Gluon saturation and Color Glass Condensate

Krakow, May 18-20, 2016

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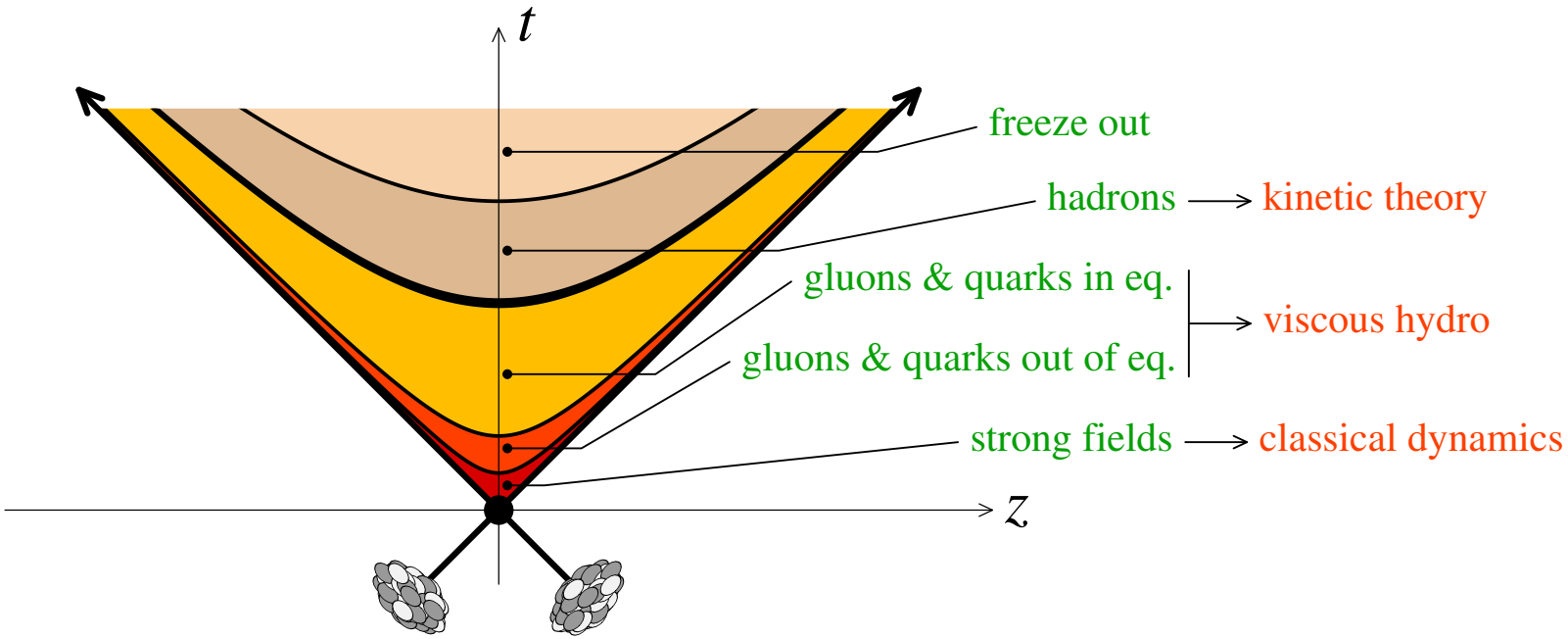
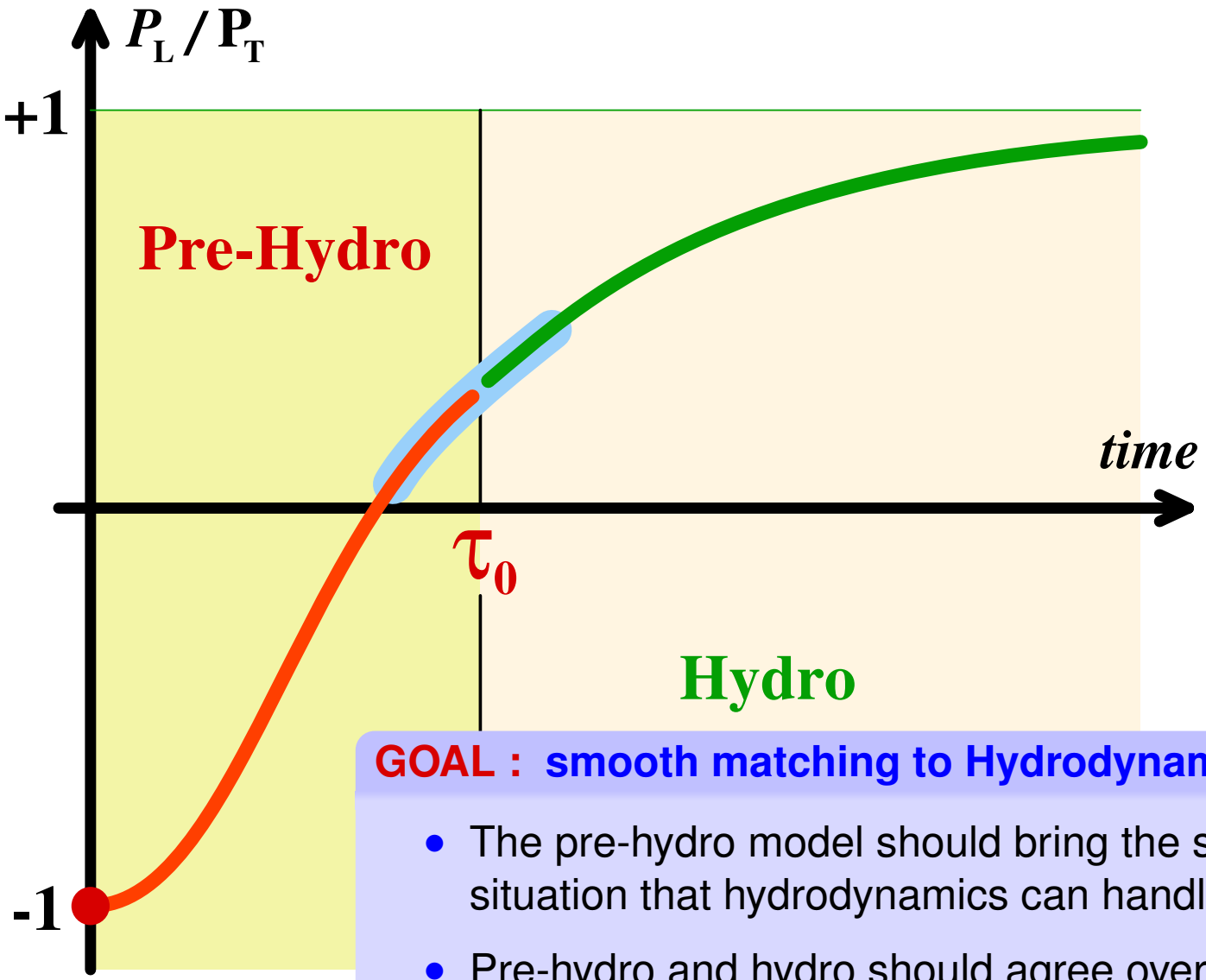
IPhT, Saclay

Stages of a nucleus-nucleus collision



- Hydrodynamics successful at describing the bulk evolution
- In this talk : **Initial state** $\Rightarrow$ **Pre-hydro evolution**



GOAL : smooth matching to Hydrodynamics

- The pre-hydro model should bring the system to a situation that hydrodynamics can handle
- Pre-hydro and hydro should agree over some range of time $\Rightarrow$ no τ_0 dependence
- NOTE : the anisotropic hydro model of Ryblewski, Strickland et al. helps for a matching at smaller τ_0

Terminology

- Weakly coupled : $g \ll 1$
- Strongly coupled : $g \gg 1$

- Weakly interacting : $g\mathcal{A} \ll 1 \qquad g^2 f(\mathbf{p}) \ll 1$

  $(2 \to 2) \gg (2 \to 3), (3 \to 2), \cdots$

- Strongly interacting : $g\mathcal{A} \sim 1 \qquad g^2 f(\mathbf{p}) \sim 1$

  $(2 \to 2) \sim (2 \to 3) \sim (3 \to 2) \sim \cdots$

Terminology

- Weakly coupled : $g \ll 1$
- Strongly

Strongly coupled	$\Rightarrow$	Strongly interacting
Weakly coupled	$\not\Rightarrow$	Weakly interacting

- Weakly interacting : $g\mathcal{A} \ll 1 \qquad g^2 f(\mathbf{p}) \ll 1$

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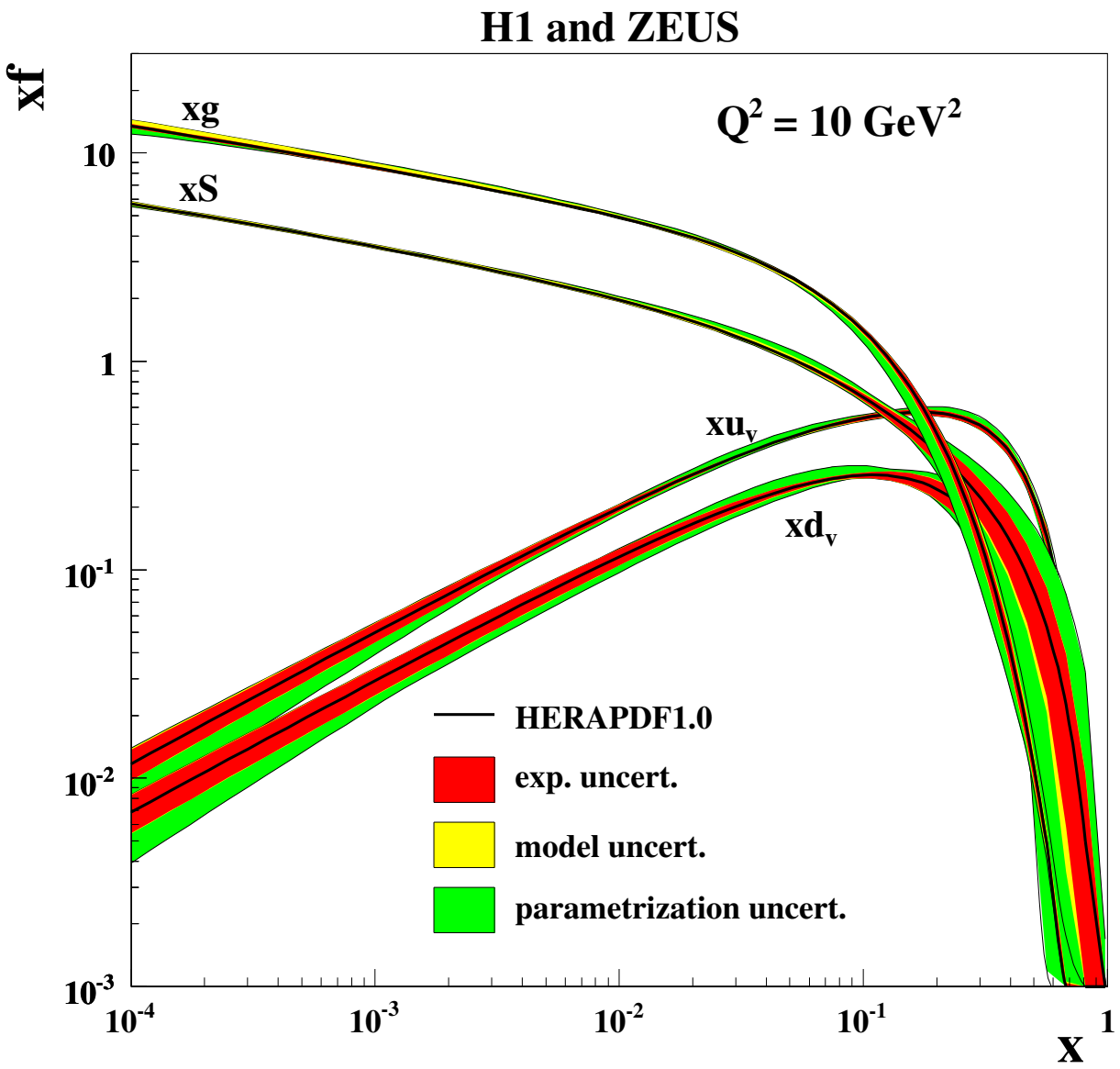
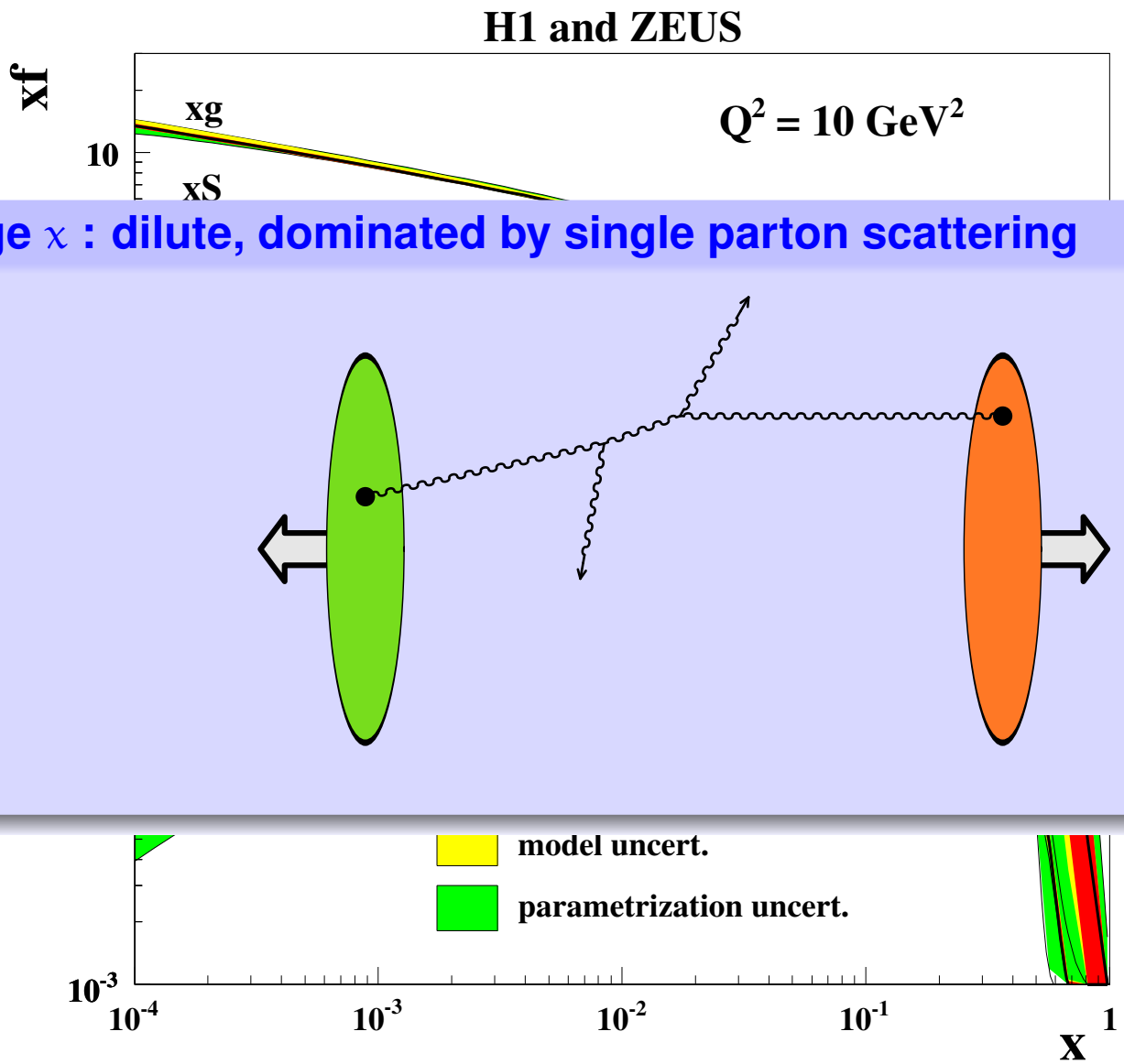
Outline

Gluon saturation

Parton distributions in a nucleon

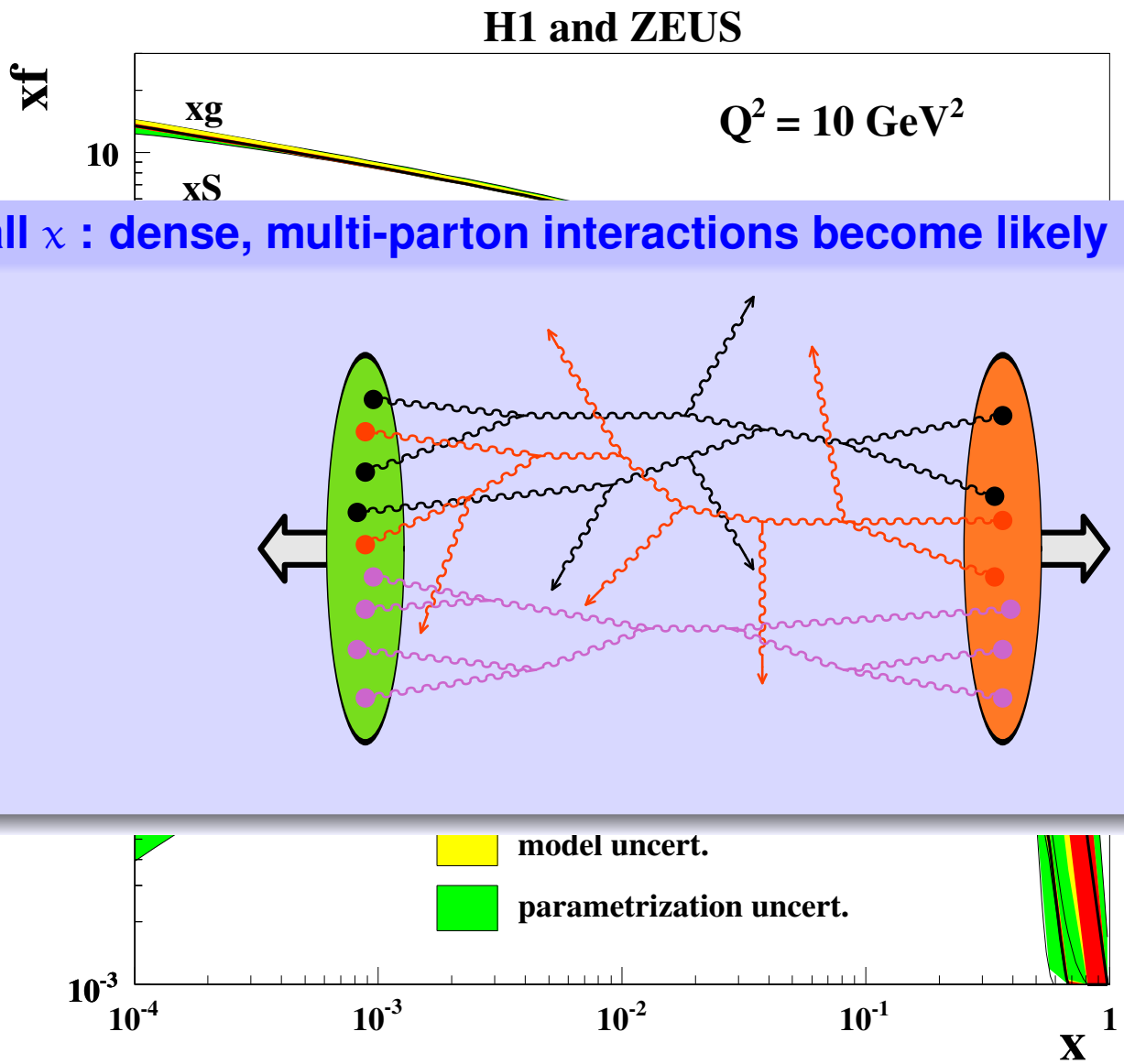


Parton distributions in a nucleon



Large x : dilute, dominated by single parton scattering

Parton distributions in a nucleon



Small χ : dense, multi-parton interactions become likely

- When their occupation number becomes large, gluons can recombine :

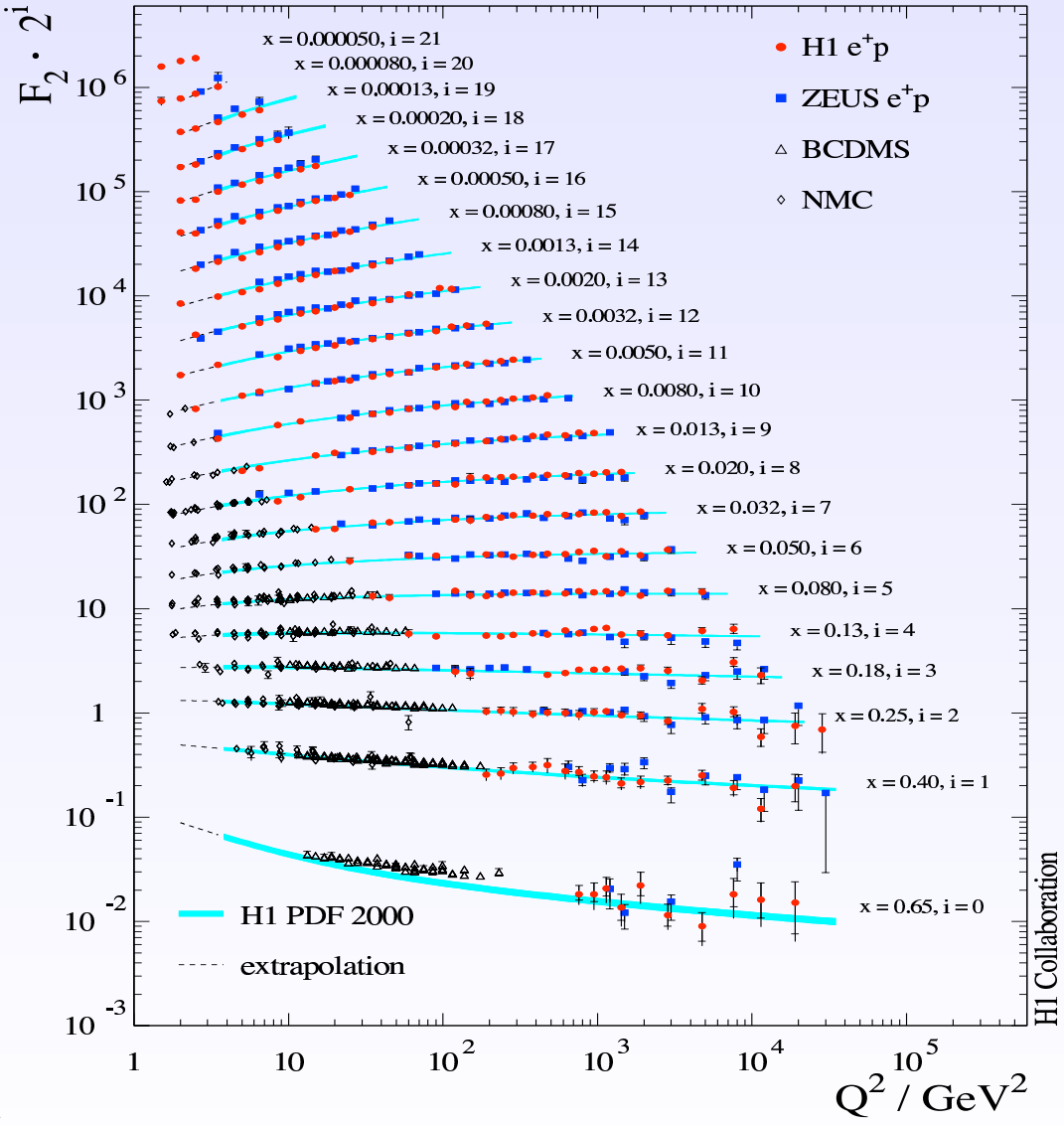
Gluon Saturation

Saturation criterion **[Gribov, Levin, Ryskin (1983)]**

$$\underbrace{\alpha_s Q^{-2}}_{\sigma_{g\,g\to g}} \times \underbrace{A^{-2/3} x G(x, Q^2)}_{\text{surface density}} \geq 1$$

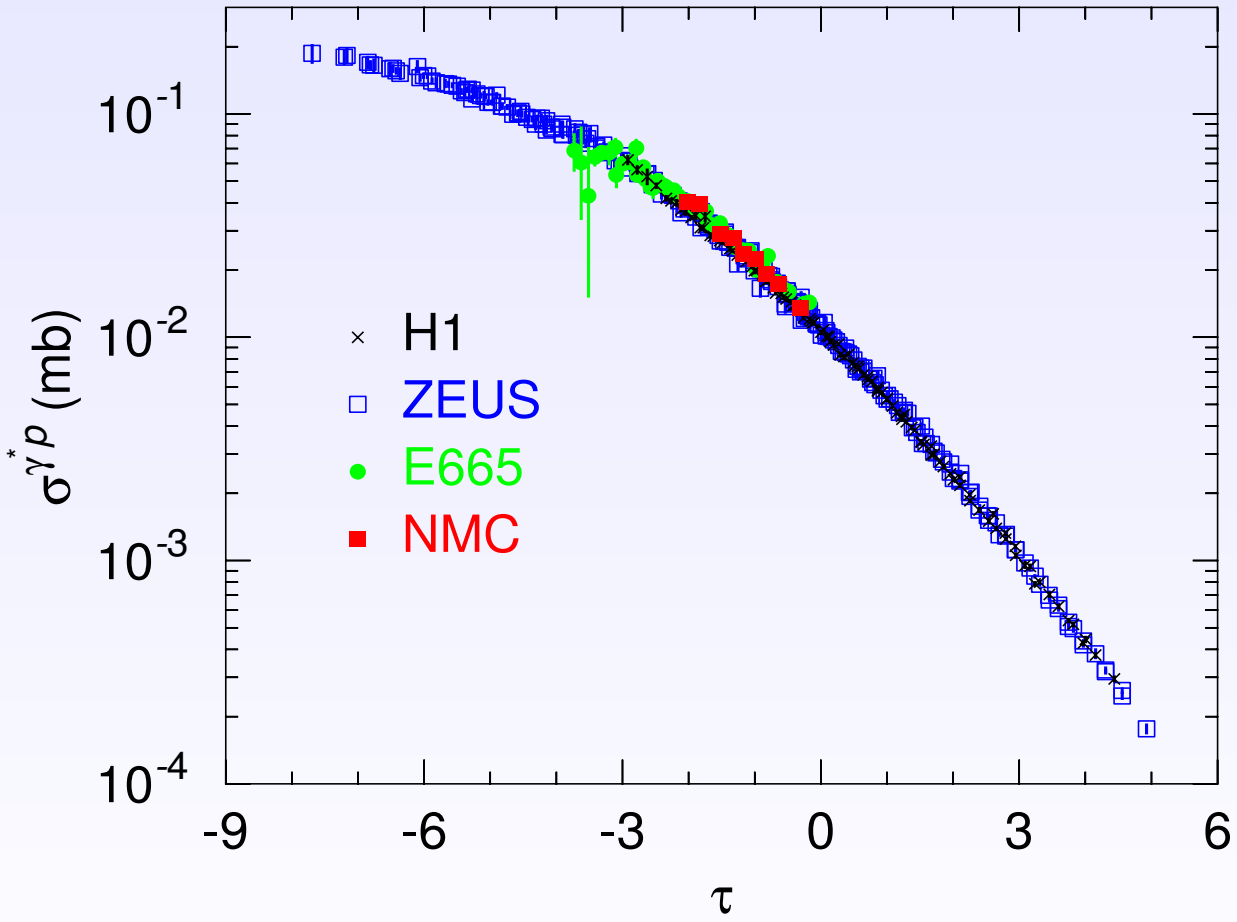
$$Q^2 \leq \underbrace{Q_s^2 \equiv \frac{\alpha_s x G(x, Q_s^2)}{A^{2/3}}}_{(\text{saturation momentum})^2} \sim A^{1/3} x^{-0.3}$$

DIS results for F_2 and DGLAP fit at NLO :

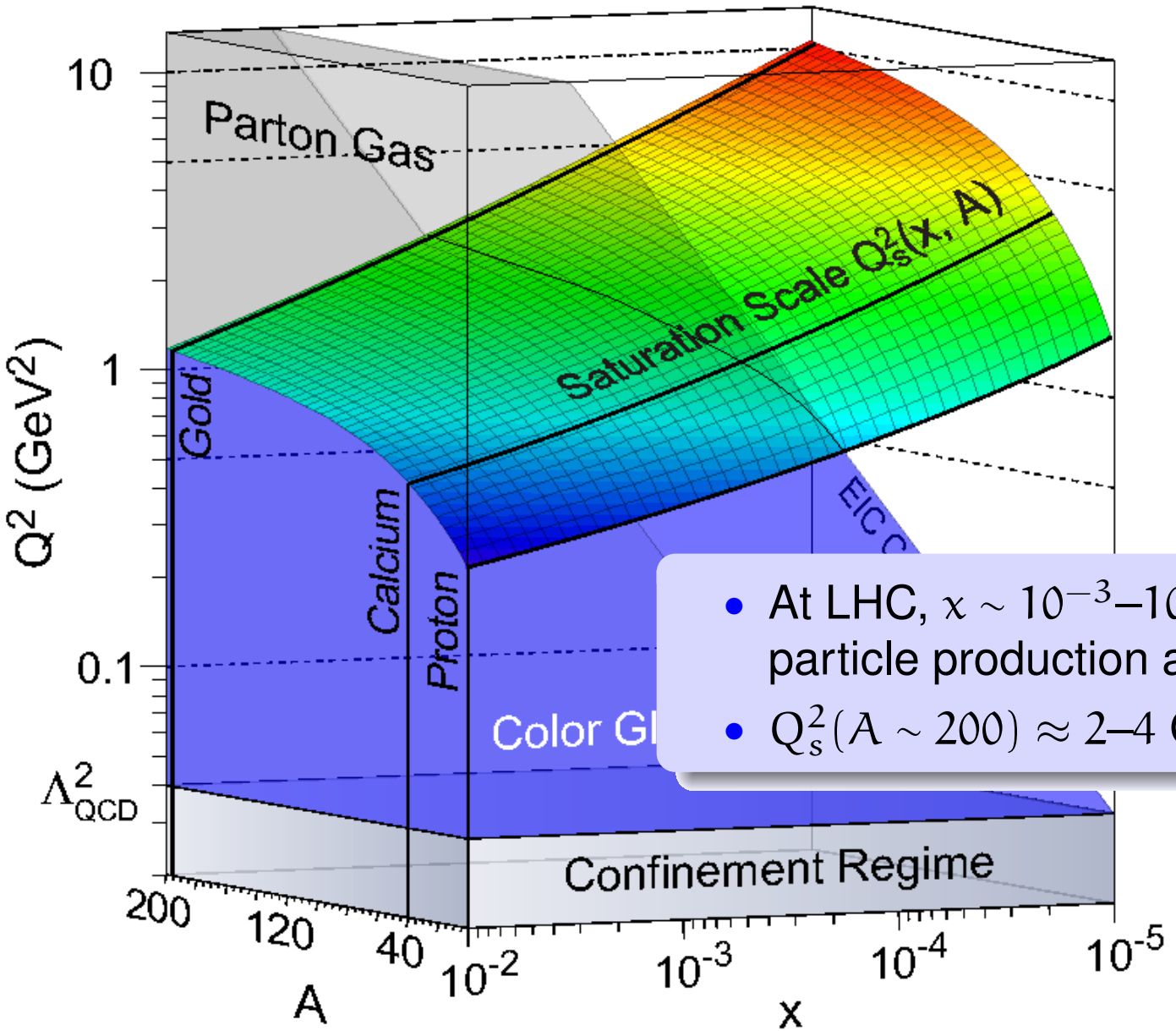


Small x data displayed differently... (Geometrical scaling)

- Small x data ($x \leq 10^{-2}$) displayed against $\tau \equiv \log(x^{0.32}\, Q^2)$:

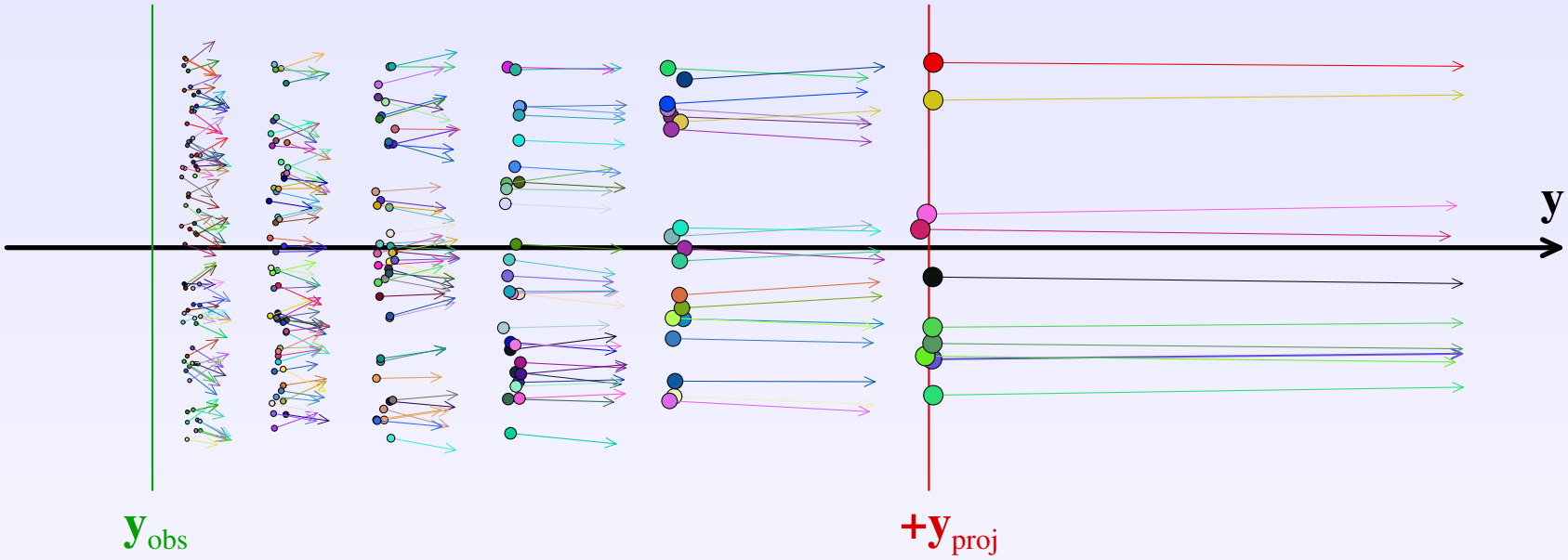


Saturation domain



- At LHC, $x \sim 10^{-3}$–10^{-4} (for bulk particle production at mid-rapidity)
- $Q_s^2(A \sim 200) \approx 2$–$4$ GeV^2

Degrees of freedom [McLerran, Venugopalan (1994)]



- $p_\perp^2 \sim Q_s^2 \sim \Lambda_{QCD}\, e^{\lambda(y_{proj}-y)}$, $p_z \sim Q_s\, e^{y-y_{obs}}$
- Fast partons : frozen dynamics, negligible $p_\perp$ $\Rightarrow$ classical sources
- Slow partons : evolve with time $\Rightarrow$ gauge fields

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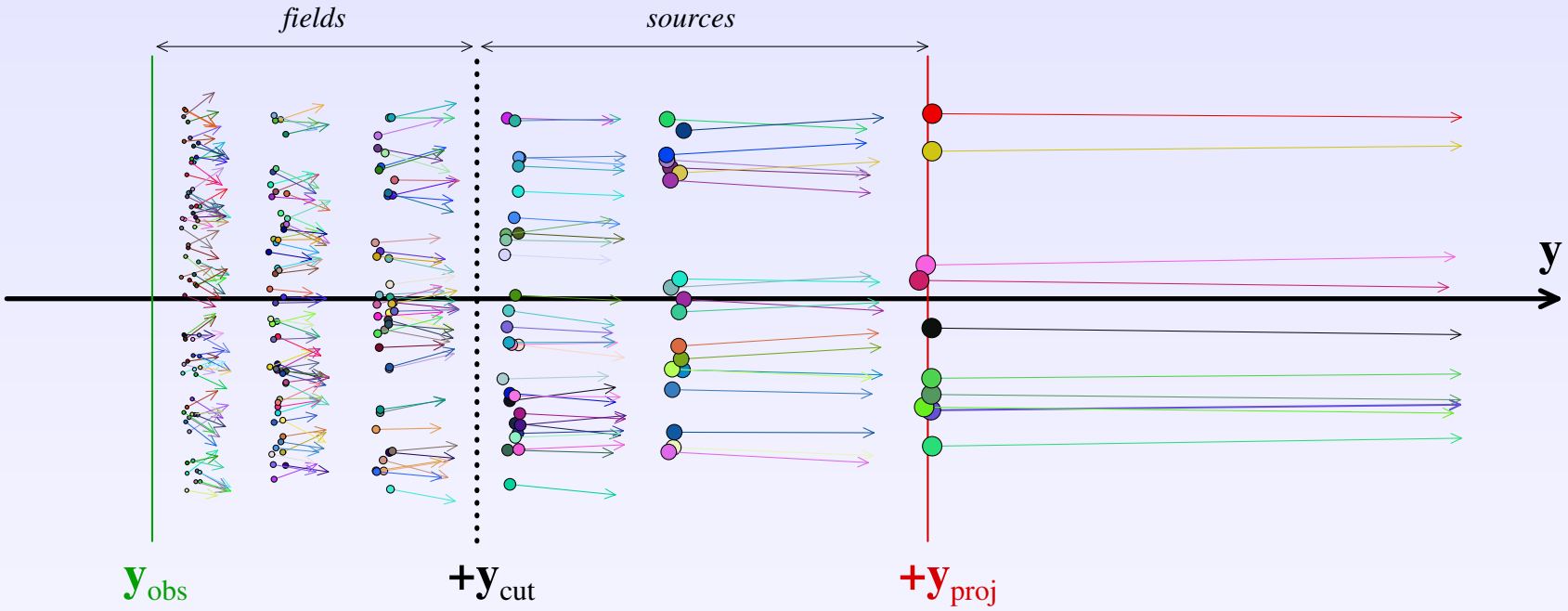

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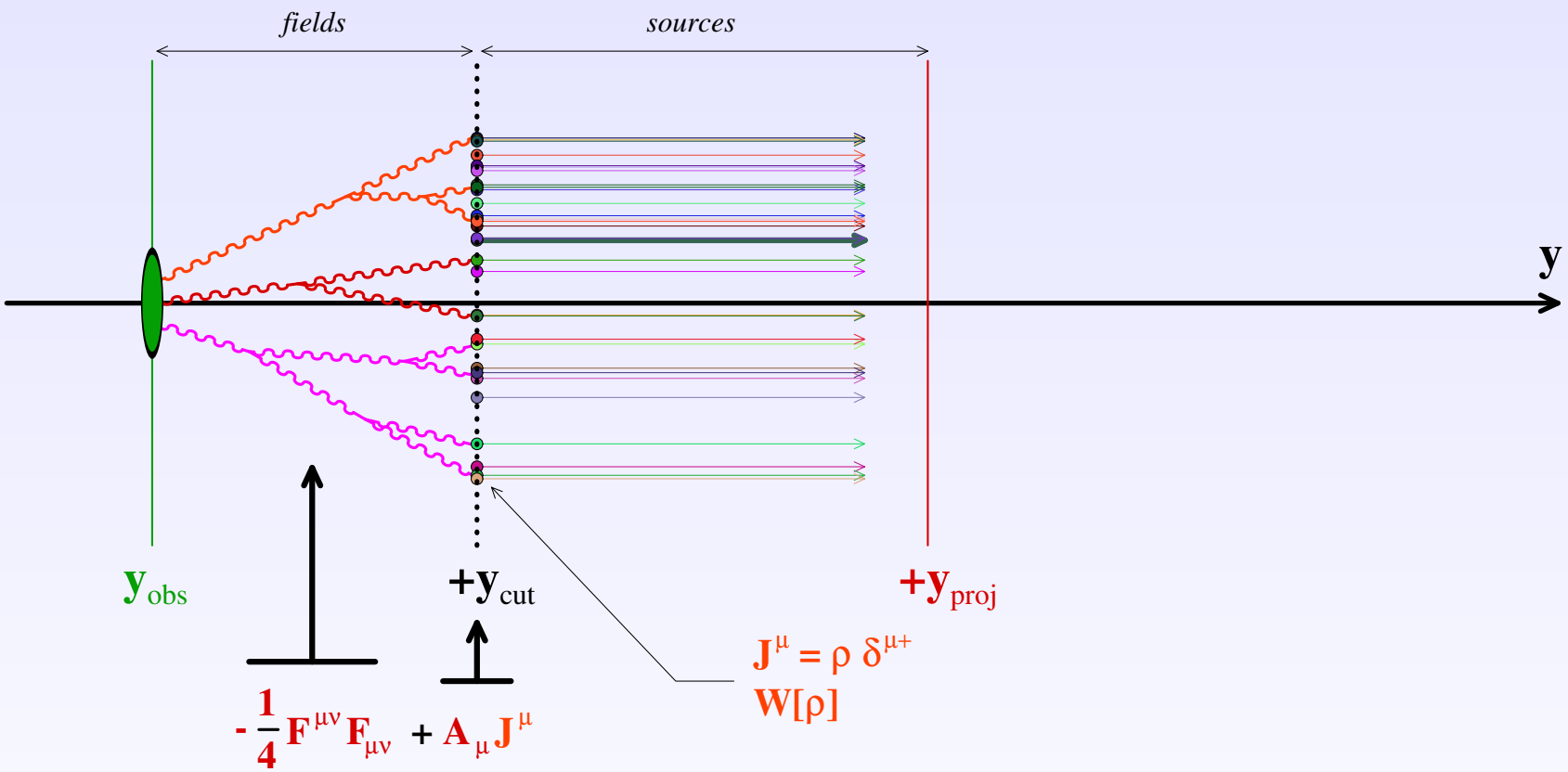

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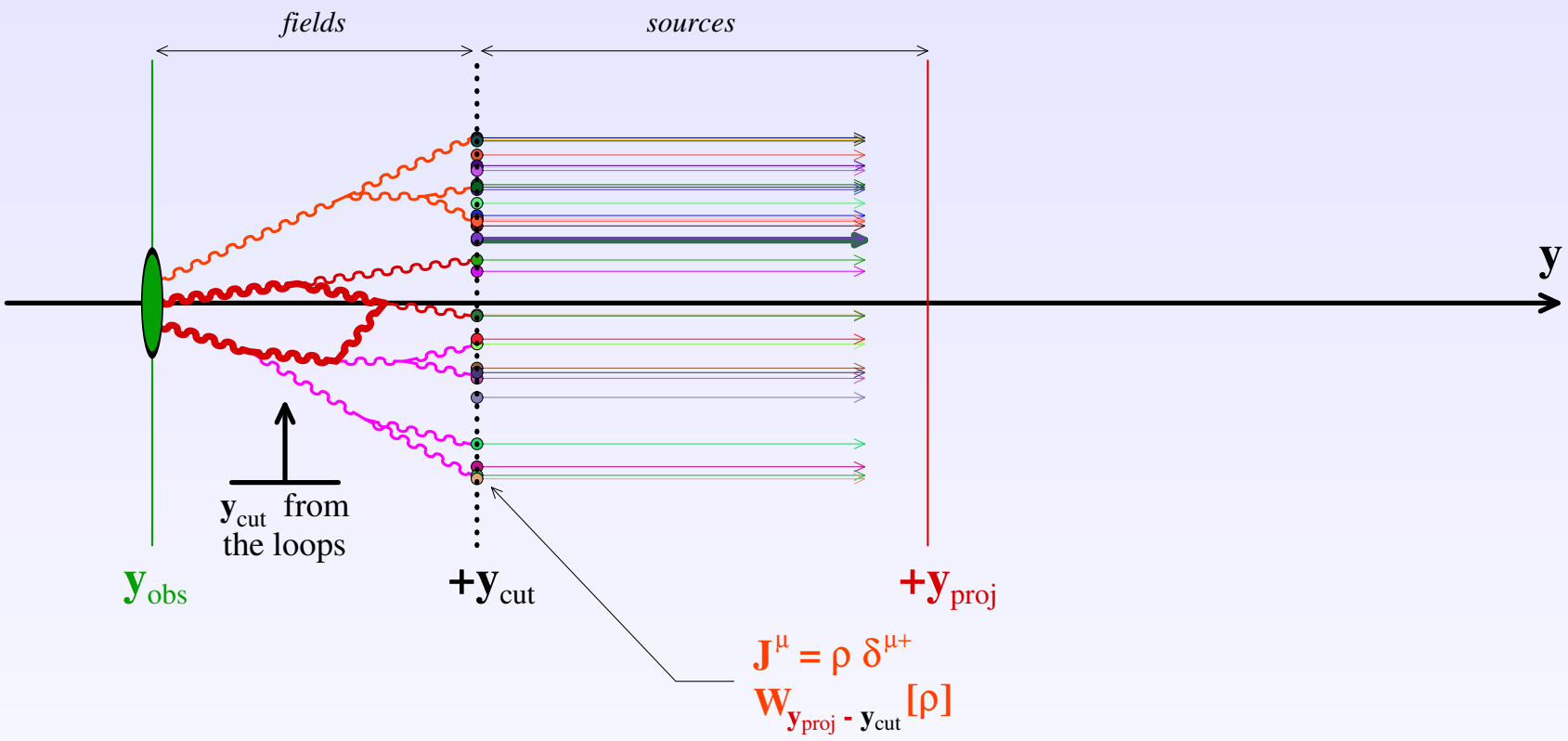
Cancellation of the cutoff dependence



- The cutoff y_{cut} is arbitrary and should not affect the result
- The probability density $W[\rho]$ changes with the cutoff
- Loop corrections cancel the cutoff dependence from $W[\rho]$

B-JIMWLK evolution equation

[Jalilian-Marian, Kovner, Leonidov, Weigert (1998)]
[Balitsky (1996)] [Iancu, Leonidov, McLerran (2001)]

B-JIMWLK equation at Leading Log

$$\frac{\partial W_Y[\rho]}{\partial Y} = \underbrace{\frac{1}{2}\int_{\vec{x}_\perp,\vec{y}_\perp} \frac{\delta}{\delta\rho_a(\vec{x}_\perp)}\chi_{ab}(\vec{x}_\perp,\vec{y}_\perp)\frac{\delta}{\delta\rho_b(\vec{y}_\perp)}}_{\mathcal{H}\quad\text{(JIMWLK Hamiltonian)}} W_Y[\rho]$$

- Mean field approx. (BK equation) : **[Kovchegov (1999)]**
- Langevin form of B-JIMWLK : **[Blaizot, Iancu, Weigert (2003)]**
- First numerical solution : **[Rummukainen, Weigert (2004)]**

B-JIMWLK evolution equation

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[Balitsky (1996)] [Iancu, Leonidov, McLerran (2001)]

B-JIMWL

Recent developments :

Running coupling correction

[Lappi, Mäntysaari (2012)]

B-JIMWLK equation at Next to Leading Log

[Kovner, Lublinsky, Mulian (2013)]

[Caron-Huot (2013)][Balitsky, Chirilli (2013)]

- Me
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B-JIMW

Recent developments :

F

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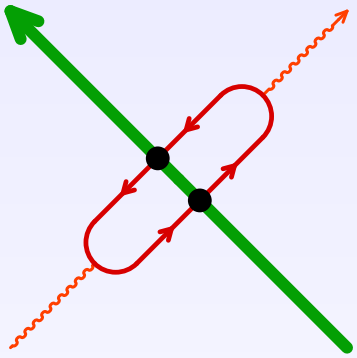
Open questions for practical uses :

- Does the NLO evolution preserve the positivity of $W[\rho]$? (non trivial if the JIMWLK Hamiltonian contains higher derivatives at NLO)
- Can the NLO B-JIMWLK equation still be mapped into a Langevin equation?

- Me [C
- Lange 3)]
- First numerical solution : [Rummukainen, Weigert (2004)]

Projectile-centric description : Balitsky-Kovchegov equation



$$\mathbf{T}(\mathbf{x}_\perp, \mathbf{y}_\perp) \equiv 1 - \frac{1}{N_c} \operatorname{tr}\left(U(\mathbf{x}_\perp) U^\dagger(\mathbf{y}_\perp)\right)$$

$$\frac{\partial \langle \mathbf{T} \rangle}{\partial Y} \sim \alpha_s \int \cdots \Big[\langle \mathbf{T} \rangle - \underbrace{\langle \mathbf{T}\mathbf{T} \rangle}_{\approx \langle \mathbf{T} \rangle \langle \mathbf{T} \rangle} \Big]$$

- preserves unitarity
- dynamical geometrical scaling
- input: model for $\langle \mathbf{T} \rangle$ at the initial Y_0 : Golec-Biernat-Wusthof, McLerran-Venugopalan,...

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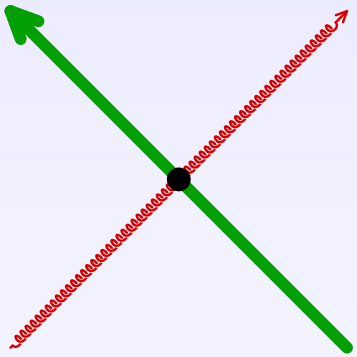

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- preserves unitarity
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- input: model for $\langle \mathbf{T} \rangle$ at the initial Y_0 : Golec-Biernat-Wusthof, McLerran-Venugopalan,...
- basis of the “hybrid” description in hadron-hadron reactions :
  - projectile 1 : dilute parton beam
  - projectile 2 : saturated

Target-centric description : Color Glass Condensate

- Color source distribution $\rho(\mathbf{x}_\perp)$ in the target

$$\Downarrow$$

- Color field $\mathcal{A}^\mu$ given by Yang-Mills equations : $[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}] = \delta^{\nu -}\rho$

$$\Downarrow$$

- Observable $\mathcal{O}$ evaluated on this field configuration

$$\Downarrow$$

- Expectation value obtained by averaging over ρ :

$$\langle \mathcal{O} \rangle = \int [\mathrm{D}\rho(\mathbf{x}_\perp)]\, W_Y[\rho]\, \mathcal{O}[\rho]$$

- Rapidity evolution : $\frac{\partial W_Y}{\partial Y} = \mathcal{H}\, W_Y$ (JIMWLK equation)

Relationship between the two points of view

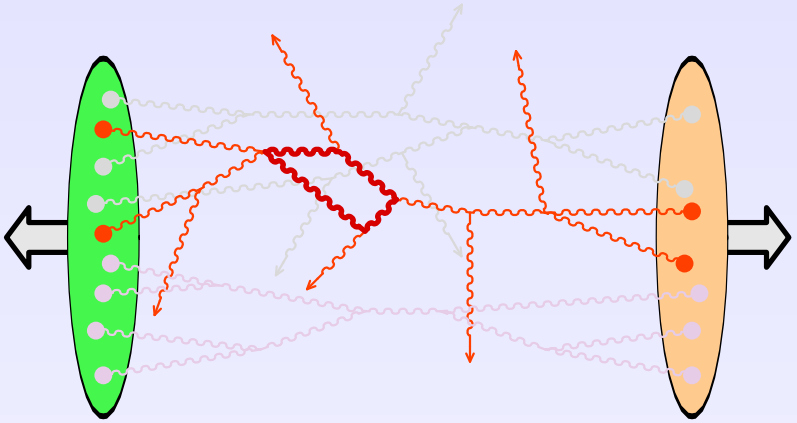
- Frame independence :

$$\langle \mathcal{O} \rangle_Y = \underbrace{\int [D\rho] \; W_0[\rho] \; \mathcal{O}_Y[\rho]}_{\text{Balitsky-Kovchegov description}} = \underbrace{\int [D\rho] \; W_Y[\rho] \; \mathcal{O}_0[\rho]}_{\text{CGC description}}$$

CGC in Heavy Ion Collisions

Power counting in the saturated regime

$$\mathcal{S} = \underbrace{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}_{\text{slow gluons}} + \int \underbrace{(J_1^\mu + J_2^\mu)}_{\text{fast partons}} A_\mu$$



In the saturated regime: $J^\mu \sim g^{-1}$

$$g^{-2}\ g^{\#\text{ of external gluons}}\ g^{2\times(\#\text{ of loops})}$$

- No dependence on the number of sources J^μ
  - ▷ infinite number of graphs at each order in g^2

Example : expansion of $T^{\mu\nu}$ in powers of g^2

$$T^{\mu\nu} \sim \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

Inclusive observables at Leading Order

[FG, Venugopalan (2006)]

- The Leading Order is the sum of all the tree diagrams
  Expressible in terms of classical solutions of Yang-Mills equations :
  $$\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = J_1^\nu + J_2^\nu$$

- Boundary conditions : $\lim_{x^0 \to -\infty} \mathcal{A}^\mu(x) = 0$
  (WARNING : this is not true for exclusive observables!)

Components of the energy-momentum tensor at LO :

$$T^{00}_{\rm LO} = \frac{1}{2}\big[\underbrace{\mathbf{E}^2 + \mathbf{B}^2}_{\text{class. fields}}\big] \qquad T^{0i}_{\rm LO} = \big[\mathbf{E} \times \mathbf{B}\big]^i$$

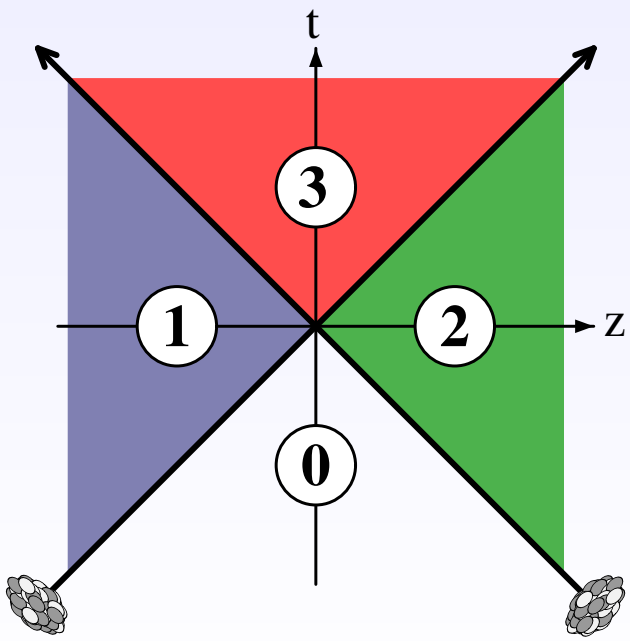
$$T^{ij}_{\rm LO} = \frac{\delta^{ij}}{2}\big[\mathbf{E}^2 + \mathbf{B}^2\big] - \big[\mathbf{E}^i\mathbf{E}^j + \mathbf{B}^i\mathbf{B}^j\big]$$

Space-time evolution of the classical field

[Kovner, McLerran, Weigert (1995)]
[Krasnitz, Venugopalan (1999)] [Lappi (2003)]

- Sources located on the light-cone :

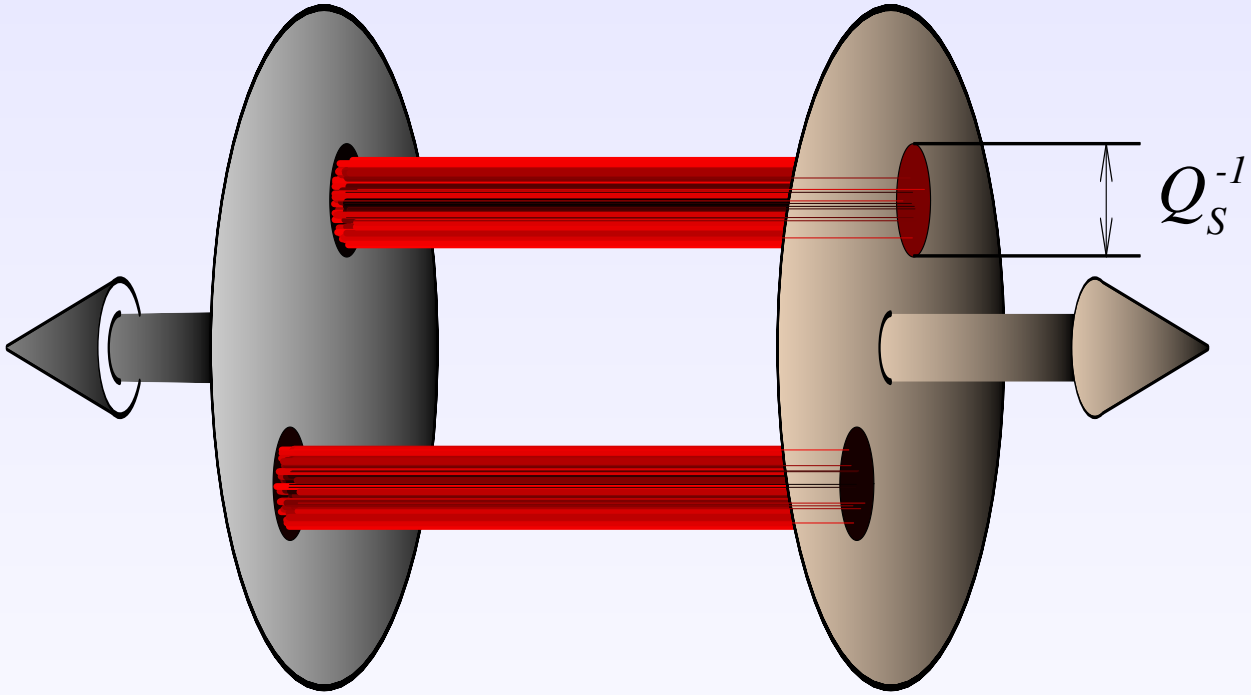
$$J^\mu = \delta^{\mu+} \underbrace{\rho_1(x^-, \mathbf{x}_\perp)}_{\sim\delta(x^-)} + \delta^{\mu-} \underbrace{\rho_2(x^+, \mathbf{x}_\perp)}_{\sim\delta(x^+)}$$



- Region 0 : $\mathcal{A}^\mu = 0$
- Regions 1,2 : $\mathcal{A}^\mu$ depends only on ρ_1 or ρ_2 (known analytically)
- Region 3 : $\mathcal{A}^\mu$ = radiated field known analytically at $\tau = 0^+$ numerical solution for $\tau > 0$

Field lines at $\tau \ll Q_s^{-1}$: Glasma flux tubes

[McLerran, Lappi (2006)]



- Seed for the long range rapidity correlations (ridge)
  [Dumitru, FG, McLerran, Venugopalan (2008)]
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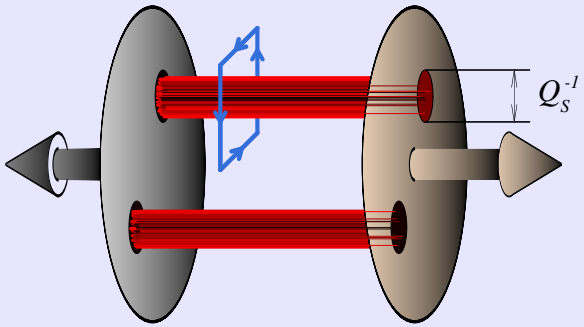

$$W \equiv \left\langle P \exp ig \int_\gamma dx^i \mathcal{A}^i \right\rangle$$

$W \sim \exp(-\text{Area})$ for Area $\times\, Q_s^2 \gtrsim 1$

$\Rightarrow$ magnetic flux bundled in domains of area $\sim Q_s^{-2}$

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Inclusive observables at Next to Leading Order

[FG, Lappi, Venugopalan (2007–2008)]

- Observables at NLO can be obtained from the LO by "fiddling" with the initial condition of the classical field :

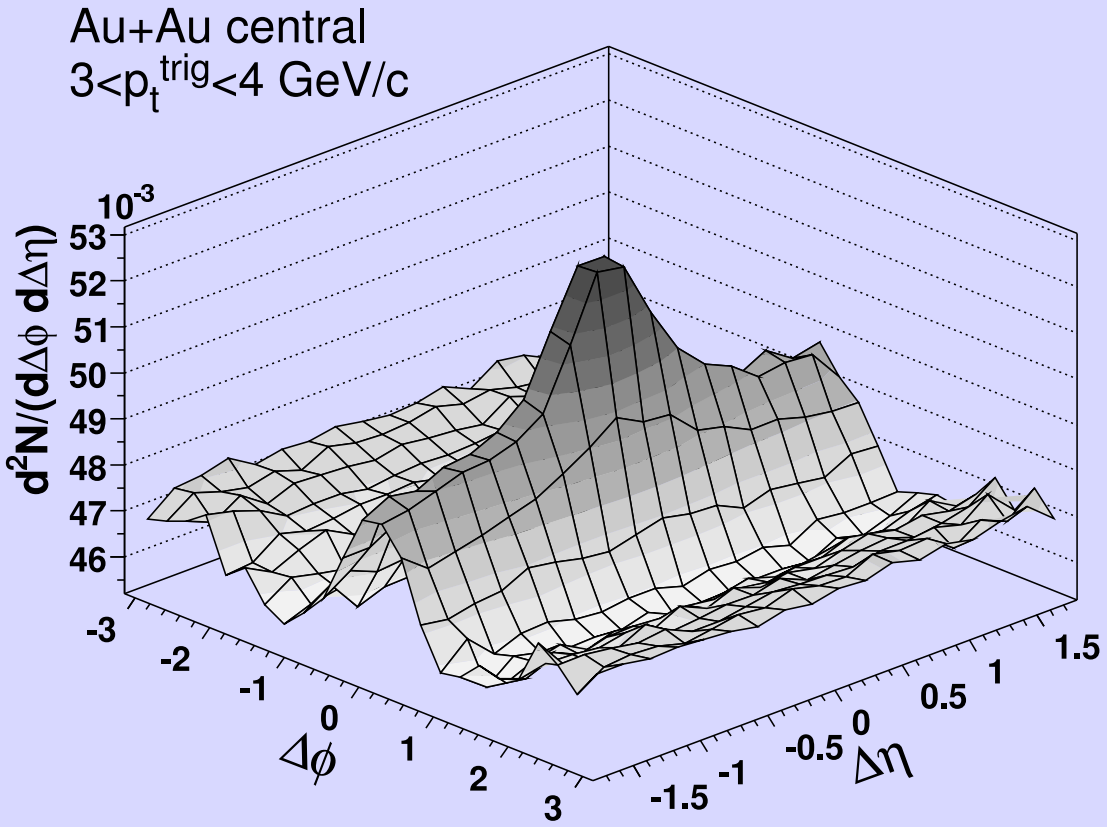
$$\mathcal{O}_{\rm NLO} = \left[\frac{1}{2} \int_{u,v} \Gamma_2(u,v) \, \frac{\partial}{\partial \mathcal{A}_{\rm init}(u)} \, \frac{\partial}{\partial \mathcal{A}_{\rm init}(v)} + \int_u \alpha(u) \, \frac{\partial}{\partial \mathcal{A}_{\rm init}(u)} \right] \mathcal{O}_{\rm LO}$$

  - NLO : the time evolution remains classical;
    $\hbar$ only enters in the initial condition
  - NNLO : $\hbar$ starts appearing in the time evolution itself
  - NOT true for exclusive observables
  - This formula is the basis for proving the factorization of the $W[\rho]$ and their universality (at Leading Log)

Ridge correlations

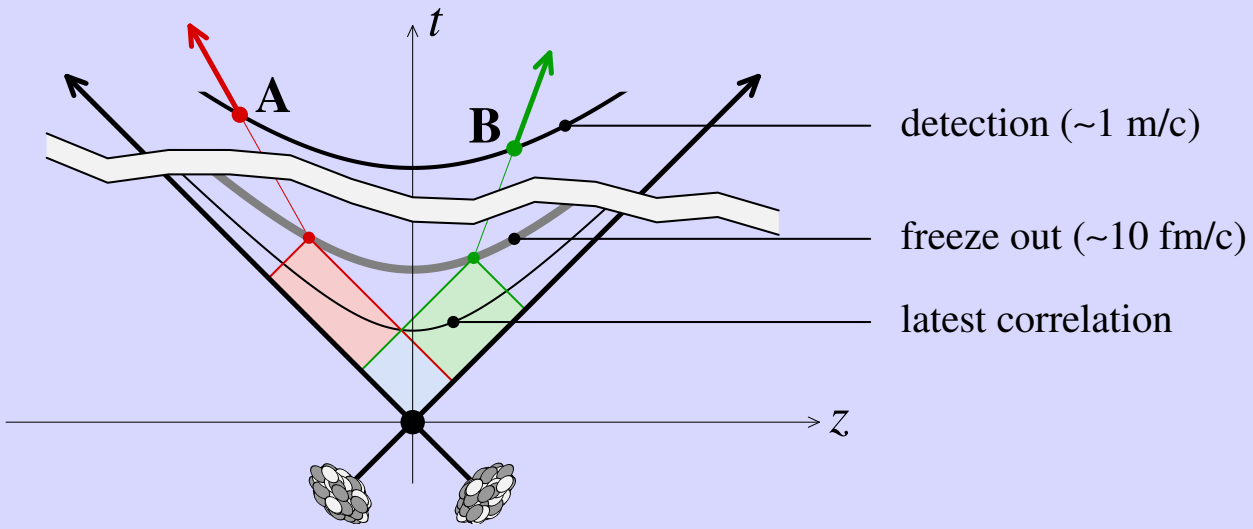
2-particle correlations in AA collisions

[STAR Collaboration, RHIC]



- Long range rapidity correlation
- Narrow correlation in azimuthal angle
- Narrow jet-like correlation near $\Delta y = \Delta\varphi = 0$

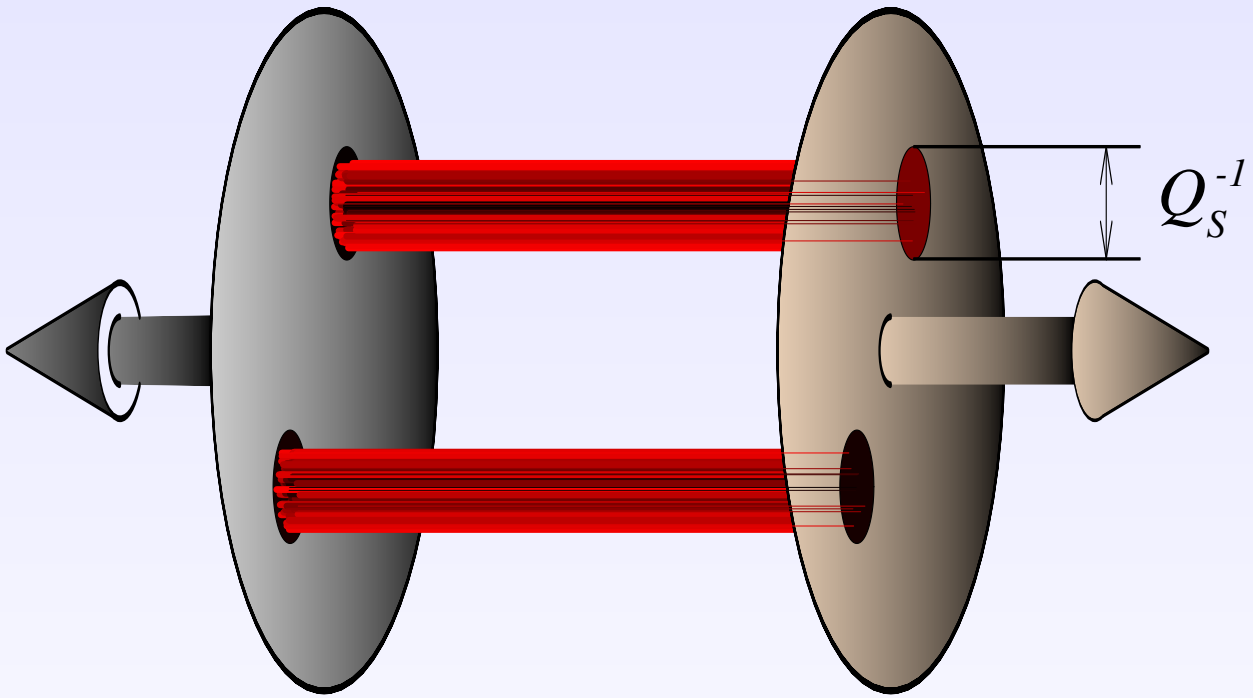
Probing early times with rapidity correlations



- By causality, long range rapidity correlations are sensitive to the dynamics of the system at early times :

$$\tau_{\text{correlation}} \leq \tau_{\text{freeze out}} \; e^{-|\Delta y|/2}$$

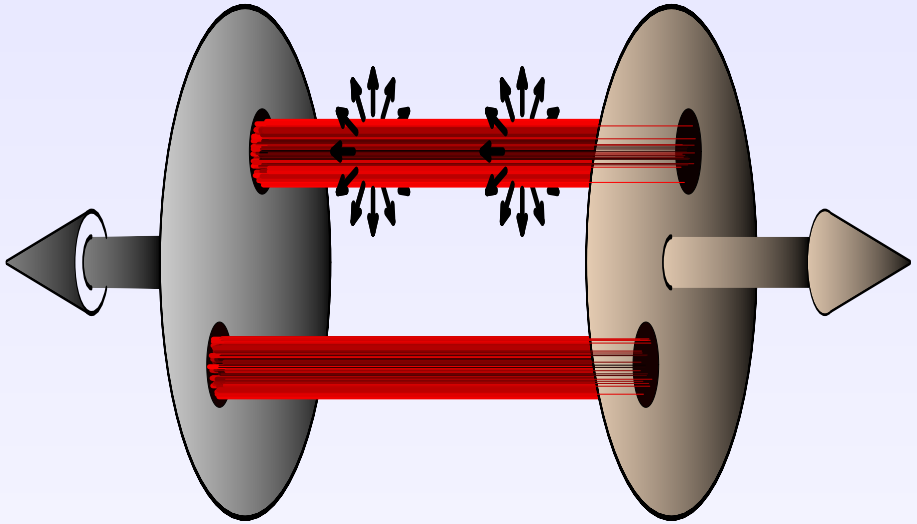
Color field at early time



- The field lines form tubes of transverse size $\sim Q_s^{-1}$
- Rapidity correlation length : $\Delta\eta \sim \alpha_s^{-1}$

2-hadron correlations from color flux tubes

- η-independent fields lead to long range correlations :



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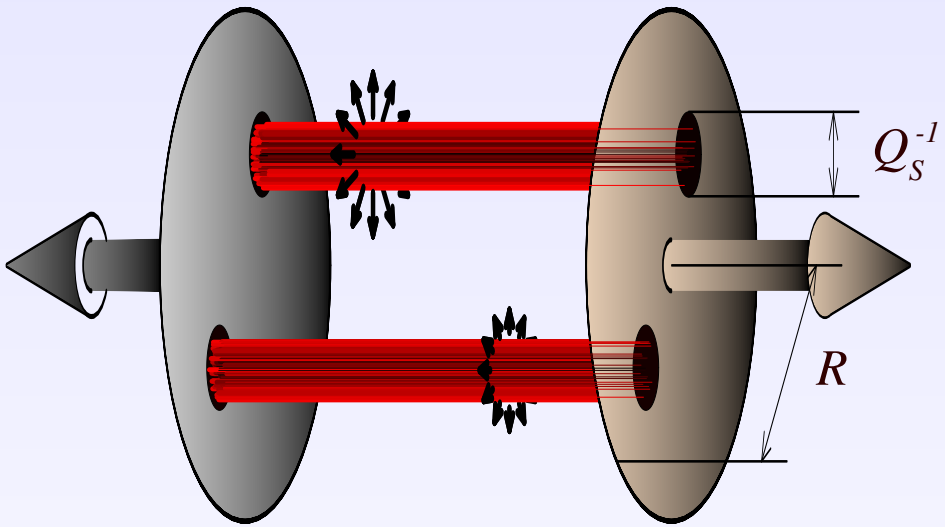

- Particles emitted by different flux tubes are not correlated
  ▷ $(RQ_s)^{-2}$ sets the strength of the correlation

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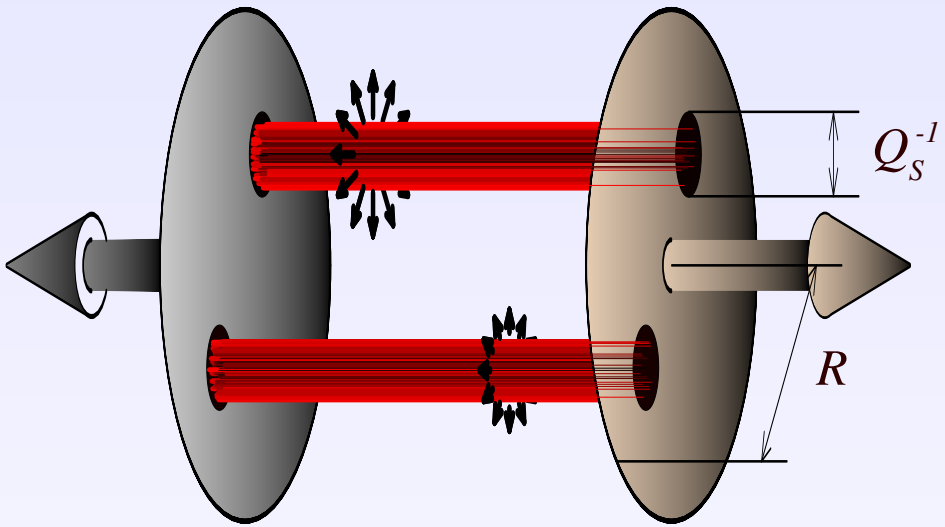

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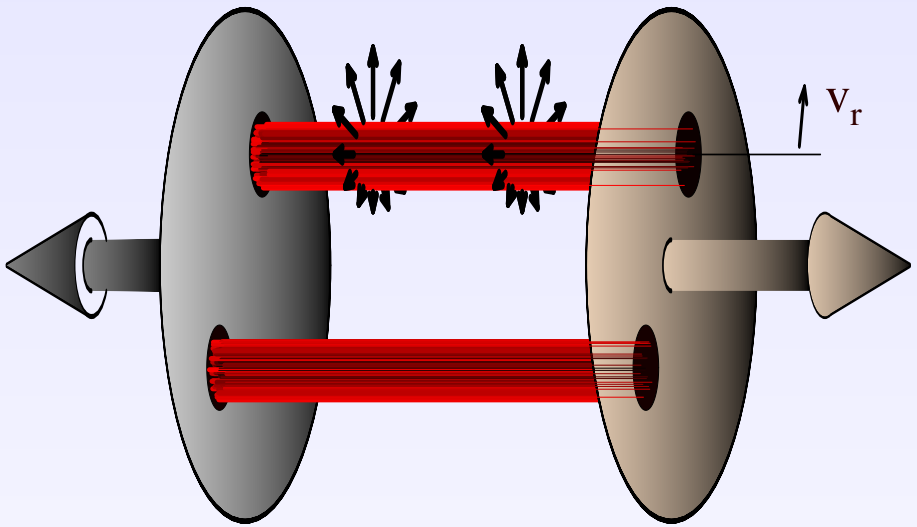
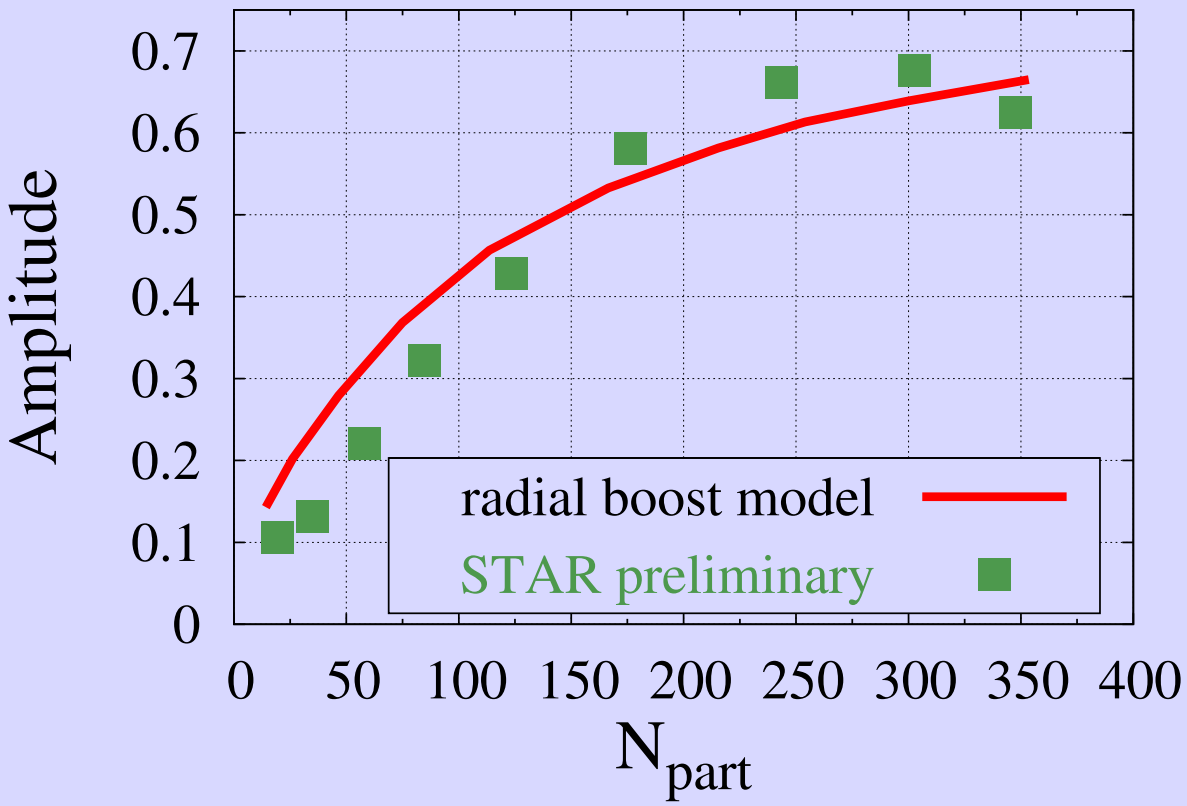


- Particles emitted by different flux tubes are not correlated
  $\triangleright$ $(RQ_s)^{-2}$ sets the strength of the correlation
- At early times, the correlation is flat in $\Delta\varphi$

  The collimation in $\Delta\varphi$ is produced later by radial flow

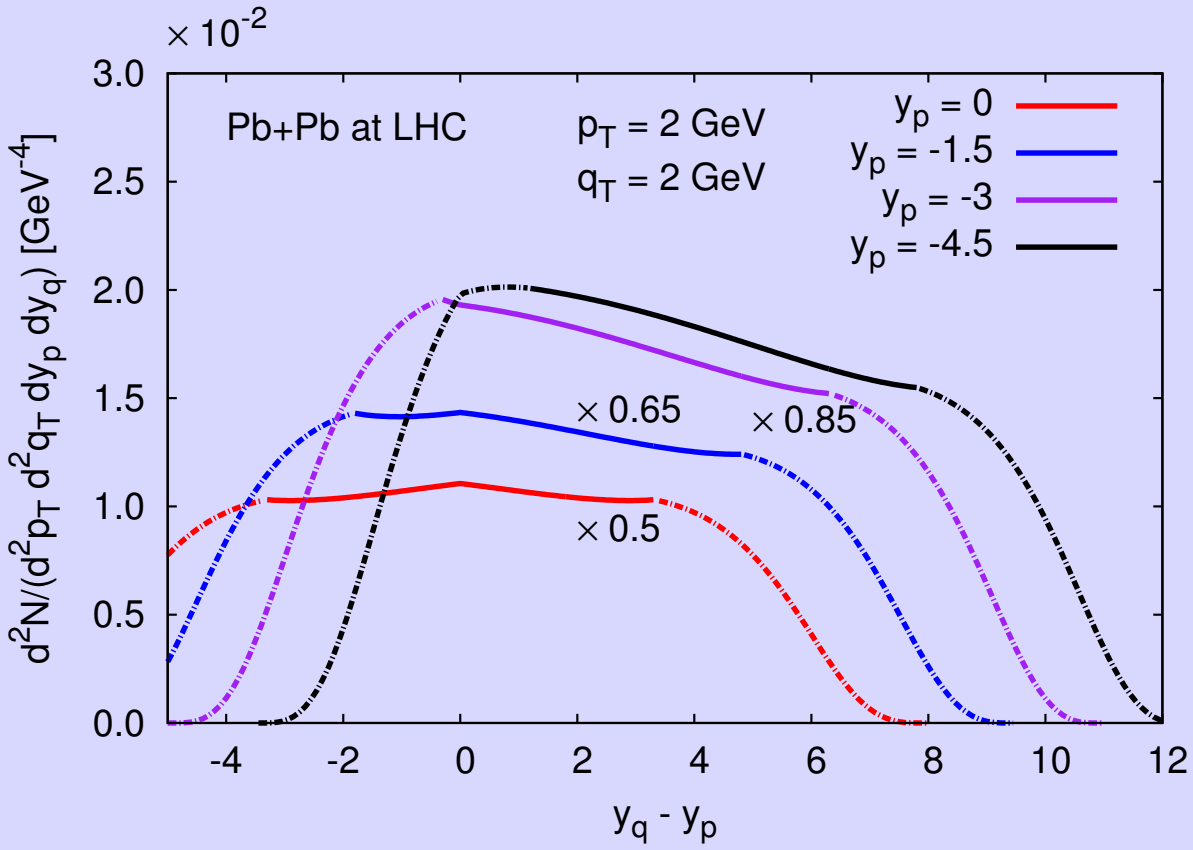
Centrality dependence



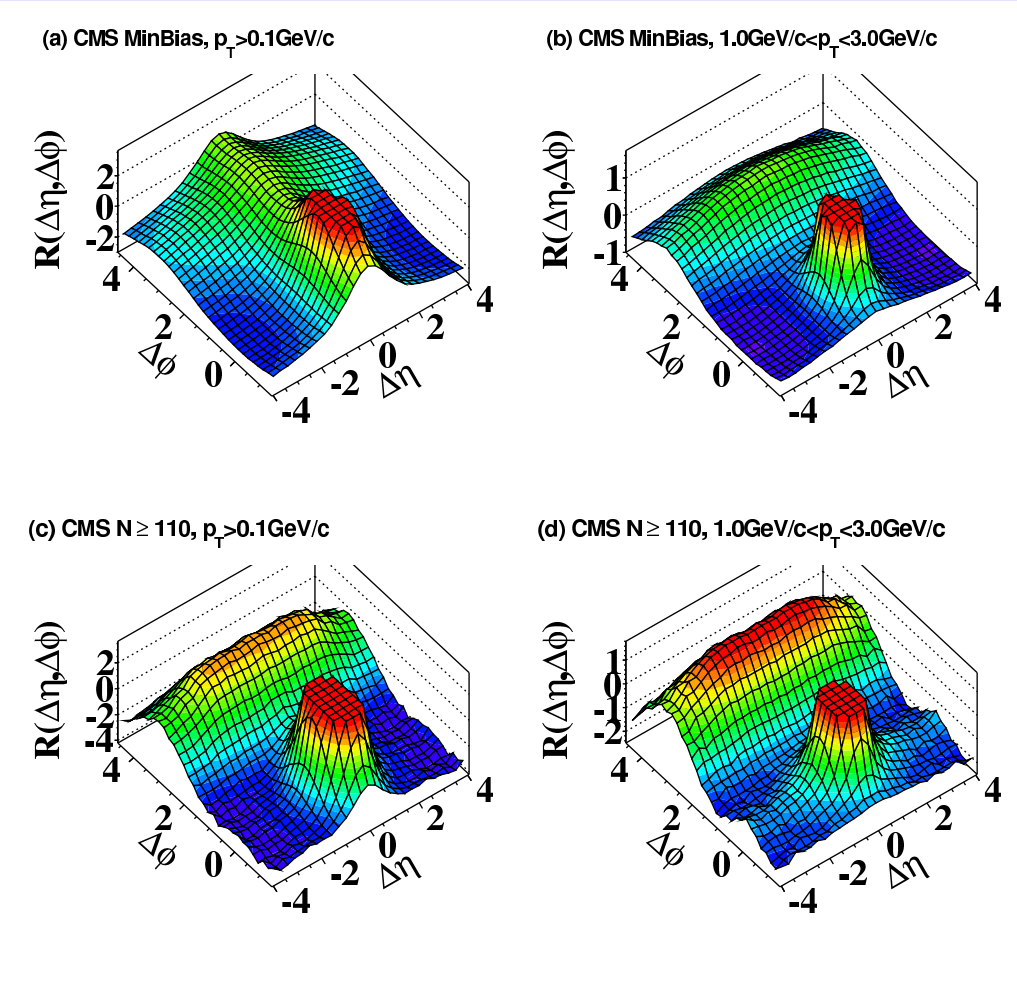
- Main effect : increase of the radial flow velocity with the centrality of the collision

Rapidity dependence

Estimate at LHC energy



High multiplicity proton-proton collisions at the LHC



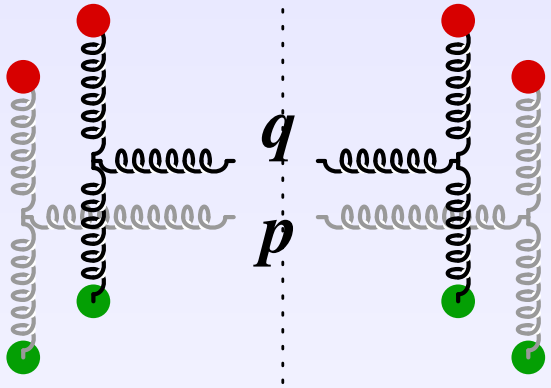
- Similar effect visible for high multiplicity p-p collisions, in an intermediate $p_\perp$ window
- Much weaker than in AA collisions

Possible origin of the angular correlation

- The long range rapidity correlations invoked in A-A collisions are also present in p-p collisions
- Whether there is a sufficient amount of radial flow to induce the azimuthal collimation is an open question
  - less particles are produced
  - the system freezes out earlier
- There is an “intrinsic” angular correlation, that exists in the absence of flow (it exists also in A-A collisions as well, but it is negligible there)

Intrinsic angular correlations

- 2-gluon inclusive spectrum before the average over $\rho_{1,2}$:

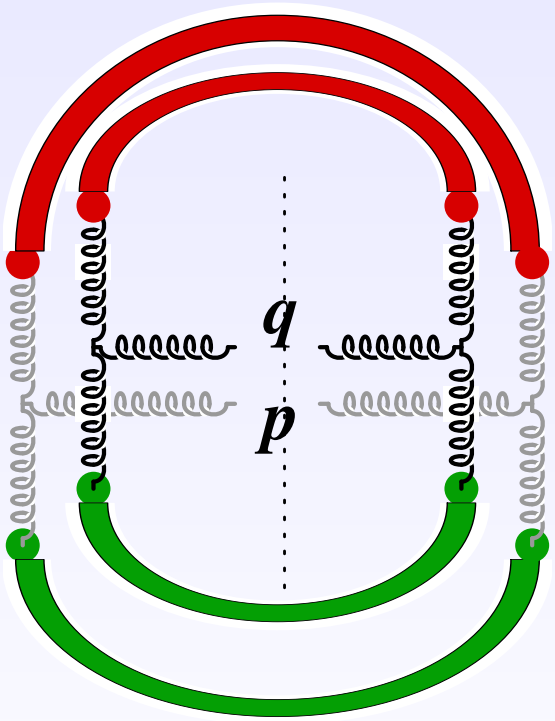


▷ this contribution dominates the 2-gluon spectrum in the regime where the parton densities are large

▷ the average over $\rho_{1,2}$ amounts to connecting the red and green lines in all the possible ways (pairwise if the sources have Gaussian distributions)

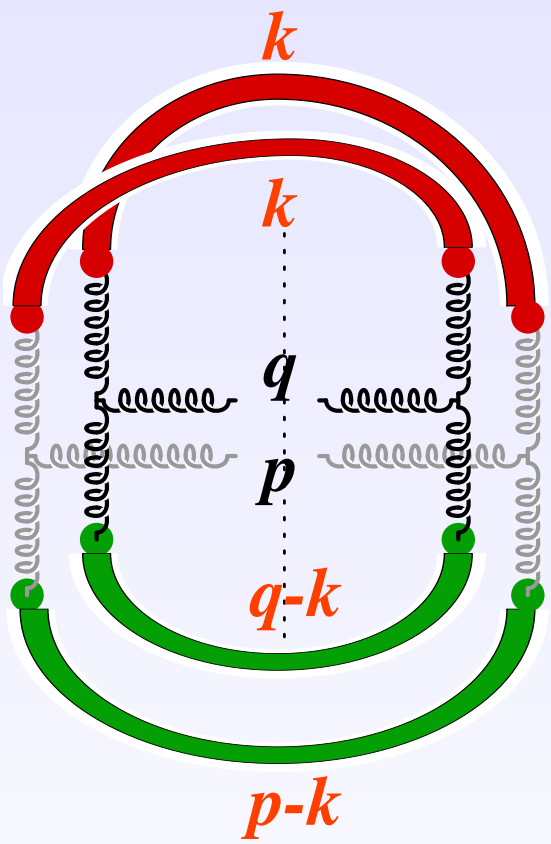
Intrinsic angular correlations

- Trivial connection (no correlation) :



Intrinsic angular correlations

- Non-trivial connection with correlations at $\Delta\varphi < \frac{\pi}{2}$:

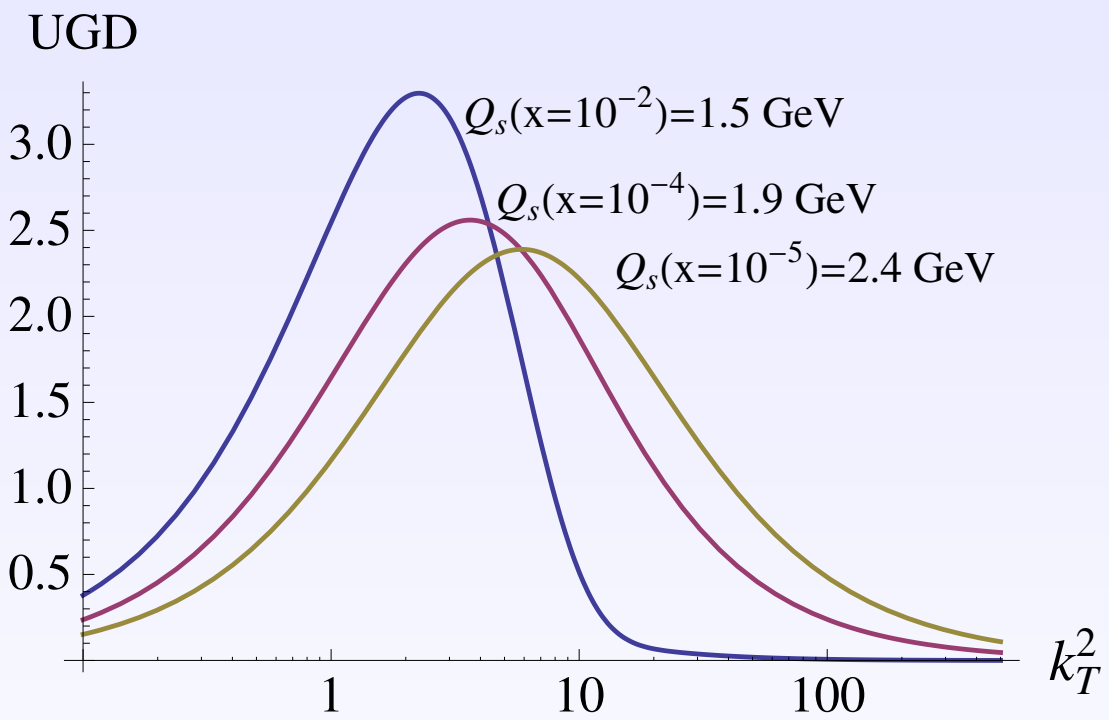


▷ Momentum assignment of the unintegrated gluon distributions:

$$[\phi_1(k_\perp)]^2 \; \phi_2(|\mathbf{p}_\perp - \mathbf{k}_\perp|) \; \phi_2(|\mathbf{q}_\perp - \mathbf{k}_\perp|)$$

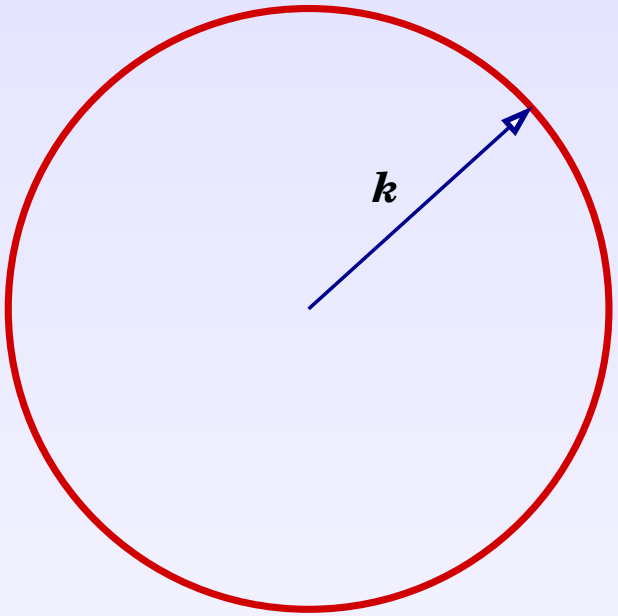
Intrinsic angular correlations

- In the saturation regime, unintegrated gluon distributions are peaked near Q_s:



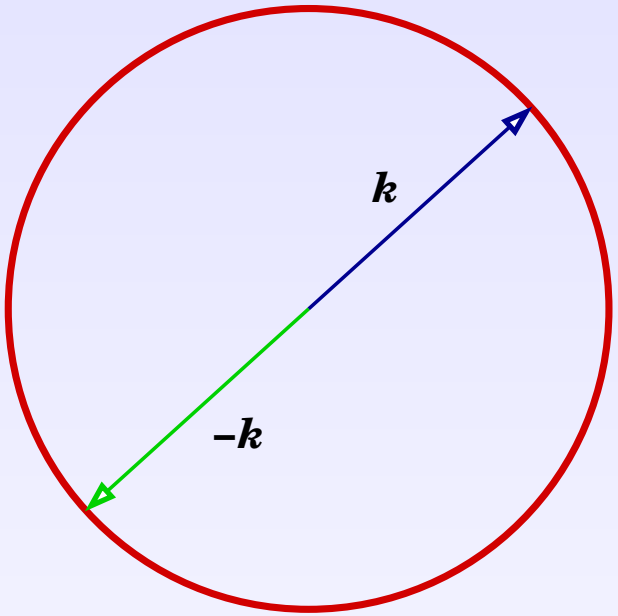
- The presence of this peak is what correlates the directions of $\vec{p}_\perp$ and $\vec{q}_\perp$ around $\Delta\phi = 0$ when we perform the integration over $\vec{k}_\perp$

Intrinsic angular correlations



- $|\vec{\mathbf{k}}_\perp| \sim Q_s$

Intrinsic angular correlations



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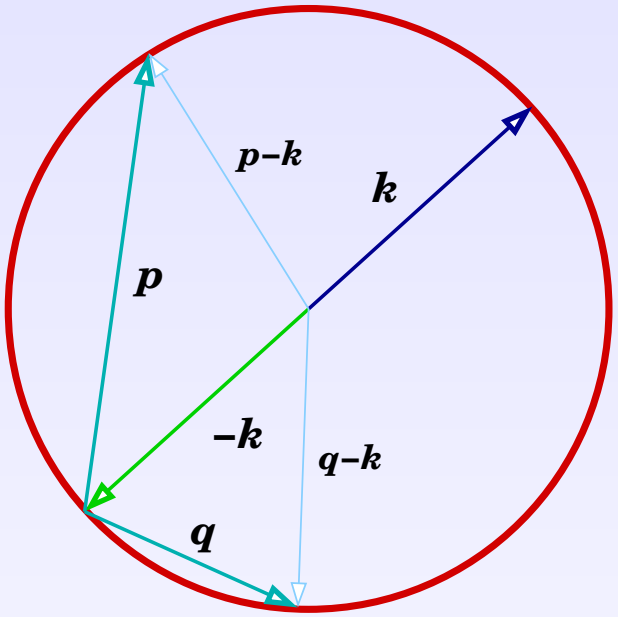
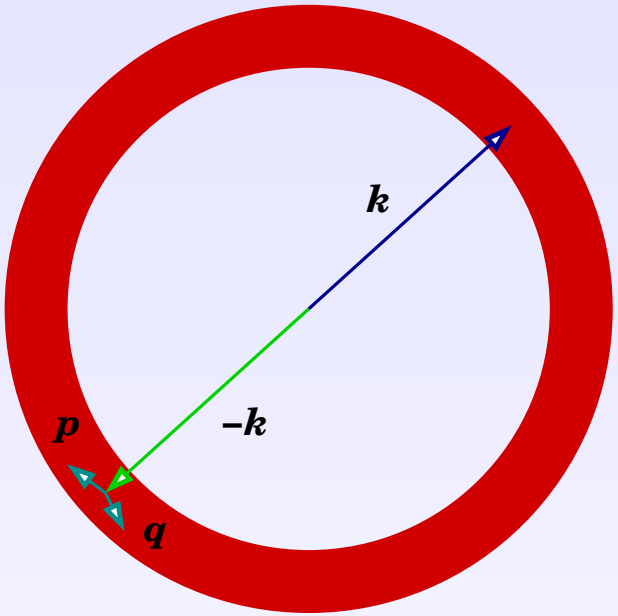
Intrinsic angular correlations



- $|\vec{k}_\perp| \sim Q_s$
- $|\vec{p}_\perp - \vec{k}_\perp| \sim |\vec{q}_\perp - \vec{k}_\perp| \sim Q_s$

Intrinsic angular correlations

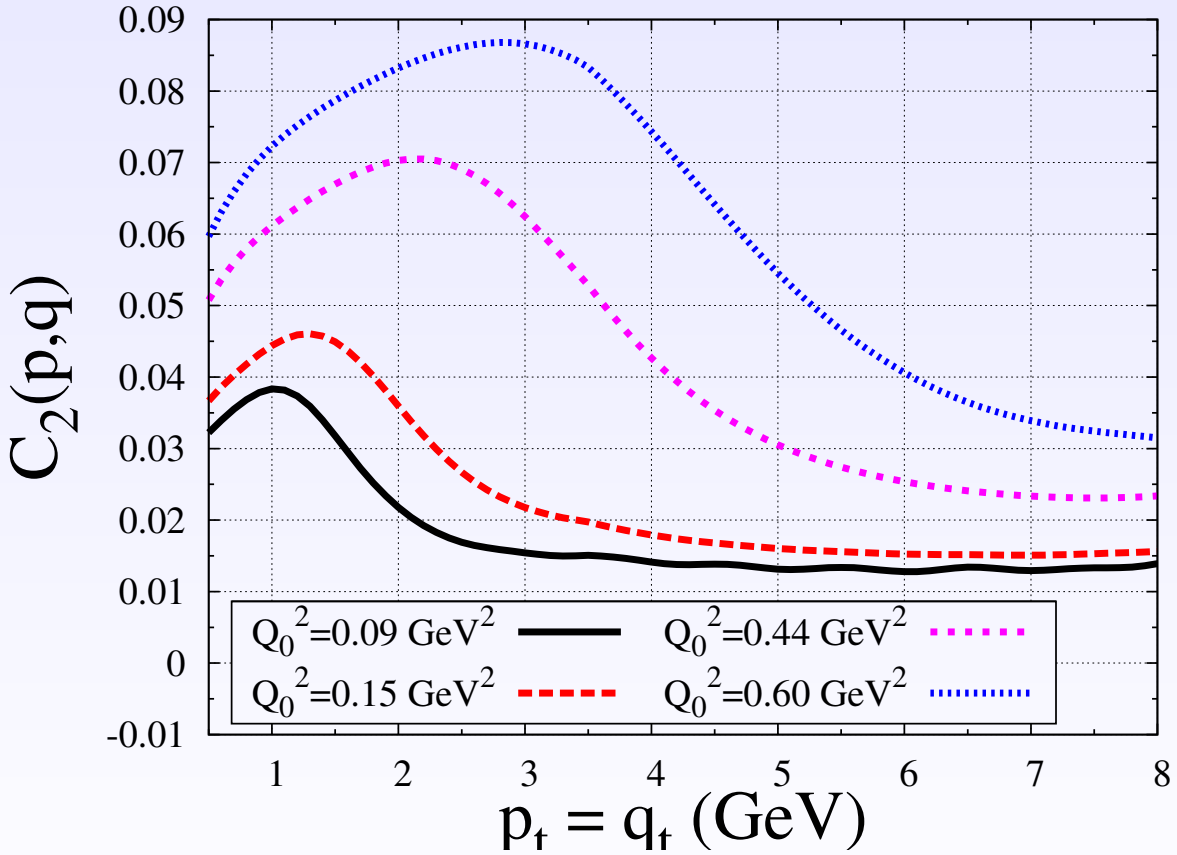


- $|\vec{k}_\perp| \sim Q_s$
- $|\vec{p}_\perp - \vec{k}_\perp| \sim |\vec{q}_\perp - \vec{k}_\perp| \sim Q_s$
- If the momenta are smaller than the width of the distributions, there is no significant angular correlation

  Similarly, for large momenta there is no correlation because the main contribution does not come from the peak of the distributions anymore

Intrinsic angular correlations

- The effect is maximal for intermediate $p_\perp, q_\perp \sim Q_s$:



Pre-Hydro Evolution

Conditions for hydrodynamics

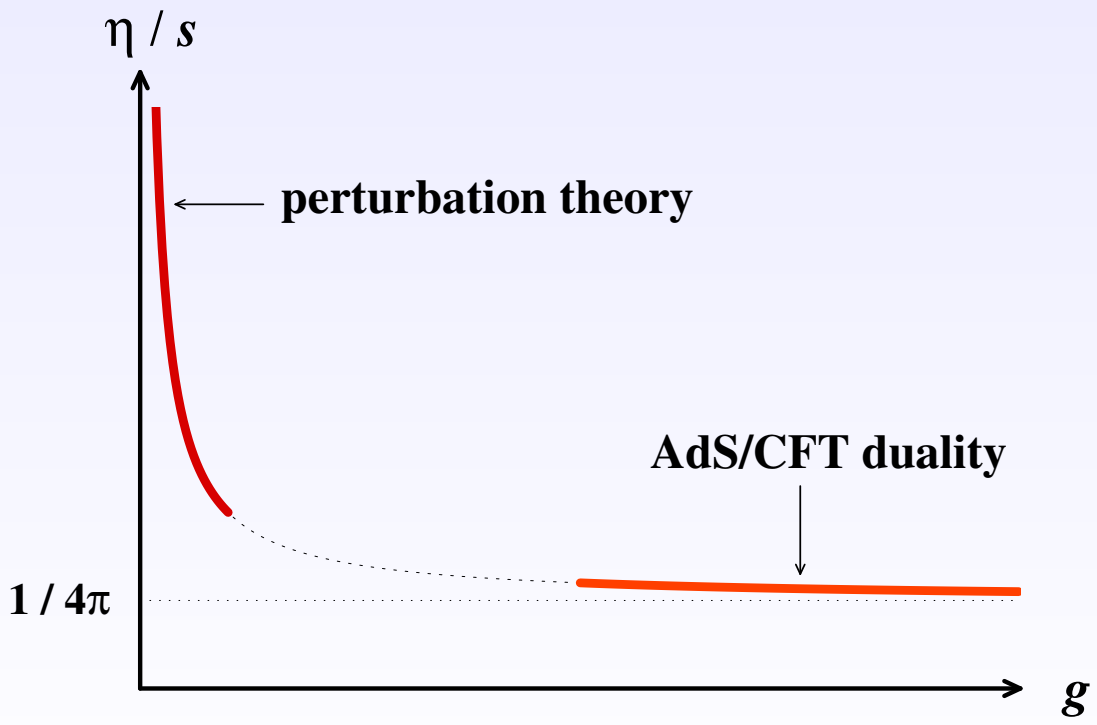
- The initial P_L/P_T should not be too small
  (for the stability of hydro codes)

- The ratio η/s should be small enough
  (for an efficient transfer from spatial to momentum anisotropy)

Shear viscosity at weak and strong coupling (in equilibrium)

Weak coupling QCD result [Arnold, Moore, Yaffe (2000)]

$$\frac{\eta}{s} \approx \frac{5.1}{g^4 \ln\left(\frac{2.4}{g}\right)}$$



Is there another possibility?

From kinetic theory :

$$\frac{\eta}{s} \sim \frac{\text{mean free path}}{\text{de Broglie wavelength}}$$

- (**de Broglie wavelength**)$^{-1} \sim Q$
- (**mean free path**)$^{-1} \sim \underbrace{g^4 Q^{-2}}_{\text{cross section}} \times \underbrace{\int_k f_k}_{\text{density}} \underbrace{(1+f_k)}_{\text{Bose enhancement}}$

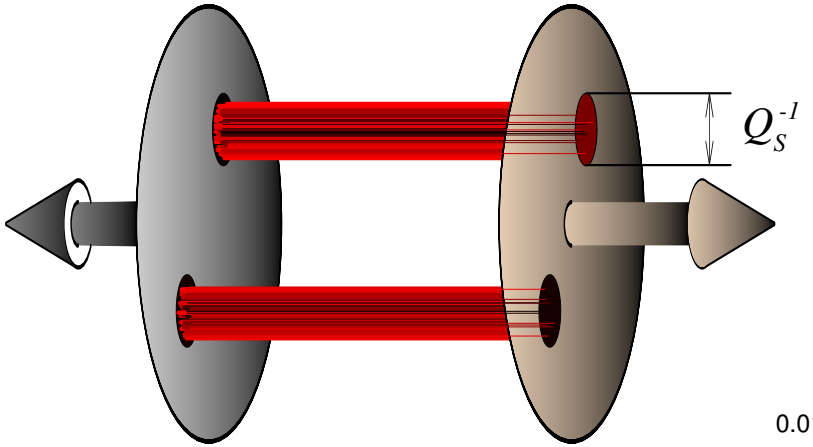
If $g \ll 1$ but $f_k \sim g^{-2}$ (weakly coupled, but strongly interacting)

$$\frac{\eta}{s} \sim g^0$$

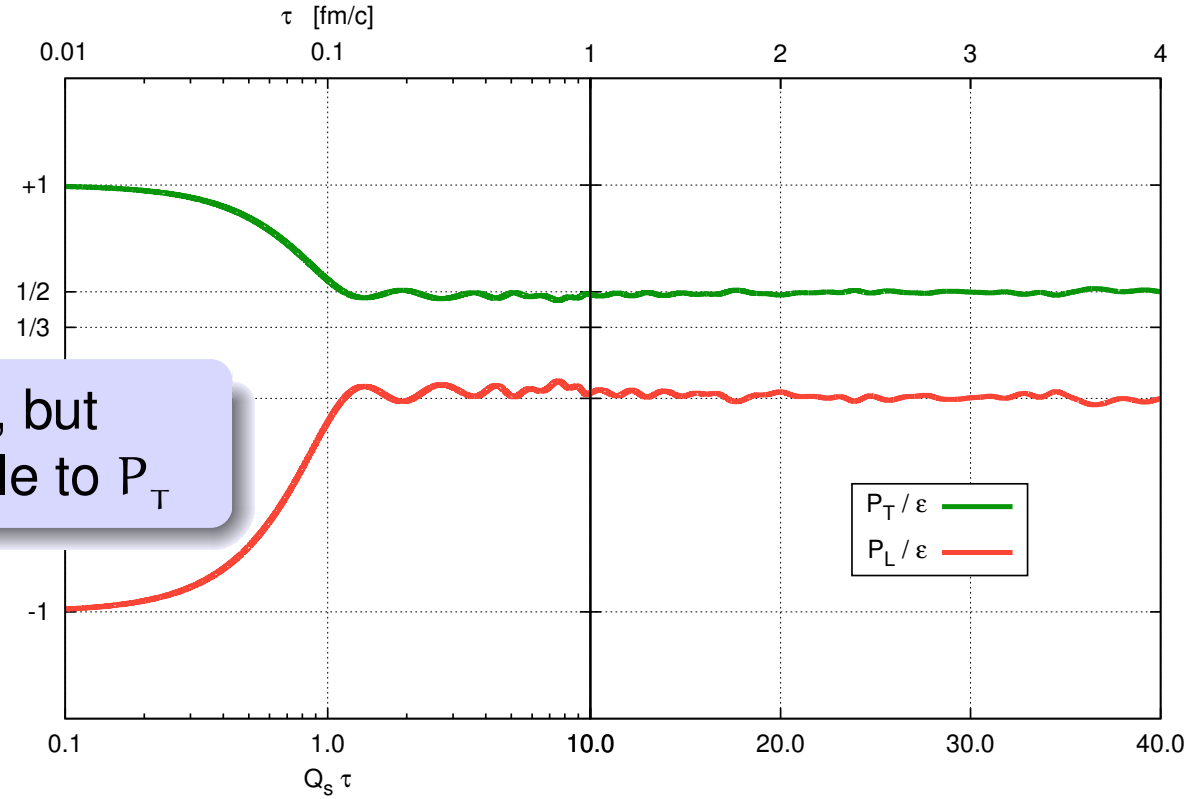
(even w/o quasiparticles, $B \sim \frac{Q^2}{g}$ has the same effect)

[Asakawa, Bass, Mueller (2006)]

CGC at LO : strong pressure anisotropy at all times

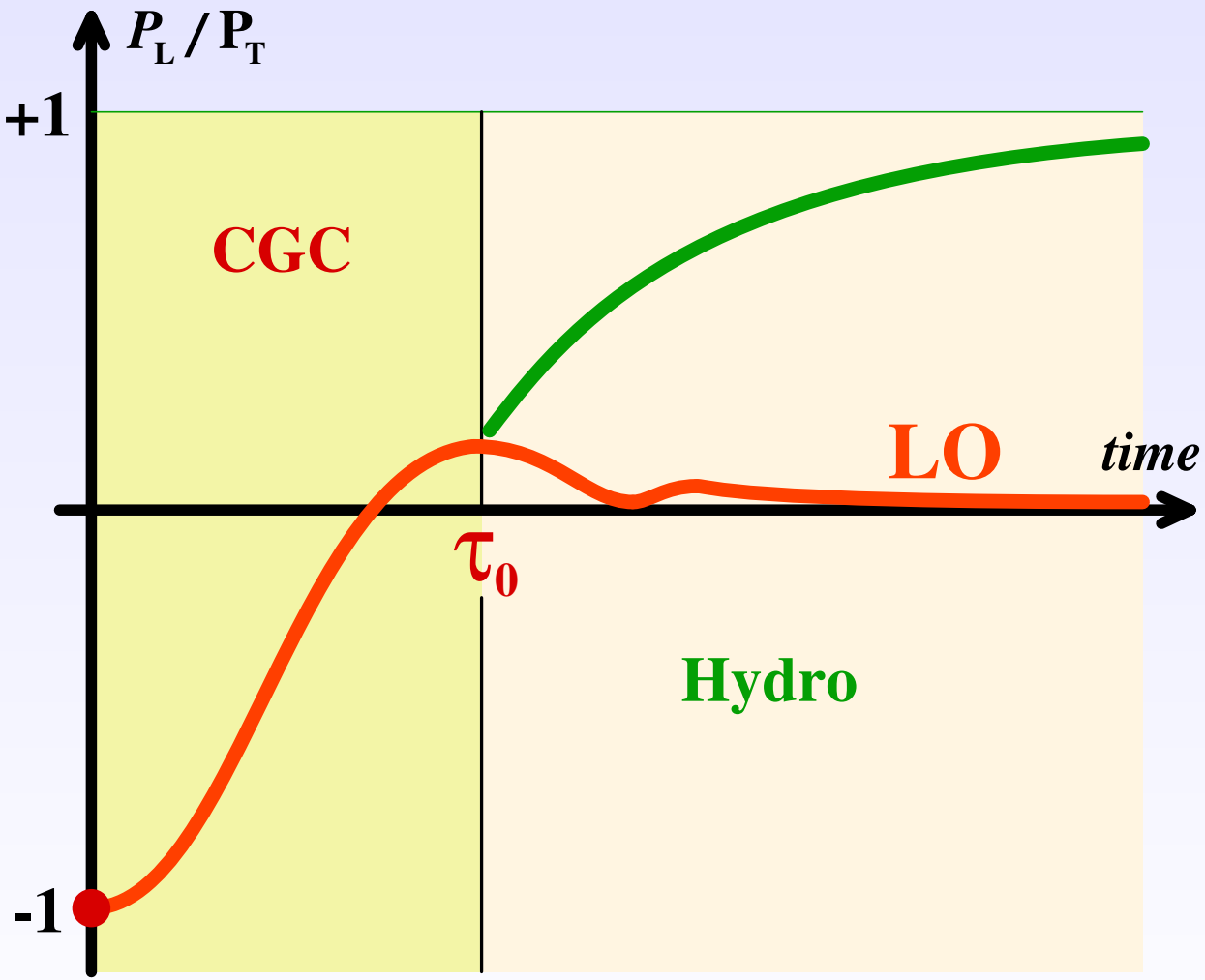


When $\mathbf{E} \parallel \mathbf{B}$:
$P_T = \epsilon, P_L = -\epsilon$



P_L rises to positive values, but never becomes comparable to P_T

CGC at LO : unsatisfactory matching to hydrodynamics



CGC at LO : unsatisfactory matching to hydrodynamics

P_L / P_T

Matching to hydro :

- Compute $T^{\mu\nu}$ from CGC
- Find time-like eigenvector : $u_\mu T^{\mu\nu} = \epsilon\, u^\nu$
- Get pressure from some equation of state $P = f(\epsilon)$
- Get viscous stress as difference between full and ideal $T^{\mu\nu}$

"CGC initial conditions" very often means :

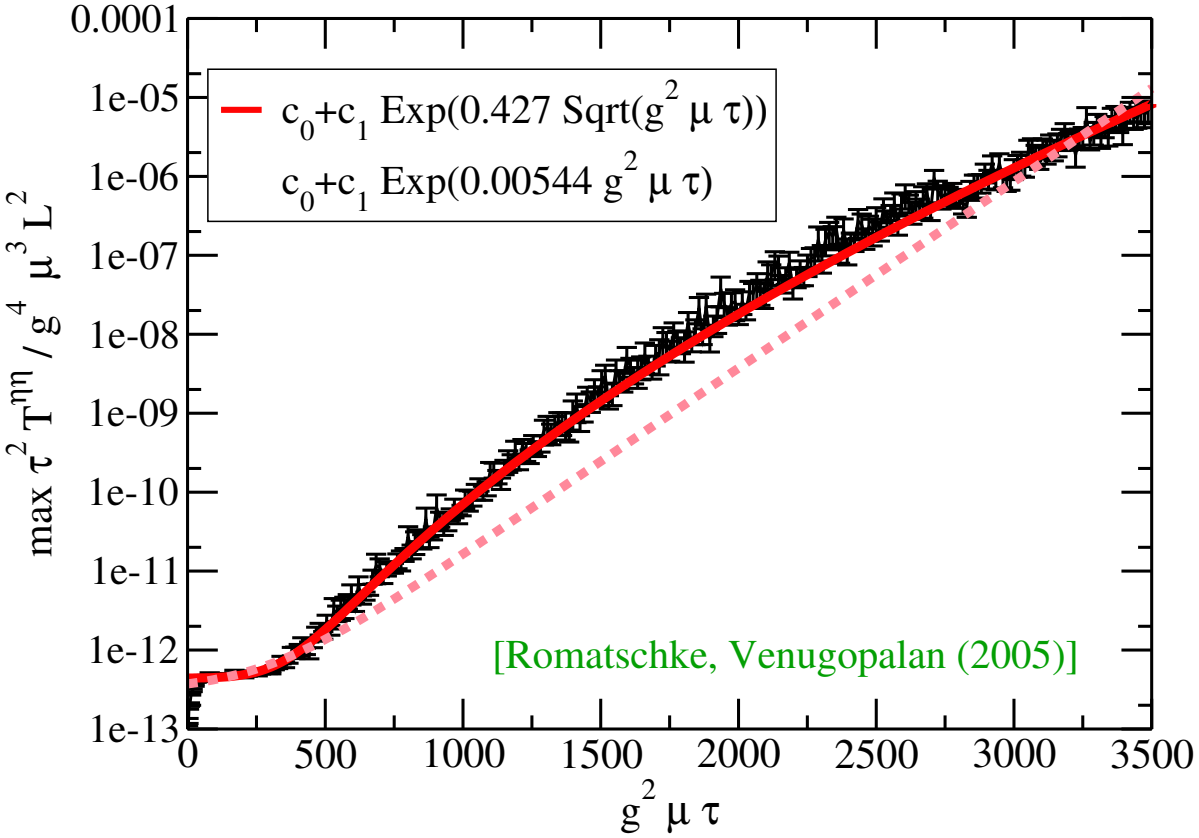
- $\epsilon = T^{00}$ from CGC (or a CGC-inspired model)
- Initial flow neglected, Viscous stress $= 0$

NOTE : glasma fields start to flow at $\tau \sim Q_s^{-1}$:

[Krasnitz, Nara, Venugopalan (2002)] [Chen, Fries (2013)]

CGC at NLO : instabilities

[Mrowczynski (1988), Romatschke, Strickland (2003), Arnold, Lenaghan, Moore (2003), Rebhan, Romatschke, Strickland (2005), Arnold, Lenaghan, Moore, Yaffe (2005), Romatschke, Rebhan (2006), Bodeker, Rummukainen (2007), Fujii, Itakura (2008),...,**Attems, Rebhan, Strickland (2012)**, **Fukushima (2013)**]



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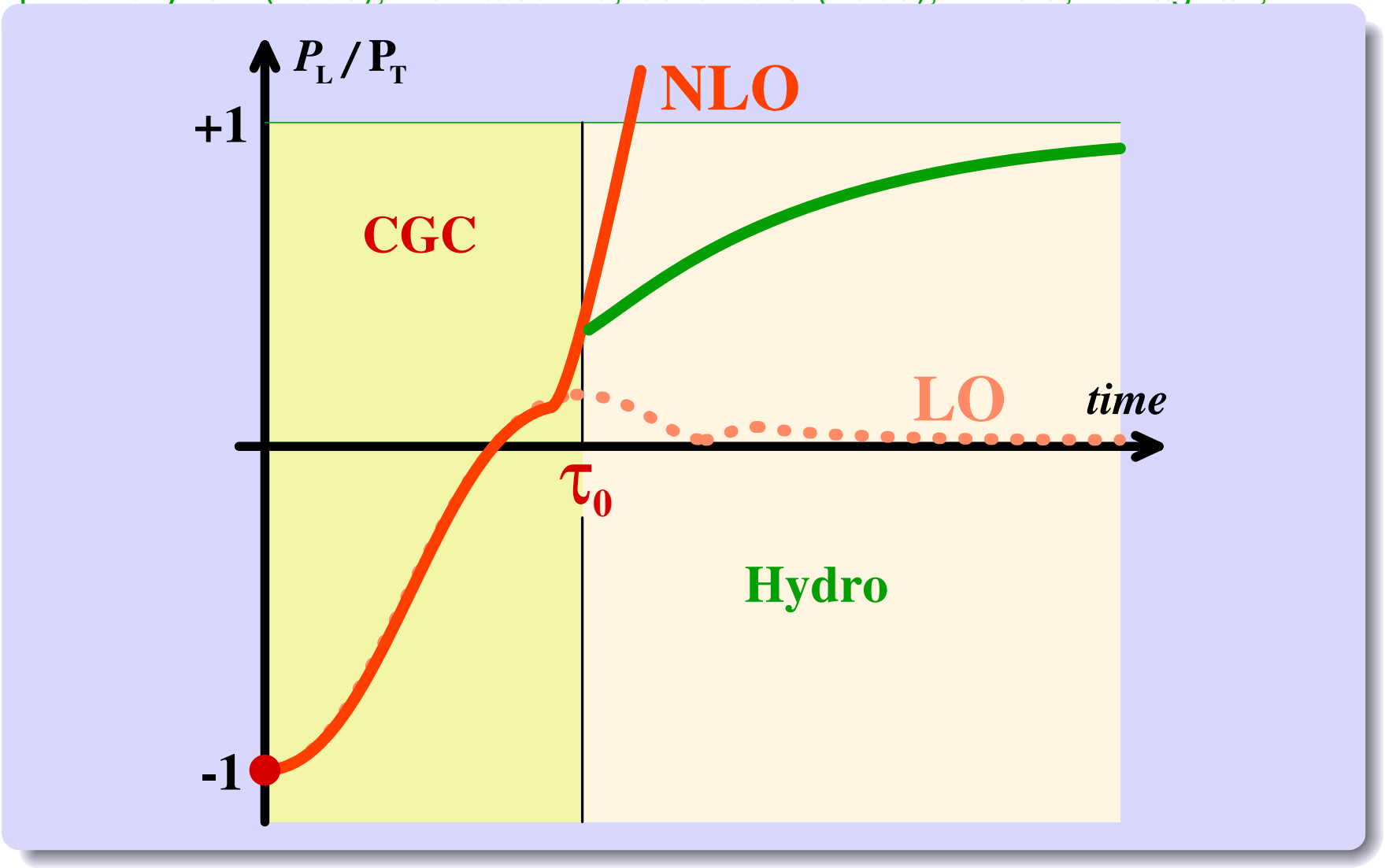
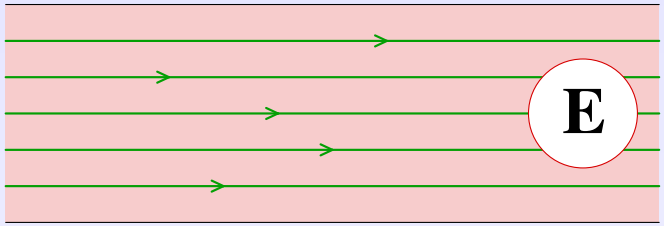
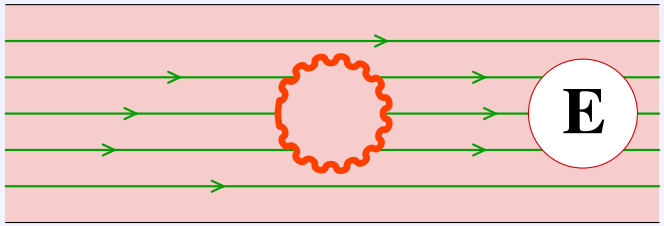


$g^2 \mu \tau$

- LO = longitudinal chromo-$\mathbf{E}$ and chromo-$\mathbf{B}$ fields



- NLO = gluon loop embedded in this field

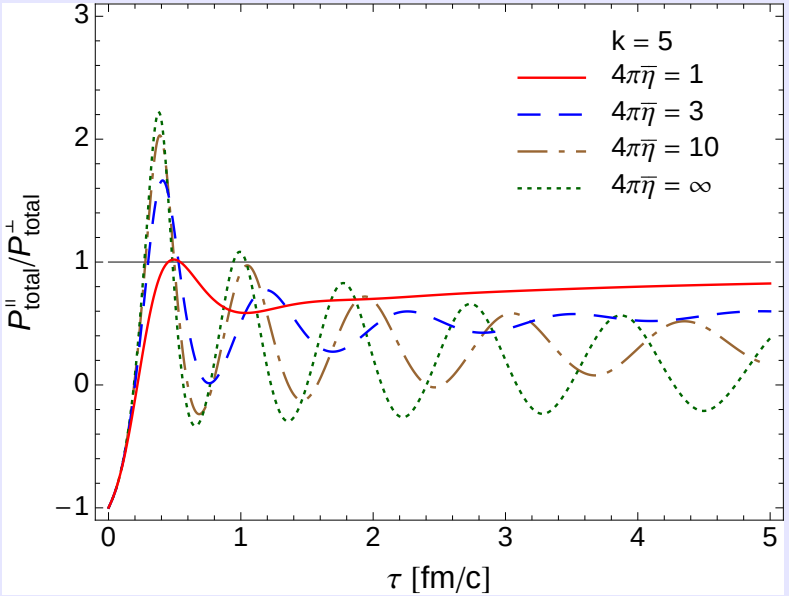


  - the loop can have an imaginary part $\Rightarrow$ gluon pair production
  - BUT : no feedback of the produced gluons on the LO field!

Color flux tube model : [Ryblewski, Florkowski (2013)]

$$(p^\mu \partial_\mu + \underbrace{g F^{\mu\nu} p_\nu}_{\text{Lorentz force}} \partial^\mu_p)\, G = \underbrace{\frac{dN}{d\Gamma}}_{\text{Schwinger}} + \underbrace{C_p[G]}_{\text{collisions}}$$

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \text{(feedback)}$$



- Field converted into particles by instability
- Nearly constant P_L/P_T
- Ratio depends on $\bar{\eta} \equiv \eta/s$

Classical statistical approximation

Three points of view / derivations

- Path integral : fields A_+, A_- on the two branches of the Schwinger-Keldysh contour. In the regime of large fields, $(A_1 \equiv A_+ - A_-) \ll (A_2 \equiv A_+ + A_-)$. Expand the action in A_1 and keep only the lowest order term.
- Diagrammatic expansion : Schwinger-Keldysh formalism in the retarded/advanced basis. Drop the 2111 vertex.
- CGC : exponentiate the operator that relates $\mathcal{O}_{\text{LO}}$ and $\mathcal{O}_{\text{NLO}}$

Practical implementation

- Classical time evolution
- Fluctuations in the initial classical field

- Dynamics fully non-linear $\Rightarrow$ no unbounded growth
- Individual classical trajectories may be chaotic $\Rightarrow$ a small initial ensemble can span a large phase space volume

Classical statistical approximation

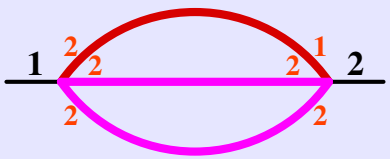
Three points of view / derivations

- CSA $\neq$ underlying theory at 2-loops and beyond
- Sources of fluctuations of the initial fields :

$$G_{22}(p) \sim \left(f_0(\mathbf{p}) + \frac{1}{2}\right)\delta(p^2)$$

quasiparticles $\hookleftarrow$ $\hookrightarrow$ vacuum fluctuations

- **Vacuum fluctuations** make the CSA **non-renormalizable**. Example of problematic graph :

Im 

$= -\dfrac{g^4}{1024\pi^3}\left(\Lambda_{\rm UV}^2 - \dfrac{2}{3}p^2\right)$

- With only **quasiparticle**-induced fluctuations :
  - Finite if $f_0(p)$ falls faster than p^{-1}
  - Super-renormalizable if $f_0(p) \sim p^{-1}$ [Aarts, Smit (1997)]

ensemble can span a large phase space volume

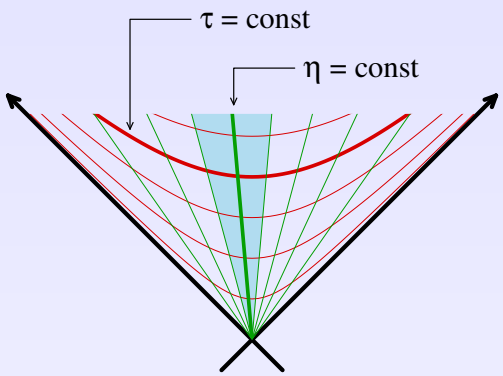
Two recent works along these lines

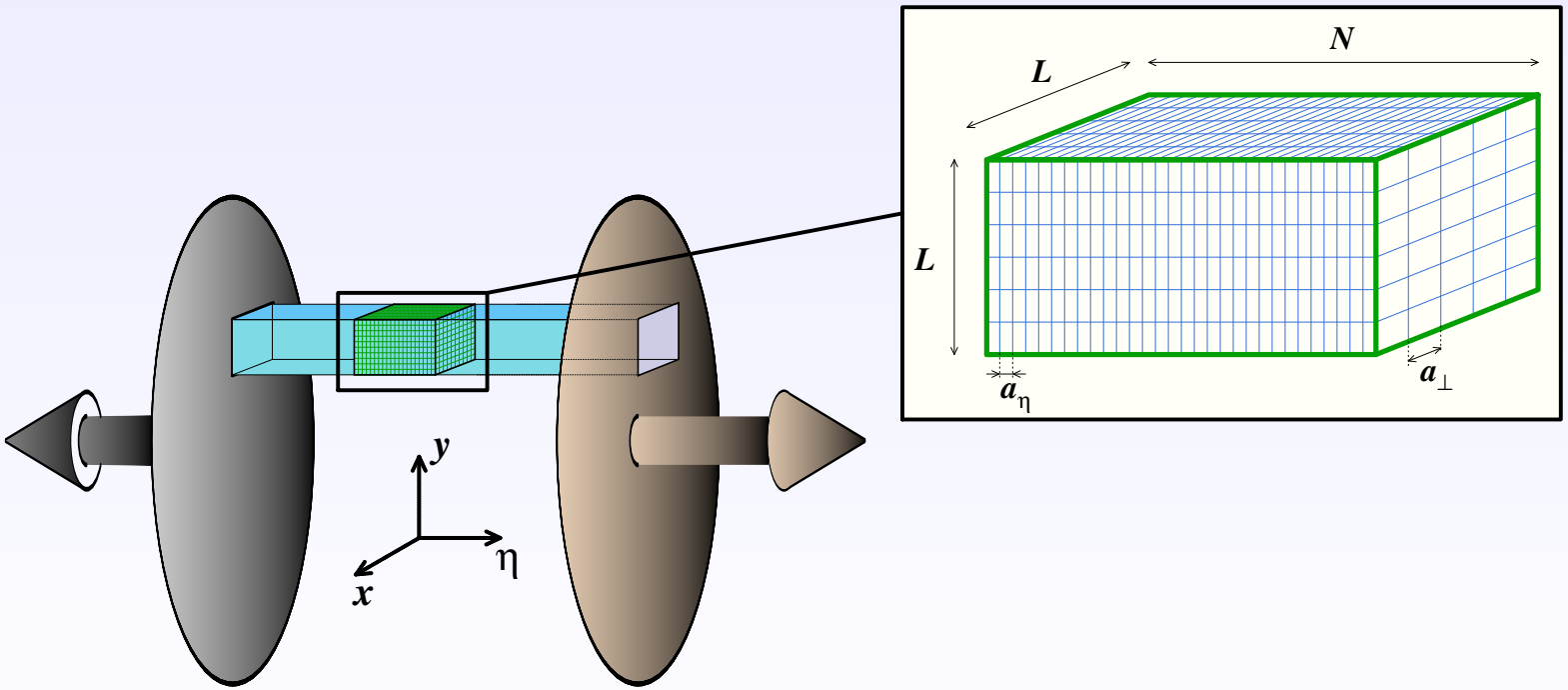
- **[Berges, Boguslavski, Schlichting, Venugopalan (2013)]**
  - Start at $Q_s\tau \gg 1$
  - Ensemble of fields that represents a highly occupied gas of gluons
  - Free parameters : initial gluon distribution
  - Fluctuations = particle-like fluctuations
  - No α_s dependence

- **[Epelbaum, FG (2013)]**
  - Start at $Q_s\tau \ll 1$
  - Ensemble of fields calculated analytically to reproduce the NLO
  - No free parameter
  - Fluctuations = vacuum fluctuations
  - Fluctuations / background field $\sim \alpha_s$

Discretization of the expanding volume

- Comoving coordinates : $\tau, \eta, \mathbf{x}_\perp$
- Simulation of a sub-volume
  + periodic boundary conditions
- $L^2 \times N$ lattice





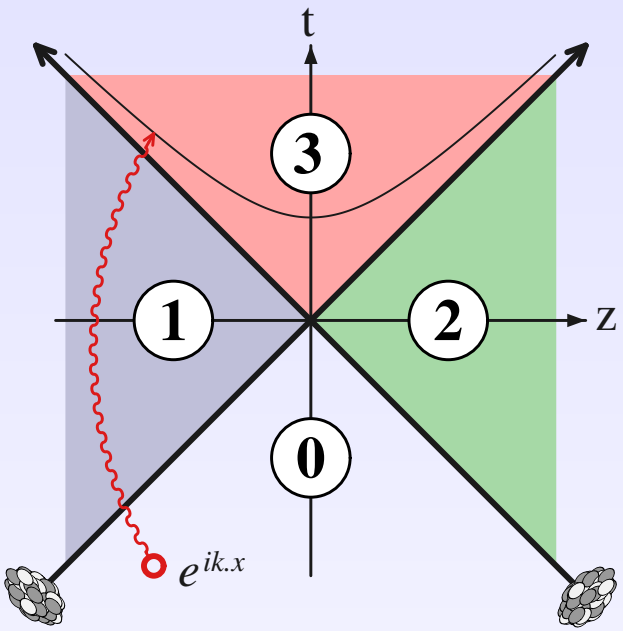
Gaussian spectra of fluctuations

EG : CGC at $\tau \ll Q_s^{-1}$ (1-loop accurate)

$$\langle \mathcal{A}^\mu \rangle = \mathcal{A}^\mu_{\rm LO} \qquad \text{Var.} = \int_{\text{modes } \mathbf{k}} \frac{1}{2}\, a_{\mathbf{k}}(\mathbf{u})\, a^*_{\mathbf{k}}(\mathbf{v})$$

$$\left[\mathcal{D}_\rho \mathcal{D}^\rho \delta^\nu_\mu - \mathcal{D}_\mu \mathcal{D}^\nu + ig\, \mathcal{F}_\mu{}^\nu\right] a^\mu_{\mathbf{k}} = 0$$

$$\lim_{x^0 \to -\infty} a_{\mathbf{k}}(x) = e^{i\mathbf{k}\cdot x}$$



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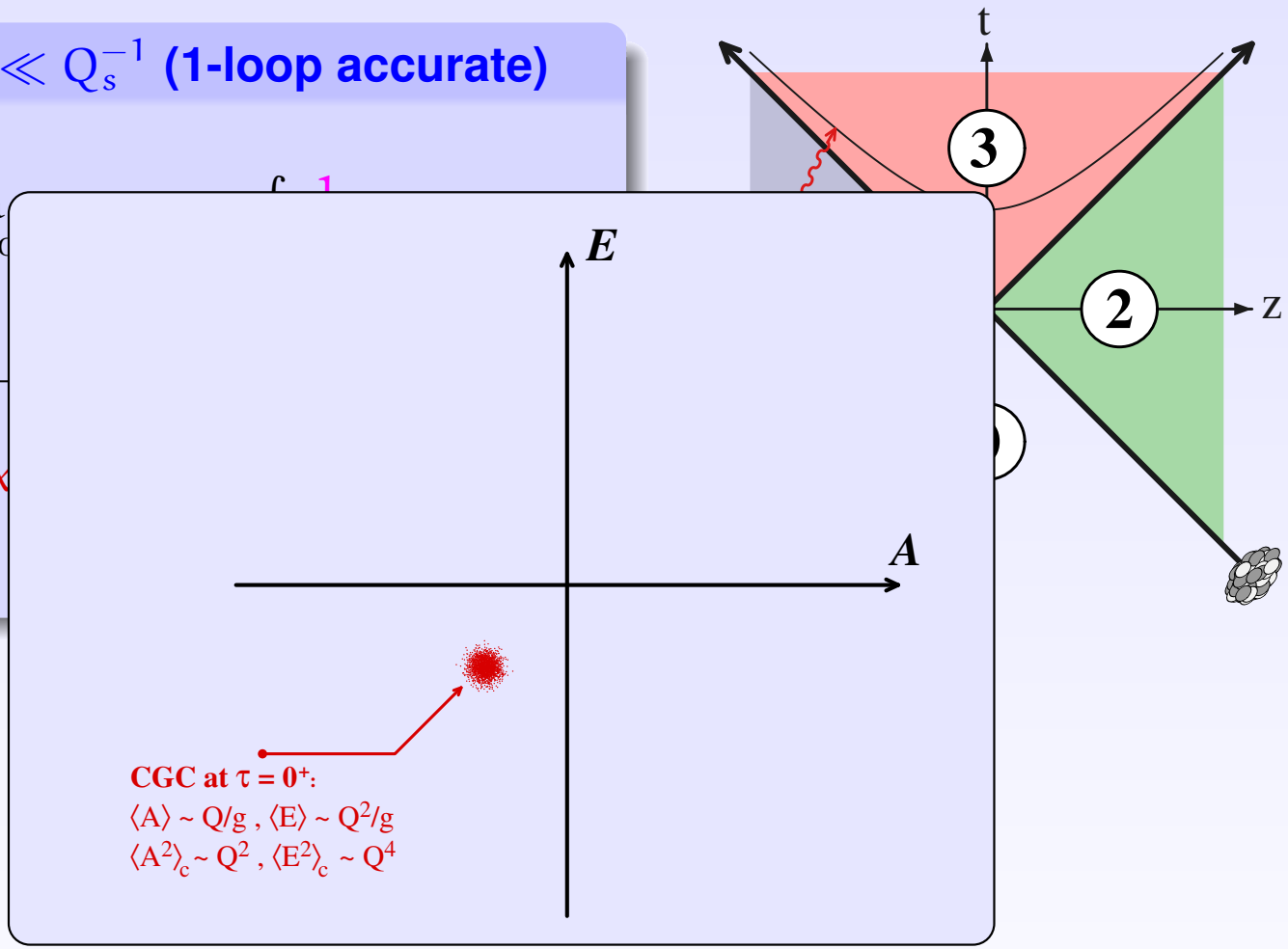

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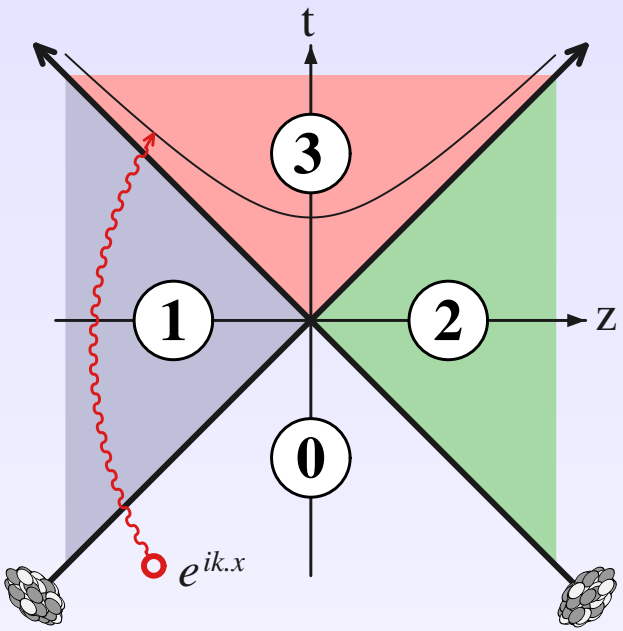

BBSV : Dense gas of free gluons at $Q_s\tau \gg 1$

$$\langle \mathcal{A}^\mu \rangle = 0 \qquad \text{Var.} = \int_{\text{modes } \mathbf{k}} f_0(\mathbf{k})\, a_{\mathbf{k}}(\mathbf{u}) a^*_{\mathbf{k}}(\mathbf{v}) \qquad a_{\mathbf{k}}(x) \equiv e^{i\mathbf{k}\cdot x}$$

$$f_0(\mathbf{k}) \sim g^{-2} \times \theta(Q_s - k)$$

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$$\langle \mathcal{A}^\mu \rangle = \mathcal{A}^\mu_{\mathrm{LO}}$$

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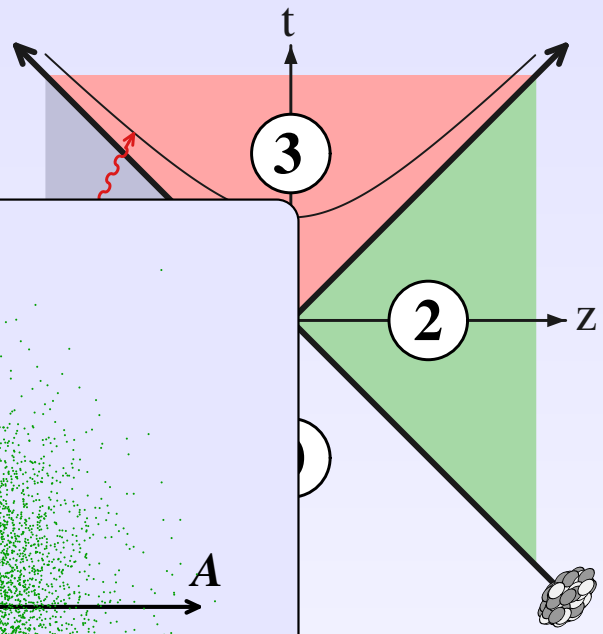
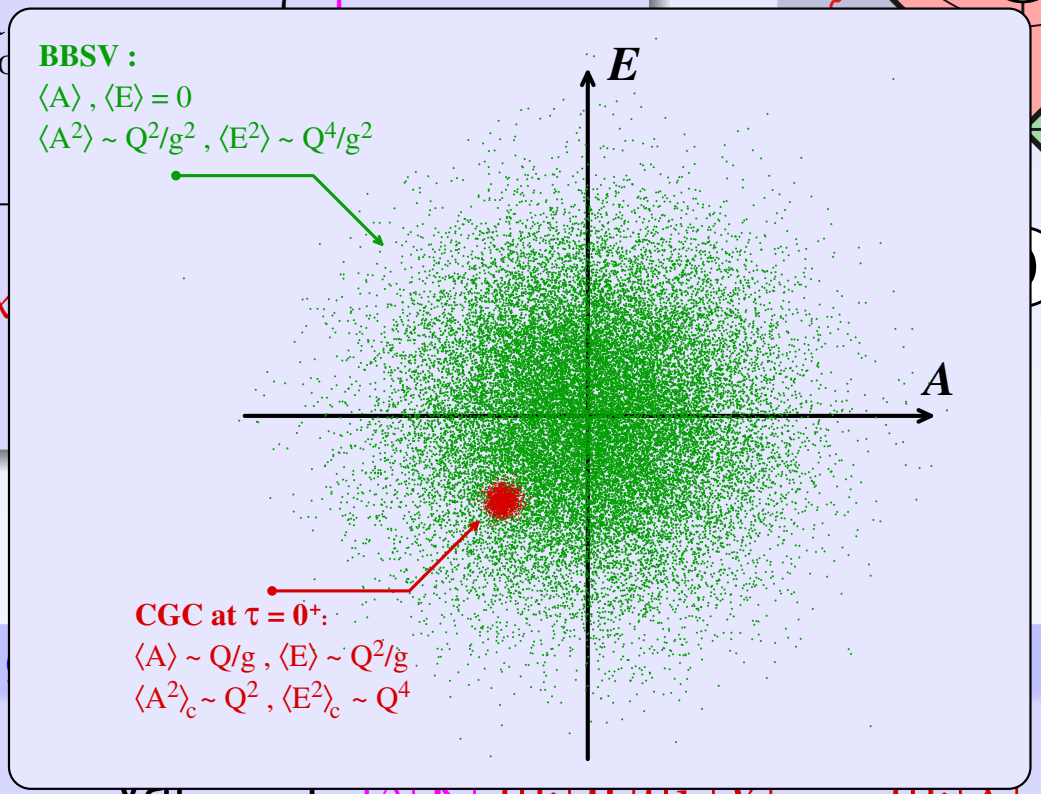


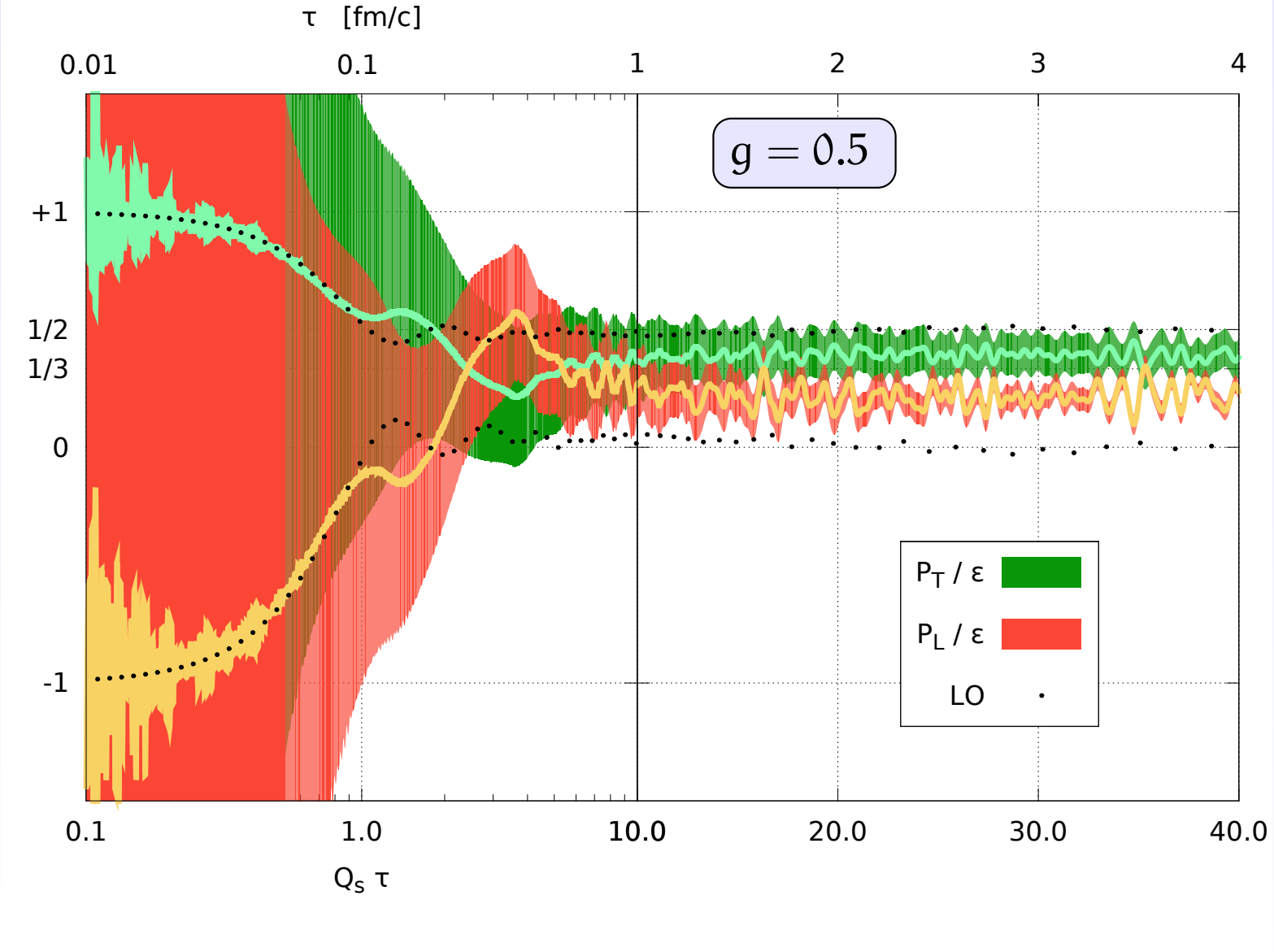


BBSV : Dense

$$\langle \mathcal{A}^\mu \rangle = 0 \qquad \mathrm{var.} = \int_{\text{modes } \mathbf{k}} f_0(\mathbf{k})\, a_{\mathbf{k}}(u) a_{\mathbf{k}}(v) \qquad a_{\mathbf{k}}(x) \equiv e^{ik\cdot x}$$

$$f_0(\mathbf{k}) \sim g^{-2} \times \theta(Q_s - k)$$

EG : CGC initial conditions at $\tau \ll Q_s^{-1}$



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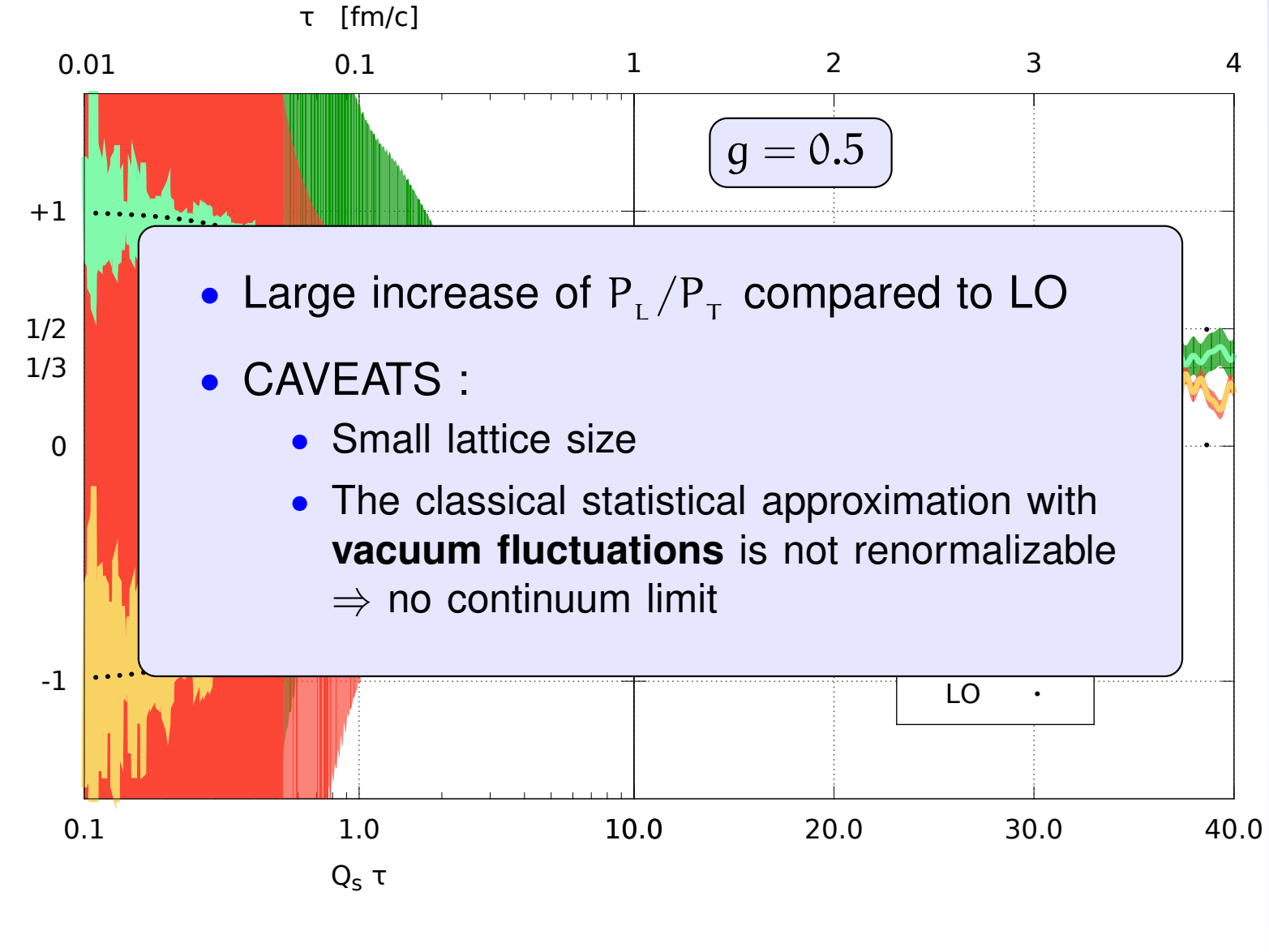
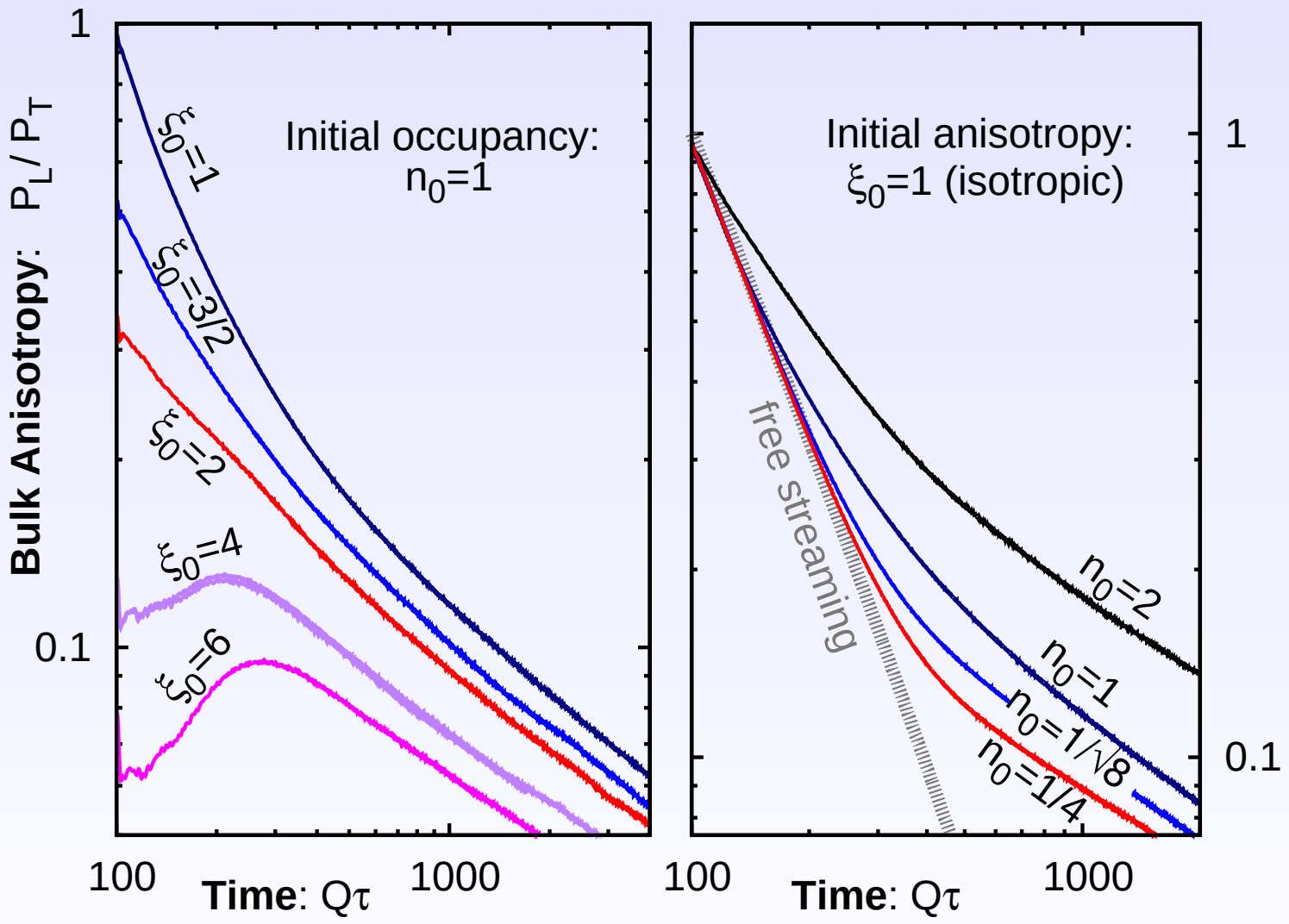


- Large increase of P_L/P_T compared to LO
- CAVEATS :
  - Small lattice size
  - The classical statistical approximation with **vacuum fluctuations** is not renormalizable $\Rightarrow$ no continuum limit

BBSV : dense gas of free gluons at $Q_s\tau \gg 1$

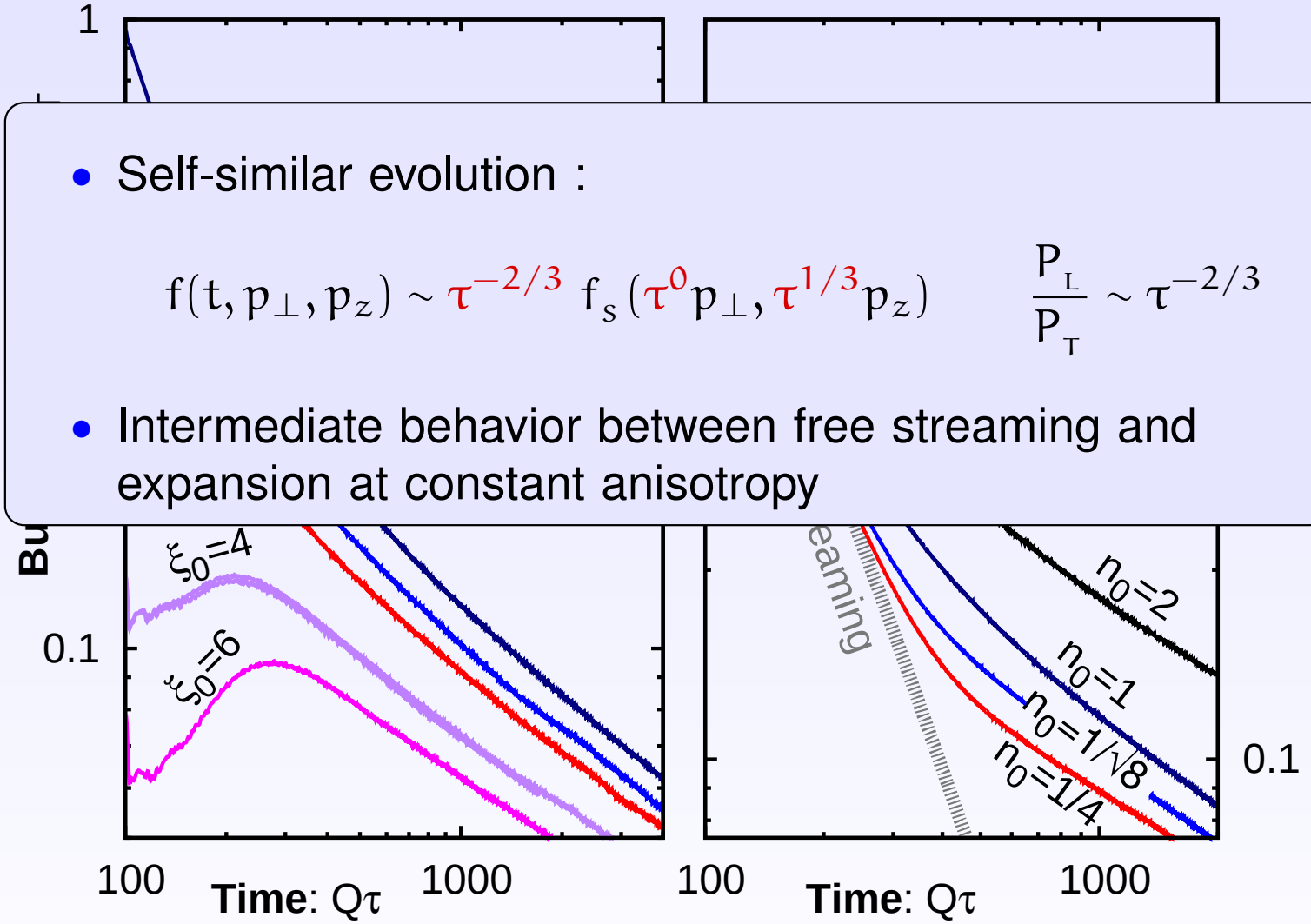


BBSV : dense gas of free gluons at $Q_s\tau \gg 1$

- Self-similar evolution :

$$f(t, p_\perp, p_z) \sim \tau^{-2/3}\, f_s\left(\tau^0 p_\perp, \tau^{1/3} p_z\right) \qquad \frac{P_L}{P_T} \sim \tau^{-2/3}$$

- Intermediate behavior between free streaming and expansion at constant anisotropy



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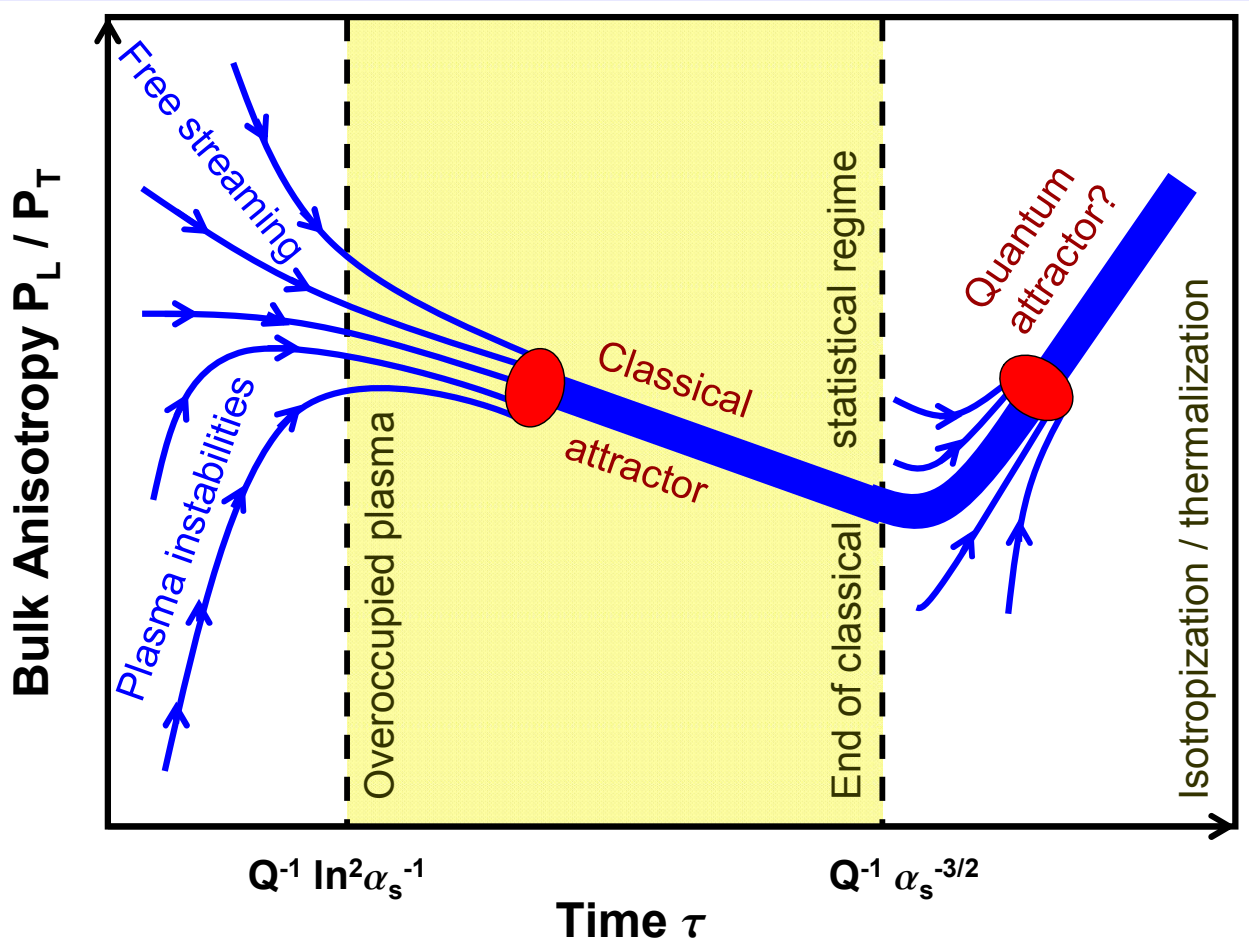

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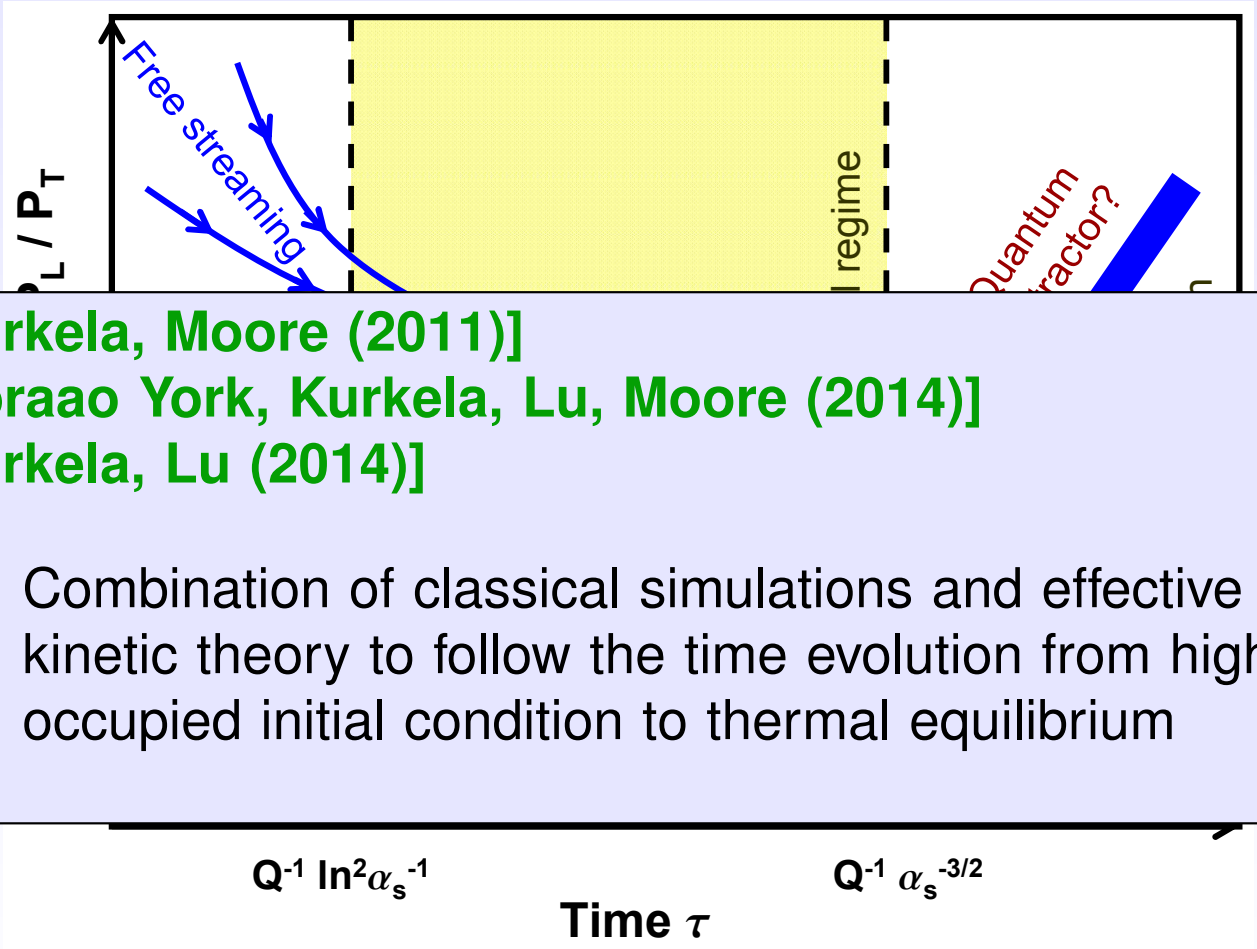


[Kurkela, Moore (2011)]
[Abraao York, Kurkela, Lu, Moore (2014)]
[Kurkela, Lu (2014)]

- Combination of classical simulations and effective kinetic theory to follow the time evolution from highly occupied initial condition to thermal equilibrium

Lessons from Kinetic Theory

Classical wave approximation(s) in Kinetic Theory

- Boltzmann equation for the elastic process $12 \to 34$:

$$\partial_t f_1 = \frac{1}{\omega_1} \int_{2,3,4} \delta^{(4)}(P_1+P_2-P_3-P_4)\ |\mathcal{M}_{12,34}|^2 \ \Big[f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4)\Big]$$

- Pure CSA : keep only the cubic terms in f

$$f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \quad \to \quad f_3 f_4 (f_1+f_2) - f_1 f_2 (f_3+f_4)$$

- Improved CSA : from the cubic terms, do $f \to f + \frac{1}{2}$

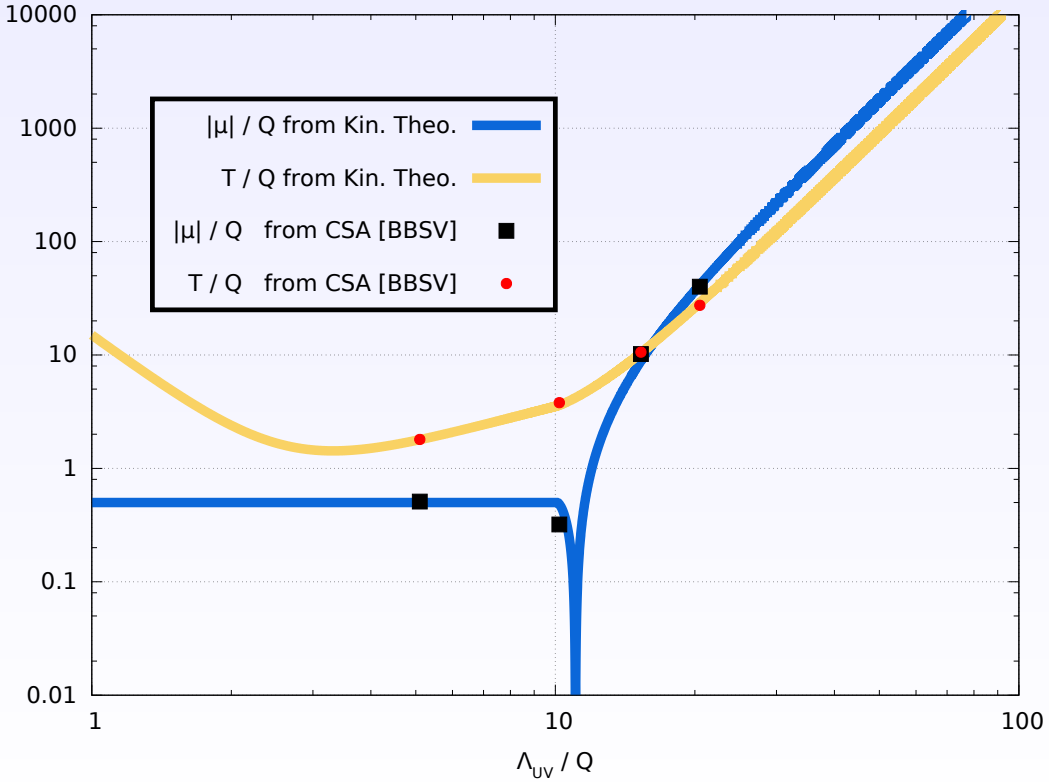
$$f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \quad \to \quad (\tfrac{1}{2}+f_3)(\tfrac{1}{2}+f_4)(1+f_1+f_2) - (\tfrac{1}{2}+f_1)(\tfrac{1}{2}+f_2)(1+f_3+f_4)$$

- The Boltzmann equation can be used to assess the effect of these approximations by comparing their solutions with that of the unapproximated equation

UV cutoff dependence of the improved CSA

[Berges, Boguslavski, Schlichting, Venugopalan (2013)]
[Epelbaum, FG, Tanji, Wu (2014)]

- Nonrenormalizable approximation of the original QFT
- UV cutoff dependence, that can also be seen in kinetic theory
- At late times, $f(\mathbf{p}) = \dfrac{T}{\omega_{\mathbf{p}} - \mu} - \dfrac{1}{2}$, but T and μ depend on Λ_{UV}



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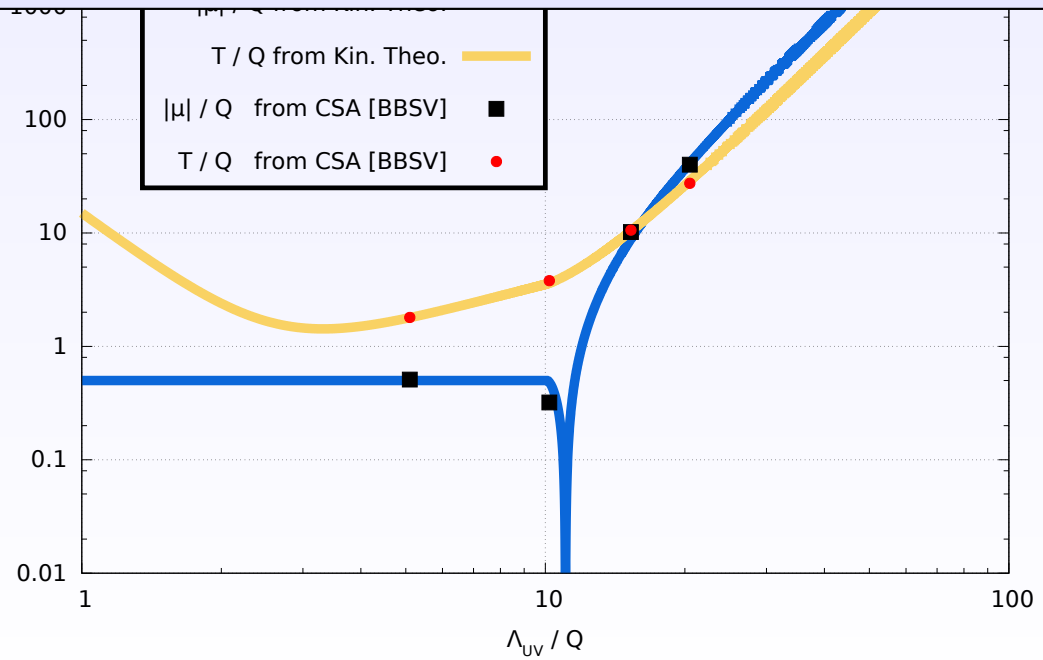
[Berges, Boguslavski, Schlichting, Venugopalan (2013)]
[Epelbaum, FG, Tanji, Wu (2014)]

- Nonrenormalizable approximation of the original QFT
- UV cutoff dependence, that can also be seen in kinetic theory
- [illegible]

  - Strong cutoff dependence if $\Lambda_{uv} \gg$ physical scales
  - Mild sensitivity if $\Lambda_{uv} \sim [3-6] \times$ (physical scales)



Pure CSA : misses large angle scatterings in anisotropic systems

[Epelbaum, FG, Jeon, Moore, Wu (2015)]

- Pure CSA in kinetic theory :

$$\partial_t f_4 \sim g^4 \int_{123} \cdots \left[f_1 f_2 (f_3 + f_4) - f_3 f_4 (f_1 + f_2) \right] + \cdots \left[\cancel{f_1 f_2 - f_3 f_4} \right]$$

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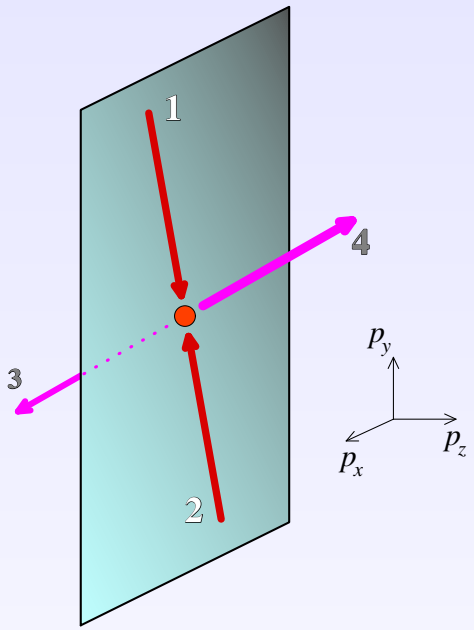

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- If the distribution is very anisotropic, trying to produce the particle 4 at large angle results in $f_3 \approx f_4 \approx 0 \quad \Rightarrow$ nothing left
- The same argument applies also to any inelastic $n \to n'$ scattering

The CSA without vacuum fluctuations underestimates large angle scatterings when the distribution is anisotropic, and may lead to wrong conclusions regarding isotropization

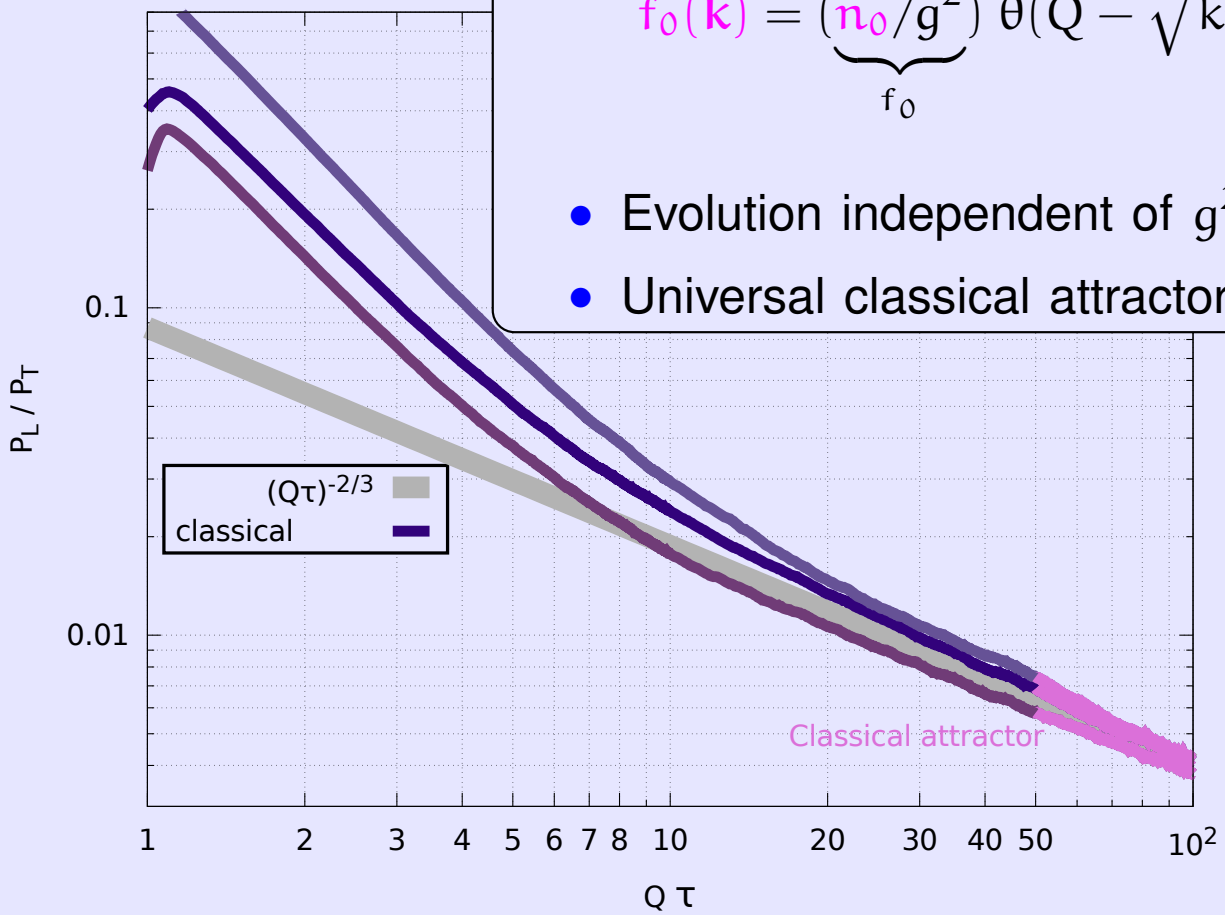
Pure CSA : misses large angle scatterings in anisotropic systems

[Epelbaum, FG, Jeon, Moore, Wu (2015)]

- Initial conditions:

$$f_0(\mathbf{k}) = \underbrace{(n_0/g^2)}_{f_0}\,\theta(Q - \sqrt{k_\perp^2 + \xi_0\, k_z^2})$$

- Evolution independent of g^2
- Universal classical attractor



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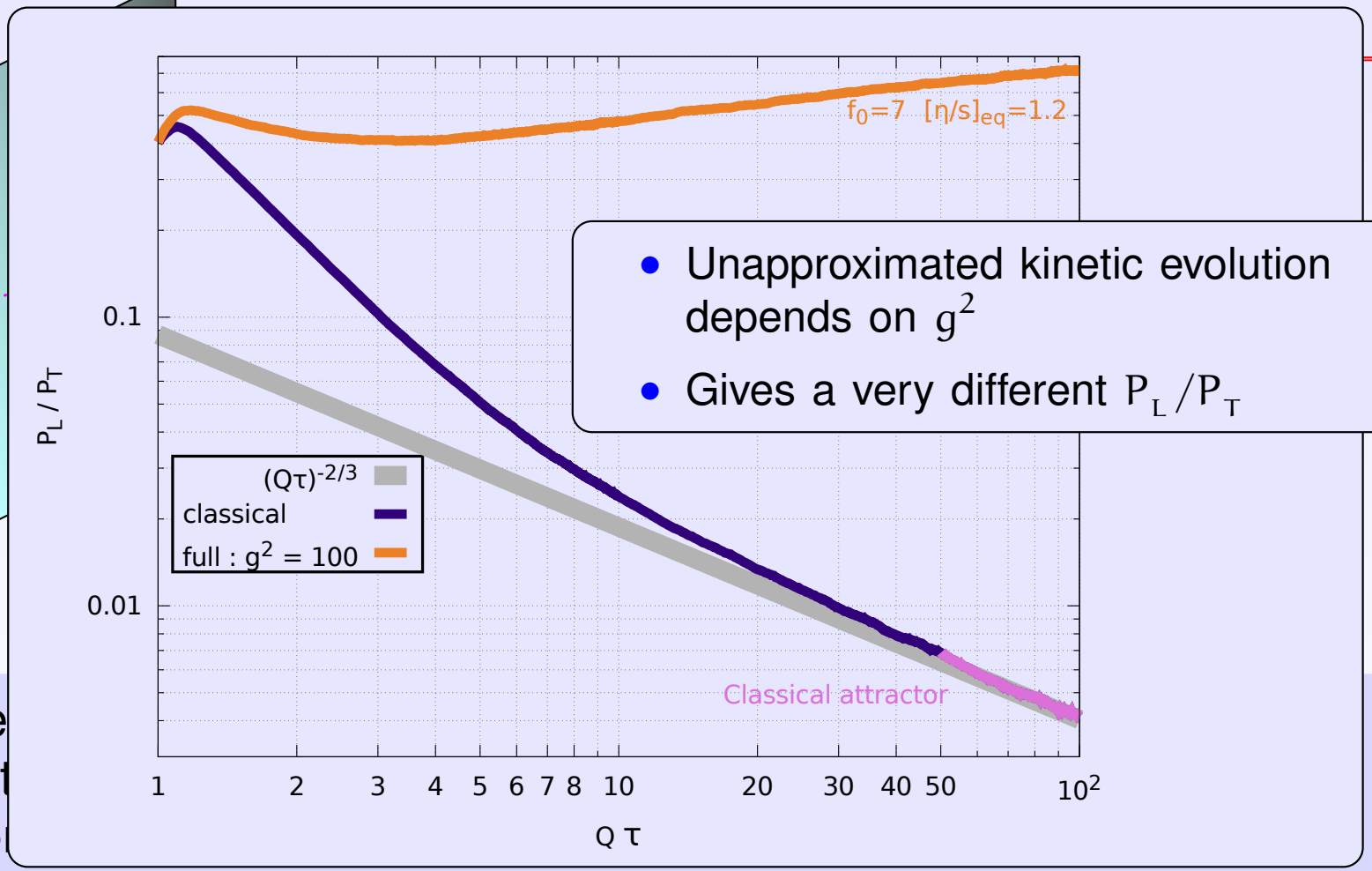

- Unapproximated kinetic evolution depends on g^2
- Gives a very different P_L/P_T

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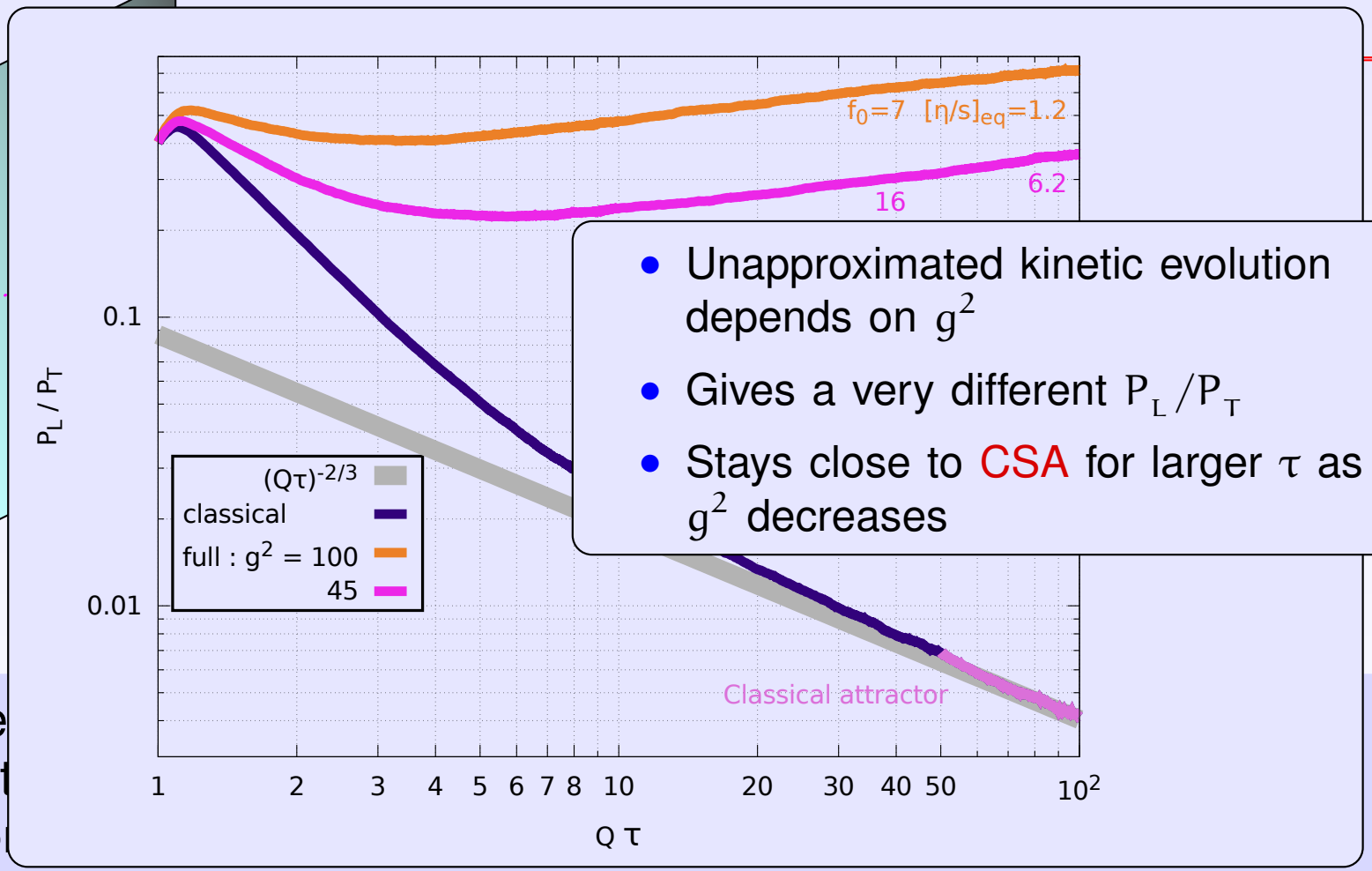

- Unapproximated kinetic evolution depends on g^2
- Gives a very different P_L/P_T
- Stays close to CSA for larger τ as g^2 decreases

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Pure CSA : misses large angle scatterings in anisotropic systems

[Epelbaum, FG, Jeon, Moore, Wu (2015)]

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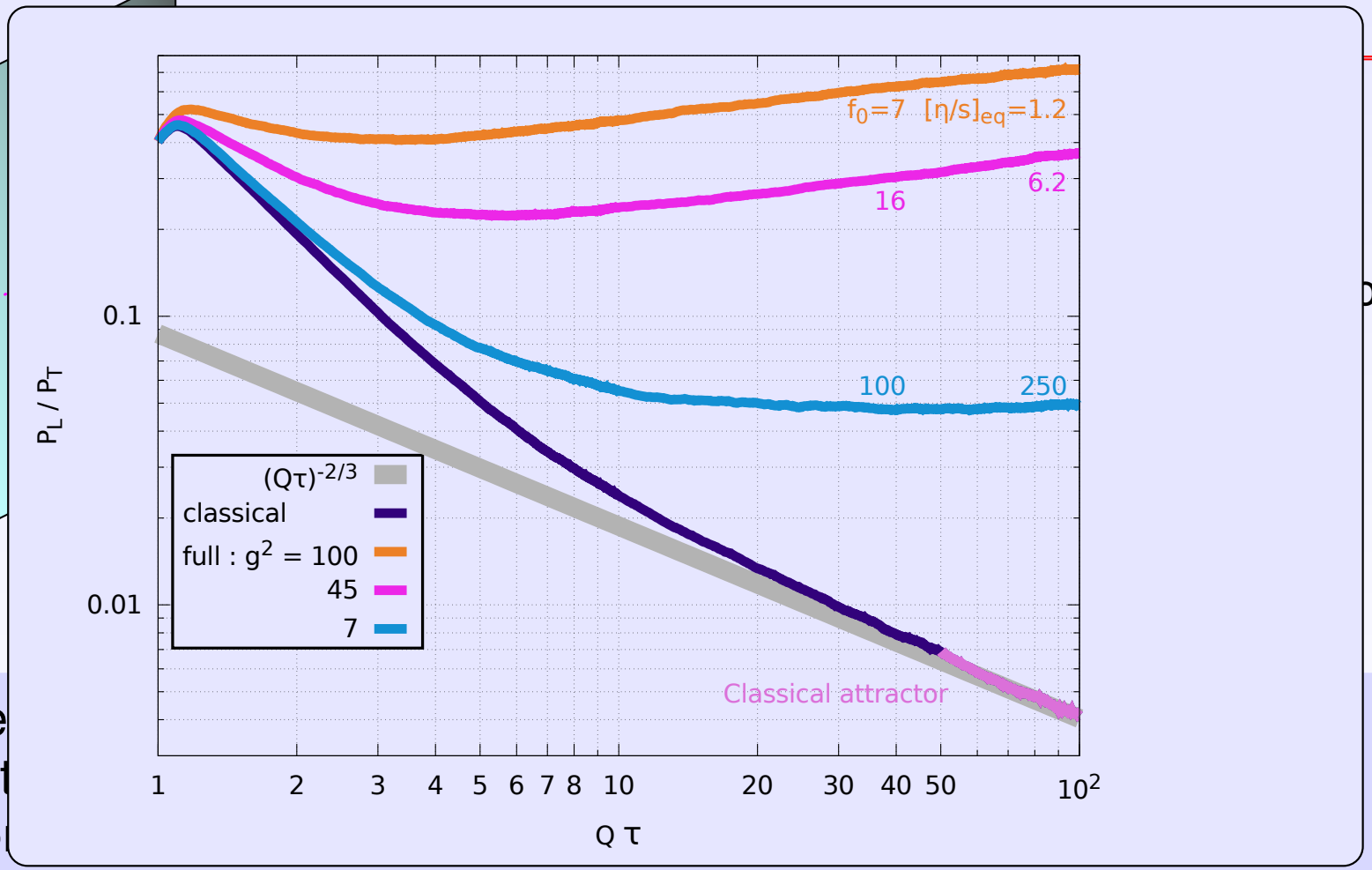

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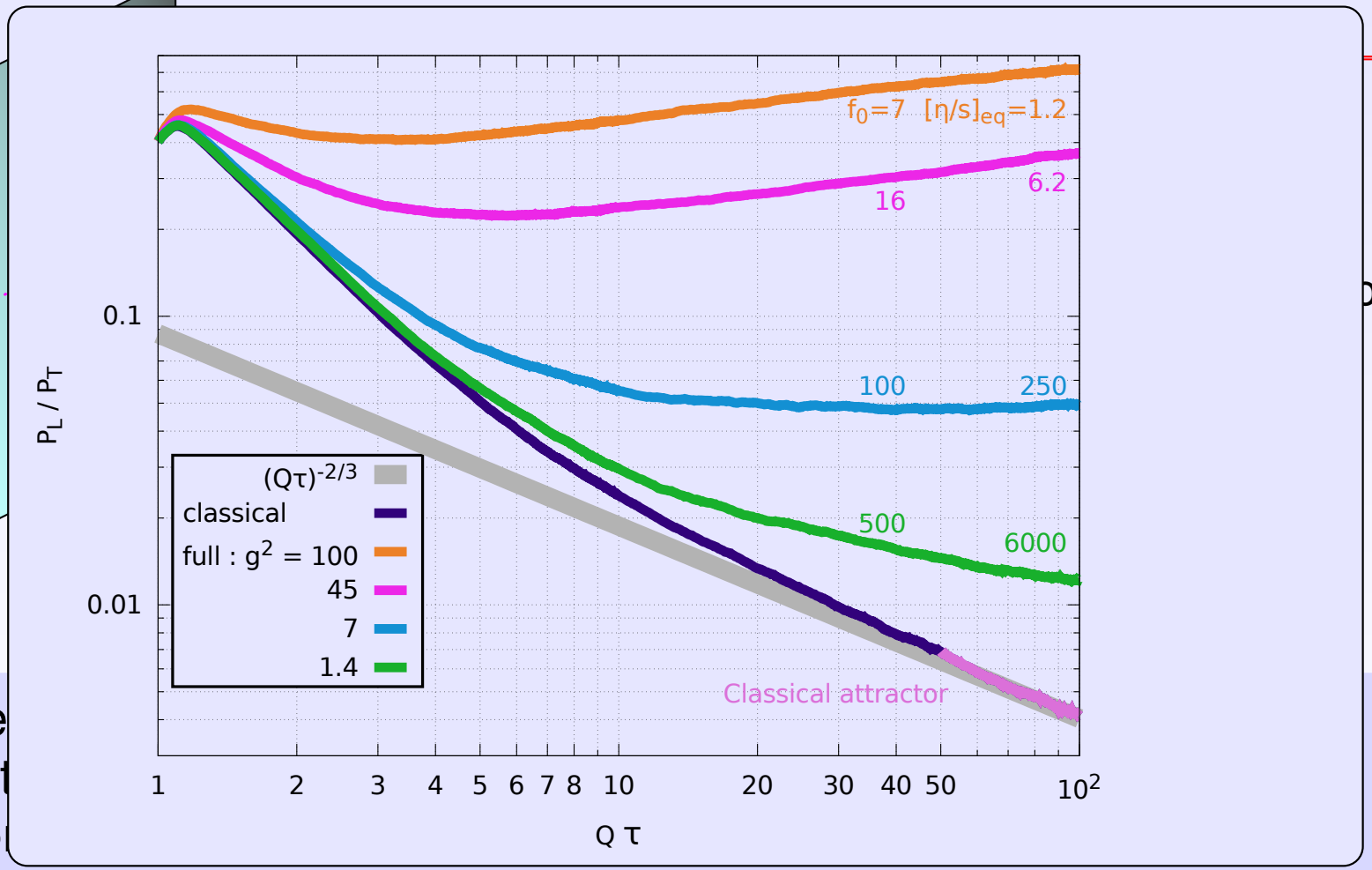

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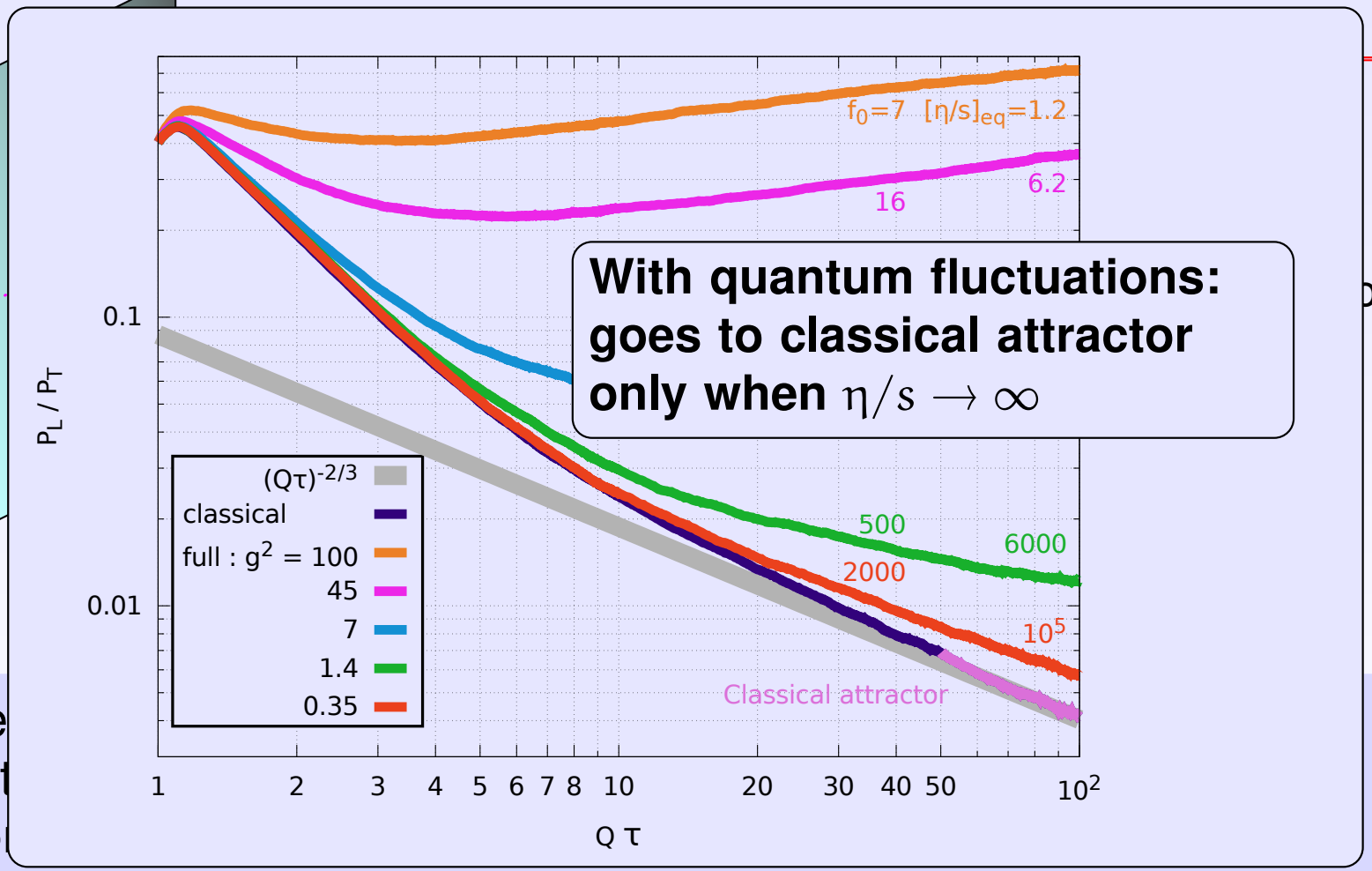

With quantum fluctuations: goes to classical attractor only when $\eta/s \to \infty$

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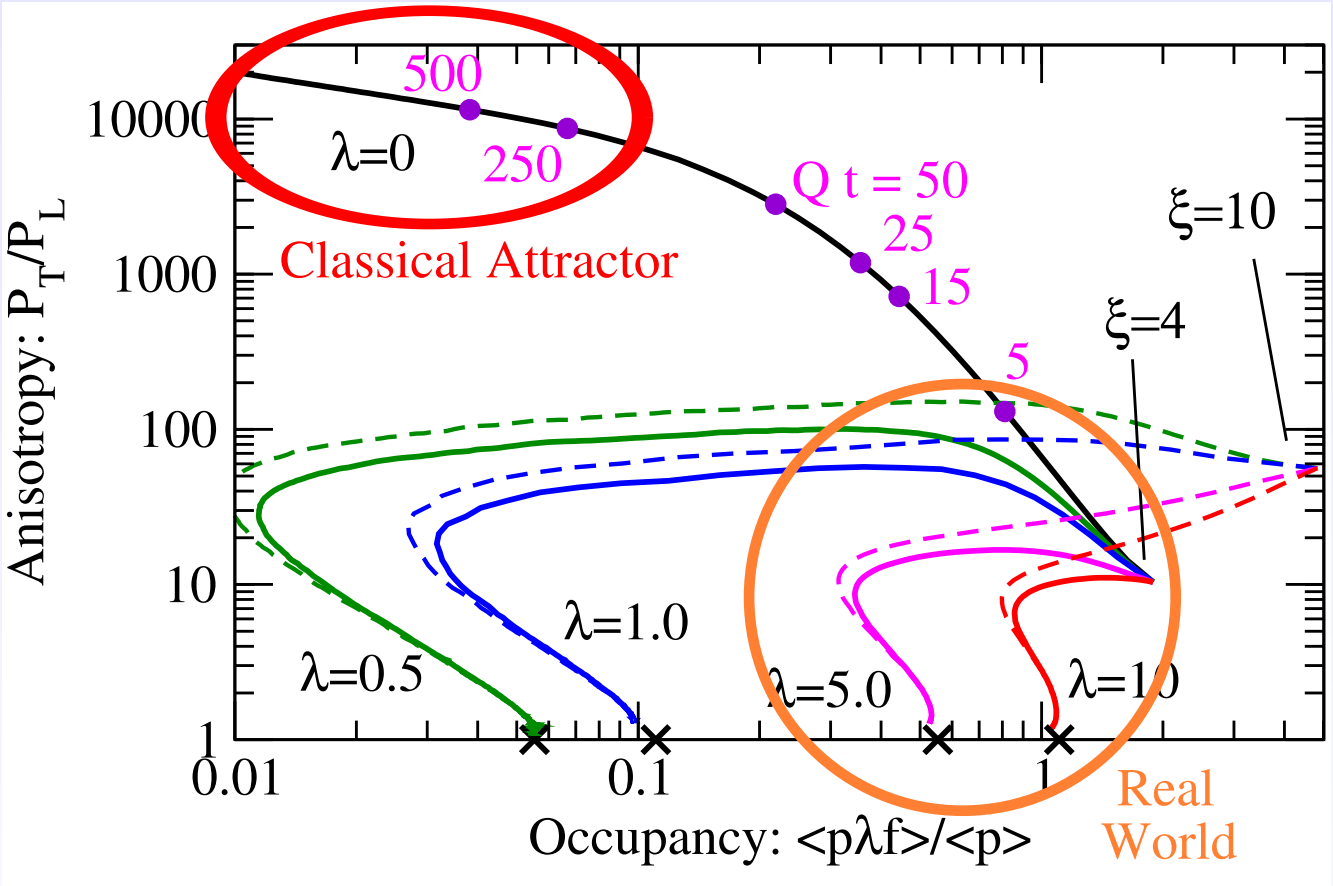
Kinetic theory results for gluons [Kurkela, Zhu (2015)]



- At $g^2 N_c = 1/2$, the pure CSA approximation breaks down at $Q\tau \approx 2$, while naive estimates suggest that it is valid up to $Q\tau \approx \alpha_s^{-3/2} \approx 350$

Summary

Summary

- LO : no pressure isotropization, NLO : instabilities
- Resummation beyond NLO : Classical statistical approximation
- Two implementations... and two different results :
  - Pure CSA : particle-like initial conditions
    universal classical attractor, P_L/P_T decreases forever,
    underestimates large angle scatterings,
    breaks long before reaching the attractor even for quite large η/s
  - Improved CSA : vacuum-like CGC initial conditions
    roughly constant P_L/P_T,
    non-renormalizable approximation, very sensitive to UV cutoff
- In the present situation, classical field simulations need to be corroborated and validated by other approaches
- Highly needed : ways to overcome the problems of the classical statistical approximation (Kinetic theory, 2-PI,...)