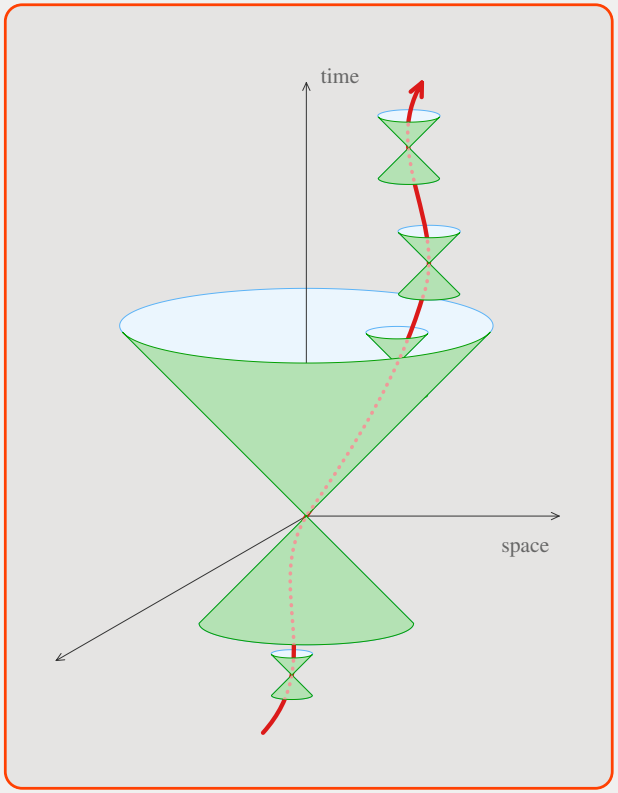
Large Fields at Small Coupling

A tractable non-perturbative regime of QFT

François Gelis

December 19th, 2016



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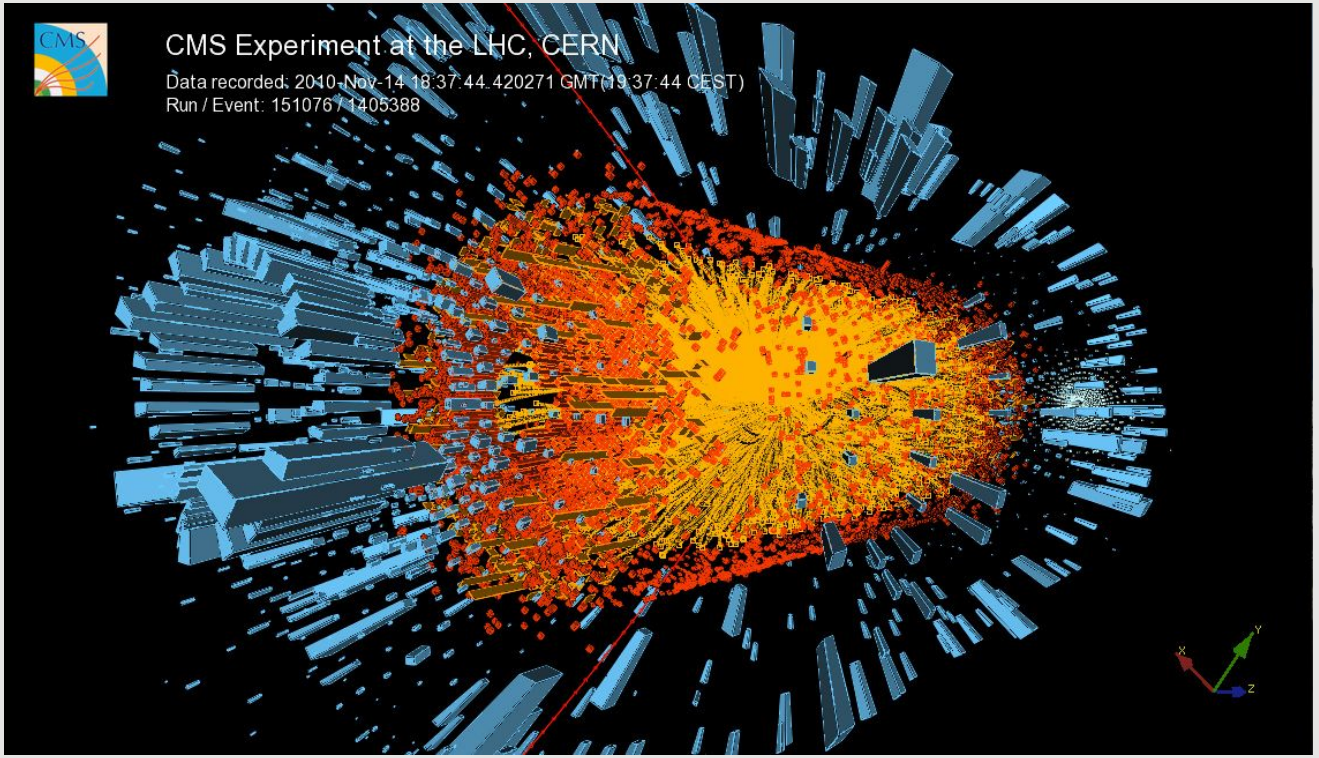
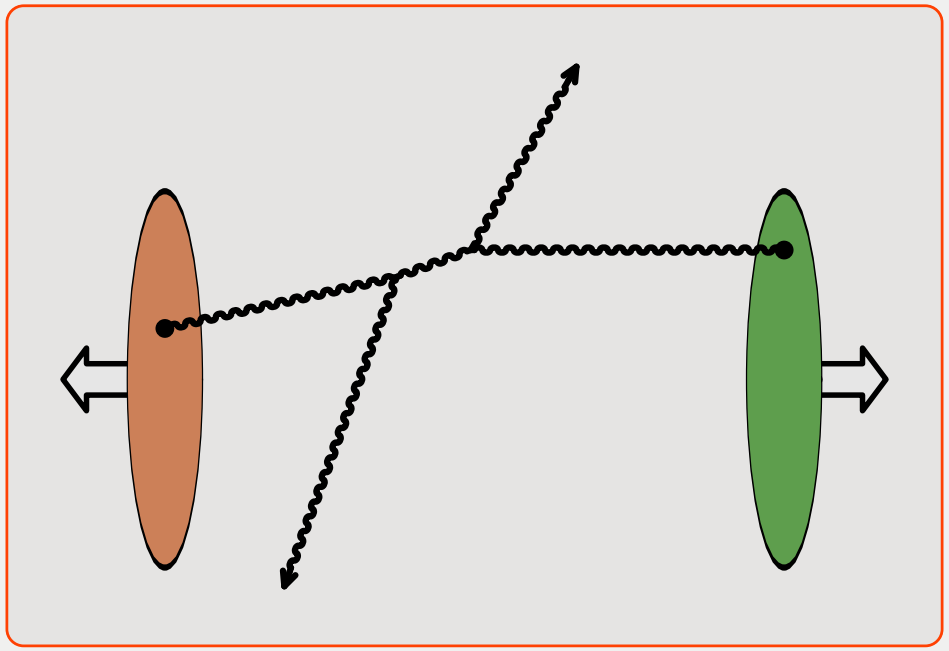
Introduction

Heavy Ion Collisions



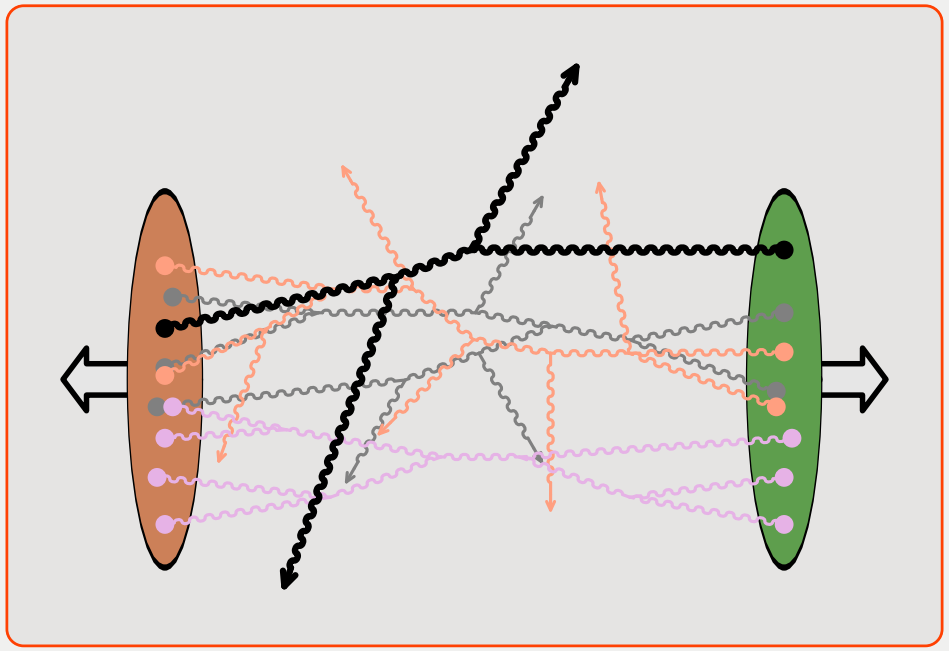
- Very high multiplicity (~ 20000 produced particles)
- Most of them rather soft ($P \lesssim 2$ GeV)

Initial state and Parton distributions



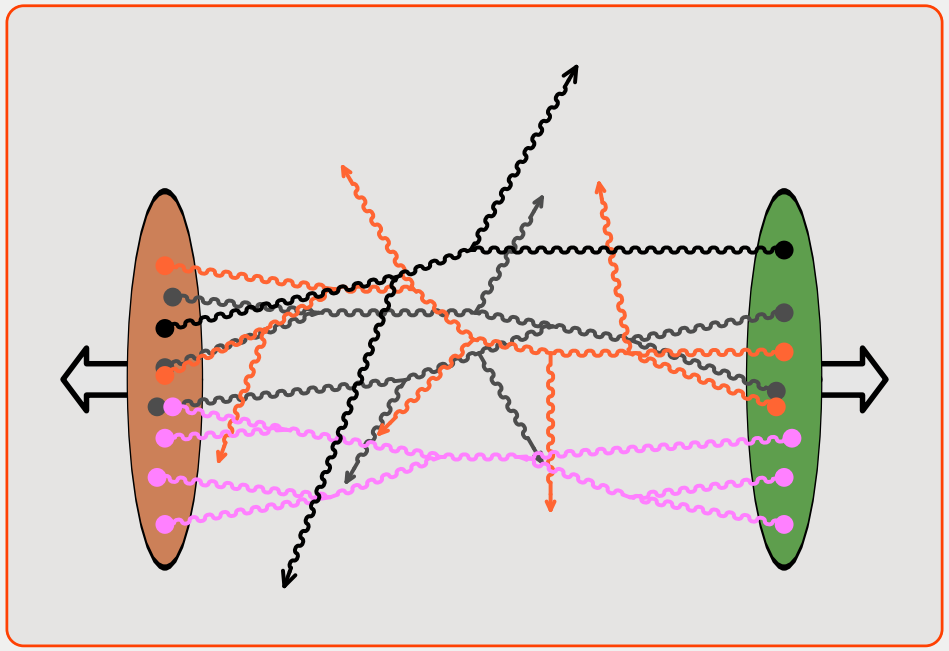
- **Factorization :** (partonic cross-section) $\otimes$ (parton distribution)
  Applicable to high momentum rare processes

Initial state and Parton distributions



- **Factorization :** (partonic cross-section) $\otimes$ (parton distribution)
  Applicable to high momentum rare processes
- **Underlying event :** cannot be calculated in this framework

Initial state and Parton distributions

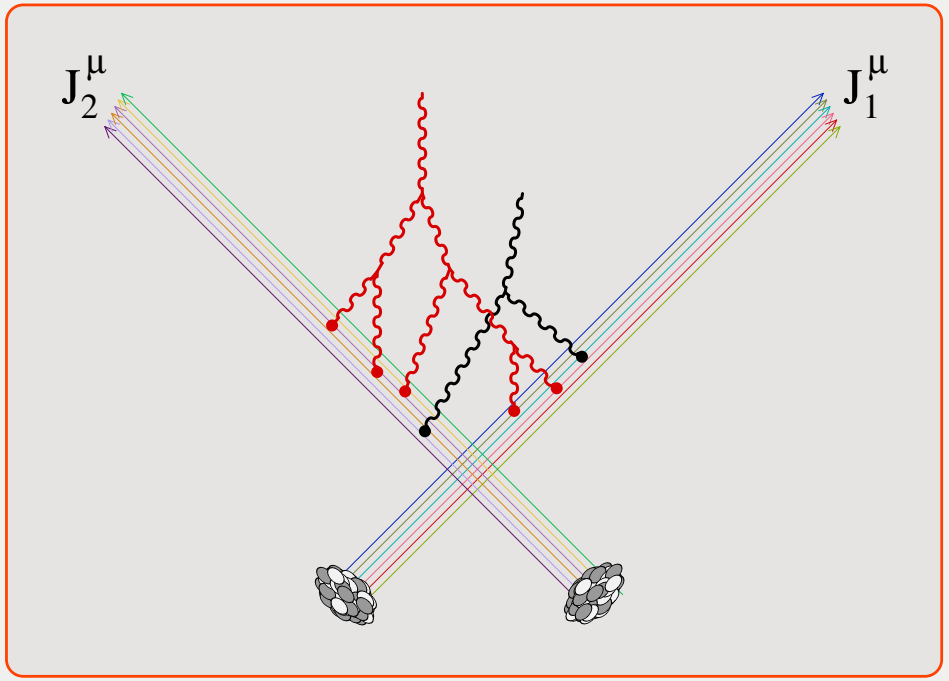


- **Factorization :** (partonic cross-section) $\otimes$ (parton distribution)
  Applicable to high momentum rare processes
- **Underlying event :** cannot be calculated in this framework
- **In a Heavy Ion Collision**, this is the most interesting part...

Effective description by an external source

Snapshot of the constituents by color currents :

$$\mathcal{S} \equiv \int d^4x \left(-\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu(x) A_\mu(x) \right)$$



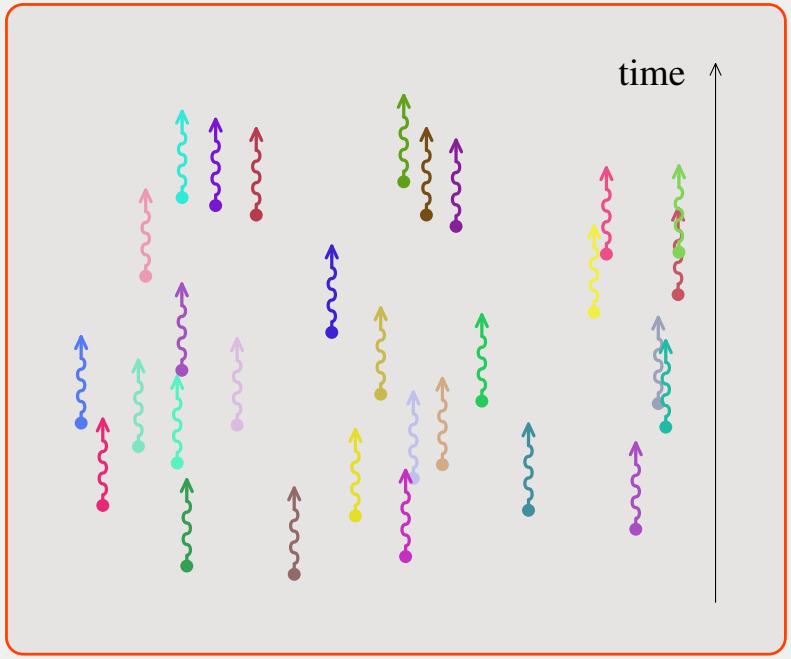
- Time dilation: static current
- Many constituents: J^μ large
- Current conservation: $[\mathcal{D}_\mu, J^\mu] = 0$

Quantum Field Theories with (Strong) Sources

TEXTBOOK CASE : WEAK SOURCE REGIME ($g\,J \ll 1$)

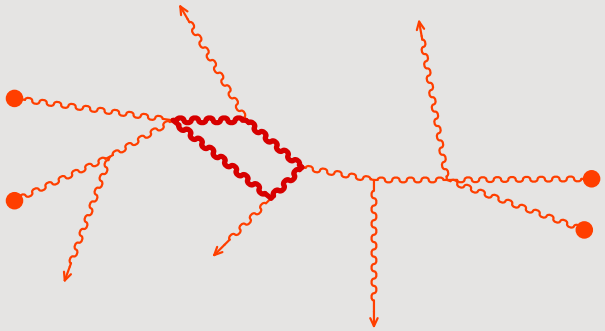
Gluon multiplicity : Poisson distribution

$$P(n) = \frac{\overline{N}^n \, e^{-\overline{N}}}{n!} \qquad \frac{d\overline{N}}{d^3\mathbf{p}} \sim \left|\widetilde{J}(\mathbf{p})\right|^2$$



- Exactly solvable when $g\,J \to 0$
  - No interaction after production
  - No thermalization

Order of magnitude of connected graphs



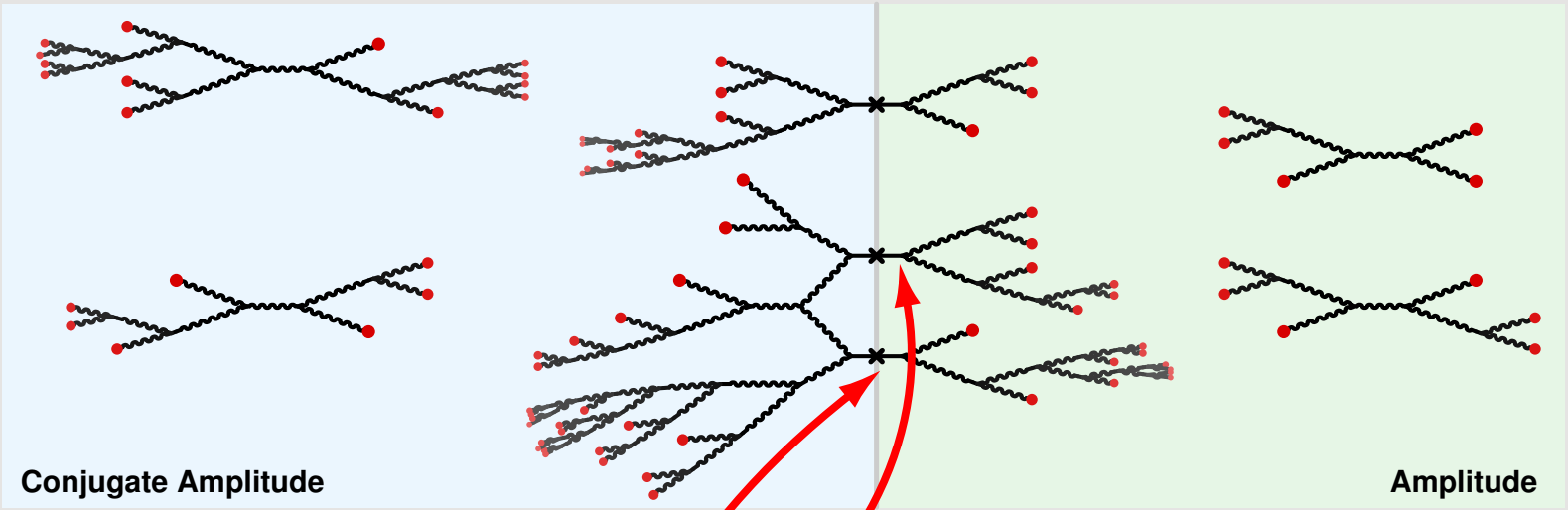
$$\sim \underbrace{g^{n_E-2}}_{\text{ext. lines}} \underbrace{(\hbar g^2)^{n_L}}_{\text{loops}} \underbrace{(g\,J)^{n_j}}_{\text{sources}}$$

- $g\,J \gtrsim 1$: strong source regime

  $\Rightarrow$ Non-perturbative dependence on $g\,J$

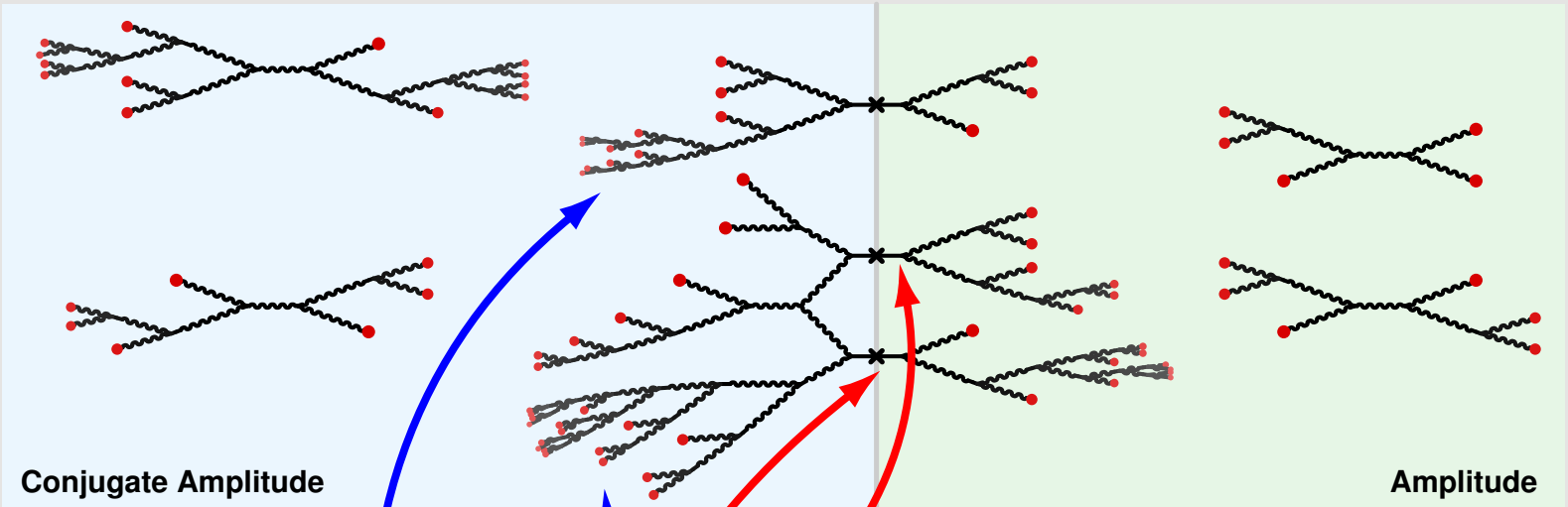
- **What happens when $g\,J \gtrsim 1$?**
  - Non-trivial correlations?
  - Thermalization?

Fully specified final states : there be dragons...



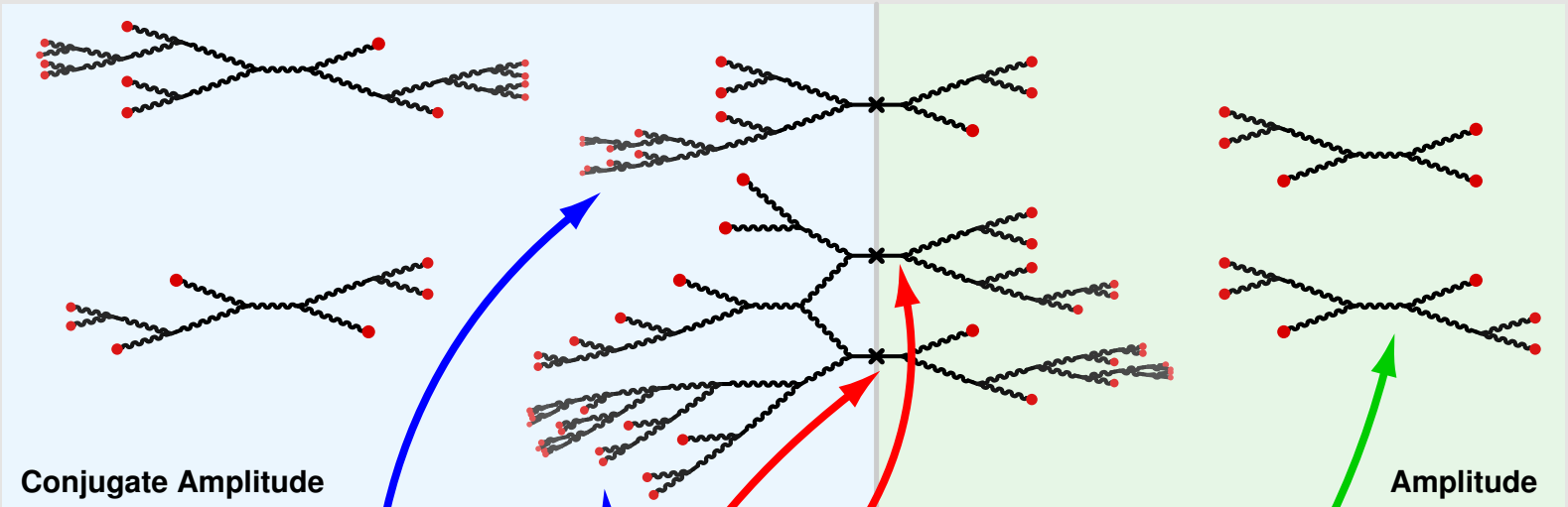
- Correlations among the produced particles

Fully specified final states : there be dragons...



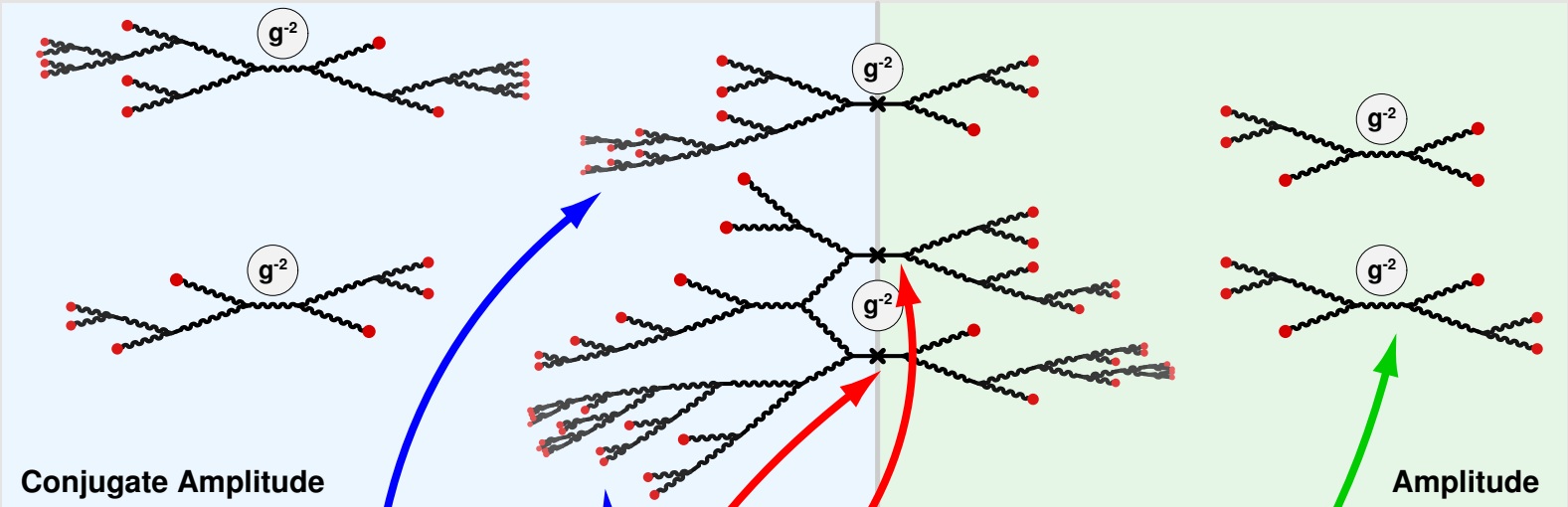
- Correlations among the produced particles
- Many disconnected graphs

Fully specified final states : there be dragons...



- Correlations among the produced particles
- Many disconnected graphs
- Vacuum graphs do not cancel

Fully specified final states : there be dragons...



- Correlations among the produced particles
- Many disconnected graphs
- Vacuum graphs do not cancel
- Pathological Taylor expansion : $g^{-2(\#\text{ connected components})}$

More modest goal : inclusive observables

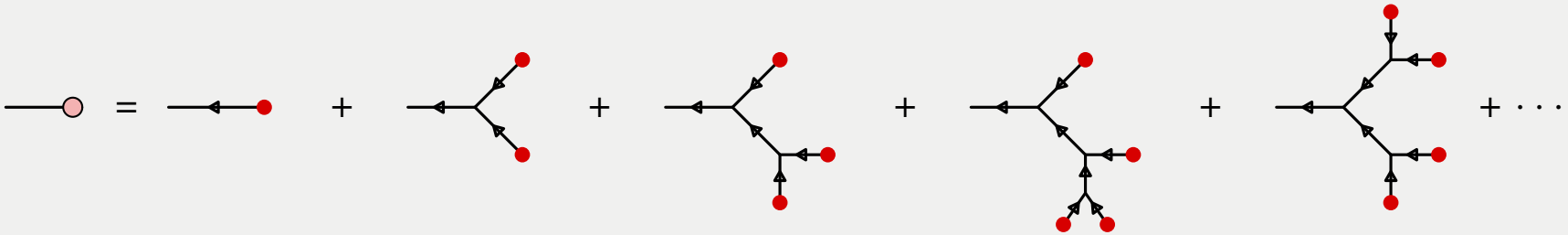
Mean gluon multiplicity

$$\left.\frac{d\overline{N}}{d^3\mathbf{p}}\right|_{LO} \sim \left|\widetilde{\mathcal{A}}(\mathbf{p})\right|^2$$

$$\underbrace{\left[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}\right] = J^\nu}_{\text{Yang-Mills eq.}} \qquad \lim_{t\to-\infty} \mathcal{A} = 0$$

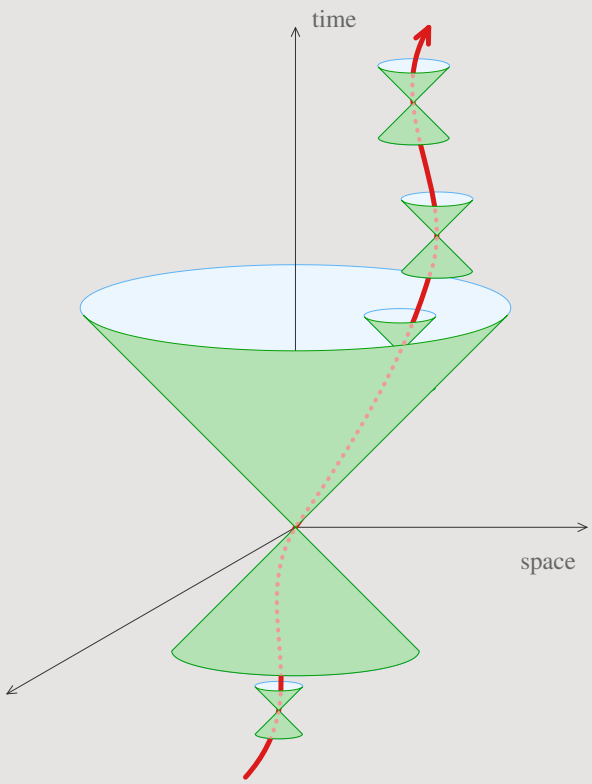
- Sum of *connected graphs* (vacuum graphs cancel)
- Expressible in terms of the classical field with *retarded boundary conditions*

$\overline{N}_{LO} =$ 



Inclusive observables : generic features

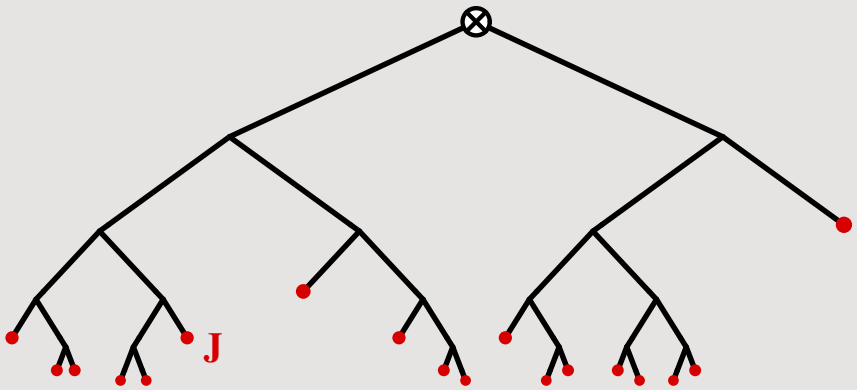
Retarded propagation



- **Inclusive** measurement :
  - Average of an observable over *all* final states
  - No constraint on the final state
  - No boundary condition for the fields at $t = +\infty$
- **Retarded = Causal evolution**
- Numerically straightforward

Inclusive observable at order $\hbar^0$

$\mathcal{O}_{\mathrm{LO}}[\mathrm{J}] =$



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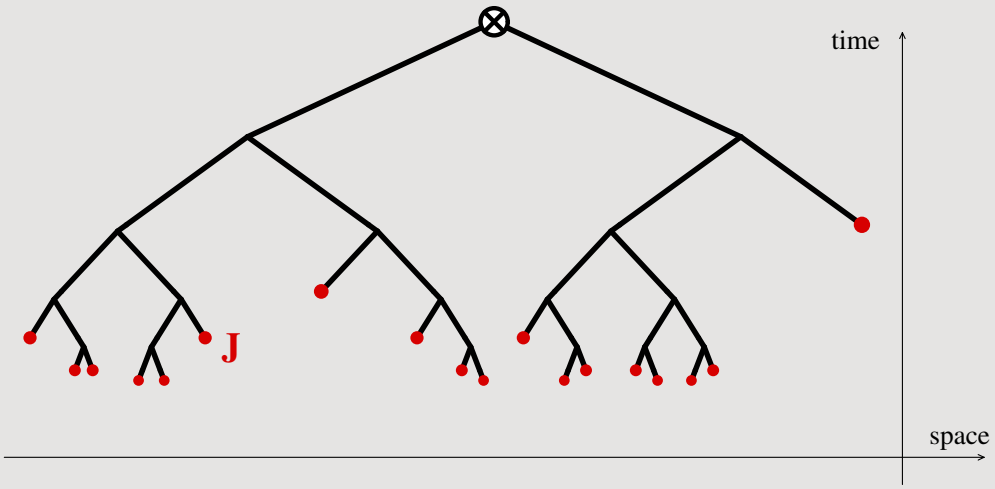
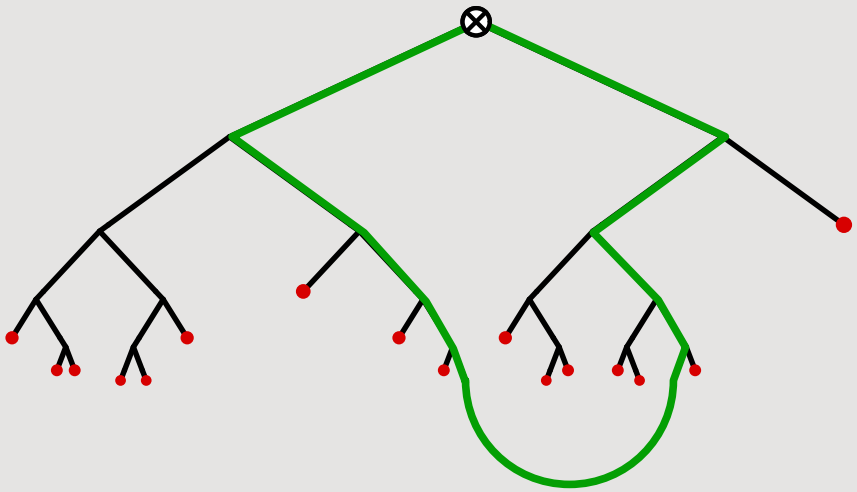


How to calculate the Next to Leading Order ?

Inclusive observable at order $\hbar^1$

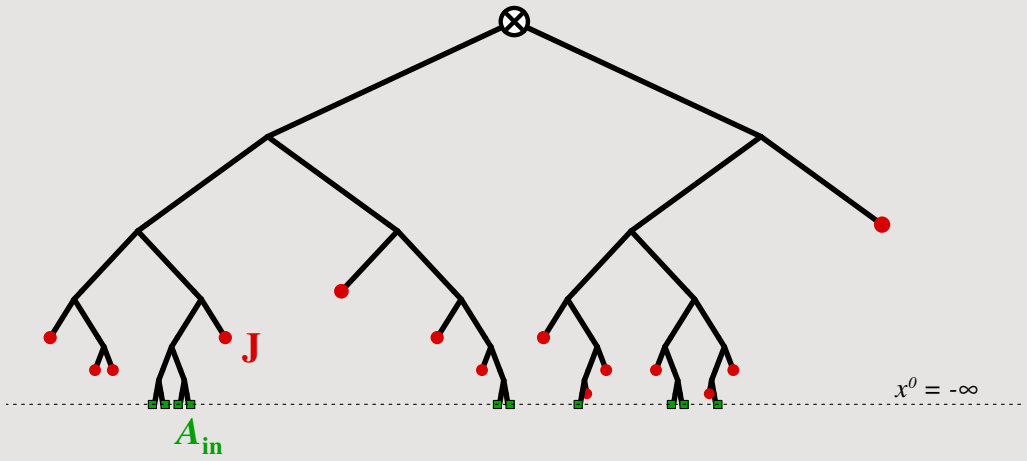
$\mathcal{O}_{\mathrm{LO}}[J] =$



How to calculate the Next to Leading Order ?

Step 1 : generalize to an arbitrary initial field at $\mathrm{t} = -\infty$

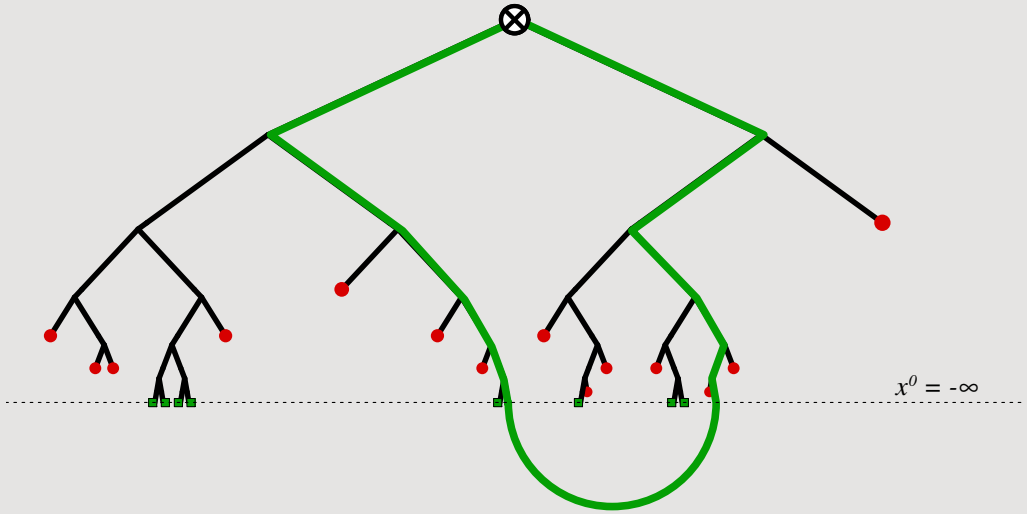
$$\mathcal{O}_{\mathrm{LO}}[\mathrm{J}, \mathcal{A}_{\mathrm{in}}] =$$



How to calculate the Next to Leading Order ?

Step 2 : add one loop

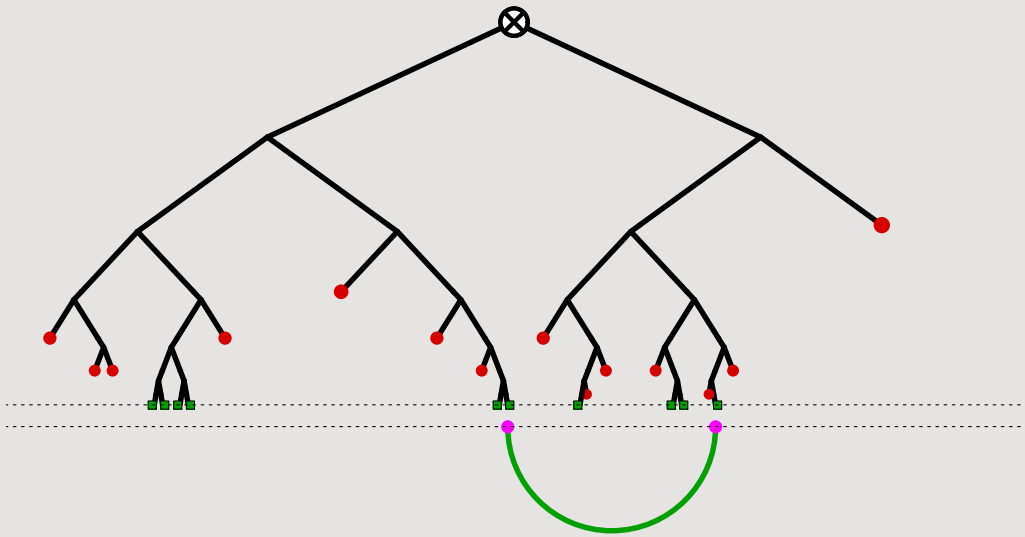
$$\mathcal{O}_{\mathrm{NLO}}[\mathrm{J}, \mathcal{A}_{\mathrm{in}}] =$$



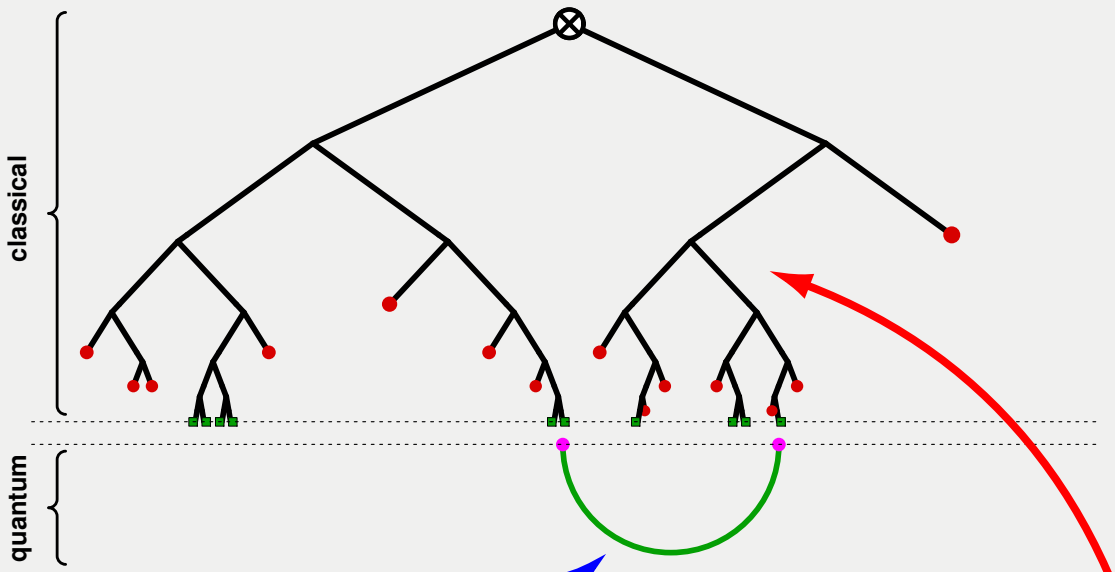
How to calculate the Next to Leading Order ?

Step 3 : view the loop as an operator acting on $\mathcal{O}_{\mathrm{LO}}$

$\mathcal{O}_{\mathrm{NLO}}[\mathrm{J}, \mathcal{A}_{\mathrm{in}}] =$



Next to Leading Order



$$\mathcal{O}_{\underline{\mathrm{NLO}}}[J, \mathcal{A}_{\mathrm{in}}] = \left[\frac{\hbar}{2} \int d^3x d^3y \; \Gamma(x, y) \; \frac{\delta}{\delta \mathcal{A}_{\mathrm{in}}(x)} \frac{\delta}{\delta \mathcal{A}_{\mathrm{in}}(y)} \right] \mathcal{O}_{\mathrm{LO}}[J, \mathcal{A}_{\mathrm{in}}]$$

Remarks

- $\Gamma(x, y)$ is *universal*, and known analytically :

$$\Gamma(x, y) = \int \frac{d^3 p}{(2\pi)^3 2E_p} \, e^{ip\cdot(x-y)}$$

- *LO contains NLO* (in a somewhat obfuscated way...)
- Applications :
  - Renormalization Group evolution of the effective theory
  - Study of thermalization

Another take on *LO contains NLO* : Moyal equation

- Liouville-von Neumann equation : $i\,\hbar\,\dfrac{\partial\widehat{\rho}_\tau}{\partial\tau} = \left[\widehat{H},\widehat{\rho}_\tau\right]$
- Wigner transform : $W_\tau(\mathbf{x},\mathbf{p}) \equiv \displaystyle\int \mathrm{d}\mathbf{s}\; e^{i\mathbf{p}\cdot\mathbf{s}}\,\langle \mathbf{x}+\frac{\mathbf{s}}{2}|\widehat{\rho}_\tau|\mathbf{x}-\frac{\mathbf{s}}{2}\rangle$
- LvN equation is equivalent to Moyal equation

$$\begin{aligned}\frac{\partial W_\tau}{\partial\tau} &= \mathcal{H}(\mathbf{x},\mathbf{p})\,\frac{2}{i\,\hbar}\,\sin\left(\frac{i\,\hbar}{2}\left(\overleftarrow{\partial}_{\mathbf{p}}\overrightarrow{\partial}_{\mathbf{x}}-\overleftarrow{\partial}_{\mathbf{x}}\overrightarrow{\partial}_{\mathbf{p}}\right)\right)\,W_\tau(\mathbf{x},\mathbf{p})\\ &= \underbrace{\{\mathcal{H},W_\tau\}}_{\text{Poisson bracket}} + \mathcal{O}(\hbar^2)\end{aligned}$$

- At $\mathcal{O}(\hbar)$, the evolution is still classical (the $\hbar^1$ corrections come from the quantum nature of the *initial state*)

What happens if the

classical dynamics is chaotic ?

Instabilities

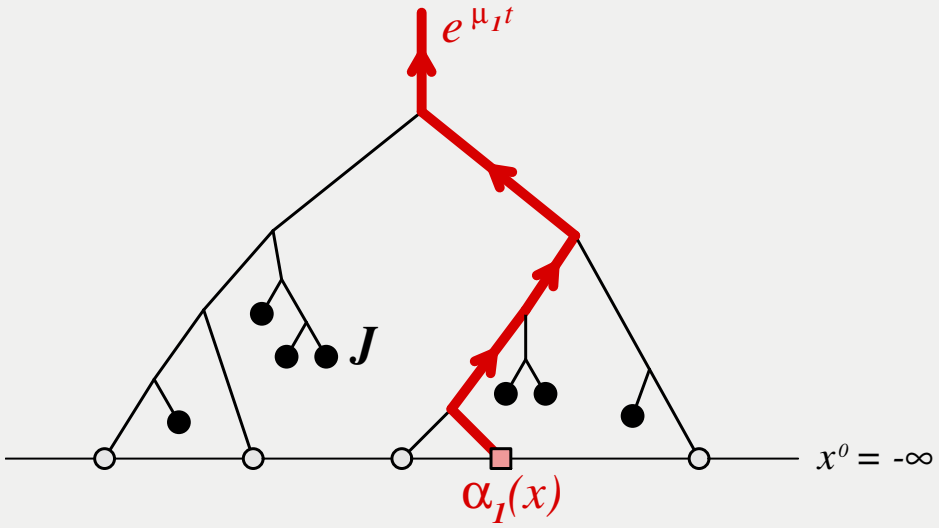
- The derivatives $\delta\mathcal{O}_{\rm LO}/\delta\mathcal{A}_{\rm in}$ are large if the classical solutions have instabilities (they measure the sensitivity to the initial condition)
- This behaviour is ubiquitous in field theory:
  - Scalar field with a ϕ^4 interaction : parametric resonance
  - Yang-Mills theory : Weibel instability
- Consequence : $\mathcal{O}_{\rm NLO}$ growths (exponentially) with time, and eventually becomes larger than $\mathcal{O}_{\rm LO}$

  $\Longrightarrow$ breakdown of the perturbative expansion

Improved power counting

- For an unstable mode:

$$\alpha(x) \underset{x^0 \to +\infty}{\sim} e^{\mu x^0} \qquad (\mu = \text{Lyapunov exponent})$$



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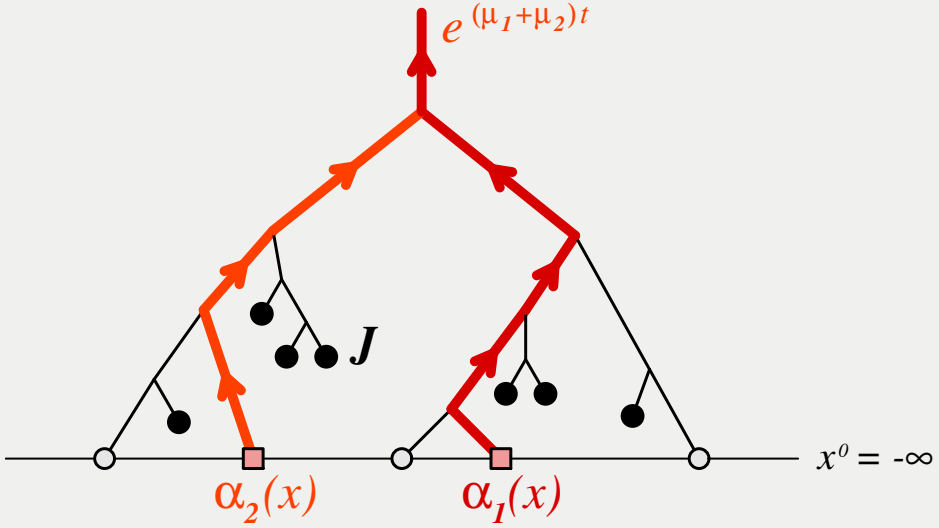

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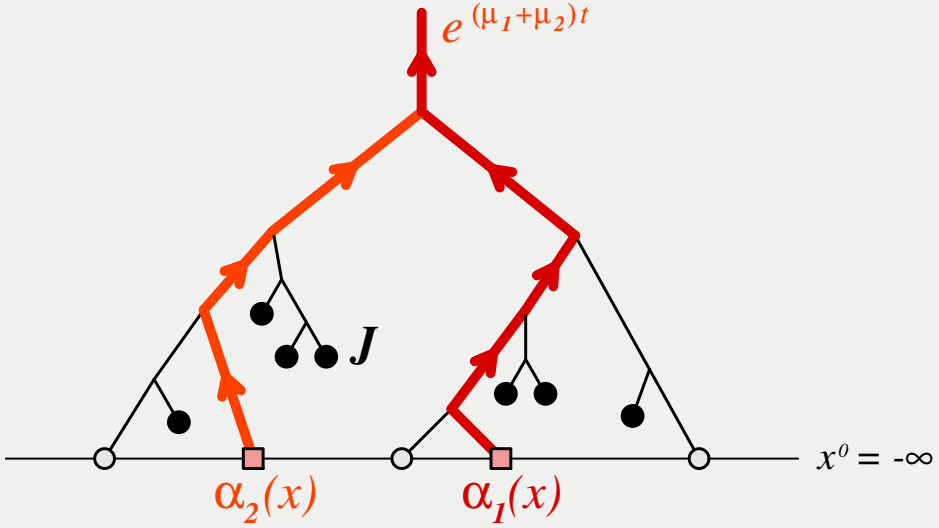
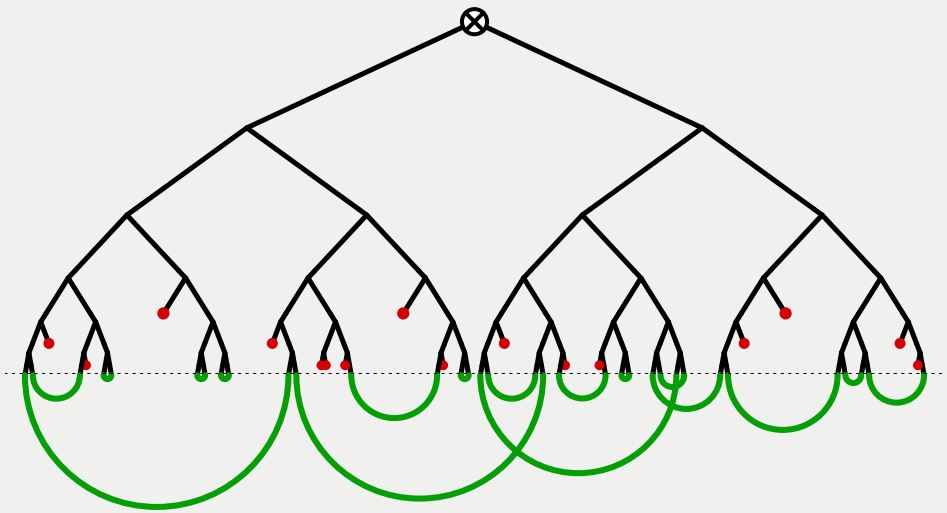


- $\mathcal{O}_{\text{NLO}} \sim e^{2\mu t}$
- At order n, there are terms $\sim e^{2n\mu t}$

Resummation of the leading terms

Resummation

$$\mathcal{O}_{\text{RESUM}} \equiv \exp\left[\frac{\hbar}{2}\int d^3x d^3y \;\; \Gamma(x,y) \; \frac{\delta}{\delta\mathcal{A}_{\text{in}}(x)}\frac{\delta}{\delta\mathcal{A}_{\text{in}}(y)}\right]\mathcal{O}_{\text{LO}}$$



$\mathcal{O}_{\text{RESUM}} = \mathcal{O}_{\text{LO}} + \mathcal{O}_{\text{NLO}} +$ subset of all higher orders

Leading terms : Classical Statistical Approximation

$$\underbrace{\exp\left[\frac{\hbar}{2}\int_{x,y}\underbrace{\Gamma_2(x,y)\,\frac{\delta}{\delta\mathcal{A}_{\rm in}(x)}\frac{\delta}{\delta\mathcal{A}_{\rm in}(y)}}_{\text{"Laplacian"}}\right]}_{\text{Diffusion operator on the classical phase-space}}\mathcal{O}_{\rm LO}[\mathcal{A}_{\rm in}]$$

$$=\int[Da]\,\exp\left[-\frac{1}{2\hbar}\int_{x,y}a(x)\Gamma_2^{-1}(x,y)a(y)\right]\mathcal{O}_{\rm LO}[\mathcal{A}_{\rm in}+a]$$

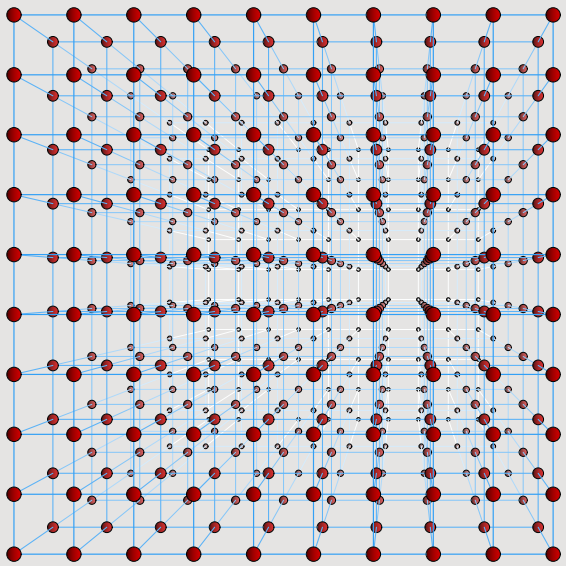
- In this resummation, the observable is obtained as an average over classical fields with fluctuating initial conditions
- The exponentiation of the 1-loop result promotes the classical vacuum $\mathcal{A}_{\rm in}\equiv 0$ into the coherent quantum state $\left|0_{\rm in}\right\rangle$

Numerical implementation

Hamiltonian lattice formalism

Space

$\Rightarrow$ **3D cubic lattice**

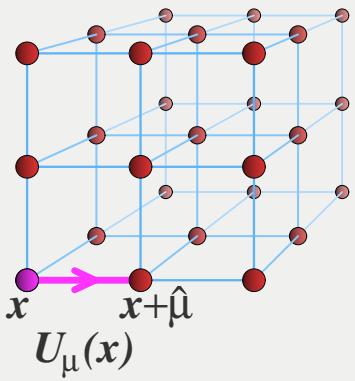


- Discrete space, continuous time
- Hamilton equations :

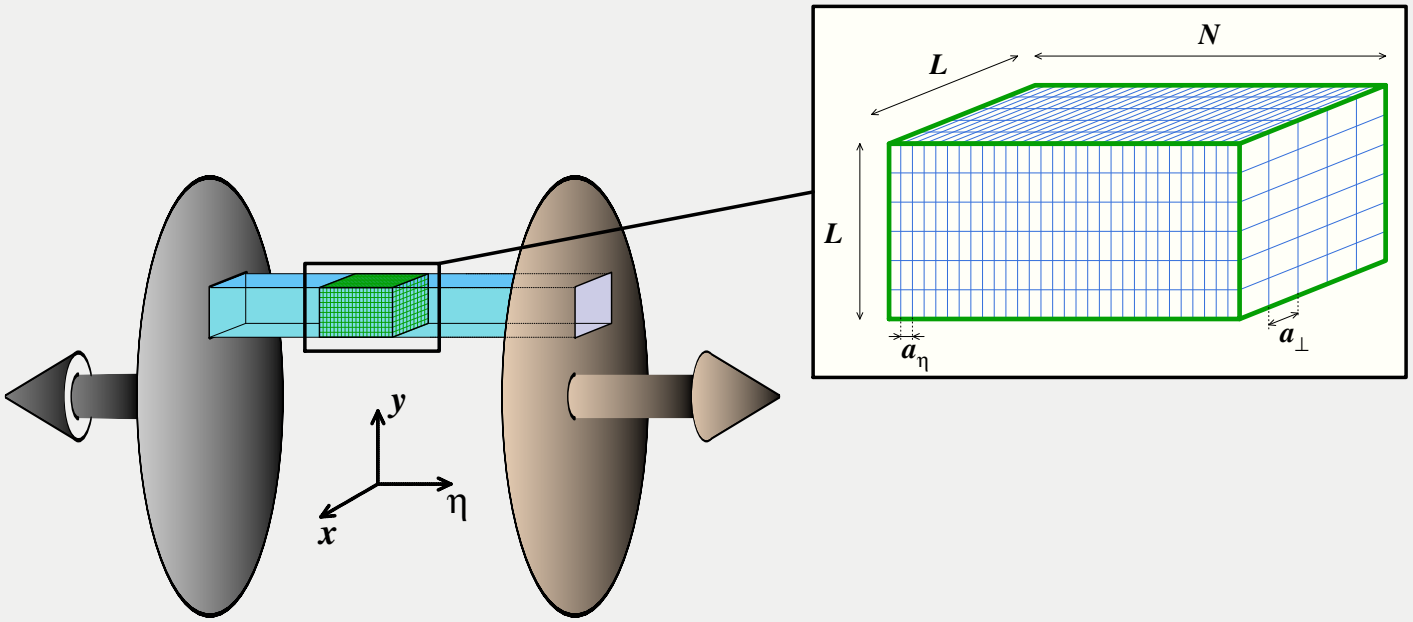
$$\partial_t \mathcal{A} = \mathcal{E}$$
$$\partial_t \mathcal{E} = F(\mathcal{A})$$

- **Yang-Mills case :**

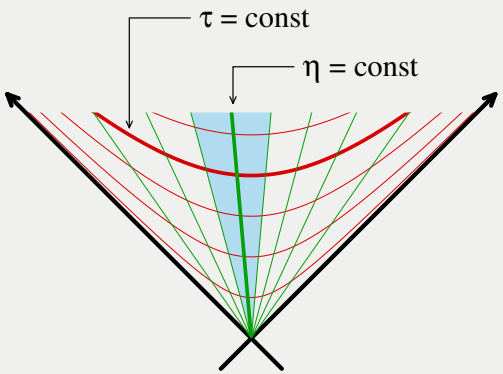
  Use link variables instead of $\mathcal{A}$ to preserve residual gauge symmetry



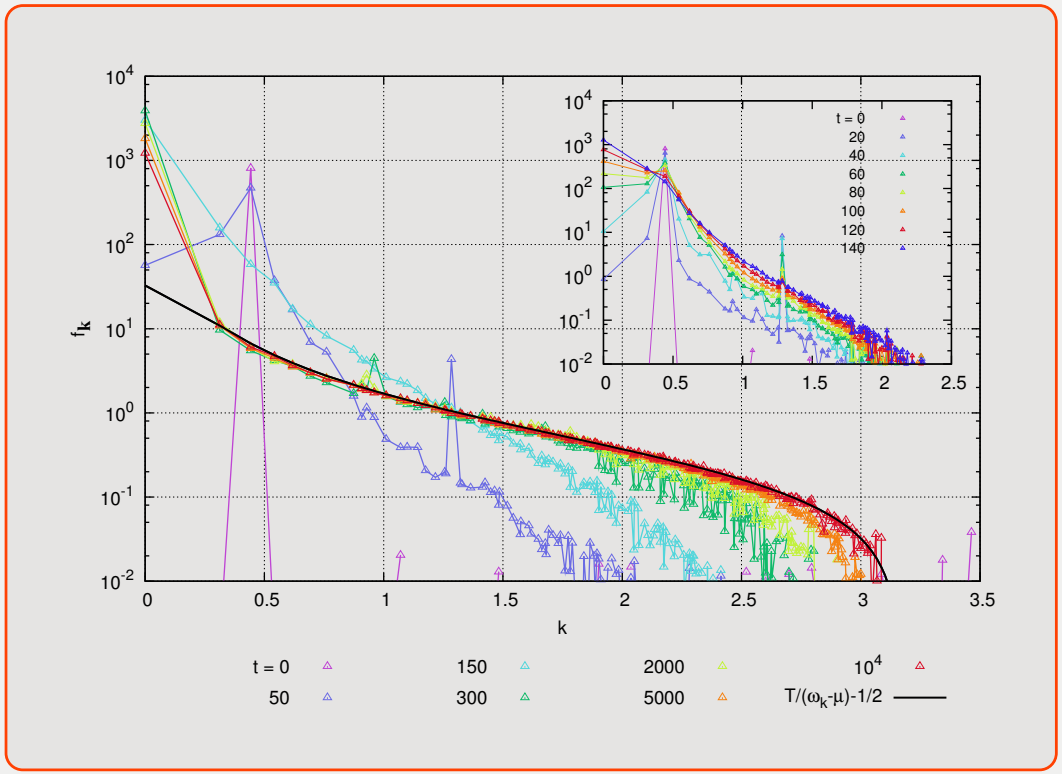
Discretization of the expanding volume



- Comoving coordinates : $\tau, \eta, x_\perp$
- Only a small volume is simulated
  + periodic boundary conditions

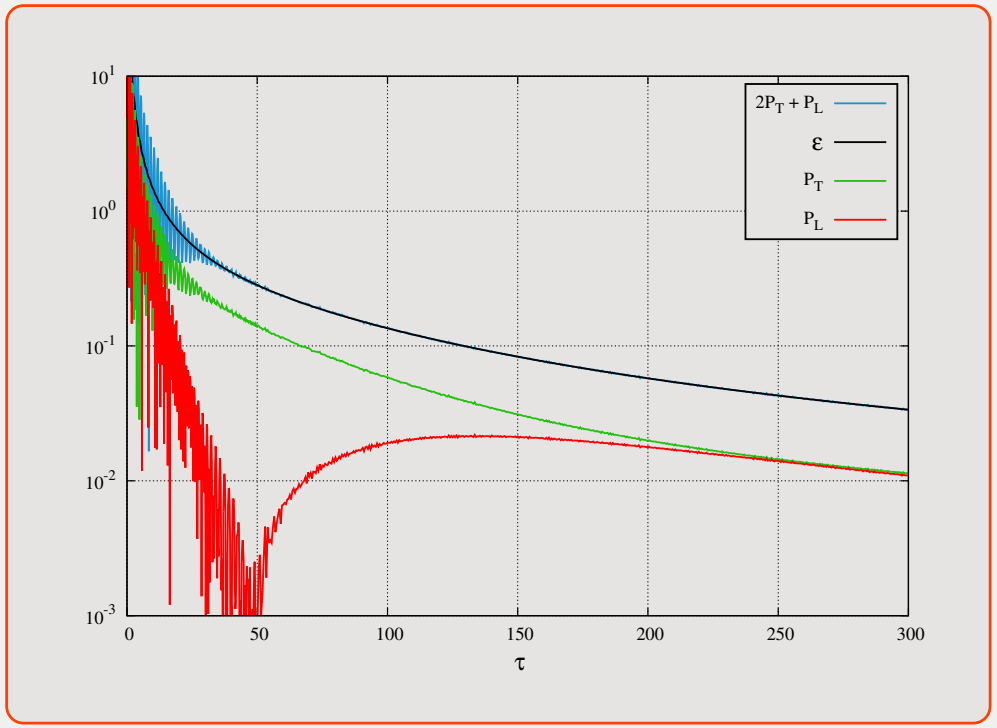


Thermalization



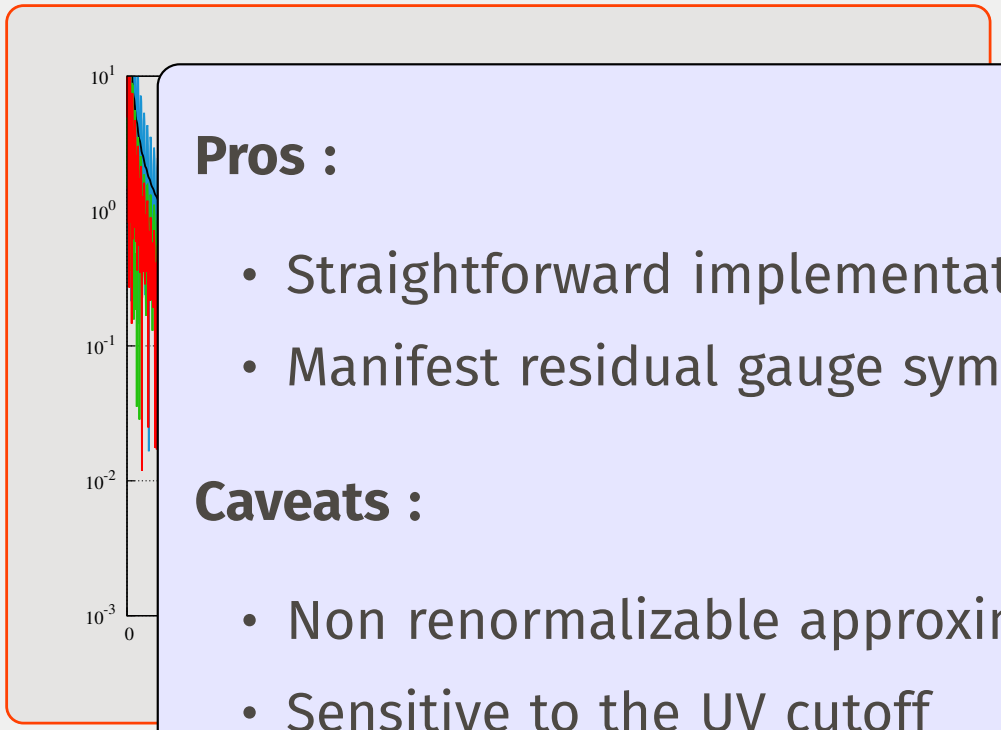
- Unstable modes grow very quickly
- Other modes are filled later
- Possibility to form a Bose-Einstein condensate
- Asymptotic distribution: classical equilibrium $T(\omega-\mu)^{-1}-\frac{1}{2}$

Pressure isotropization



- At early times, P_L drops much faster than P_T (redshifting of the longitudinal momenta due to the expansion)
- Drastic change of behavior when the expansion rate becomes smaller than the growth rate of the unstability
- Eventually, isotropic pressure tensor : $P_L \approx P_T$

Pressure isotropization



Pros :

- Straightforward implementation
- Manifest residual gauge symmetry

Caveats :

- Non renormalizable approximation
- Sensitive to the UV cutoff

- At early times, P_L drops much faster than P_T (rodshifting of the dinal momenta the expansion)

change of
or when the
on rate becomes
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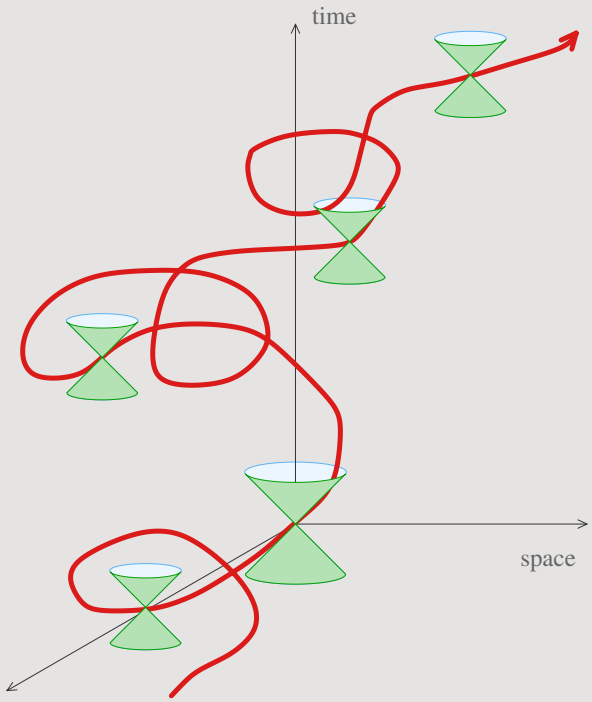
Thank you !!

What if... we wanted to calculate exclusive quantities?

Exclusive observables

Feynman propagator



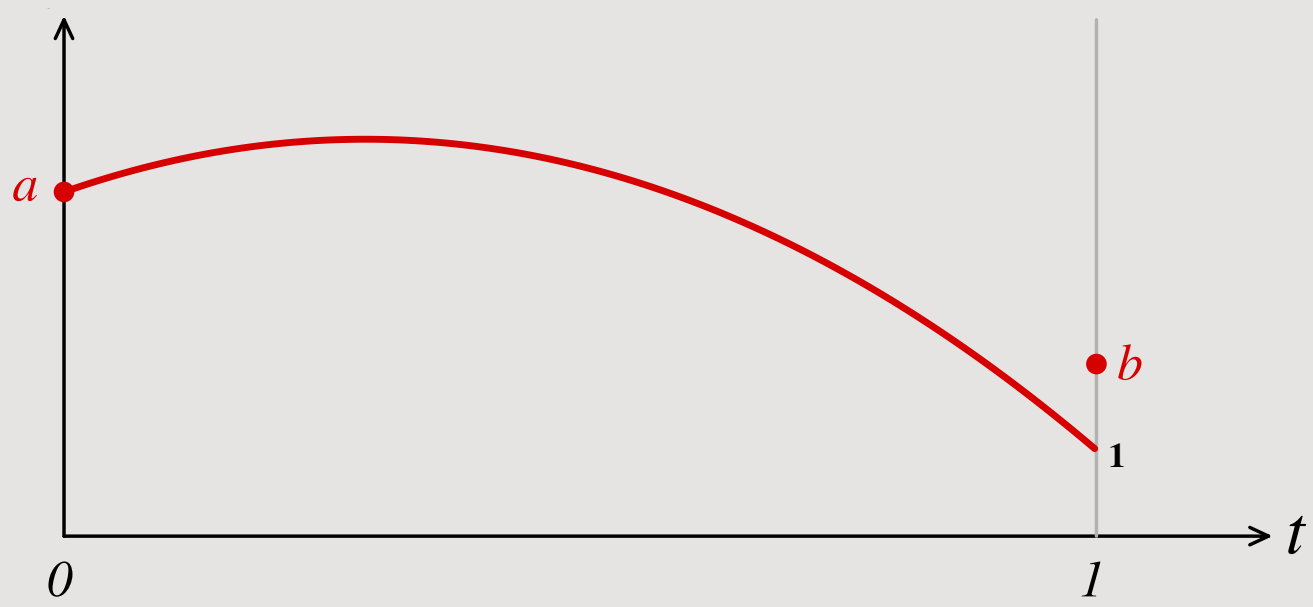
- **Exclusive** measurement :
  - Select specific final states
  - Boundary condition on the fields at $t = +\infty$
- **Feynman propagator**
  = Non causal evolution
- Numerically untractable

Point-and-shoot problem

Differential equation with mixed boundary conditions

$$\ddot{y} = f(y, \dot{y}) \quad , \qquad y(0) = a \, , \quad y(1) = b$$

Gentle nonlinear case



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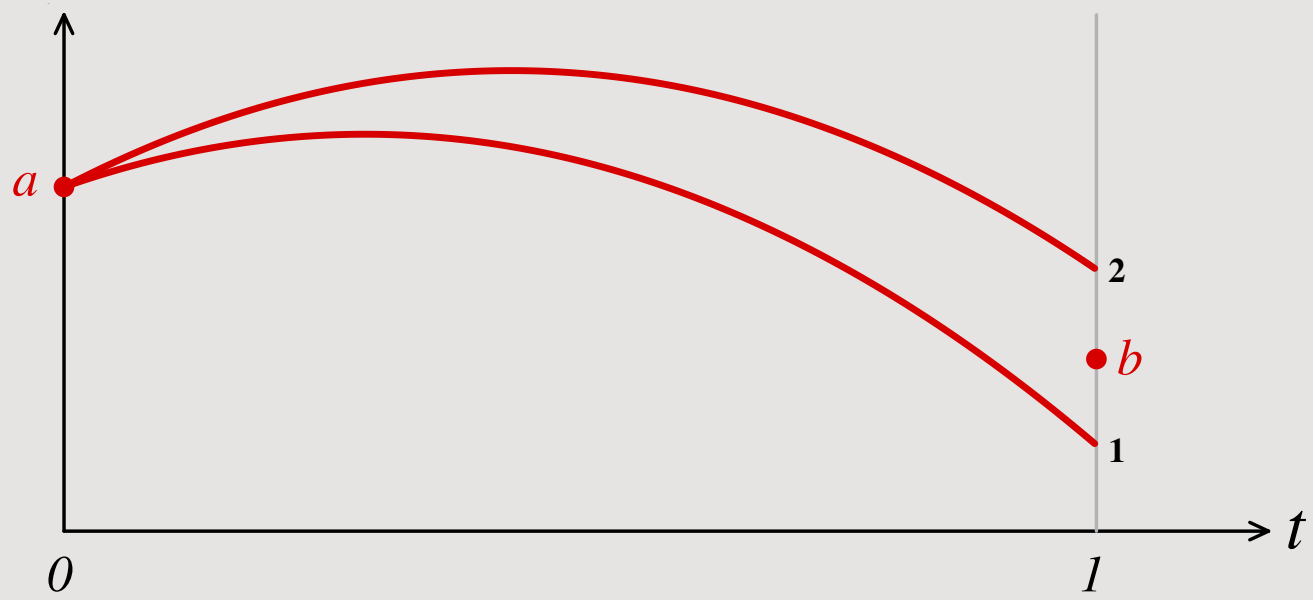

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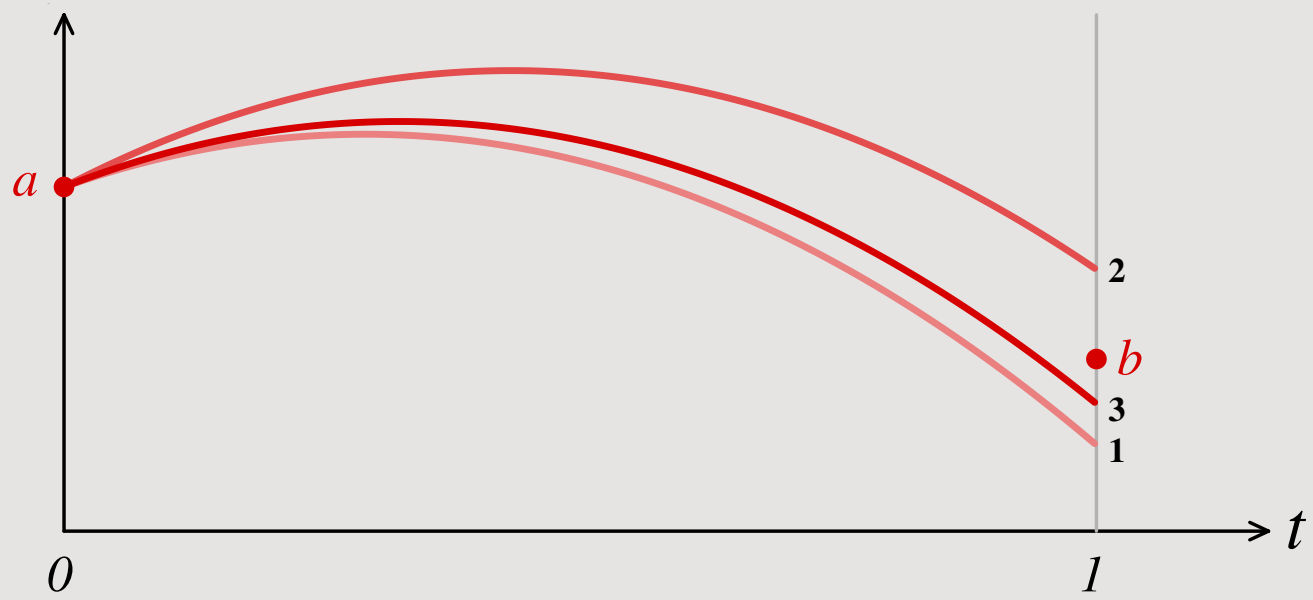

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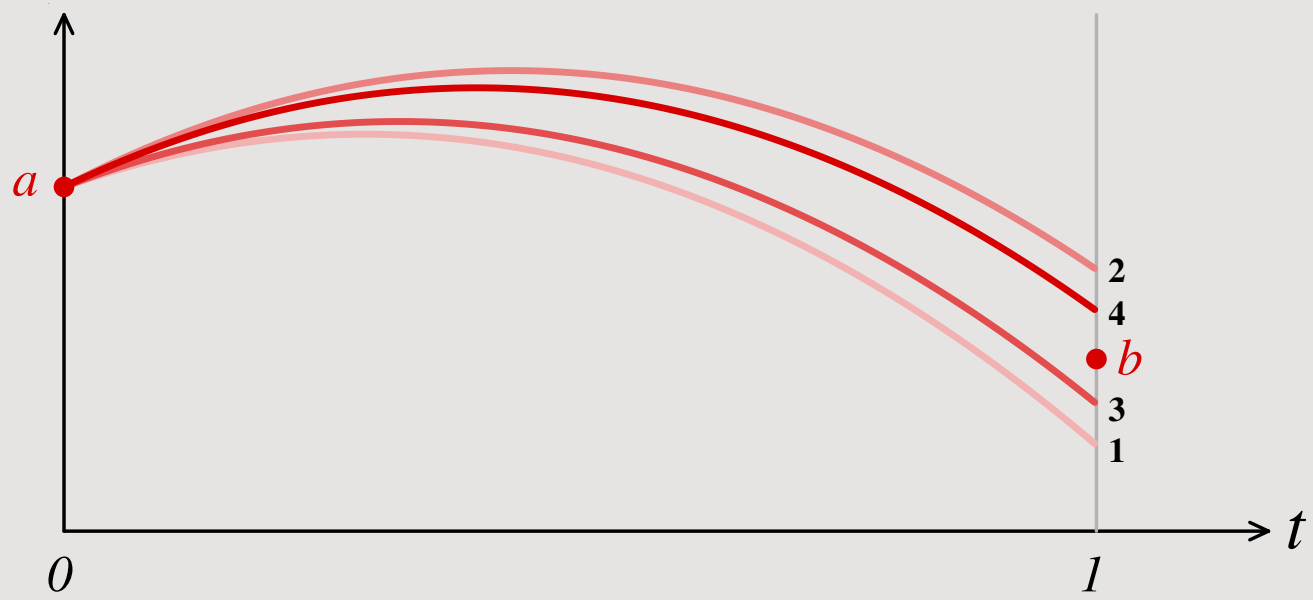

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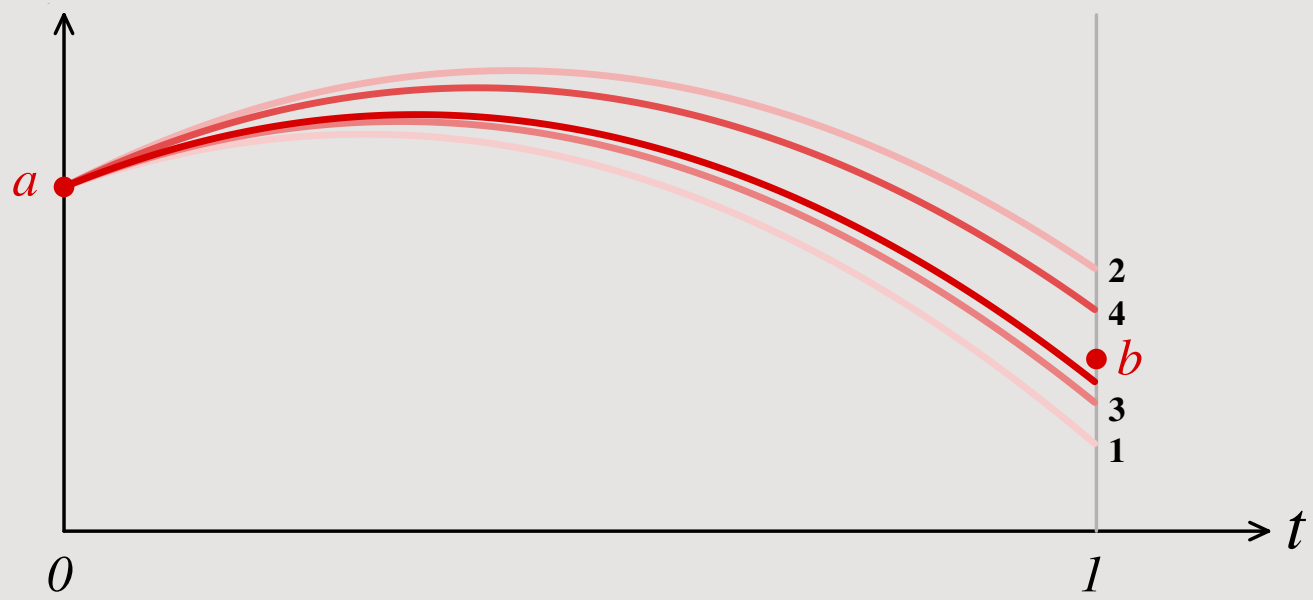
Gentle nonlinear case

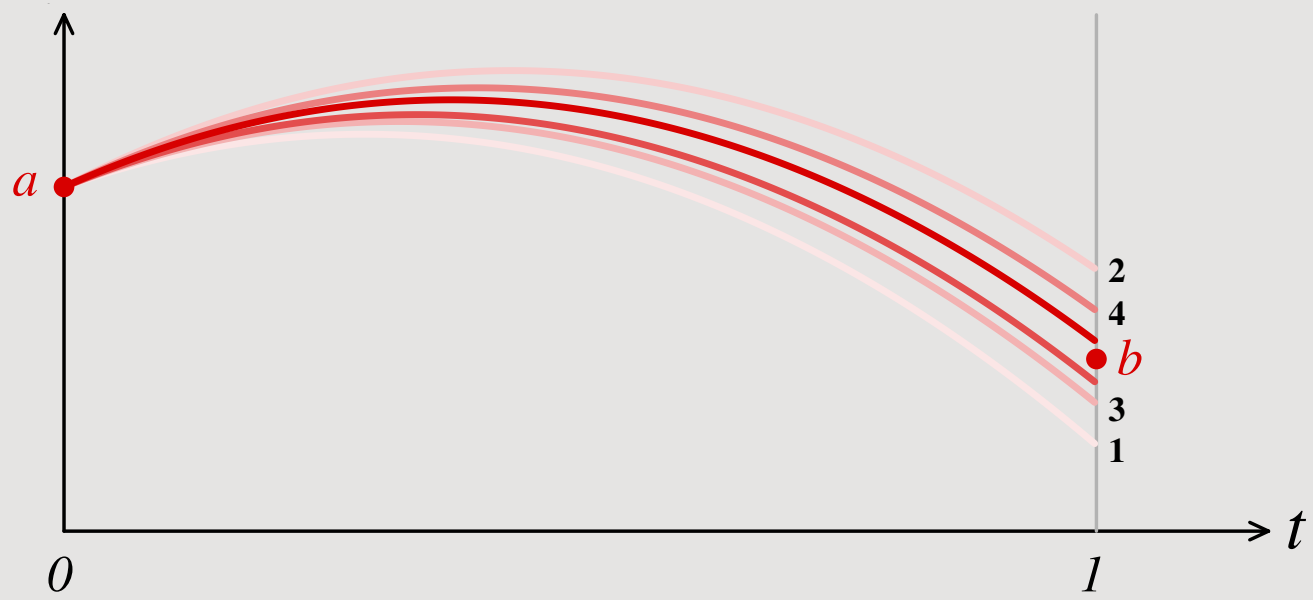


Point-and-shoot problem

Differential equation with mixed boundary conditions

$$\ddot{y} = f(y, \dot{y}) \quad , \qquad y(0) = a \, , \quad y(1) = b$$

Gentle nonlinear case



Differential equation with mixed boundary conditions

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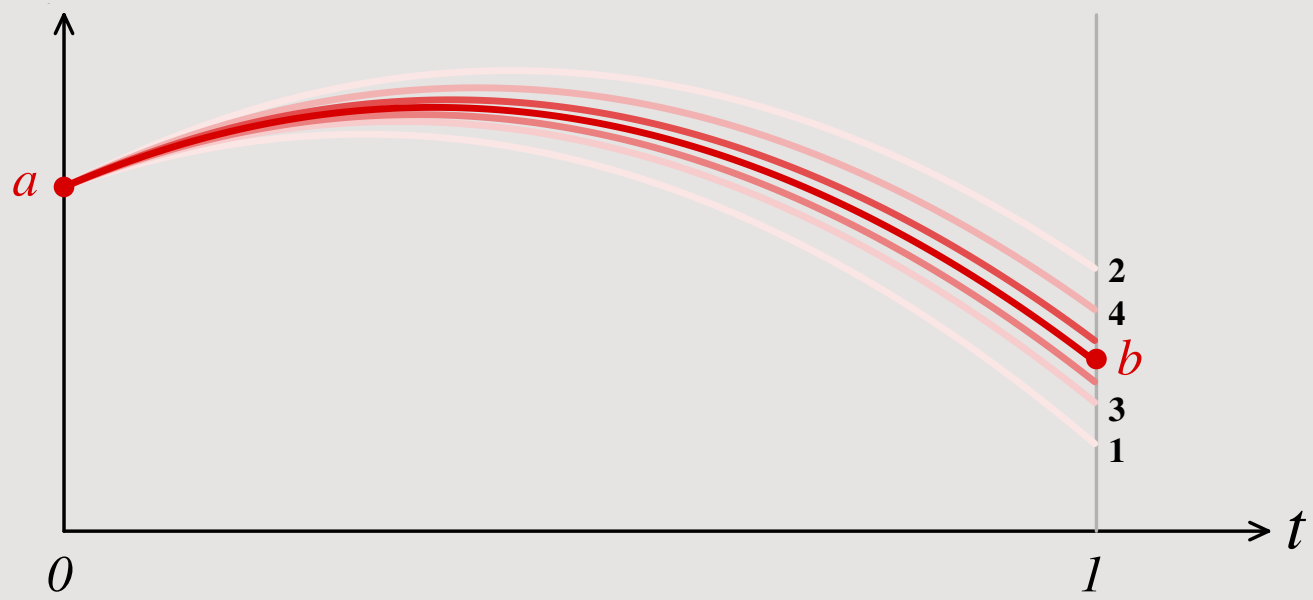
Gentle nonlinear case

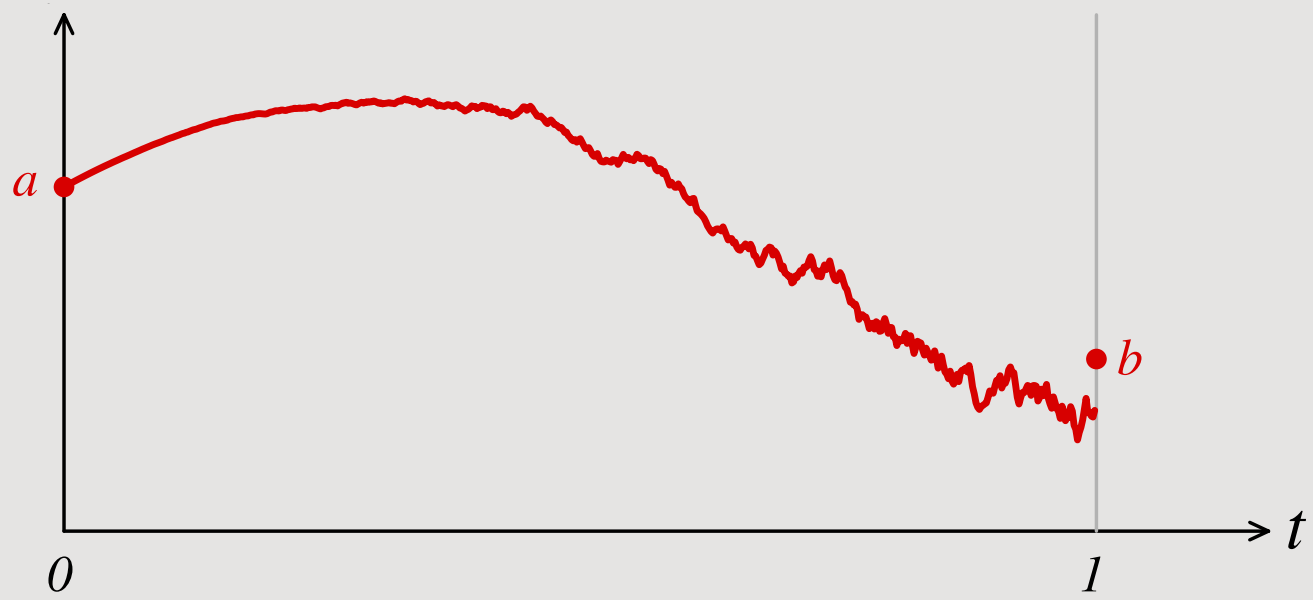


Point-and-shoot problem

Differential equation with mixed boundary conditions

$$\ddot{y} = f(y, \dot{y}) \quad , \qquad y(0) = a \, , \quad y(1) = b$$

Chaotic nonlinear case



Point-and-shoot problem

Differential equation with mixed boundary conditions

$$\ddot{y} = f(y, \dot{y}) \quad , \qquad y(0) = a \, , \quad y(1) = b$$

Chaotic nonlinear case