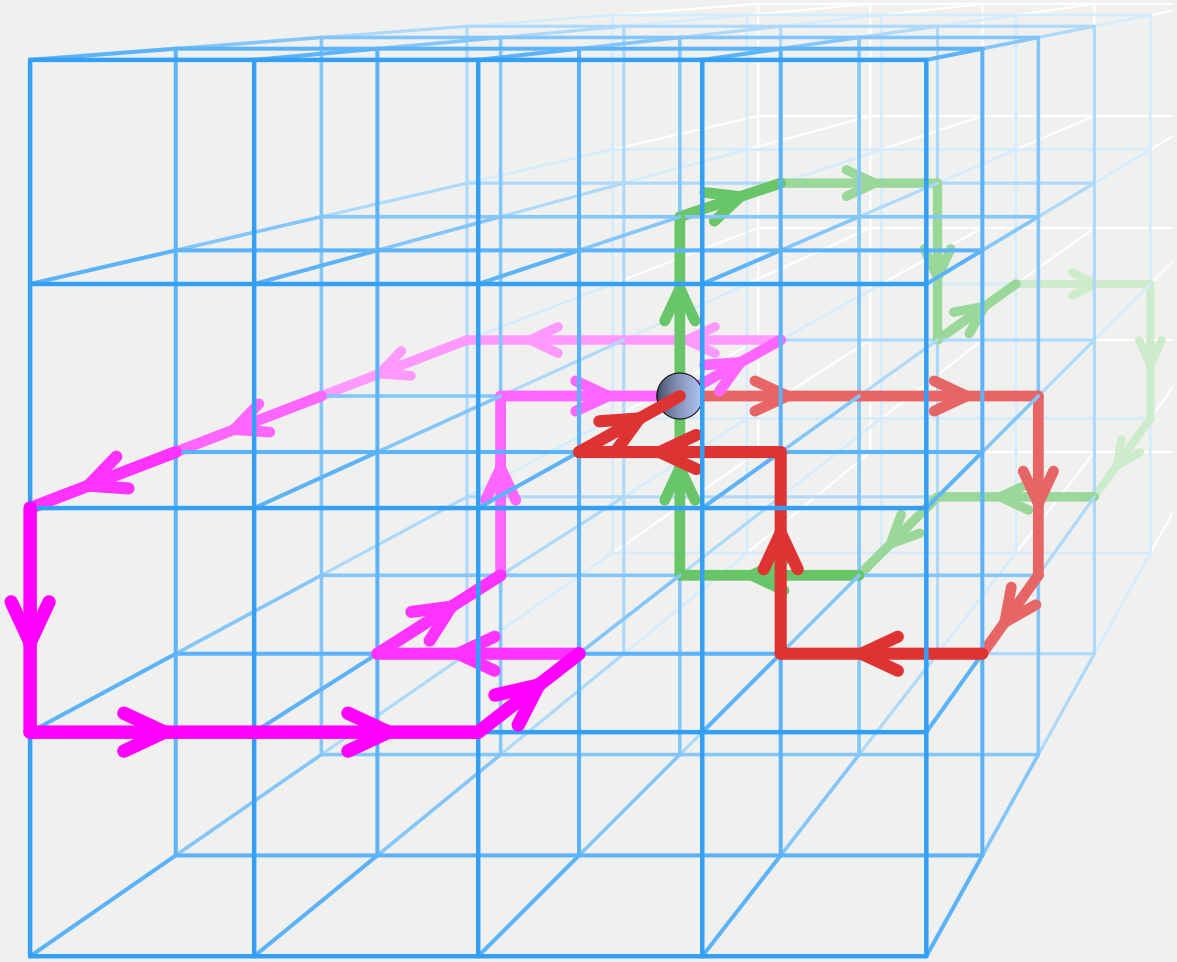


Lattice worldline formalism [Schmidt, Stamatescu, 2002]

- When the background field results of a lattice computation (and is given in terms of link variables on the lattice), we may write

$$\mathrm{tr}\,\ln\left(-D^iD^i+m^2\right)=-\sum_{n=0}^{\infty}\frac{1}{n}\frac{1}{(2\tilde{d})^n}\sum_{x\in\text{ lattice}}\sum_{\gamma\in\Gamma_n(x,x)}\prod_{\ell\in\gamma}U_\ell$$

  - n plays the role of the fictitious time T
  - $U_\ell =$ link variable on the edge ℓ
  - $\Gamma_n(x,x) =$ set of loops (from x to x) of length n (in lattice units)
  - $\tilde{d}=d+\frac{1}{2}m^2a^2$ ($\tilde{d}>d$ suppresses the long loops)



Worldline instanton approximation [Dunne, Schubert, 2005]

- Define $\tau \equiv Tu$ and $m^2 T = s$. Then:

$$\mathcal{V}_{E,1\text{ loop}} = -\int_0^\infty \frac{ds}{s}\, e^{-s} \int_{x^i(0)=x^i(1)} [Dx^i(u)]\, \exp\left(-\int_0^1 du \left(\frac{m^2}{4s}\frac{\dot{x}^2}{4} + ie\dot{x}^i A^i(x)\right)\right)$$

- The integral over s gives a Bessel function:

$$\mathcal{V}_{E,1\text{ loop}} = -2 \int_{x^i(0)=x^i(1)} [Dx^i(u)]\, K_0\left(\left(m\int_0^1 du\, \dot{x}^2\right)^{\frac{1}{2}}\right) \exp\left(-ie\int_0^1 du\, \dot{x}^i A^i(x)\right)$$

- In the regime where $m^2 \int_0^1 du\, \dot{x}^2 \gg 1$, approximate $K_0(z) \approx \sqrt{\pi/2}\, e^{-z}/\sqrt{z}$ and perform a stationary phase approximation. We need extrema of

$$\mathcal{S} \equiv m\left(\int_0^1 du\, \dot{x}^2\right)^{1/2} + ie\int_0^1 du\, \dot{x}^i A^i(x)$$

- They are the closed paths $x^i(u)$ that obey

$$m \frac{\ddot{x}^i}{\sqrt{\int_0^1 du\, \dot{x}^2}} = ieF^{ij}\dot{x}^j$$

  Note: $\dot{x}^i\dot{x}^i = \text{const}$
- For each extremum

$$\operatorname{Im} \mathcal{V}_{E,1\text{ loop}} \sim e^{-\mathcal{S}_{\text{extremum}}}$$

- The prefactor is obtained by integrating Gaussian deviations about the extremal path

Example: Sauter field $E(x^3) = E/\cosh^2(kx^3)$

- Use the gauge potential

$$A^4 = -i\,\frac{E}{k}\,\tanh(kx^3)$$

- Equations of motion for the stationary solutions

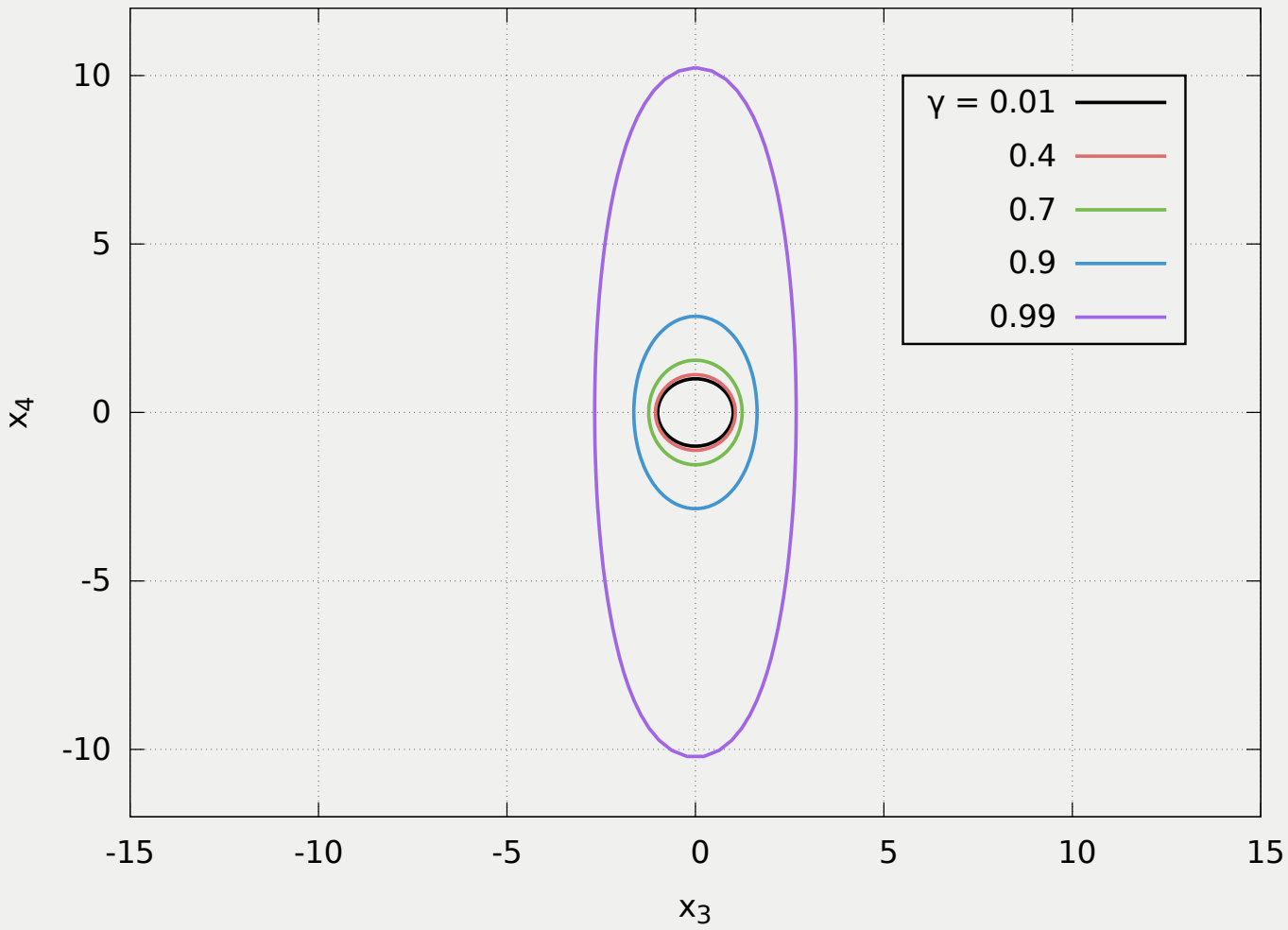
$$\dot{x}^3 = v\sqrt{1-\gamma^{-2}\tanh^2(kx^3)}\,, \qquad \dot{x}^4 = -\gamma^{-1}v\,\tanh(kx^3)$$

  ($\gamma \equiv mk/(eE)$)
- Countable infinity of periodic solutions:

$$x^3(u) = \frac{m}{eE}\,\frac{1}{\gamma}\,\mathrm{arcsinh}\left(\frac{\gamma}{\sqrt{1-\gamma^2}}\,\sin(2\pi n\,u)\right)$$

$$x^4(u) = \frac{m}{eE}\,\frac{1}{\gamma\sqrt{1-\gamma^2}}\,\arcsin\left(\gamma\,\cos(2\pi n\,u)\right)$$

  ($n =$ winding index of the solution)
- Extended field ($\gamma \to 0$): circular solutions
- $\gamma \to 1$: very elongated orbits, action becomes infinite
  $\to$ no pair production (field coherence length too small)


γ = 0.01
0.4
0.7
0.9
0.99
10
5
0
-5
-10
-15
-10
-5
0
5
10
15
x4
x3

Dynamically assisted Schwinger effect

[Schutzhold, Gies, Dunne, 2008]
[Di Piazza, Lotstedt, Milstein, Keitel, 2009]
[Orthaber, Hebenstreit, Alkofer, 2011]
[Monin, Voloshin, 2012]
[Taya, Fujii, Itakura, 2014]
+ many others

Comparison between exact and perturbative results

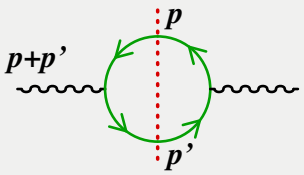
- Consider the Sauter temporal pulse: $A^3(t) = E\tau \tanh(t/\tau)$
- Exact spectrum:

$$\frac{dN_1}{d^3\mathbf{p}} = \frac{V}{(2\pi)^3}\left(\frac{\sinh\left[\pi(\lambda+\mu-\nu)\right]\sinh\left[\pi(\lambda-\mu+\nu)\right]}{\sinh\left(2\pi\mu\right)\sinh\left(2\pi\nu\right)}\right)$$

$$\mu \equiv \frac{\tau}{2}\sqrt{m^2+p_\perp^2+(p_z-2eE\tau)^2}$$

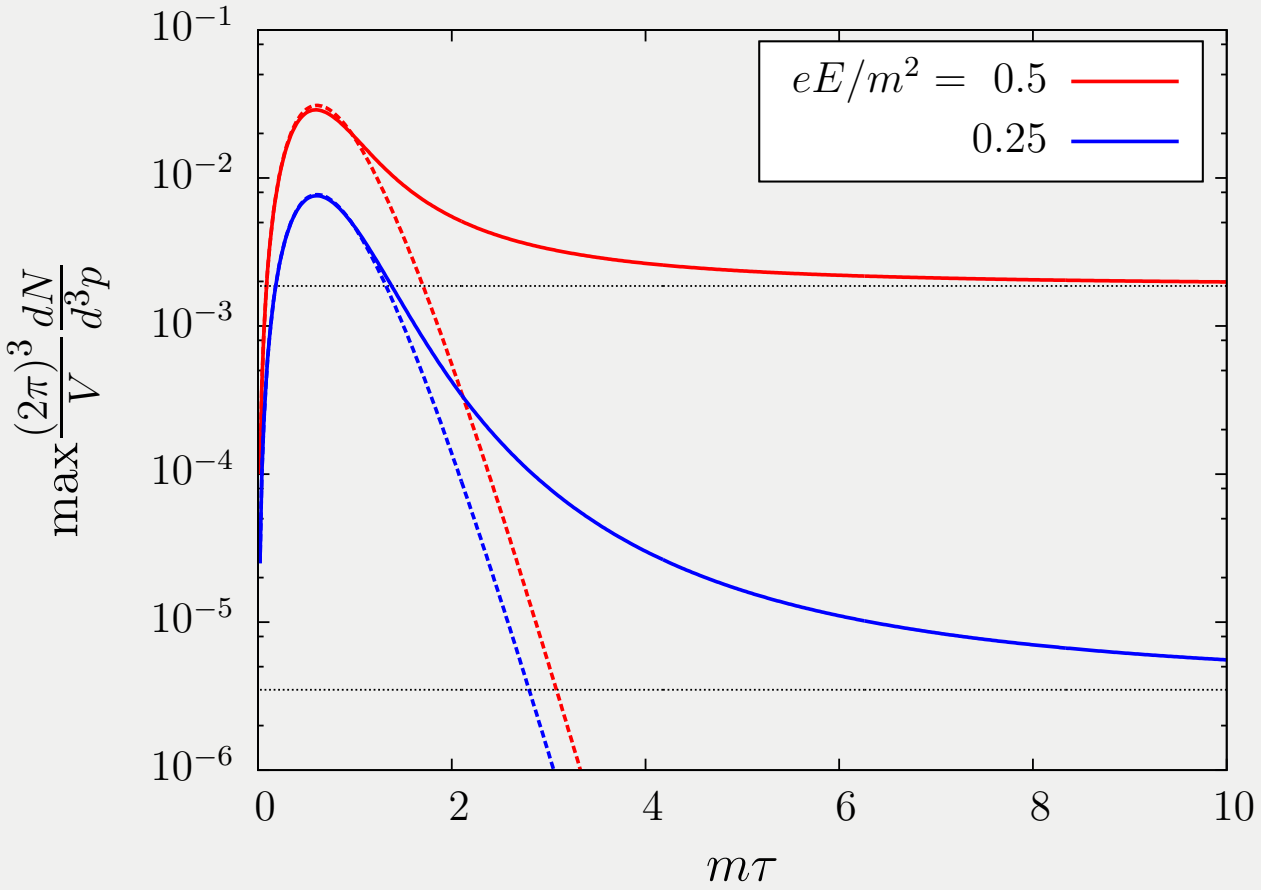
$$\nu \equiv \frac{\tau E_{\mathbf{p}}}{2}, \quad \lambda \equiv eE\tau^2$$

- One-photon spectrum:



$$\rightarrow \frac{dN_1^{(1\gamma)}}{d^3\mathbf{p}} = \frac{Ve^2E^2}{(2\pi)^3}\left[1-\left(\frac{p_z}{E_{\mathbf{p}}}\right)^2\right]\frac{\pi^2\tau^4}{\sinh^2(\pi E_{\mathbf{p}}\tau)}$$

Numerical comparison for subcritical fields



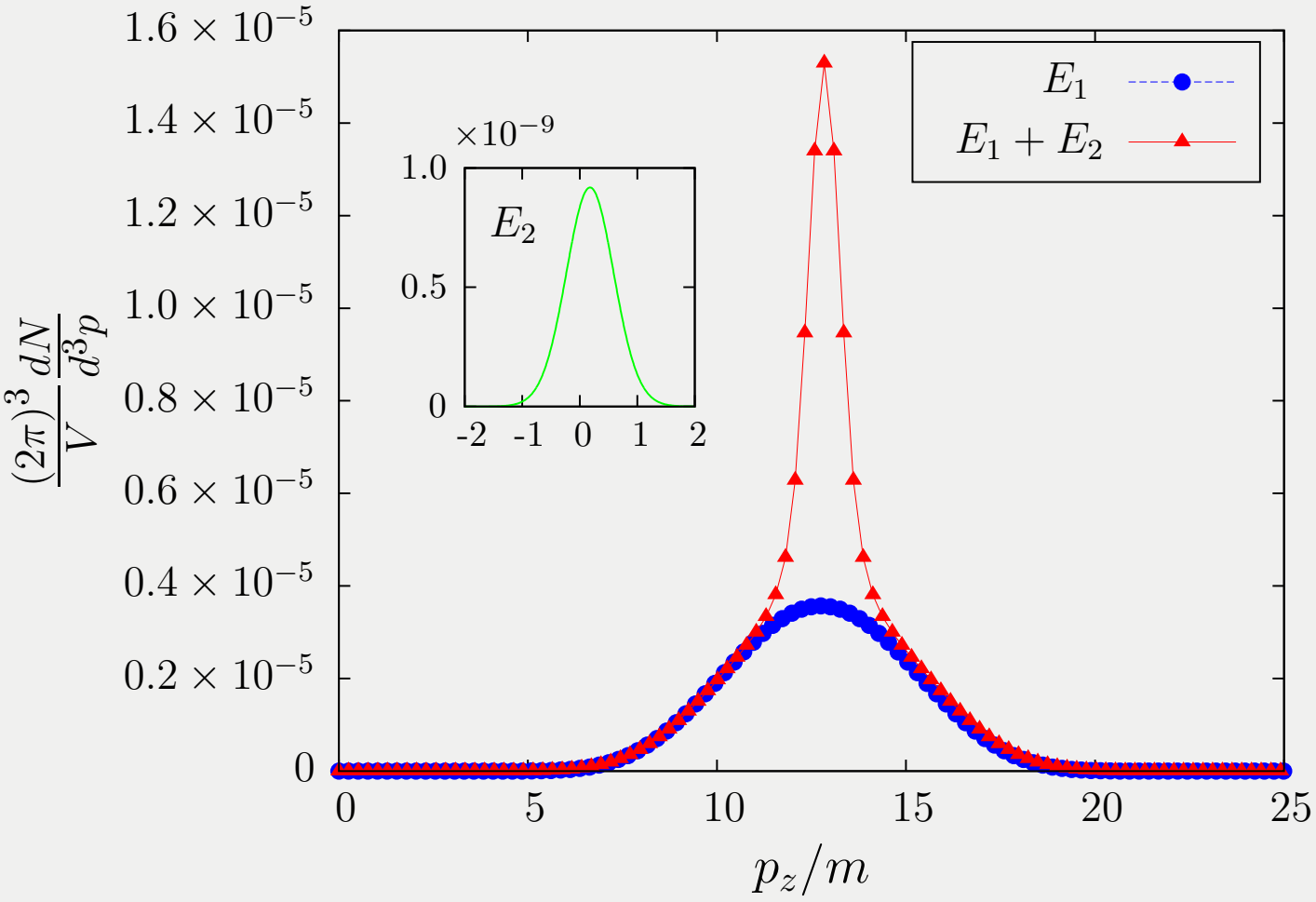
- Solid lines: exact result. Dashed lines: one-photon result
- Black dotted line: constant field result ($\exp(-\pi m^2/(eE))$)
- Note: considerably enhanced spectrum in the regime $m\tau \sim 1$

Superposition of fields

- Consider the sum of two Sauter fields, with $E_1 \gg E_2$ and $\tau_1 \gg \tau_2$:

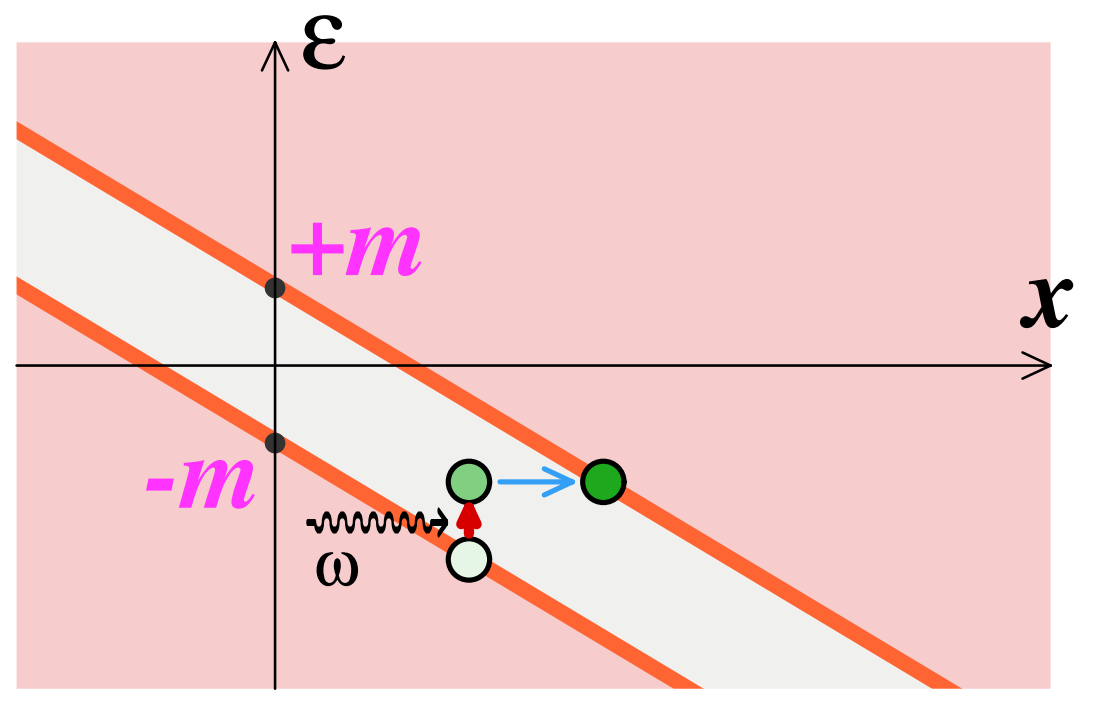
$$E_z(t) = \frac{E_1}{\cosh^2\left(\frac{t}{\tau_1}\right)} + \frac{E_2}{\cosh^2\left(\frac{t}{\tau_2}\right)}$$

  - E_1: strong and slow (one-photon process forbidden)
  - E_2: weak and fast (one-photon process possible)
- Non-trivial effects since the spectrum is non-linear in the field

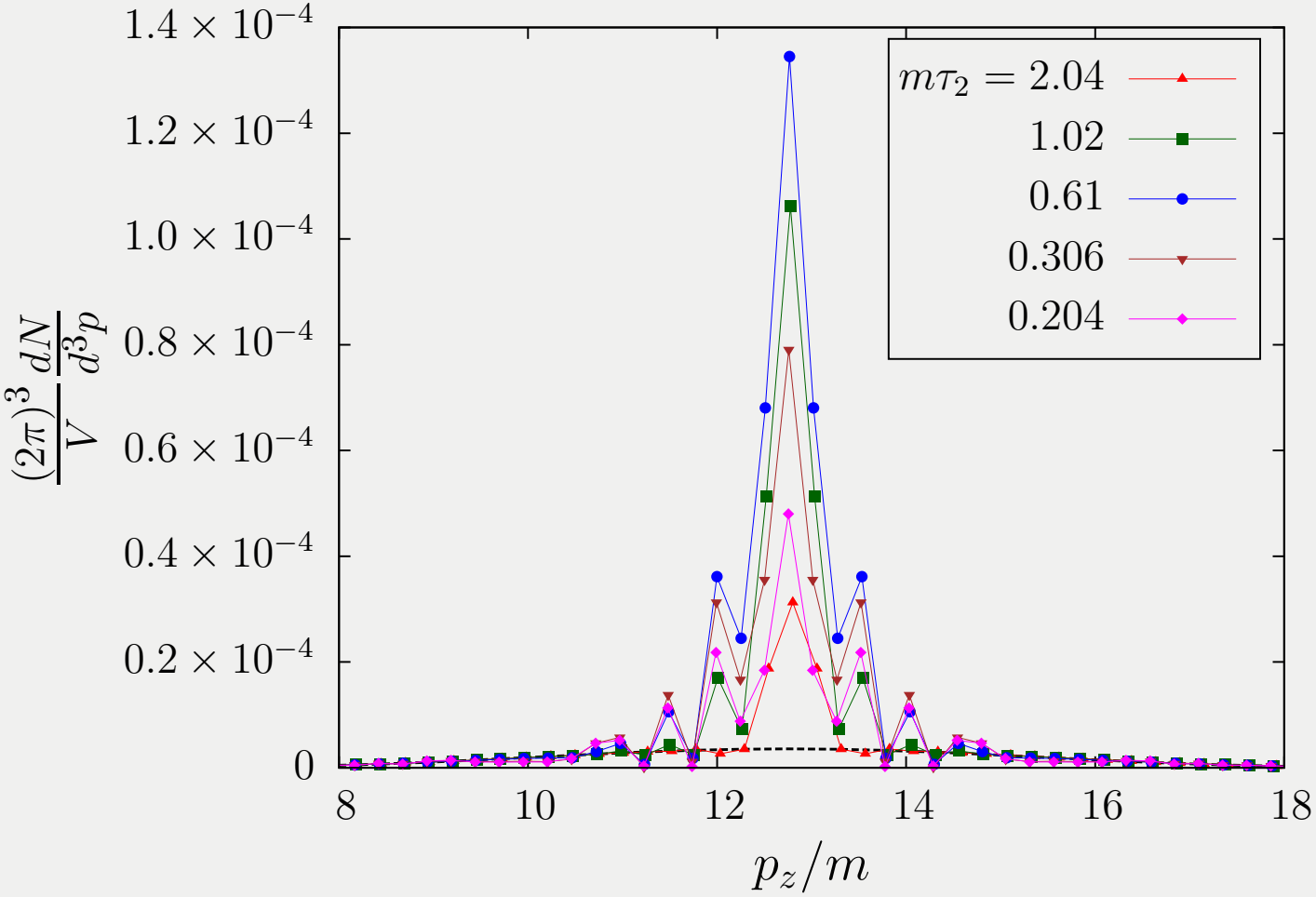


- $eE_1 = 0.25\, m_e^2$, $\tau_1 = 10^{-4}$ eV^{-1}
- $eE_2 = 0.025\, m_e^2$, $\tau_2 = 7 \times 10^{-6}$ eV^{-1}.

Qualitative interpretation



- Slope due to the strong and slow field E_1 (no appreciable slope from the weak field)
- A single photon from the weak and fast field E_2 raises a hole excitation (the more, the better)
- Tunneling distance is reduced, which affects exponentially the resulting spectrum



- $eE_1 = 0.25\, m_e^2$, $m_e\tau_1 = 510$
- $eE_2 = 0.025\, m_e^2$, variable $m_e\tau_2$
- Black dashed line: E_1 alone
- Maximal enhancement when $m_e\tau_2 \approx 0.6$ (roughly, $\tau_2 \sim (2m_e)^{-1}$)

Summary

- Very interesting playground for studying QFT in a non-perturbative regime and testing novel methods (with a few exact results to compare with)
- A number of approaches have been applied to this problem:
  - Mode functions on the lattice
  - Worldline formalism
  - *Quantum kinetic equations*
  - *Wigner formalism*
  - *Holography, AdS/CFT*
- For high enough frequency, perturbative (one-photon) result dominates over constant field result
- Dynamical enhancement can achieved by superimposing slow and fast fields