Isotropization in Heavy Ion Collisions at High Energy

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T. Epelbaum, FG :
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arXiv:1307.2214

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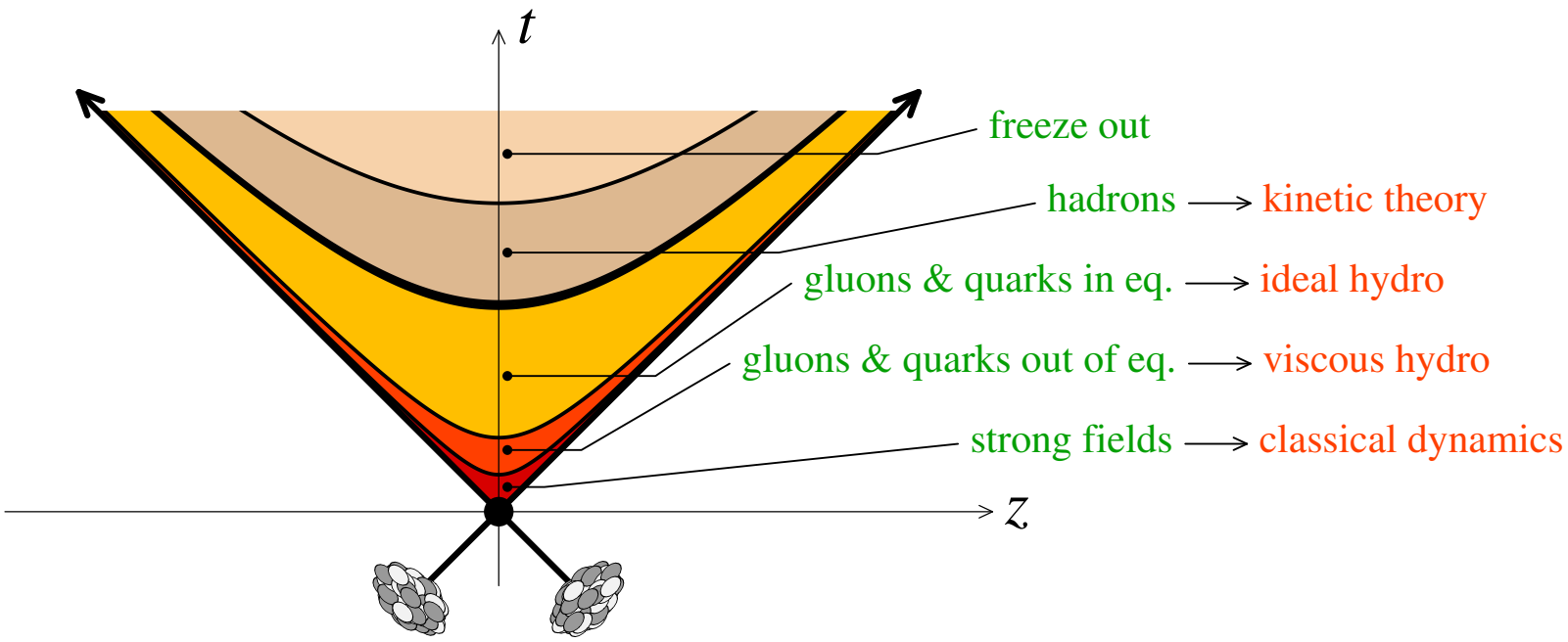
Outline

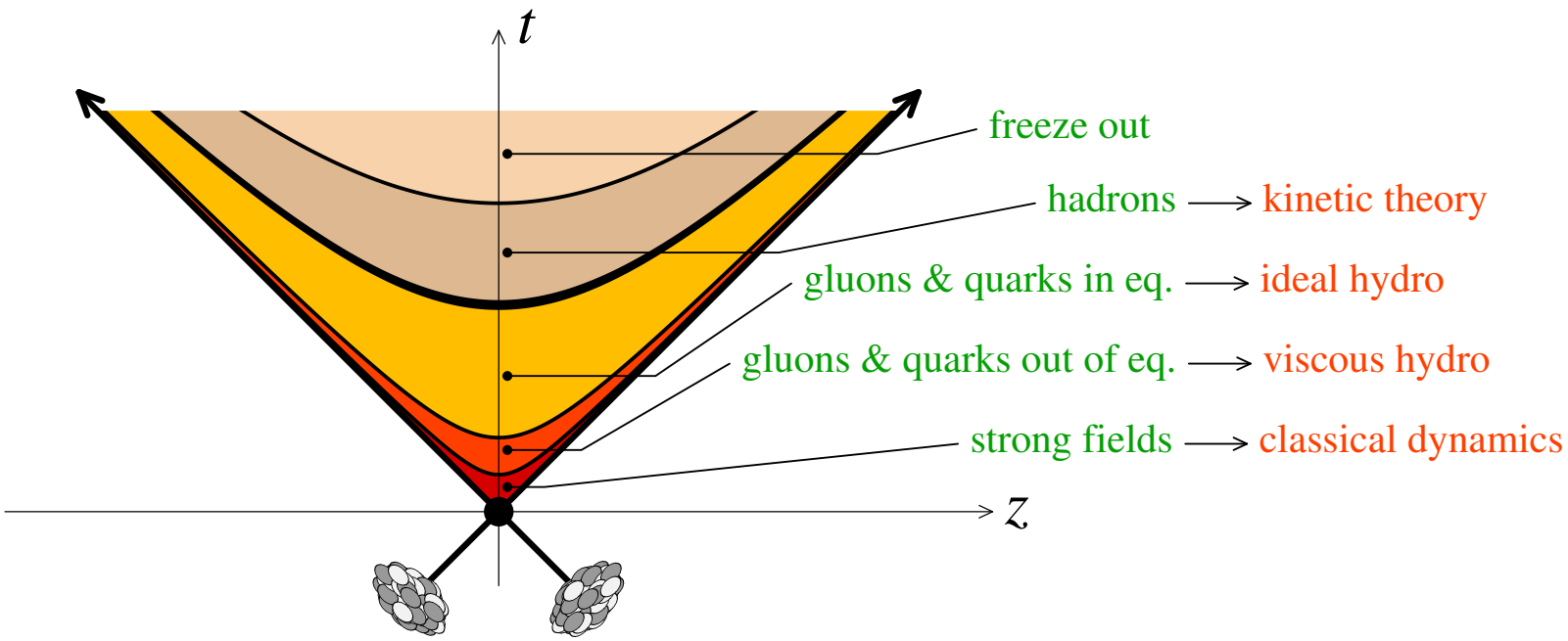
1. The surprising success of relativistic hydrodynamics
2. Color Glass Condensate in heavy ion collisions
3. Initial stages of heavy ion collisions
4. Non Renormalizability issues

Relativistic hydrodynamics

Stages of a nucleus-nucleus collision

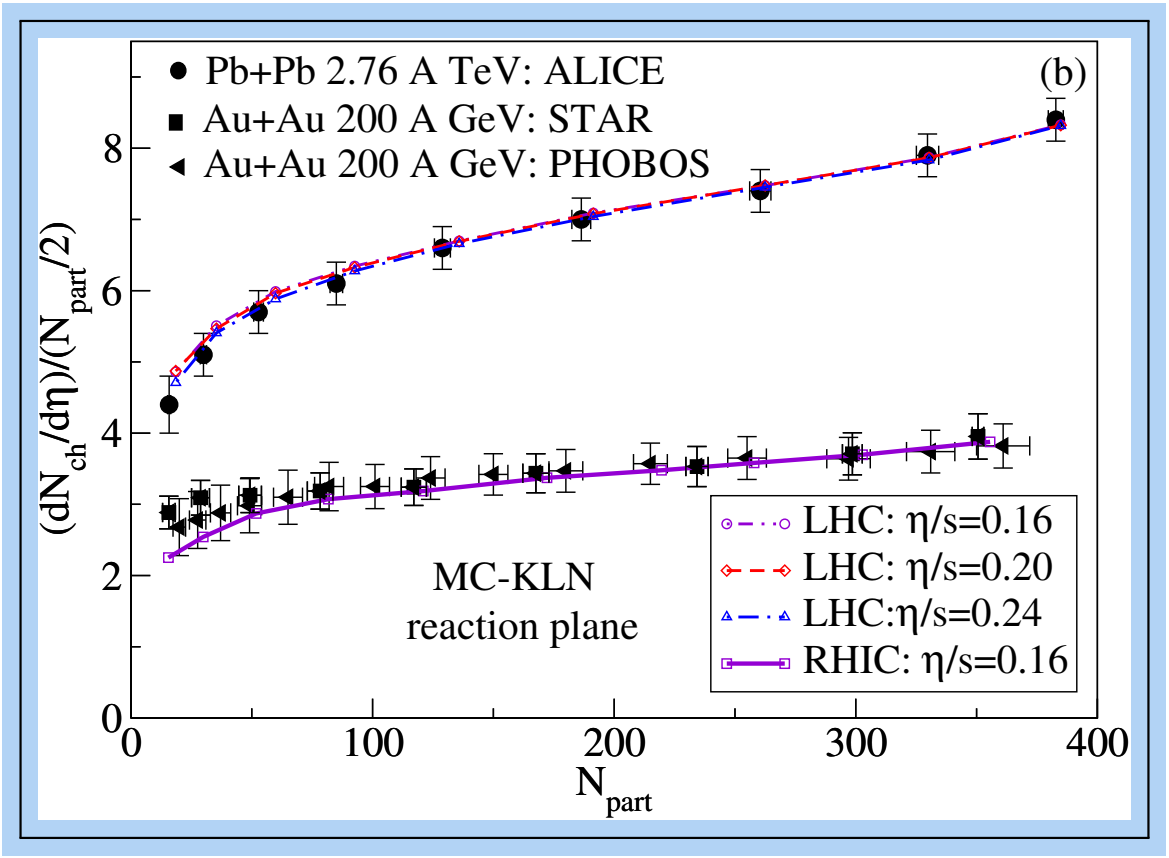


Stages of a nucleus-nucleus collision



- Well described as a fluid expanding into vacuum according to relativistic hydrodynamics

Evidence for hydrodynamical behavior



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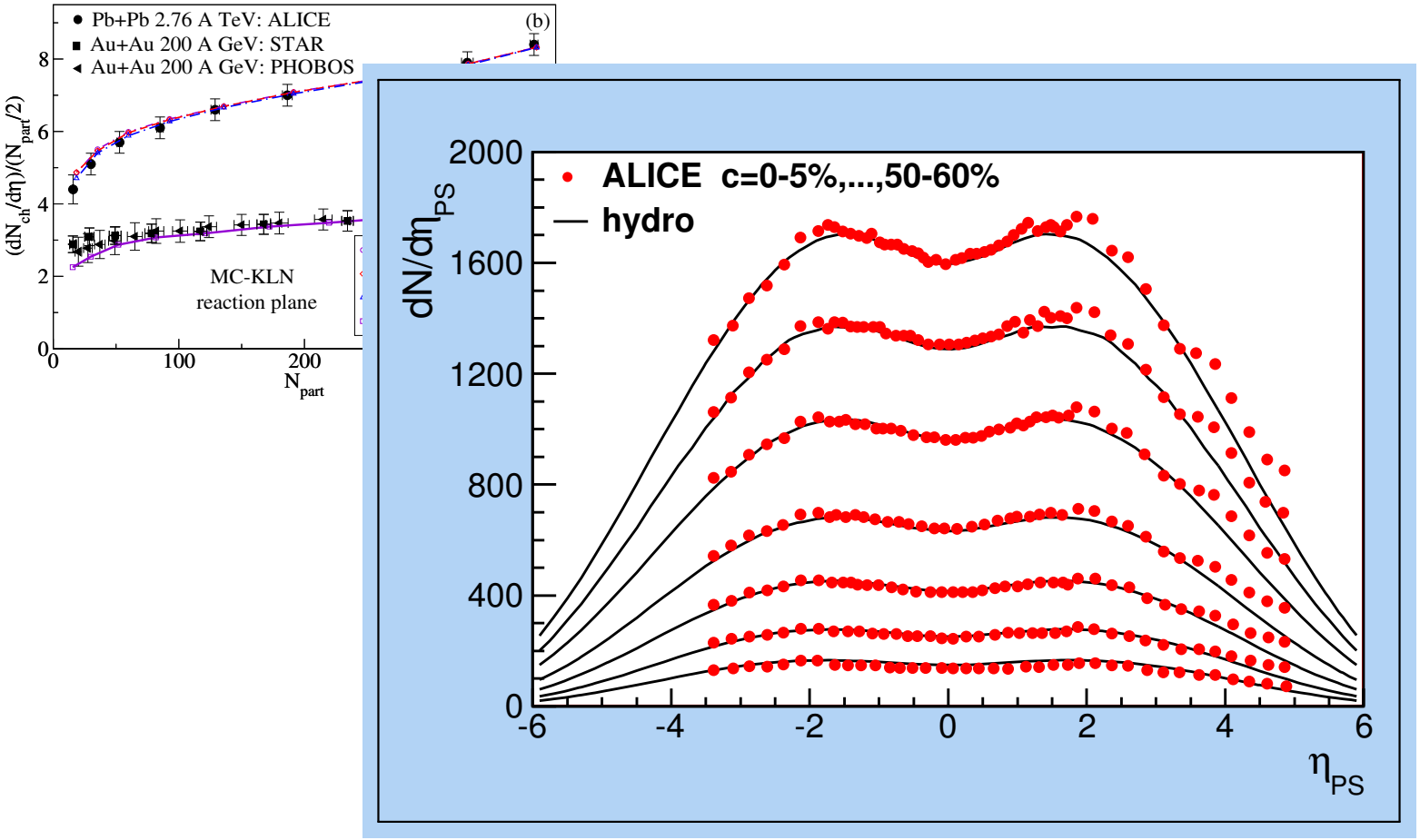

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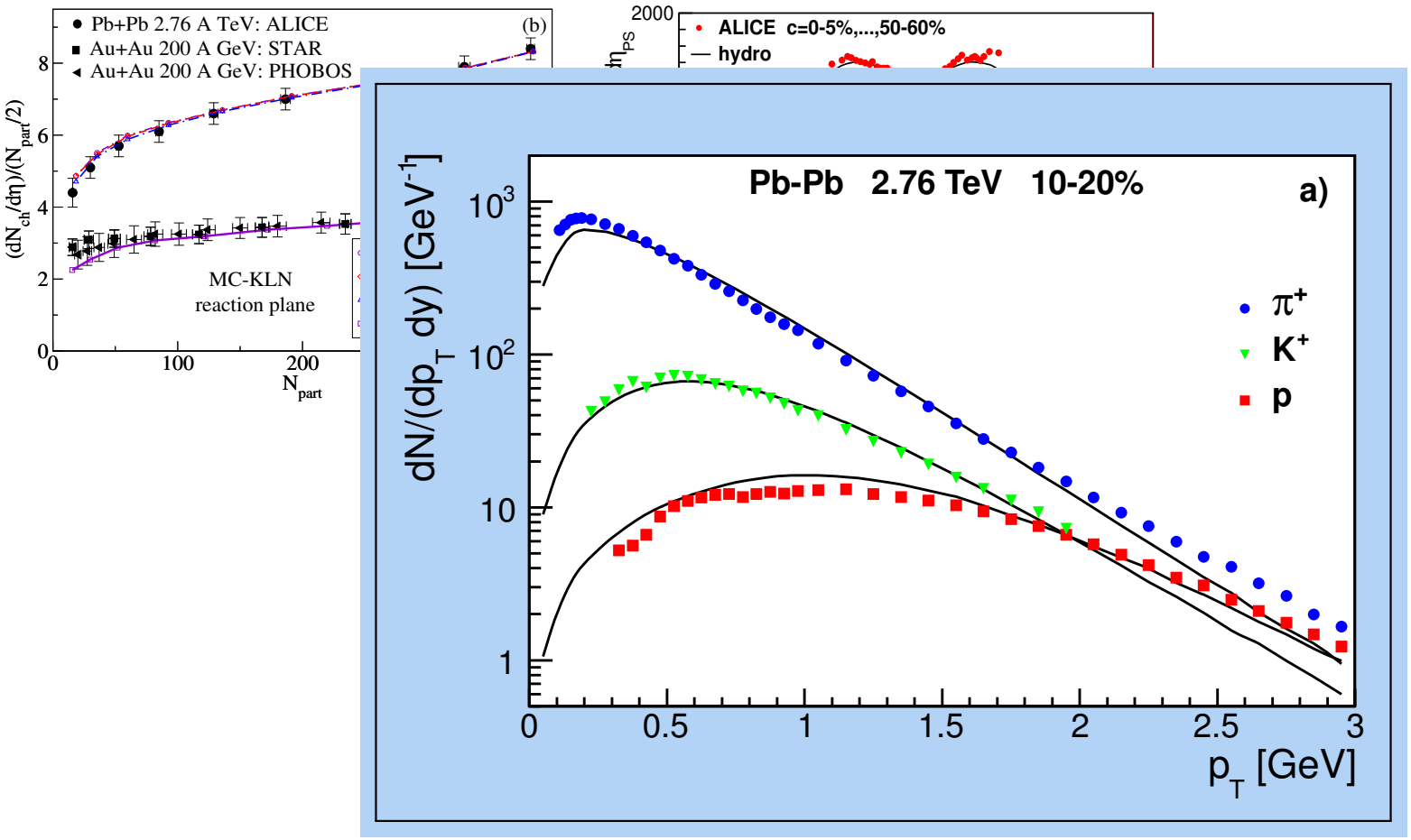

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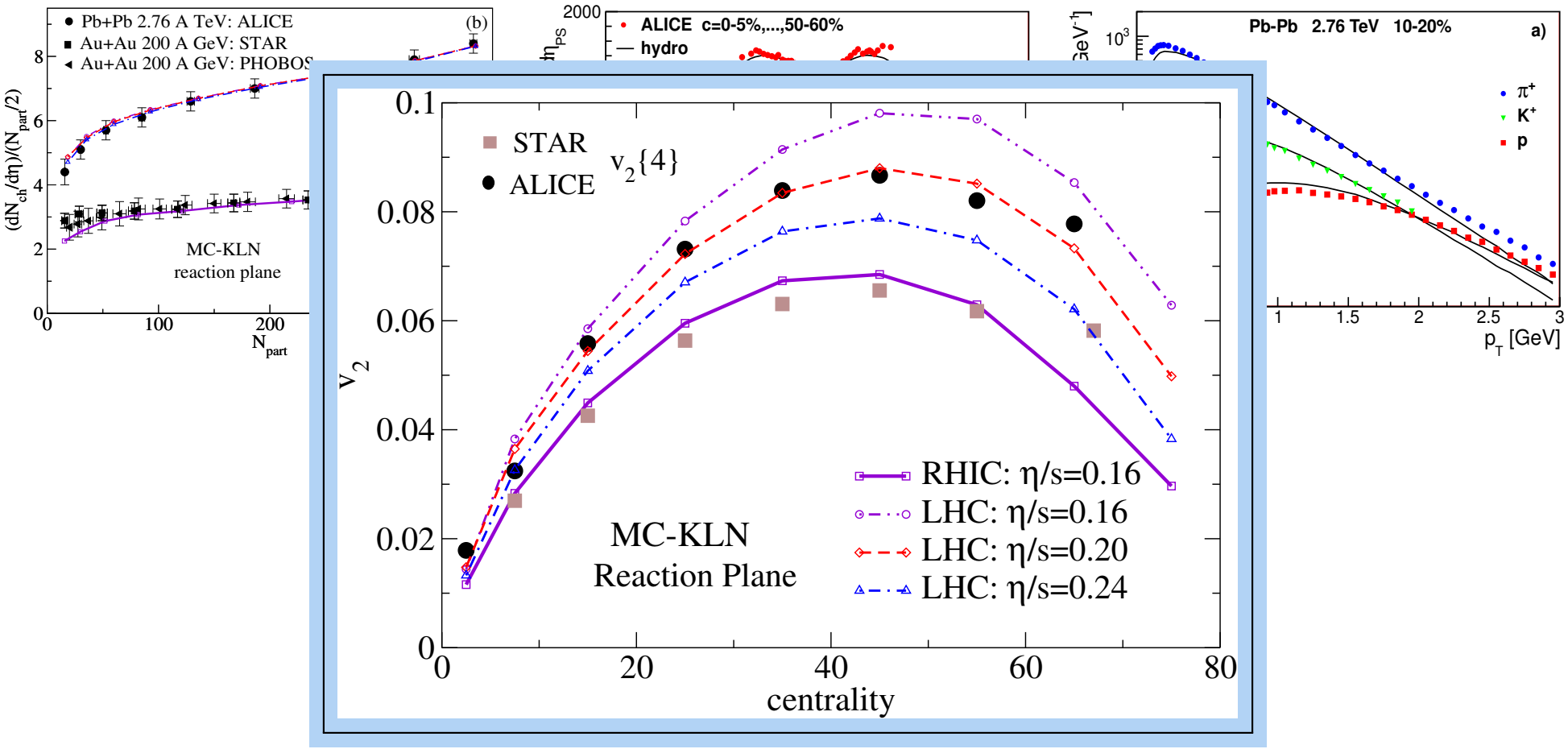

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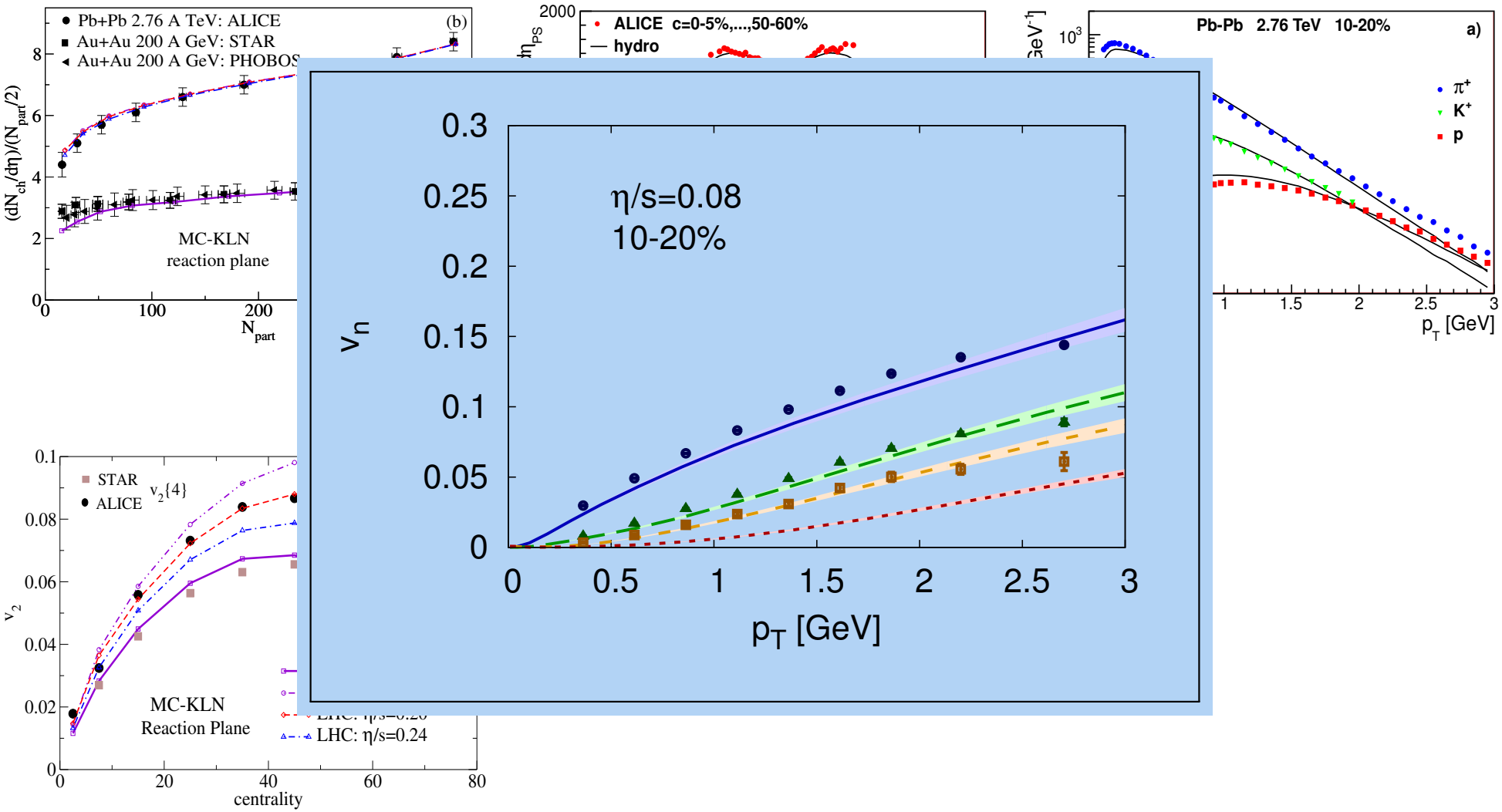

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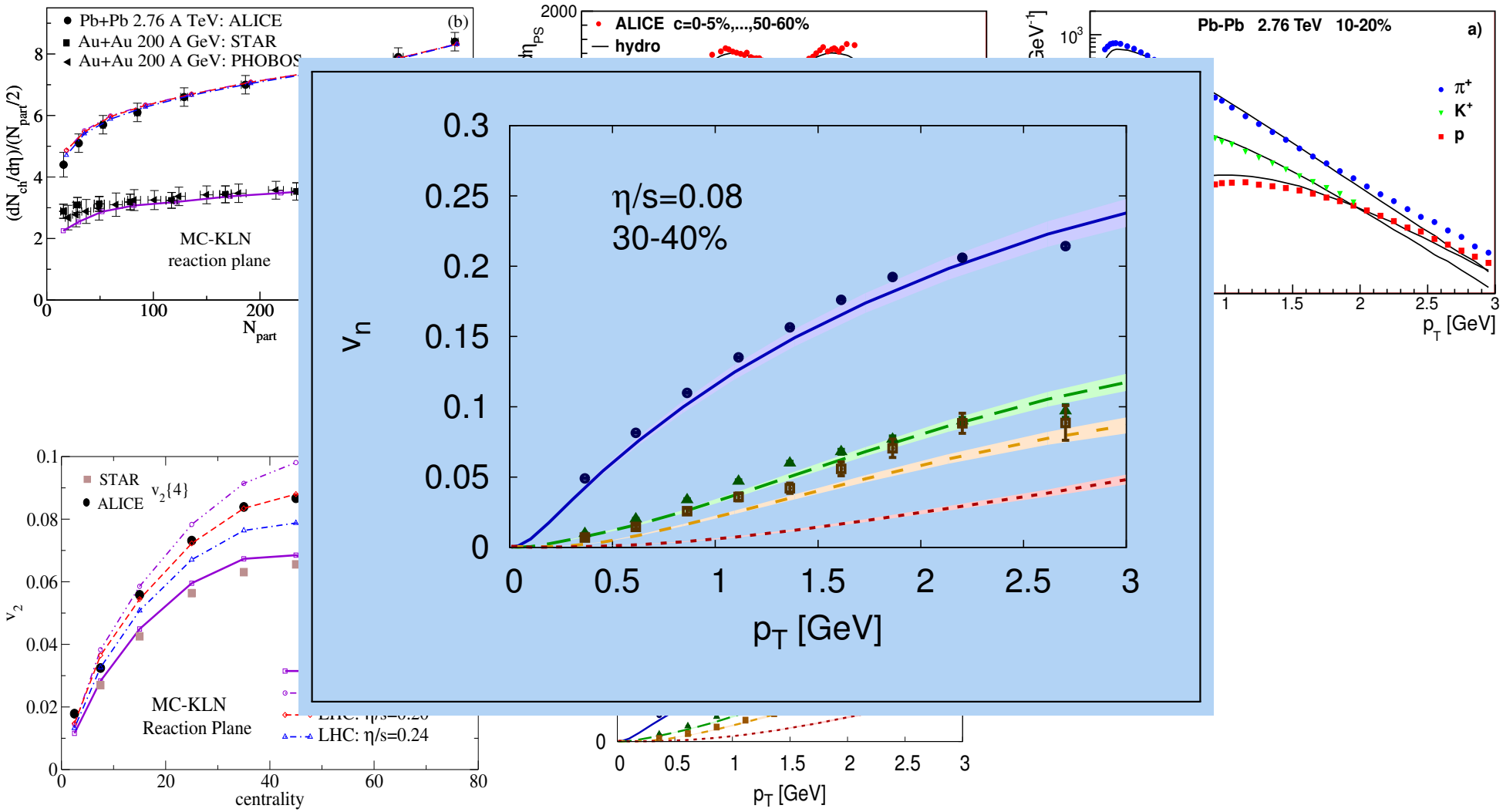

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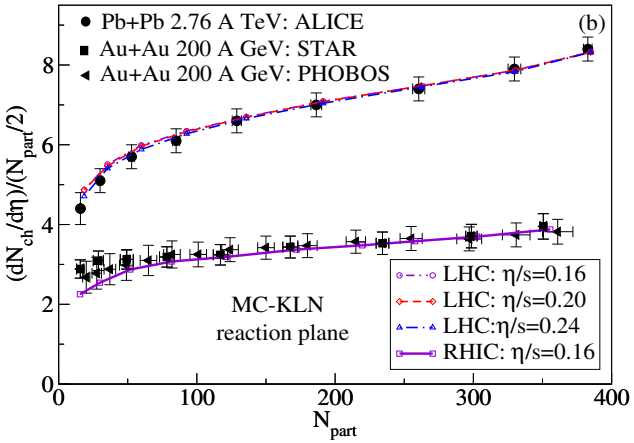

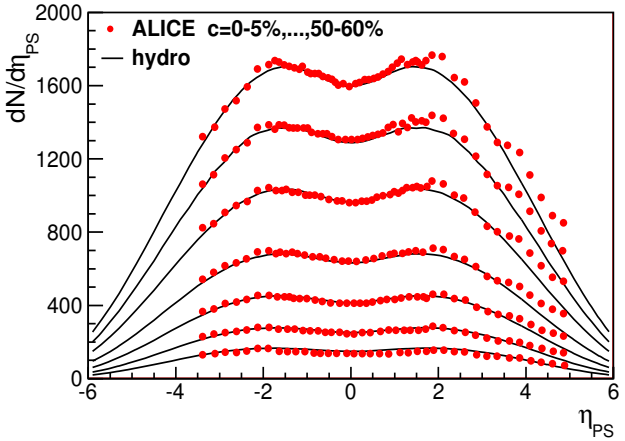

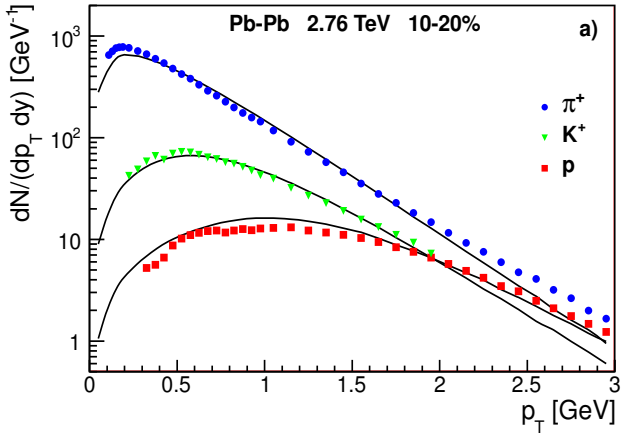

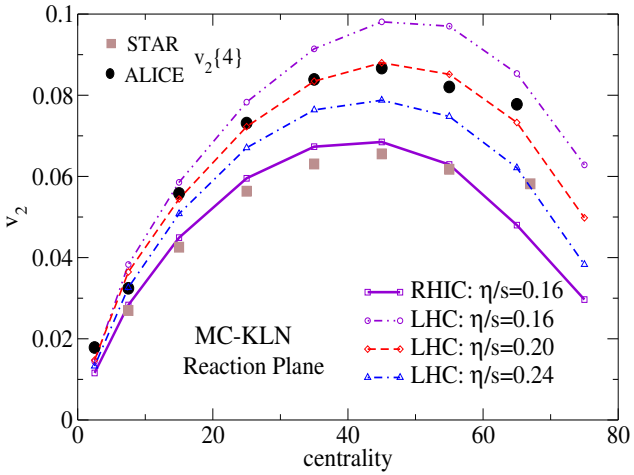

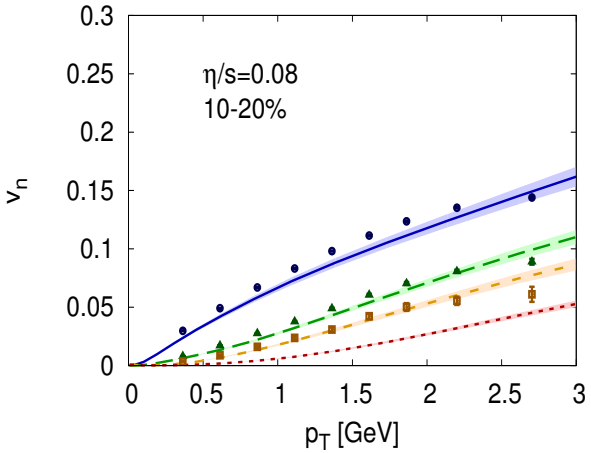

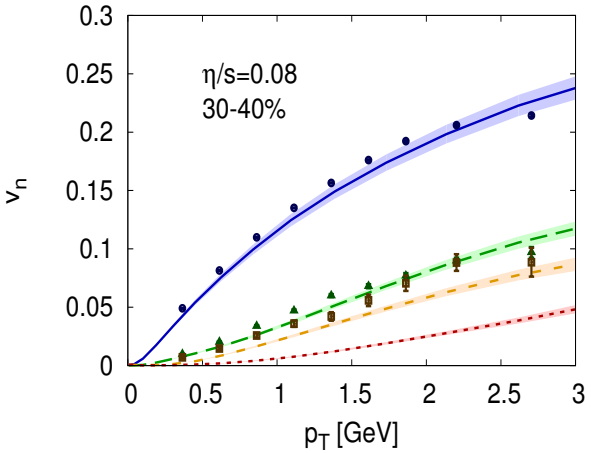

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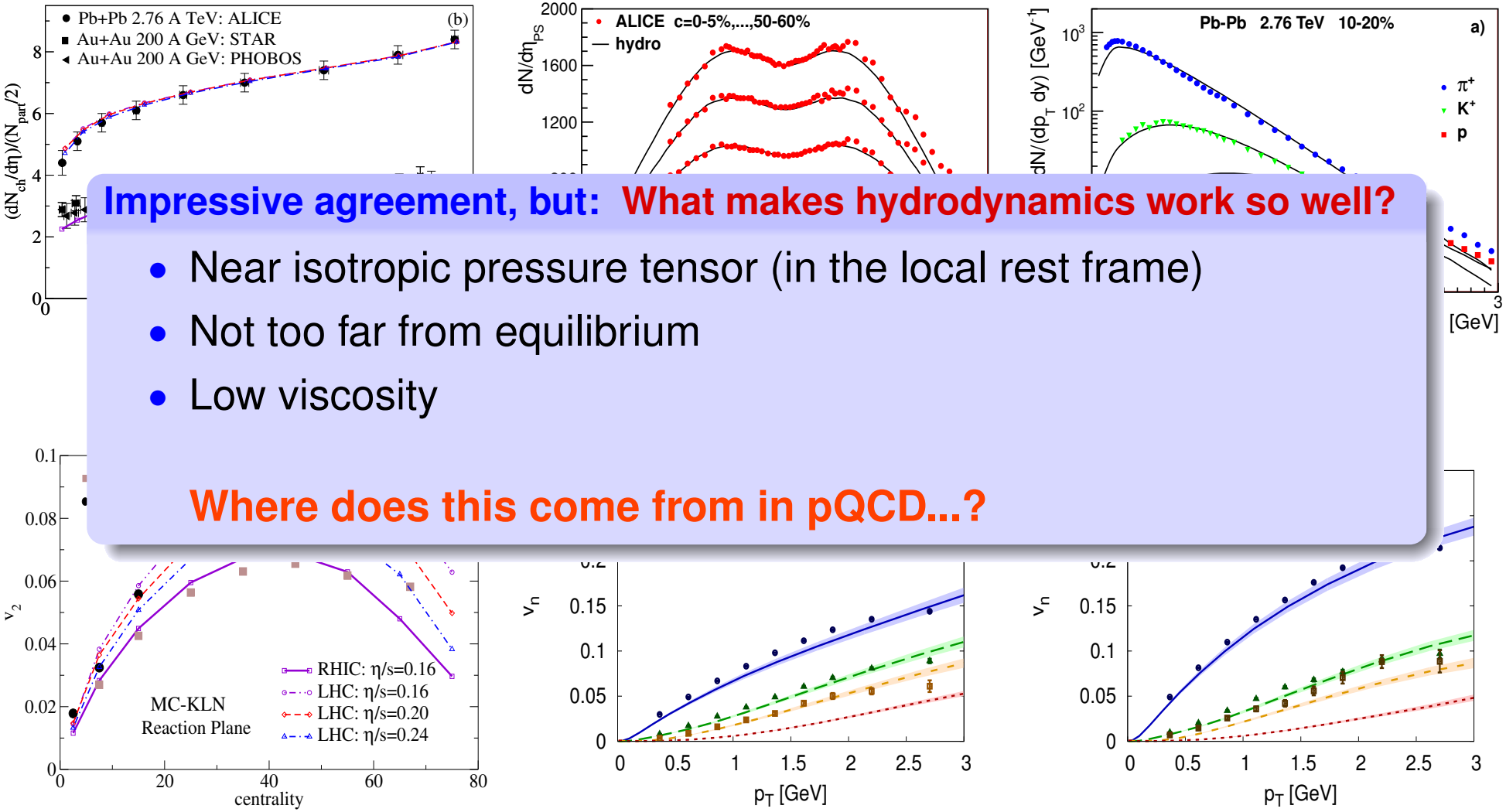

Impressive agreement, but: What makes hydrodynamics work so well?

- Near isotropic pressure tensor (in the local rest frame)
- Not too far from equilibrium
- Low viscosity

Where does this come from in pQCD...?

What is hydrodynamics?

- Hydrodynamics is a macroscopic description based on energy-momentum conservation :

$$\partial_\mu T^{\mu\nu} = 0$$

True in any quantum field theory

But does not provide a closed set of equations to fully describe the evolution of the system (4 equations, while $T^{\mu\nu}$ has 10 independent components)

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- In general : $T^{\mu\nu} = T^{\mu\nu}_{\text{ideal}} \oplus \eta\, \nabla^\mu u^\nu \oplus \zeta\, (\nabla_\rho u^\rho) \oplus \cdots$
- Microscopic inputs : $\epsilon = f(P)$ (EoS), $\eta, \zeta, \cdots$ (transport coeff.)

Why is it hard to justify in QCD?

Just after the collision, $T^{\mu\nu}$ is far from ideal

$$T^{\mu\nu}_{\substack{\text{QCD}\\ \text{rest frame}}} = \begin{pmatrix} \epsilon & & & \\ & \epsilon & & \\ & & \epsilon & \\ & & & -\epsilon \end{pmatrix} \qquad T^{\mu\nu}_{\substack{\text{ideal}\\ \text{rest frame}}} = \begin{pmatrix} \epsilon & & & \\ & \frac{\epsilon}{3} & & \\ & & \frac{\epsilon}{3} & \\ & & & \frac{\epsilon}{3} \end{pmatrix}$$

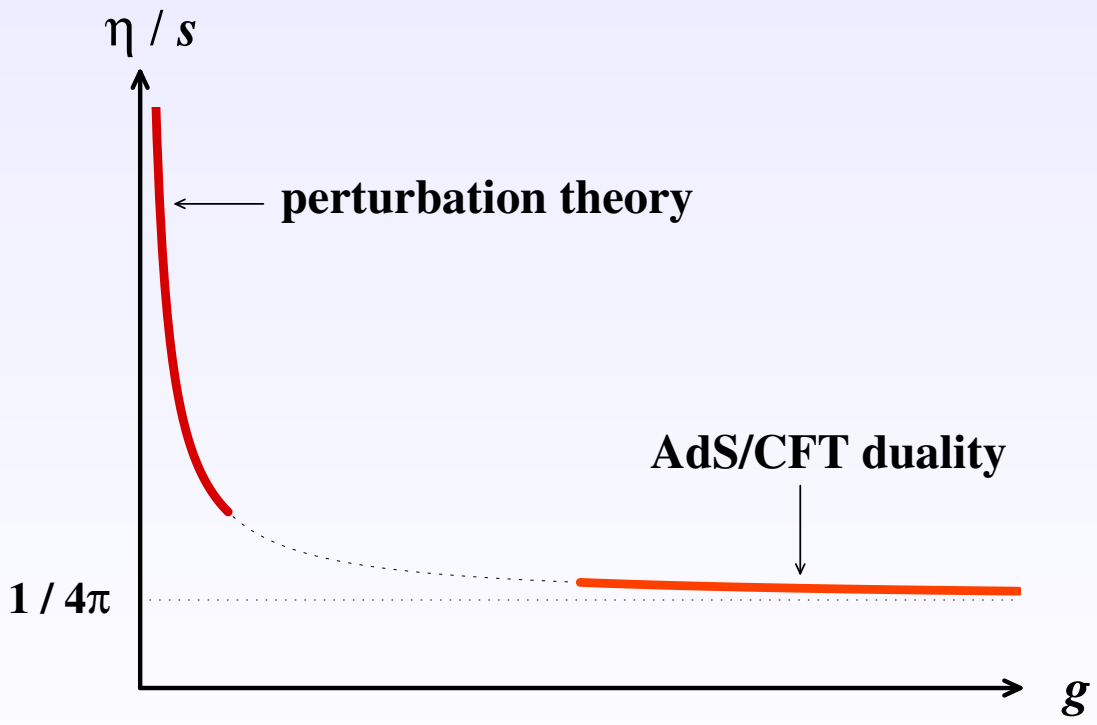
$\Rightarrow$ Very large deviation from ideal hydro at early times

- Can a QCD-based model explain how $T^{\mu\nu}$ evolves to the hydrodynamical form?
- There should be an overlap between this model and hydrodynamics, so that the final results do not depend on the time τ_0 at which one switches over to hydrodynamics

Why is it hard to justify in QCD?

Large shear viscosity at weak coupling in QCD

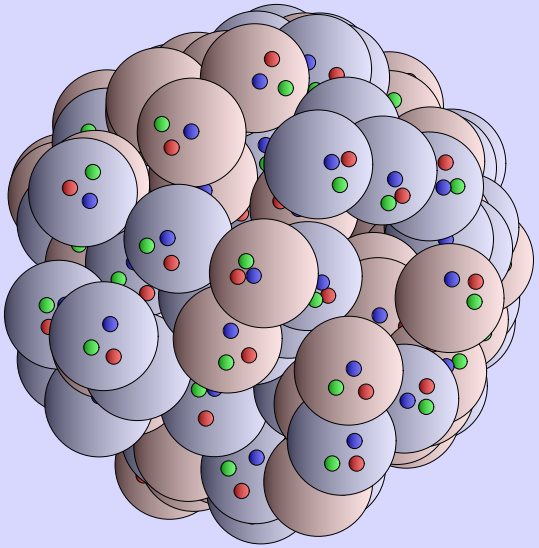
$$\frac{\eta}{s} = \frac{5.12}{g^4 \ln\left(\frac{2.42}{g}\right)}$$



Color Glass Condensate

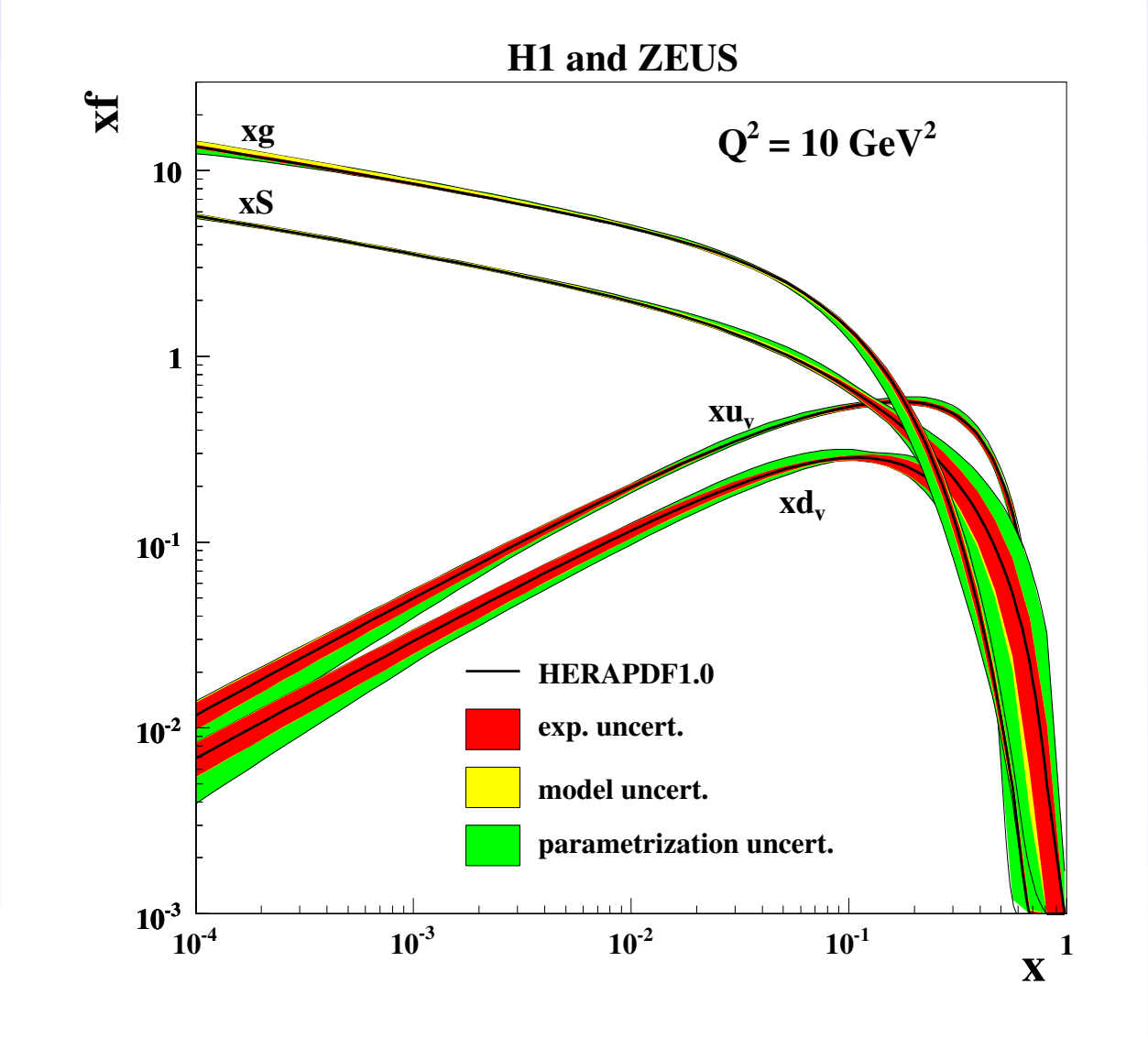
What do we need to know about nuclei?

Nucleus at rest



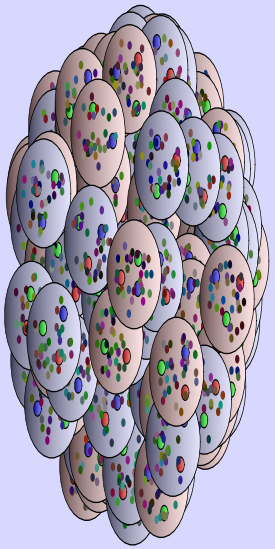
- At low energy : valence quarks

Parton distributions in a nucleon



What do we need to know about nuclei?

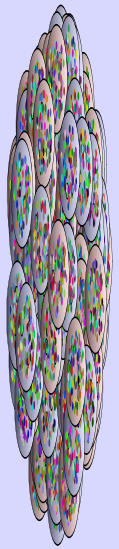
Slightly boosted nucleus



- At low energy : valence quarks
- At higher energy :
  - Lorenz contraction of longitudinal sizes
  - Time dilation ▷ slowing down of the internal dynamics
  - Gluons start becoming important

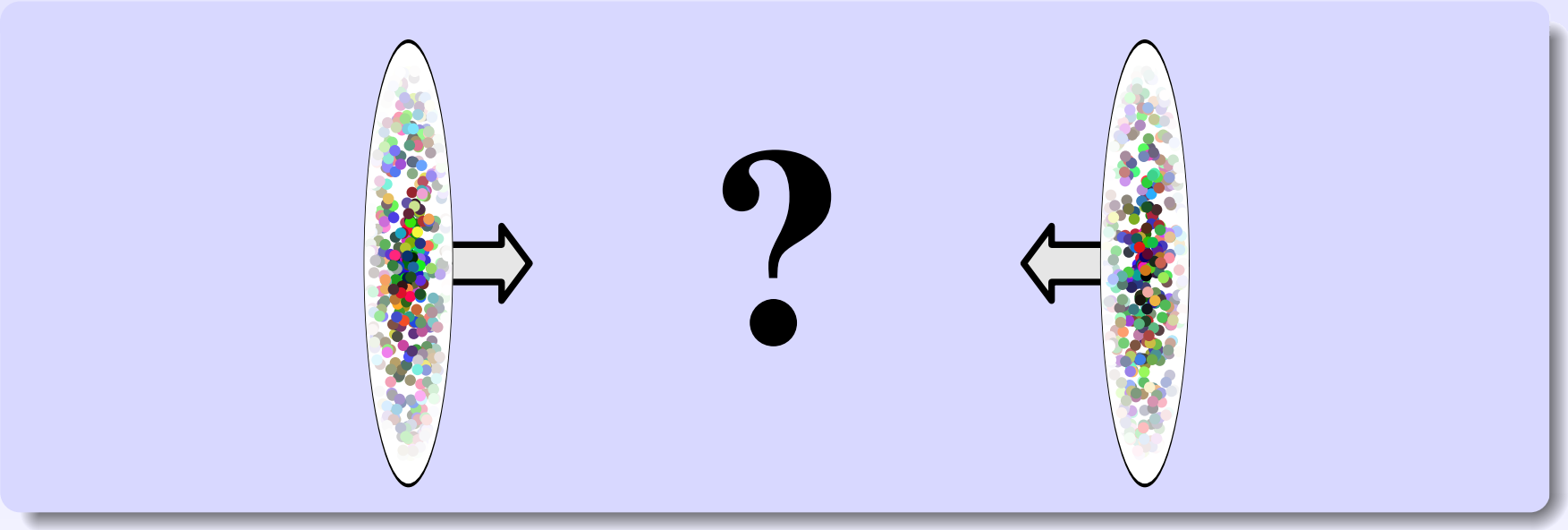
What do we need to know about nuclei?

High energy nucleus



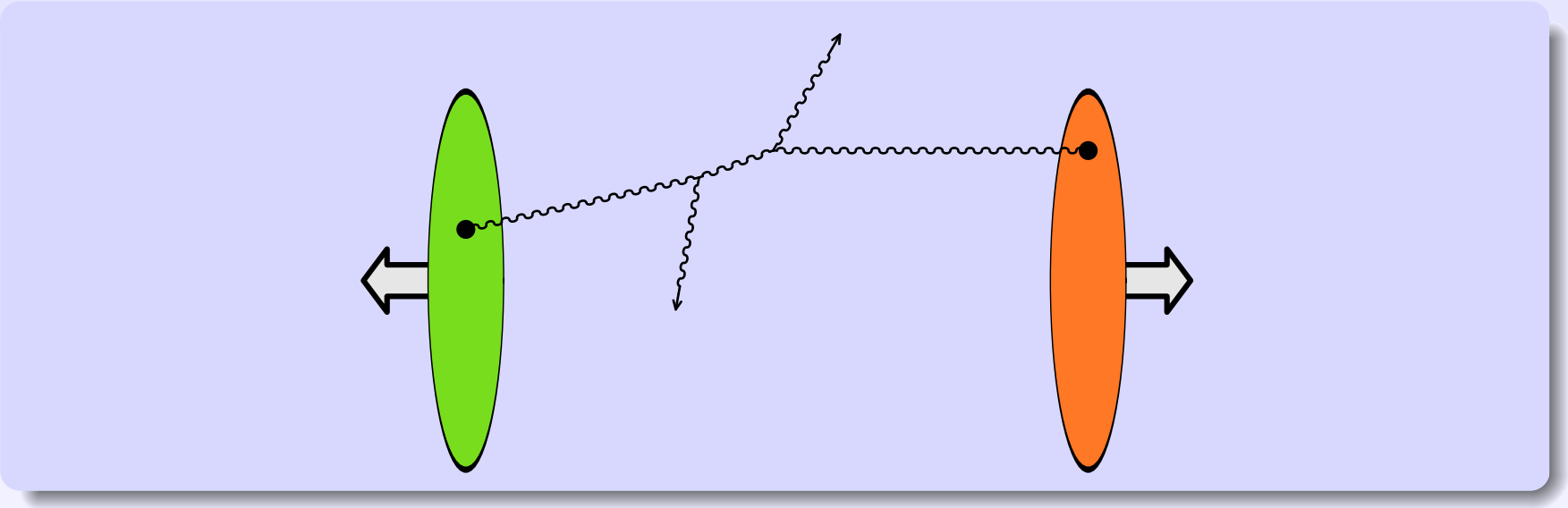
- At low energy : valence quarks
- At higher energy :
  - Lorenz contraction of longitudinal sizes
  - Time dilation ▷ slowing down of the internal dynamics
  - Gluons start becoming important
- At very high energy : gluons dominate

Multiple scatterings and gluon recombination



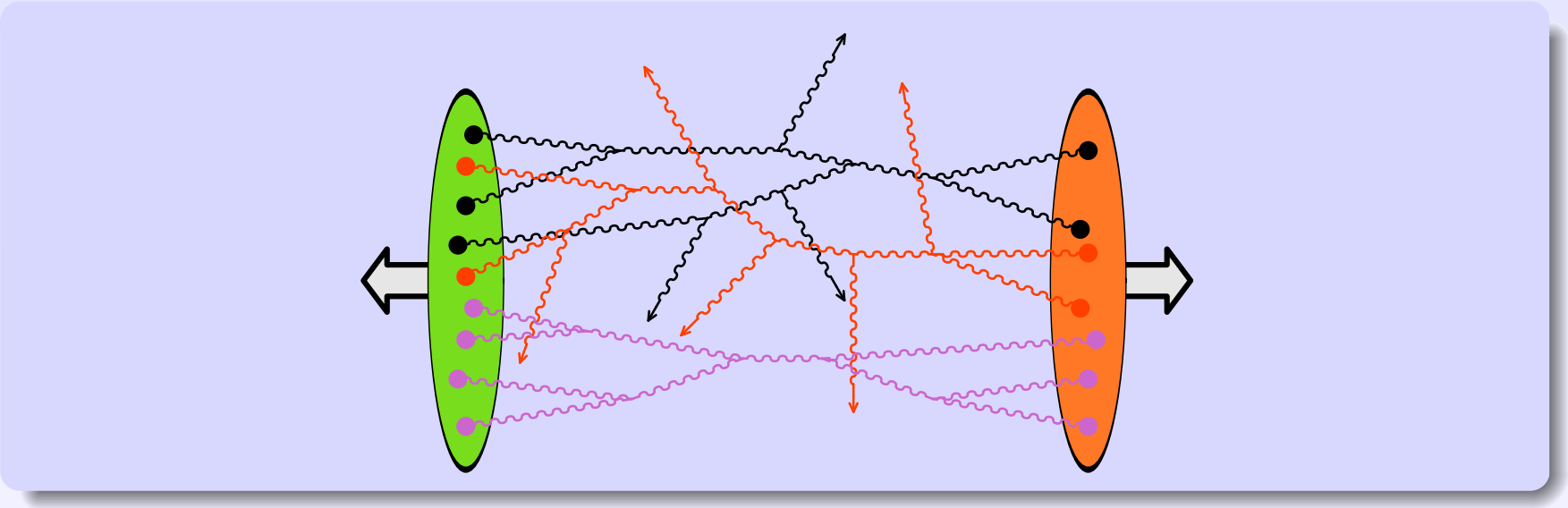
- Main difficulty: How to treat collisions involving a large number of partons?

Multiple scatterings and gluon recombination



- Dilute regime : one parton in each projectile interact
  - ▷ single parton distributions, standard perturbation theory

Multiple scatterings and gluon recombination



- Dense regime : multiparton processes become crucial
  - ▷ gluon recombinations are important (saturation)
  - ▷ multi-parton distributions
  - ▷ alternative approach : treat the gluons in the projectiles as external currents

$$\mathcal{L} = -\frac{1}{4}F^2 + A \cdot (J_1 + J_2)$$

(gluons only, field A for $k^+ < \Lambda$, classical source J for $k^+ > \Lambda$)

Color Glass Condensate

CGC = effective theory of small x gluons

- The fast partons ($k^+ > \Lambda^+$) are frozen by time dilation
  ▷ described as static color sources on the light-cone :

$$J^\mu = \delta^{\mu+}\rho(x^-, \vec{x}_\perp) \qquad (0 < x^- < 1/\Lambda^+)$$

- The color sources ρ are random, and described by a probability distribution $W_{\Lambda^+}[\rho]$

- Slow partons ($k^+ < \Lambda^+$) cannot be considered static over the time-scales of the collision process
  ▷ must be treated as standard gauge fields
  ▷ eikonal coupling to the current J^μ : $A_\mu J^\mu$

Terminology

- Weakly coupled : $g \ll 1$

- Weakly interacting : $g\mathcal{A} \ll 1 \qquad g^2 f(\mathbf{p}) \ll 1$

  $(2 \to 2) \gg (2 \to 3), (3 \to 2), \cdots$

- Strongly interacting : $g\mathcal{A} \sim 1 \qquad g^2 f(\mathbf{p}) \sim 1$

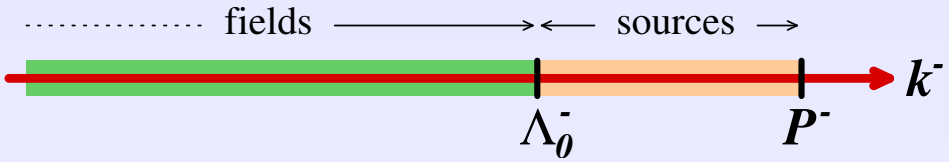
  $(2 \to 2) \sim (2 \to 3) \sim (3 \to 2) \sim \cdots$

  No well defined quasi-particles

CGC = weakly coupled, but strongly interacting effective theory

Power counting

- CGC effective theory with cutoff at the scale Λ_0 :

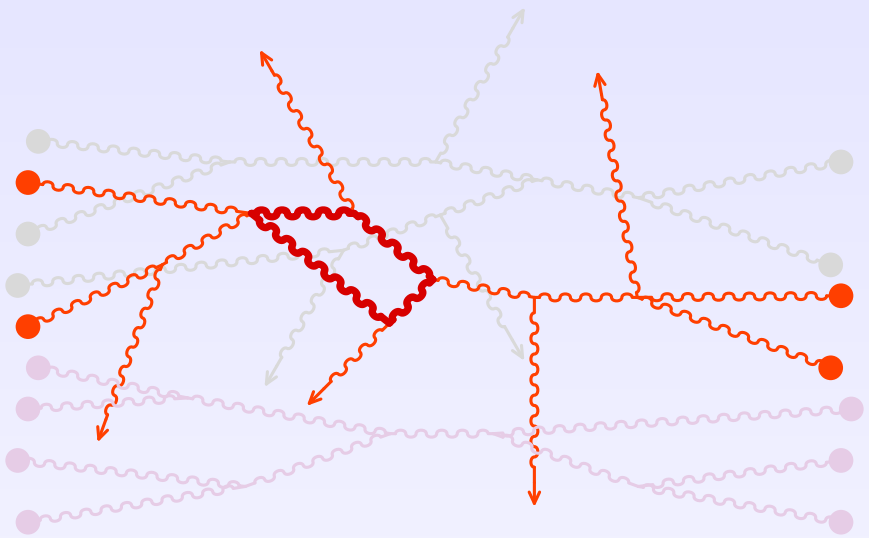


$$\mathcal{S} = \underbrace{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}_{\mathcal{S}_{\rm YM}} + \int \underbrace{(J_1^\mu + J_2^\mu)}_{\text{fast partons}} \; A_\mu$$

- Expansion in g^2 in the saturated regime:

$$T^{\mu\nu} \sim \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

Power counting



In the saturated regime: $J^\mu \sim g^{-1}$

$$g^{-2}\ g^{\#\text{ of external legs}}\ g^{2\times(\#\text{ of loops})}$$

- No dependence on the number of sources J^μ
  - ▷ infinite number of graphs at each order

Leading Order in g^2 : tree diagrams

- The Leading Order is the sum of all the tree diagrams

  Observables can be expressed in terms of classical solutions of Yang-Mills equations :

$$\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = J_1^\nu + J_2^\nu$$

- Boundary conditions for inclusive observables :

$$\lim_{x^0 \to -\infty} \mathcal{A}^\mu(x) = 0$$

Example : 00 component of the energy-momentum tensor

$$T^{00}_{\text{LO}} = \frac{1}{2}\Big[\underbrace{\mathcal{E}^2 + \mathcal{B}^2}_{\text{class. fields}}\Big]$$

Next to Leading Order in g^2 : 1-loop diagrams

Getting the NLO from tree graphs...

$$\mathcal{O}_{\rm NLO} = \left[\frac{1}{2} \int_{u,v} \Gamma_2(u,v)\, \mathbb{T}_u \mathbb{T}_v + \int_u \alpha(u)\, \mathbb{T}_u \right] \mathcal{O}_{\rm LO}$$

- $\mathbb{T}$ is the generator of the shifts of the initial value of the field :

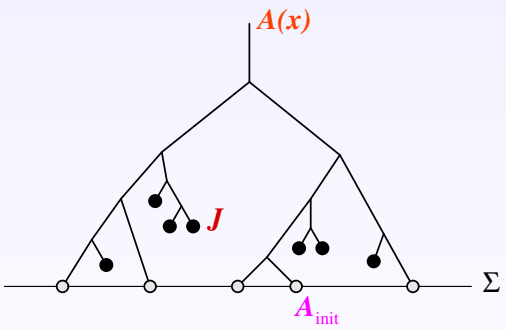
$$\mathbb{T}_u \sim \frac{\partial}{\partial \mathcal{A}_{\rm init}}$$

$$\exp\left[\int_u \alpha_u\, \mathbb{T}_u \right] \mathcal{O}\Big[\overbrace{\mathcal{A}_\tau(\ \underbrace{\mathcal{A}_{\rm init}}_{\text{init. value}}\ )}^{\text{class. field at } \tau} \Big] = \mathcal{O}\Big[\mathcal{A}_\tau(\ \underbrace{\mathcal{A}_{\rm init} + \alpha}_{\text{shifted init. value}}\ ) \Big]$$

Shift operator $\mathbb{T}$ – Definition

Equations of motion for a field $\mathcal{A}$ and a small perturbation α

$$\begin{aligned} \Box\mathcal{A} + V'(\mathcal{A}) &= J \\ [\Box + V''(\mathcal{A})]\ \alpha &= 0 \end{aligned}$$



- Getting the perturbation by shifting the initial condition of $\mathcal{A}$ at one point :

$$\alpha(x) = \int_u \alpha_u\, \mathbb{T}_u\ \mathcal{A}(x)$$

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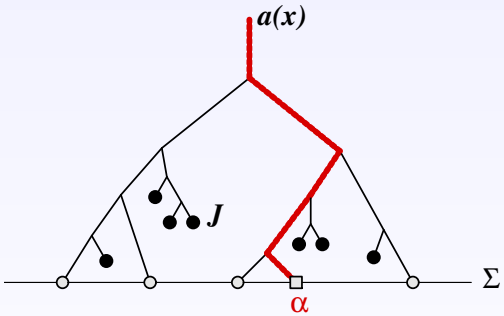

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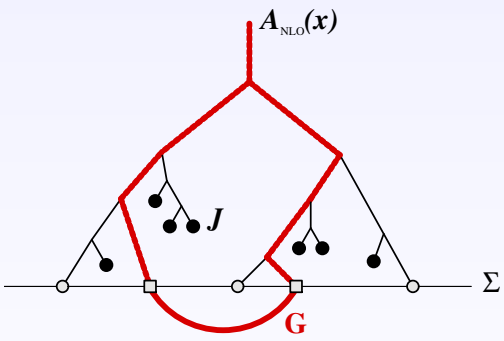

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- A loop is obtained by shifting the initial condition of $\mathcal{A}$ at two points

Formulation of QM in the classical phase-space

- To make the connection with classical mechanics, it is useful to use Moyal's formulation of QM in terms of classical variables

- Dual formulation of QM :

Density	$\hat{\rho}$		$W(Q,P)$
Evolution	$\partial_t \hat{\rho} + i[\widehat{H}, \hat{\rho}] = 0$	$\xleftrightarrow[\text{transform}]{\text{Weyl-Wigner}}$	$\partial_t W + \{\{W, H\}\} = 0$
States	coherent state		Gaussian of width $\hbar$

Moyal bracket and expansion in $\hbar$

- $\{\{A, B\}\}$ is the Wigner transform of the commutator $[\widehat{A}, \widehat{B}]$

$$\{\{A, B\}\} = \frac{2}{\hbar} A(Q, P) \ \sin\left(\frac{\hbar}{2}(\overleftarrow{\nabla}_Q \overrightarrow{\nabla}_P - \overleftarrow{\nabla}_P \overrightarrow{\nabla}_Q)\right) B(Q, P)$$

- Quantum deformation of the Poisson bracket :

$$\{\{A, B\}\} = \underbrace{\{A, B\}}_{\text{Poisson bracket}} + \mathcal{O}(\hbar^2)$$

- LO : $\mathcal{O}(\hbar^0)$
  - Moyal equation $\Longrightarrow$ Liouville equation
  - Initial state $\Longrightarrow$ δ-function in (Q, P)
- NLO : $\mathcal{O}(\hbar^1)$
  - Moyal equation $\Longrightarrow$ Liouville equation
  - Initial state $\Longrightarrow$ Gaussian of width $\hbar$

Initial state logarithms

- In the CGC, upper cutoff on the loop momentum : $k^{\pm} < \Lambda$, to avoid double counting with the sources $J^{\gamma}_{1,2}$
  $\triangleright$ logarithms of the cutoff

Central result for factorization at Leading Log

$$\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\,\mathbb{T}_u\mathbb{T}_v + \int_u \alpha(u)\,\mathbb{T}_u =$$

$$= \log\left(\Lambda^+\right)\,\mathcal{H}_1 + \log\left(\Lambda^-\right)\,\mathcal{H}_2 + \text{terms w/o logs}$$

$\mathcal{H}_{1,2} =$ JIMWLK Hamiltonians of the two nuclei

- No mixing between the logs of the two nuclei
- Since the LO $\leftrightarrow$ NLO relationship is the same for all inclusive observables, these logs have a universal structure

Factorization of the logarithms

Inclusive observables at Leading Log accuracy

$$\langle \mathcal{O} \rangle_{\text{Leading Log}} = \int \left[D\rho_1 \, D\rho_2 \right] W_1\left[\rho_1\right] \, W_2\left[\rho_2\right] \, \underbrace{\mathcal{O}_{\text{LO}}[\rho_1, \rho_2]}_{\text{fixed } \rho_{1,2}}$$

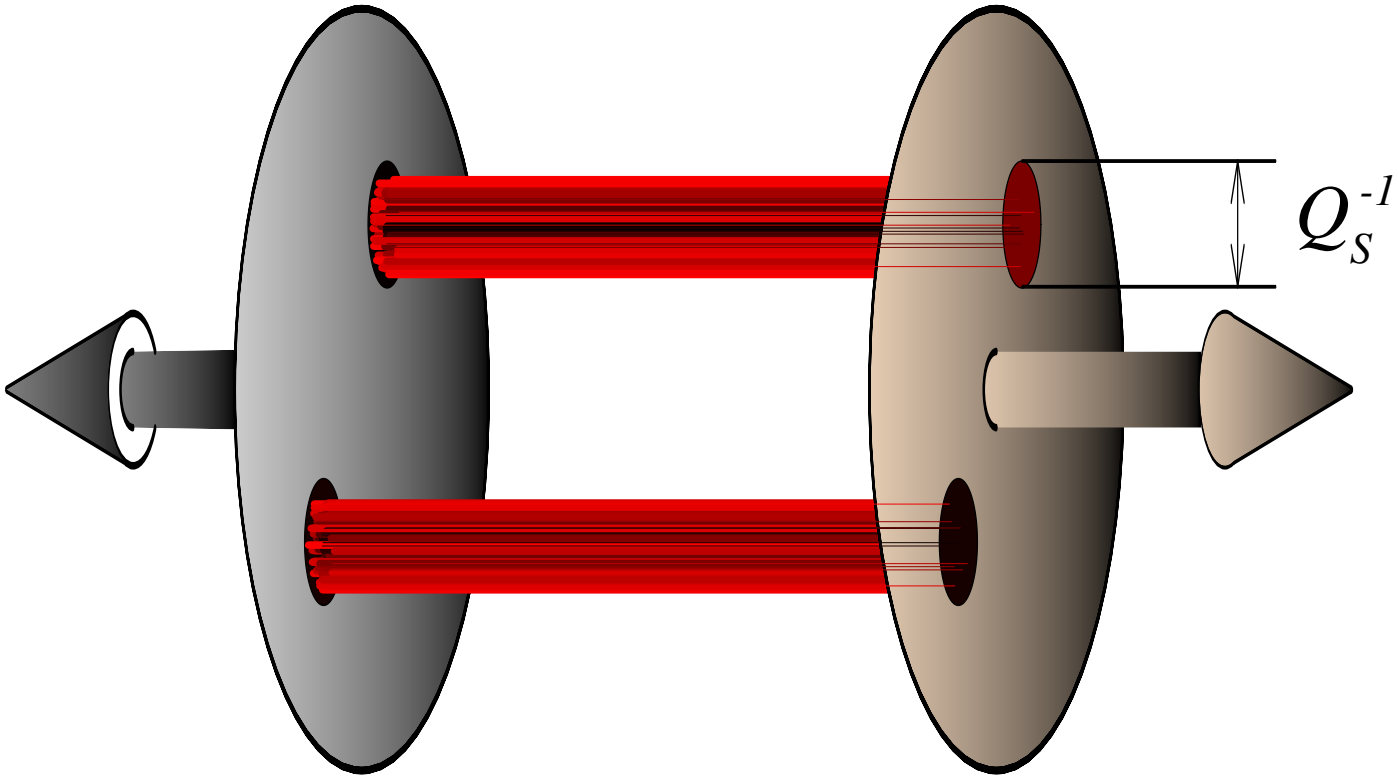
- Logs absorbed into the scale evolution of $W_{1,2}$

$$\Lambda \frac{\partial W}{\partial \Lambda} = \mathcal{H} W \qquad \text{(JIMWLK equation)}$$

- Universality : the same W's for all inclusive observables

Initial Stages of Heavy Ion Collisions

Energy momentum tensor of the initial classical field



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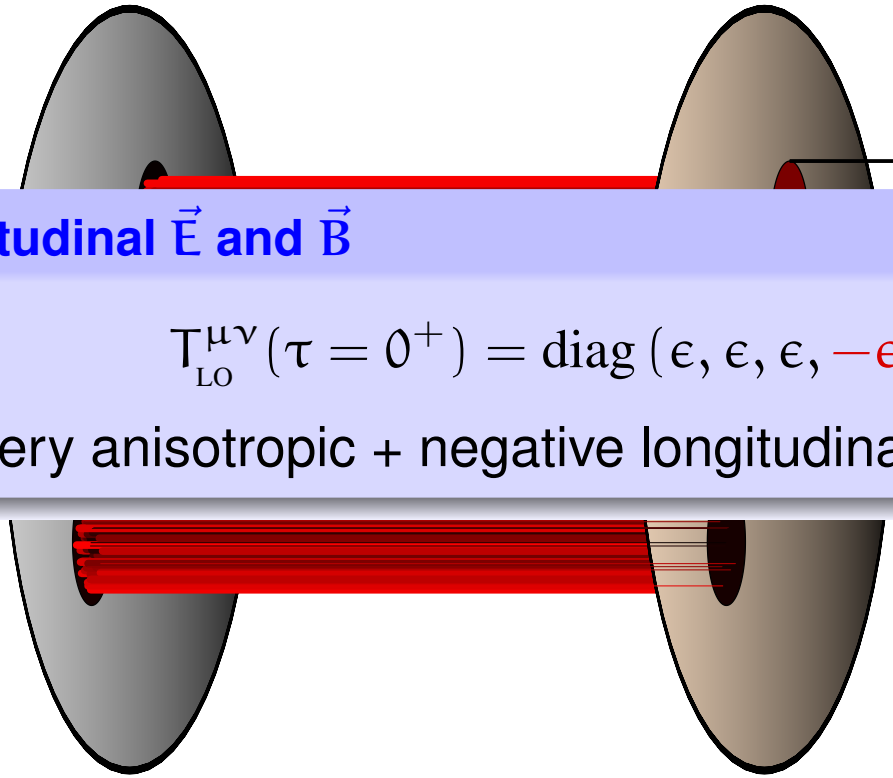
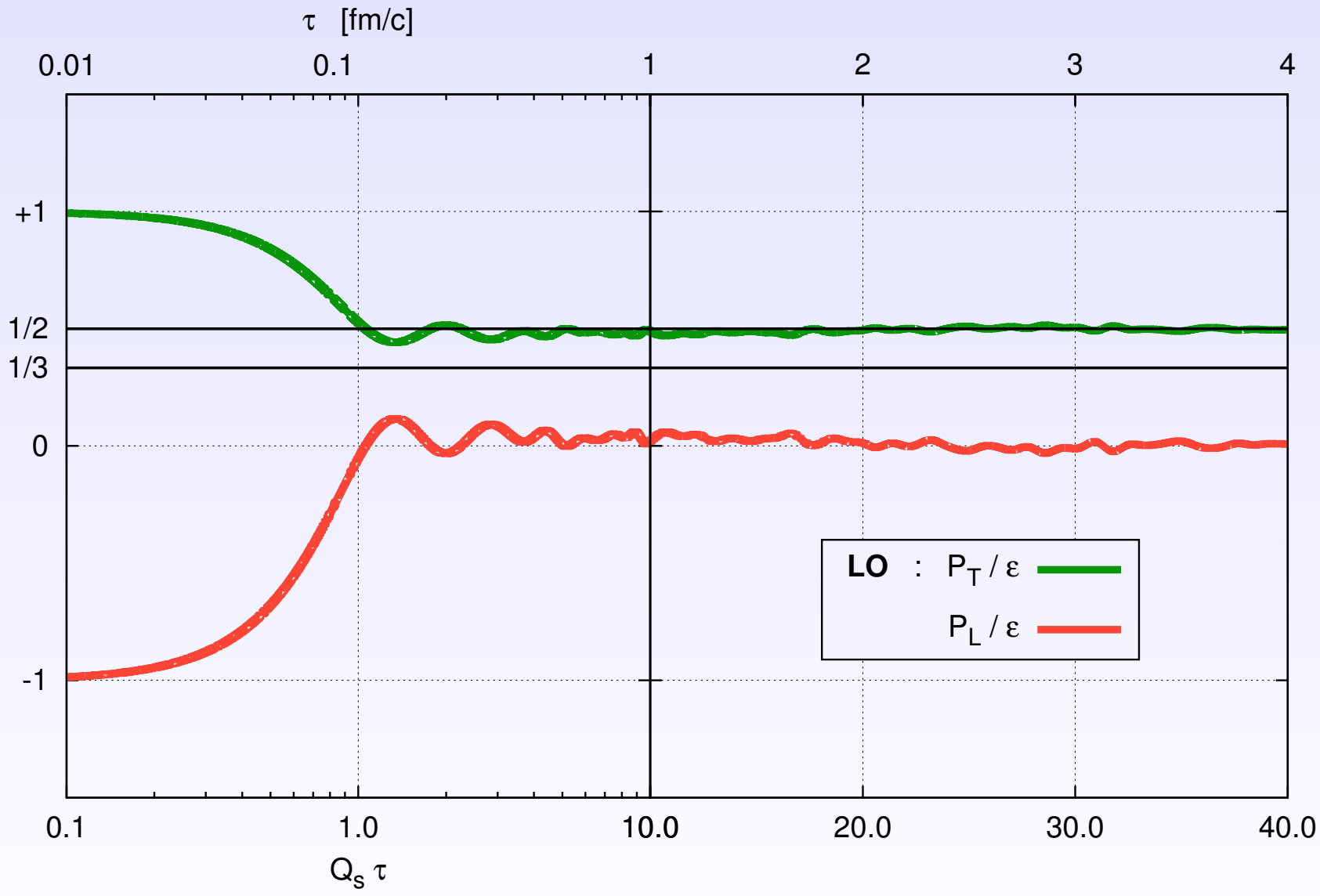


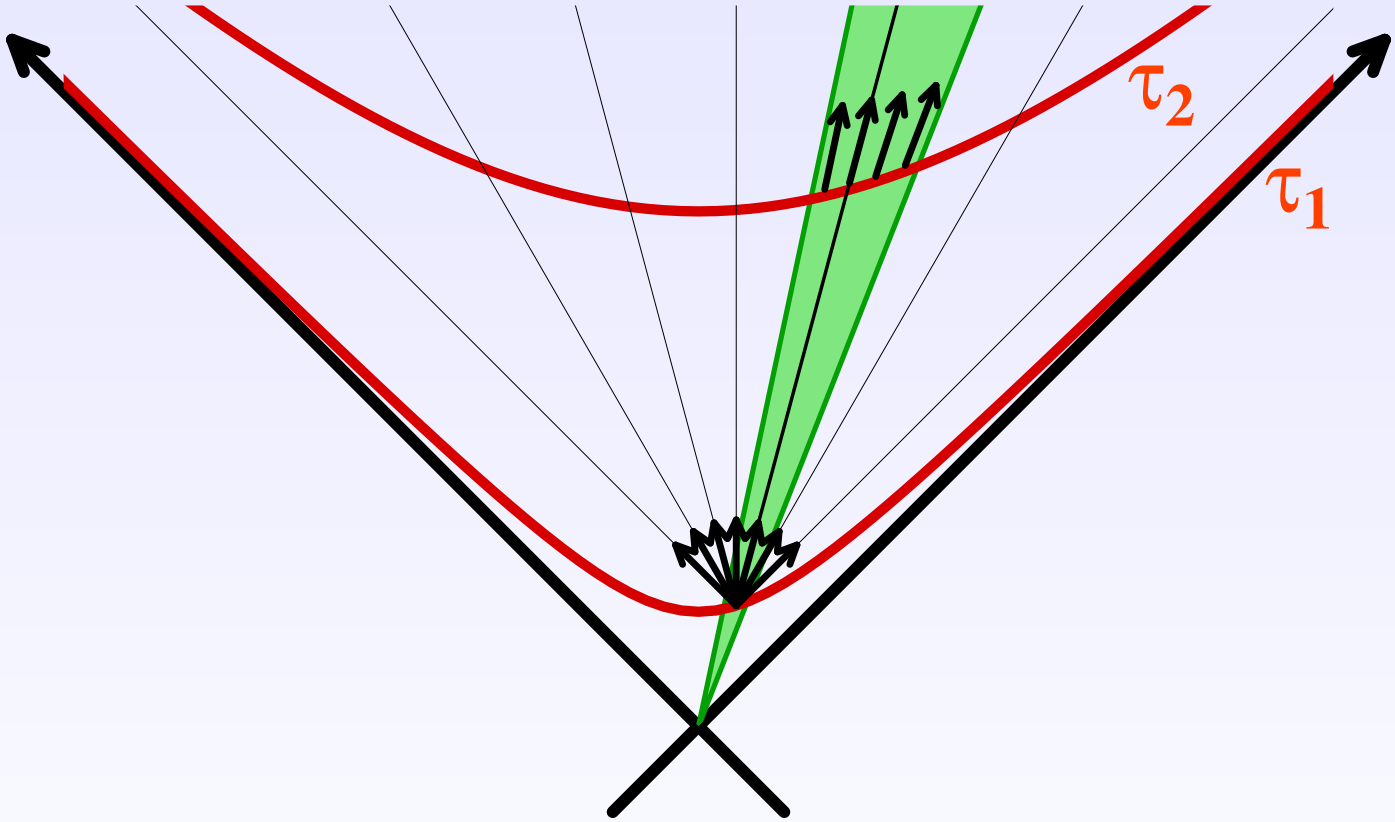
$T^{\mu\nu}$ for longitudinal $\vec{E}$ and $\vec{B}$

$$T^{\mu\nu}_{\rm LO}(\tau = 0^+) = \mathrm{diag}\,(\epsilon, \epsilon, \epsilon, -\epsilon)$$

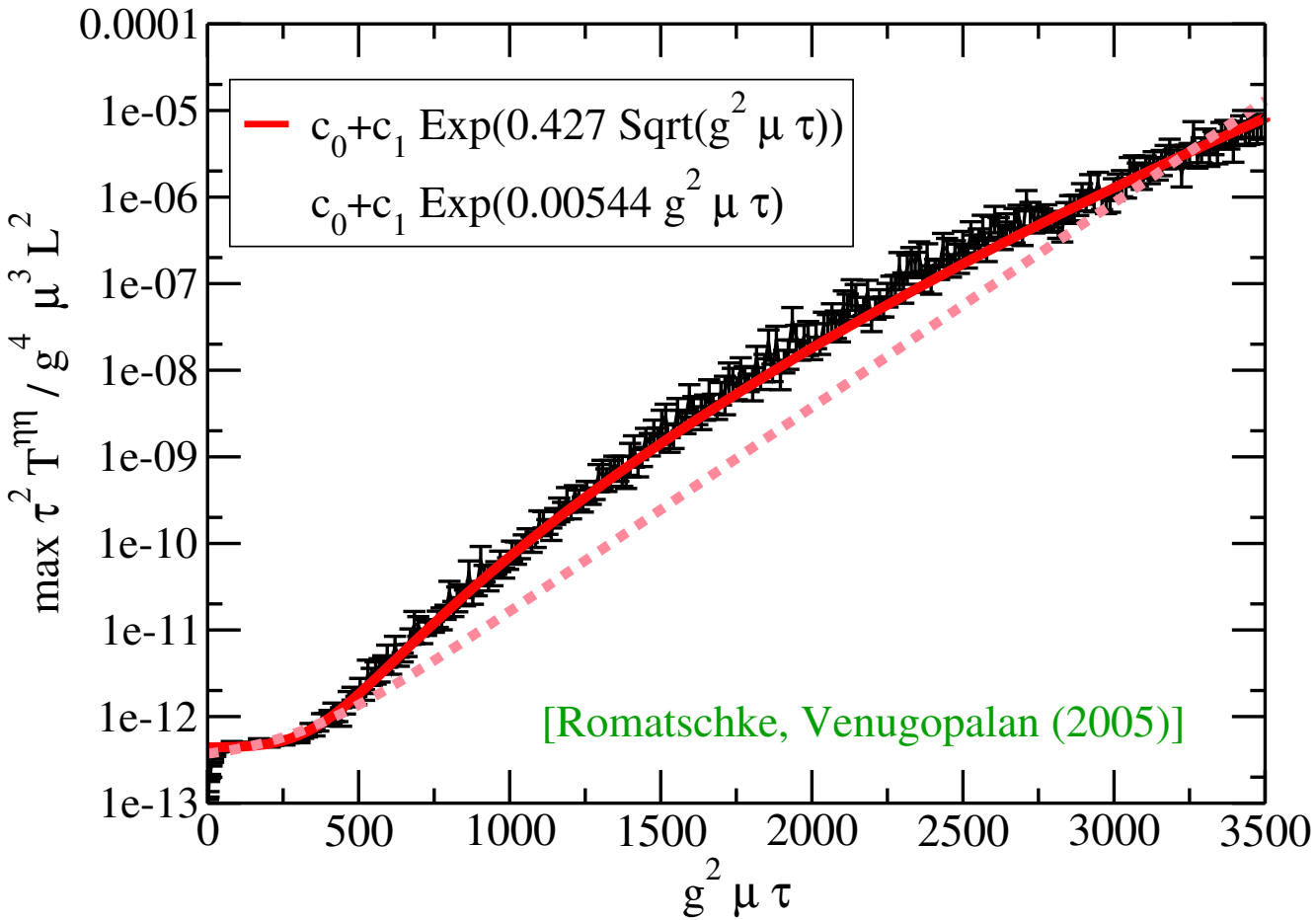
▷ very anisotropic + negative longitudinal pressure


τ [fm/c]
0.01
0.1
1
2
3
4
+1
1/2
1/3
0
-1
LO : P_T / ε
P_L / ε
0.1
1.0
10.0
20.0
30.0
40.0
Q_s τ

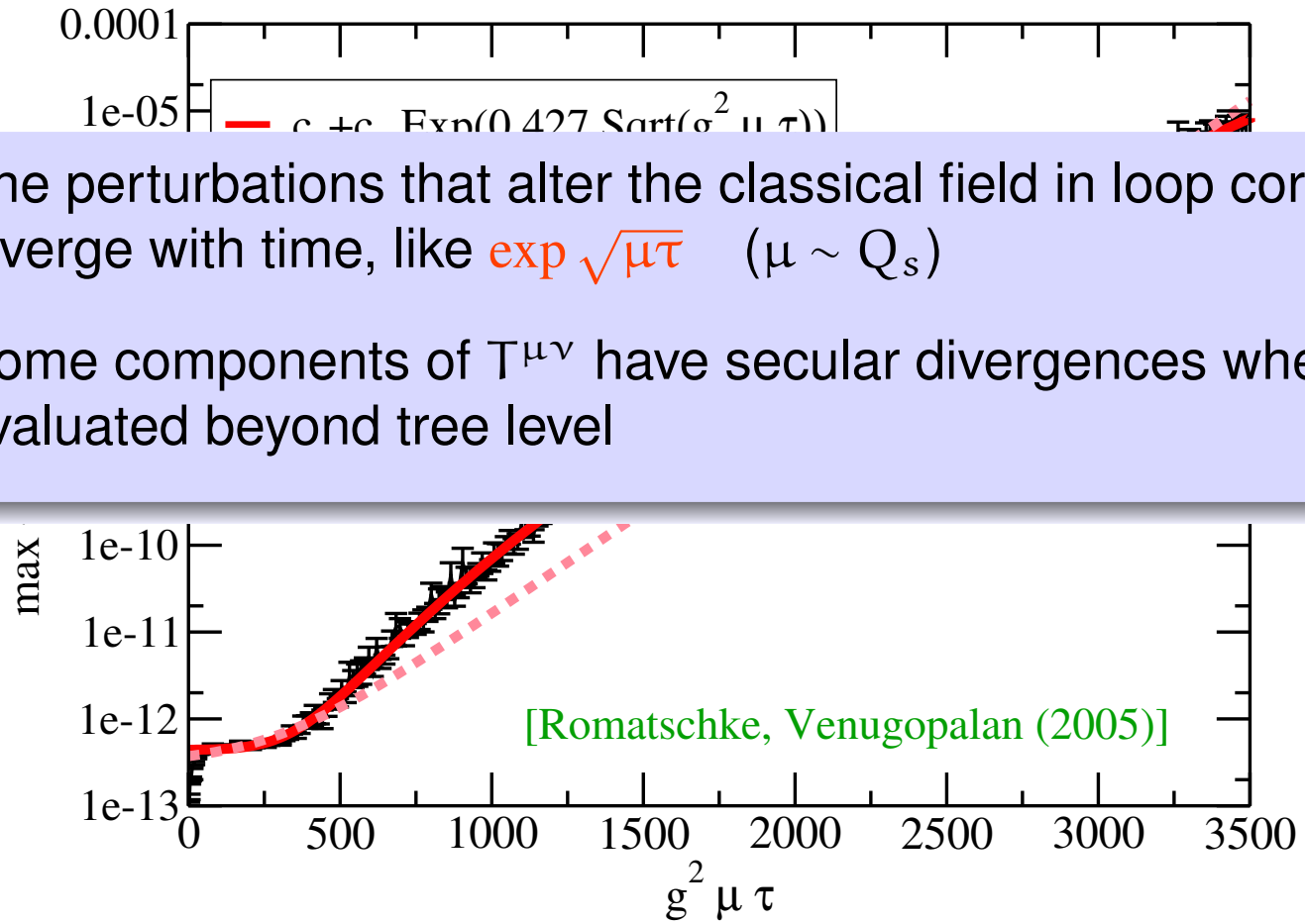
Competition between Expansion and Isotropization



Weibel instabilities for small perturbations



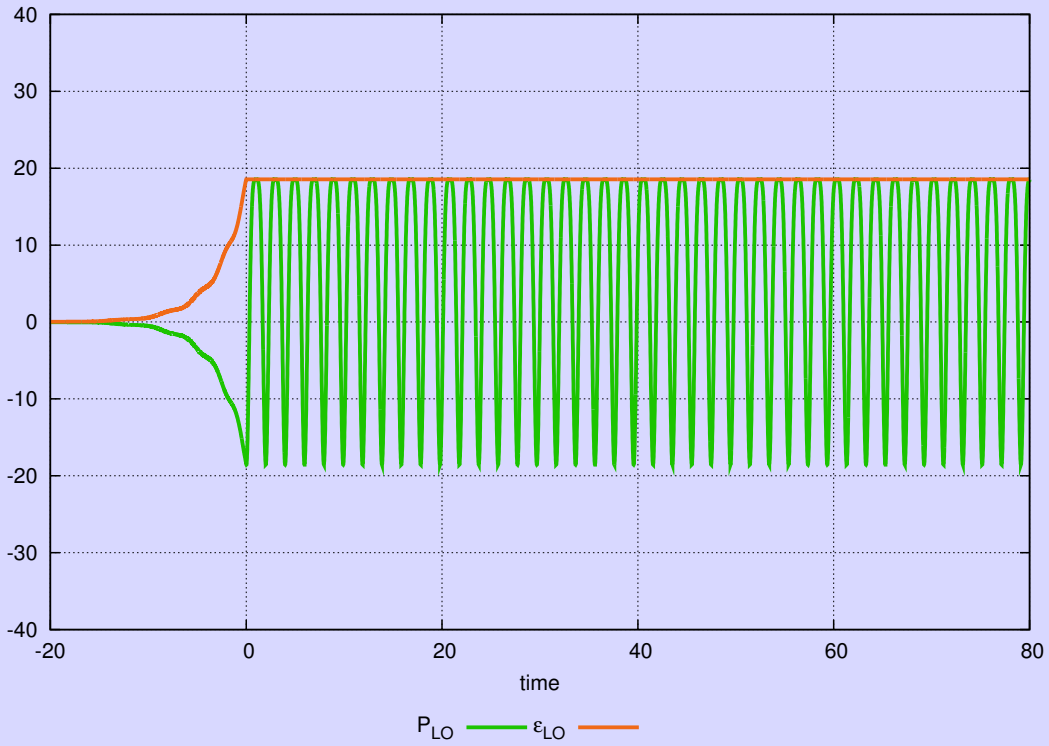
Weibel instabilities for small perturbations



- The perturbations that alter the classical field in loop corrections diverge with time, like $\exp\sqrt{\mu\tau}$ $\quad (\mu \sim Q_s)$
- Some components of $T^{\mu\nu}$ have secular divergences when evaluated beyond tree level

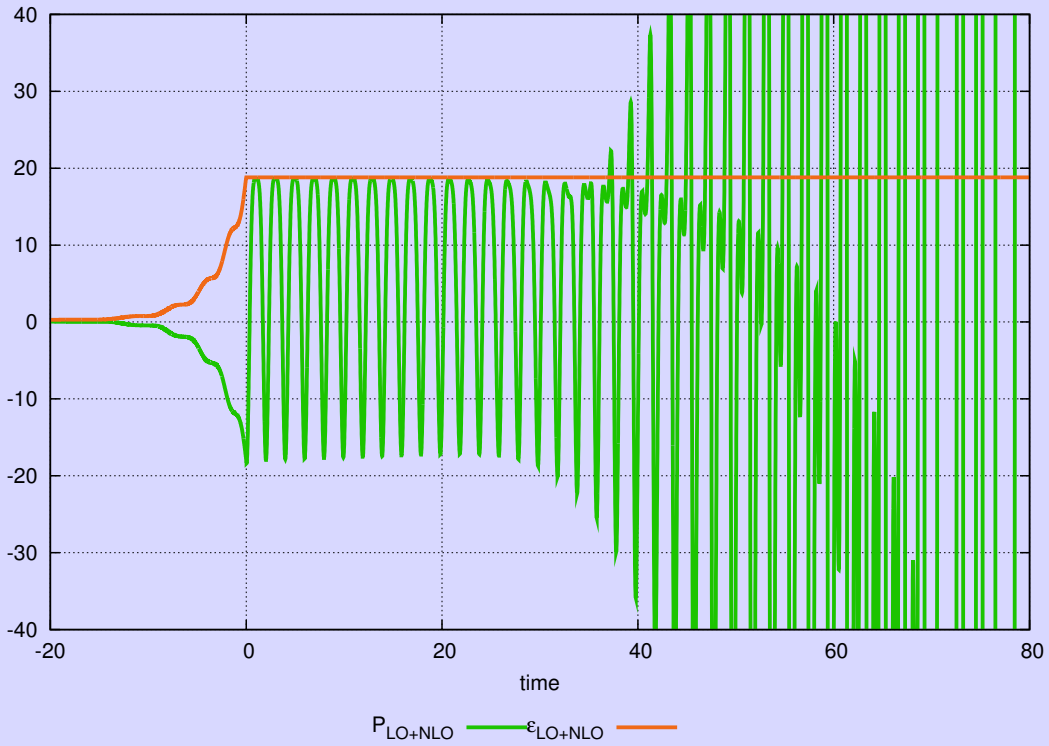
Example of pathologies in fixed order calculations (scalar theory)

LO



Example of pathologies in fixed order calculations (scalar theory)

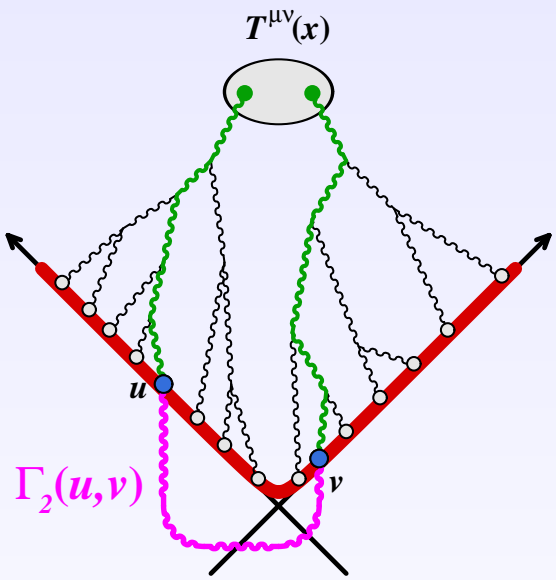
LO + NLO



- Small correction to the energy density (protected by energy conservation)
- Secular divergence in the pressure

Improved power counting and resummation

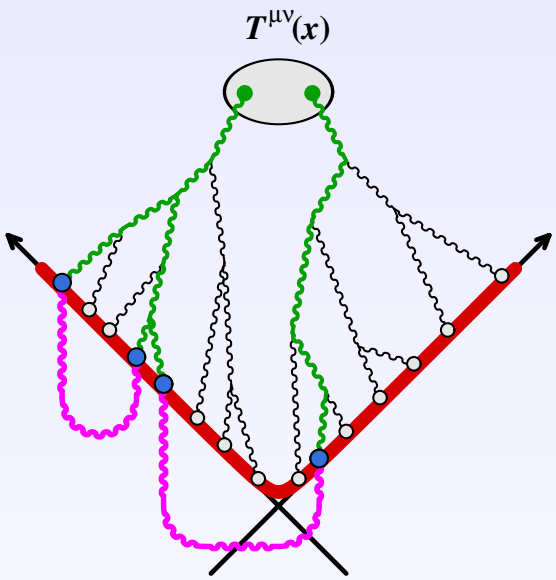
$$\text{Loop} \sim g^2 \quad , \quad \mathbb{T} \sim e^{\sqrt{\mu\tau}}$$



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$

Improved power counting and resummation

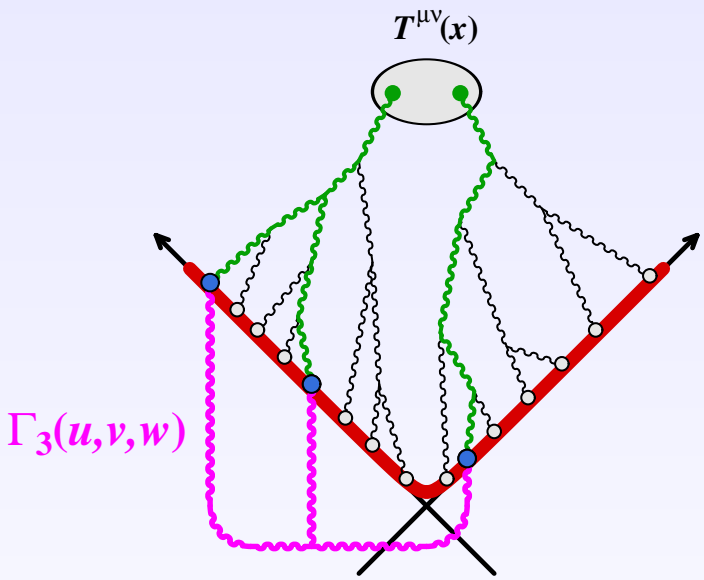
$$\text{Loop} \sim g^2 \quad , \quad \mathbb{T} \sim e^{\sqrt{\mu\tau}}$$



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$

Improved power counting and resummation

$$\text{Loop} \sim g^2 \quad , \quad \mathbb{T} \sim e^{\sqrt{\mu\tau}}$$



- 1 loop : $(ge^{\sqrt{\mu\tau}})^2$
- 2 disconnected loops : $(ge^{\sqrt{\mu\tau}})^4$
- 2 entangled loops : $g(ge^{\sqrt{\mu\tau}})^3$ $\triangleright$ subleading

Leading terms

- All disconnected loops to all orders
  - $\triangleright$ exponentiation of the 1-loop result

Resummation of the leading secular terms

$$
\begin{aligned}
T^{\mu\nu}_{\text{resummed}} &= \exp\left[\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\,\mathbb{T}_u\,\mathbb{T}_v\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}] \\
&= \underbrace{T^{\mu\nu}_{\text{LO}} + T^{\mu\nu}_{\text{NLO}}}_{\text{in full}} + \underbrace{T^{\mu\nu}_{\text{NNLO}} + \cdots}_{\text{partially}}
\end{aligned}
$$

- The exponentiation of the 1-loop result collects all the terms with the worst time behavior

$$e^{\frac{\alpha}{2}\partial_x^2}\, f(x) = \int_{-\infty}^{+\infty} dz\, \frac{e^{-z^2/2\alpha}}{\sqrt{2\pi\alpha}}\, f(x+z)$$

Resummation of the leading secular terms

$$
\begin{aligned}
T^{\mu\nu}_{\text{resummed}} &= \exp\left[\frac{1}{2}\int_{u,v} \Gamma_2(u,v)\mathbb{T}_u\mathbb{T}_v\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}] \\
&= \int [Da]\, \exp\left[-\frac{1}{2}\int_{u,v} a(u)\Gamma_2^{-1}(u,v)a(v)\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}} + a]
\end{aligned}
$$

- The exponentiation of the 1-loop result collects all the terms with the worst time behavior
- Equivalent to Gaussian fluctuations of the initial field + classical time evolution
- At $Q_s\tau_0 \ll 1$: $\mathcal{A}_{\text{init}} \sim Q_s/g$, $a \sim Q_s$

Note : Classical field + Fluctuations = Coherent state

- This Gaussian distribution of initial fields is the Wigner distribution of a **coherent state** $\big|\mathcal{A}\big\rangle$

  Coherent states are the "most classical quantum states"

  Their Wigner distribution has the minimal support permitted by the uncertainty principle ($\mathcal{O}(\hbar)$ for each mode)

- $\big|\mathcal{A}\big\rangle$ is not an eigenstate of the full Hamiltonian
  $\triangleright$ decoherence via interactions

What needs to be done?

Main steps

1. Determine the 2-point function $\Gamma_2(u,v)$ that defines the Gaussian fluctuations, for the initial time τ_0 of interest

   Note : this is an initial value problem, whose outcome is uniquely determined by the state of the system at $x^0 = -\infty$, and depends on the history of the system from $x^0 = -\infty$ to $\tau = \tau_0$

   Problem solvable only if the fluctuations are weak, $a^\mu \ll Q_s/g$

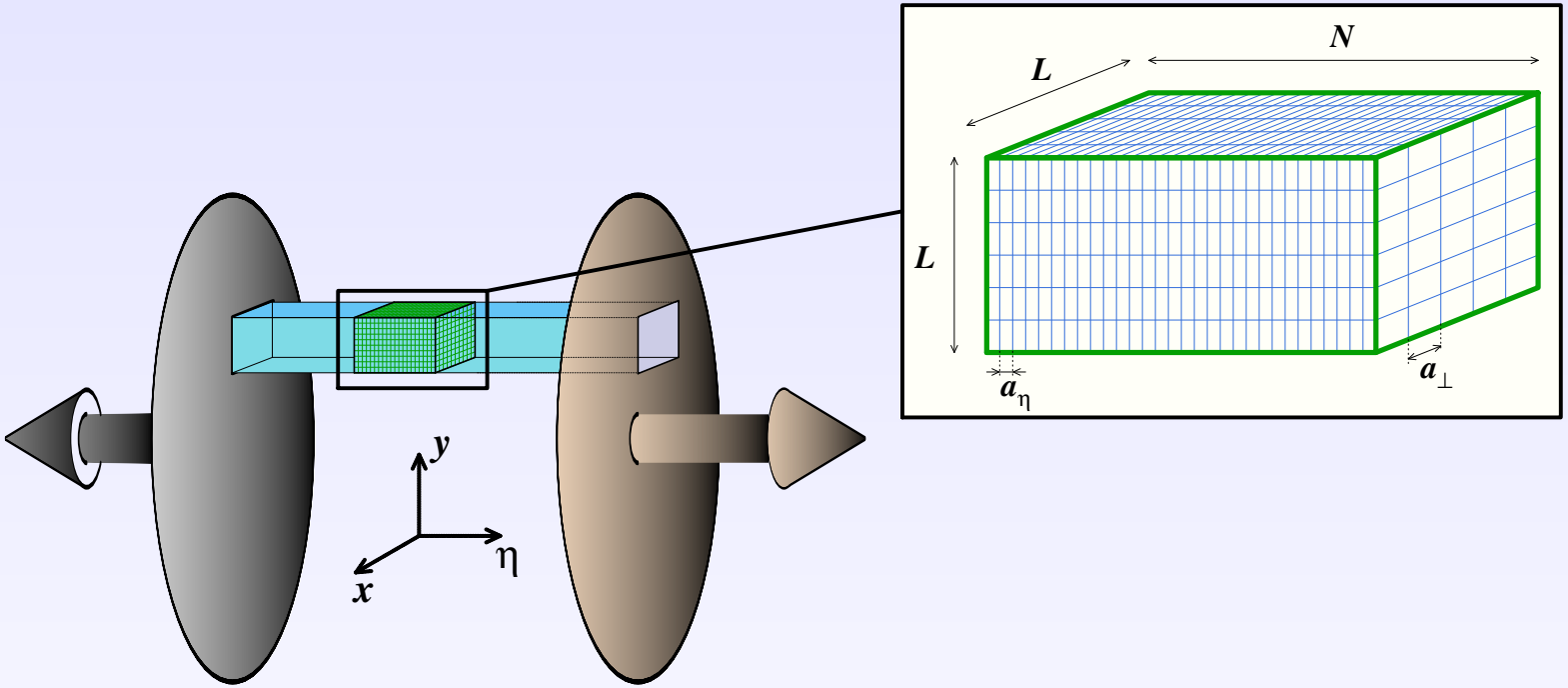
   $Q_s \tau_0 \ll 1$ necessary for the fluctuations to be Gaussian

2. Solve the classical Yang-Mills equations from τ_0 to τ_f

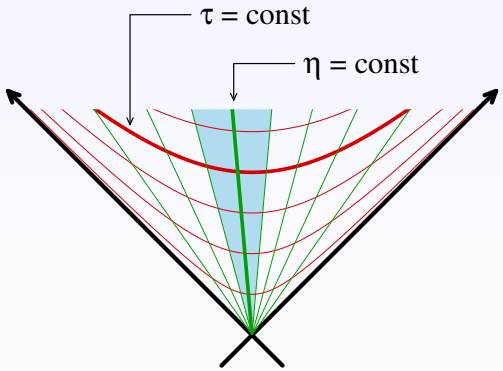
   Note : the problem as a whole is boost invariant, but individual field configurations are not $\implies$ 3+1 dimensions necessary

3. Do a Monte-Carlo sampling of the fluctuating initial conditions

Discretization of the expanding volume



- Comoving coordinates : $\tau, \eta, \mathbf{x}_\perp$
- Only a sub-volume is simulated + periodic boundary conditions
- $L^2 \times N$ lattice

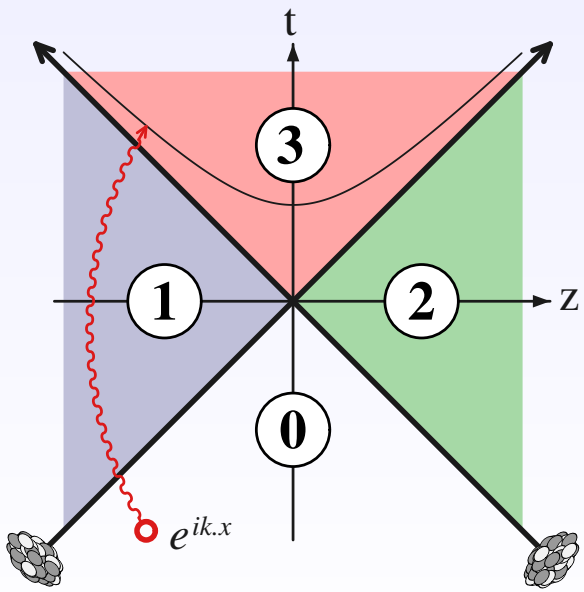


Gaussian spectrum of fluctuations

Expression of the variance (from 1-loop considerations)

$$\Gamma_2(u,v) = \int_{\text{modes } k} a_k(u) a_k^*(v)$$

$$\left[\mathcal{D}_\rho \mathcal{D}^\rho \delta^\nu_\mu - \mathcal{D}_\mu \mathcal{D}^\nu + ig\,\mathcal{F}_\mu{}^\nu\right] a^\mu_k = 0 \quad , \quad \lim_{x^0 \to -\infty} a_k(x) \sim e^{ik\cdot x}$$



0. $\mathcal{A}^\mu = 0$, trivial

1,2. $\mathcal{A}^\mu =$ pure gauge, analytical solution

3. $\mathcal{A}^\mu$ non-perturbative
$\Rightarrow$ expansion in $Q_s\tau$

- We need the fluctuations in Fock-Schwinger gauge $x^+ a^- + x^- a^+ = 0$
- Delicate light-cone crossings, since $\mathcal{F}^{\mu\nu} = \infty$ there

Mode functions for given quantum numbers : $\nu, \mathbf{k}_\perp, \lambda, c$

$$a^i = \beta^{+i} + \beta^{-i} \qquad a^\eta = \mathcal{D}^i\left(\frac{\beta^{+i}}{2+i\nu} - \frac{\beta^{-i}}{2-i\nu}\right)$$

$$e^i = -i\nu\left(\beta^{+i} - \beta^{-i}\right) \qquad e^\eta = -\mathcal{D}^i\left(\beta^{+i} - \beta^{-i}\right)$$

$$\beta^{+i} \equiv e^{\frac{\pi\nu}{2}}\Gamma(-i\nu)e^{i\nu\eta}\,\mathcal{U}_1^\dagger(\mathbf{x}_\perp)\int_{\mathbf{p}_\perp} e^{i\mathbf{p}_\perp\cdot\mathbf{x}_\perp}\,\widetilde{\mathcal{U}}_1(\mathbf{p}_\perp+\mathbf{k}_\perp)\left(\frac{p_\perp^2\tau}{2k_\perp}\right)^{i\nu}\left(\delta^{ij}-2\frac{p_\perp^i p_\perp^j}{p_\perp^2}\right)\epsilon_\lambda^j$$

$$\beta^{-i} \equiv e^{-\frac{\pi\nu}{2}}\Gamma(i\nu)e^{i\nu\eta}\,\mathcal{U}_2^\dagger(\mathbf{x}_\perp)\int_{\mathbf{p}_\perp} e^{i\mathbf{p}_\perp\cdot\mathbf{x}_\perp}\,\widetilde{\mathcal{U}}_2(\mathbf{p}_\perp+\mathbf{k}_\perp)\left(\frac{p_\perp^2\tau}{2k_\perp}\right)^{-i\nu}\left(\delta^{ij}-2\frac{p_\perp^i p_\perp^j}{p_\perp^2}\right)\epsilon_\lambda^j$$

- Linearized EOM and Gauss' law satisfied up to terms of order $(Q_s\tau)^2$
- Fock-Schwinger gauge condition ($a^\tau = e^\tau = 0$)
- Evolved from plane waves in the remote past

Computational cost

Initial Conditions

- Naive :

$$N \log(N) \times L^4 \log(L) \times N_{\text{confs}}$$

- Better algorithm :

$$N \log(N) \times L^4 \times (\log(L) + N_{\text{confs}})$$

Time evolution

$$N \times L^2 \times N_{\text{confs}} \times N_{\text{time steps}}$$

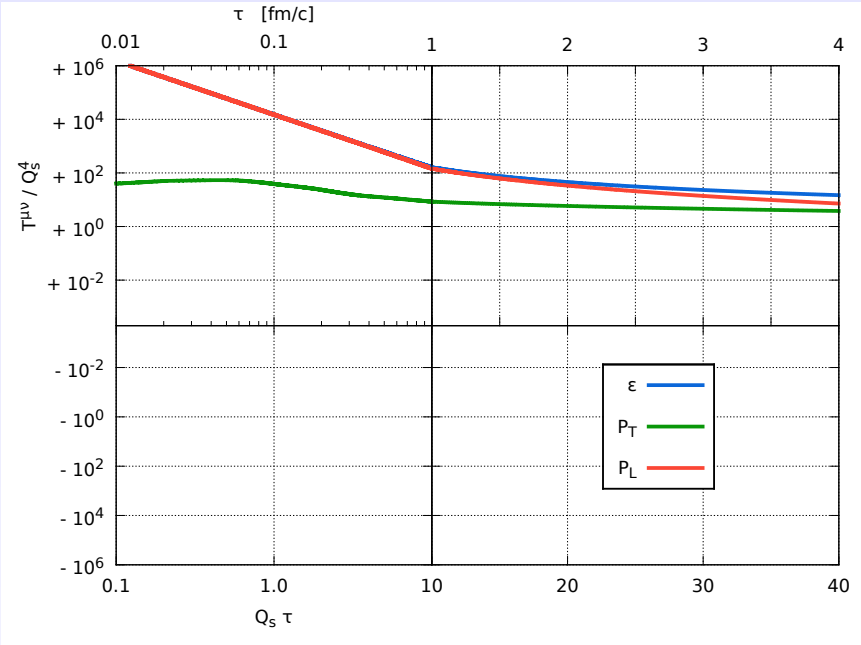
Useful statistics (at fixed L, N)

$$\sqrt{N_{\text{confs}}} \sim \frac{g^2}{(a_\perp a_\eta)^2}$$

Ultraviolet subtractions

- Fixed spacing in η
  $\iff \Lambda_z \sim \tau^{-1}$

  Bare ϵ and P_L diverge as τ^{-2} when $\tau \to 0^+$



Ultraviolet subtractions

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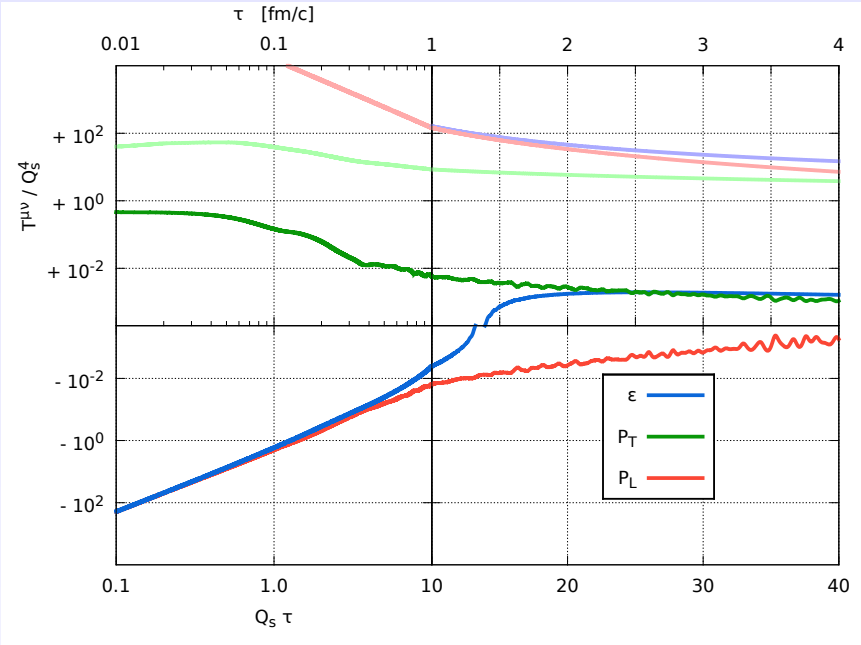
  Bare ϵ and P_L diverge as τ^{-2} when $\tau \to 0^+$

- Zero point energy $\sim \Lambda_\perp^2 \Lambda_z^2$:

  Subtracted by redoing the calculation with the sources turned off



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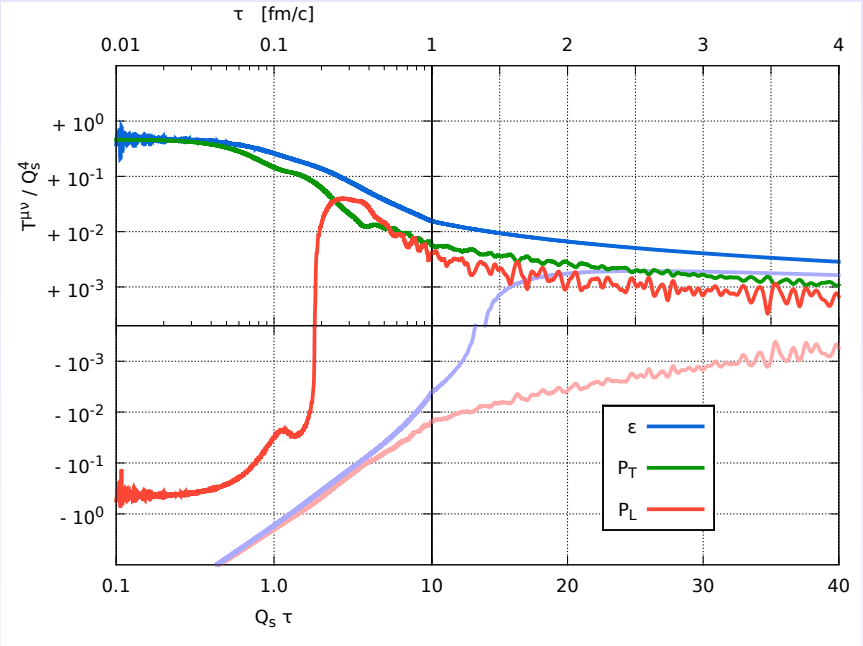


- Subleading divergences $\sim \Lambda_z^2$ in ϵ and P_L :

  Exist only at finite $\perp$ lattice spacing (not in the continuum)

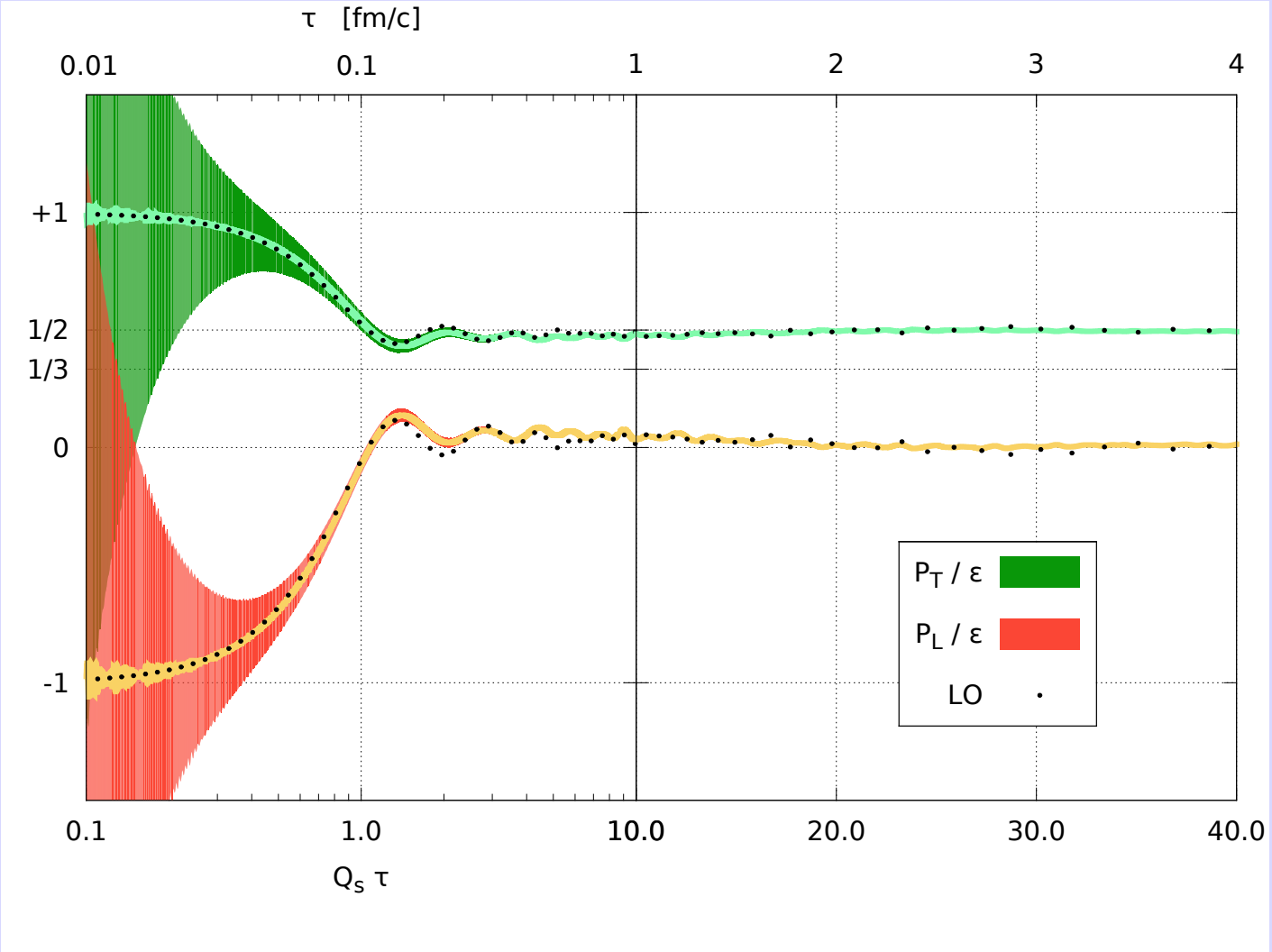
  Same counterterm in ϵ and P_L to preserve $T^\mu{}_\mu = 0$

  Must be of the form $A \times \tau^{-2}$ to preserve Bjorken's law

  At the moment, not calculated from first principles $\Rightarrow$ A fitted

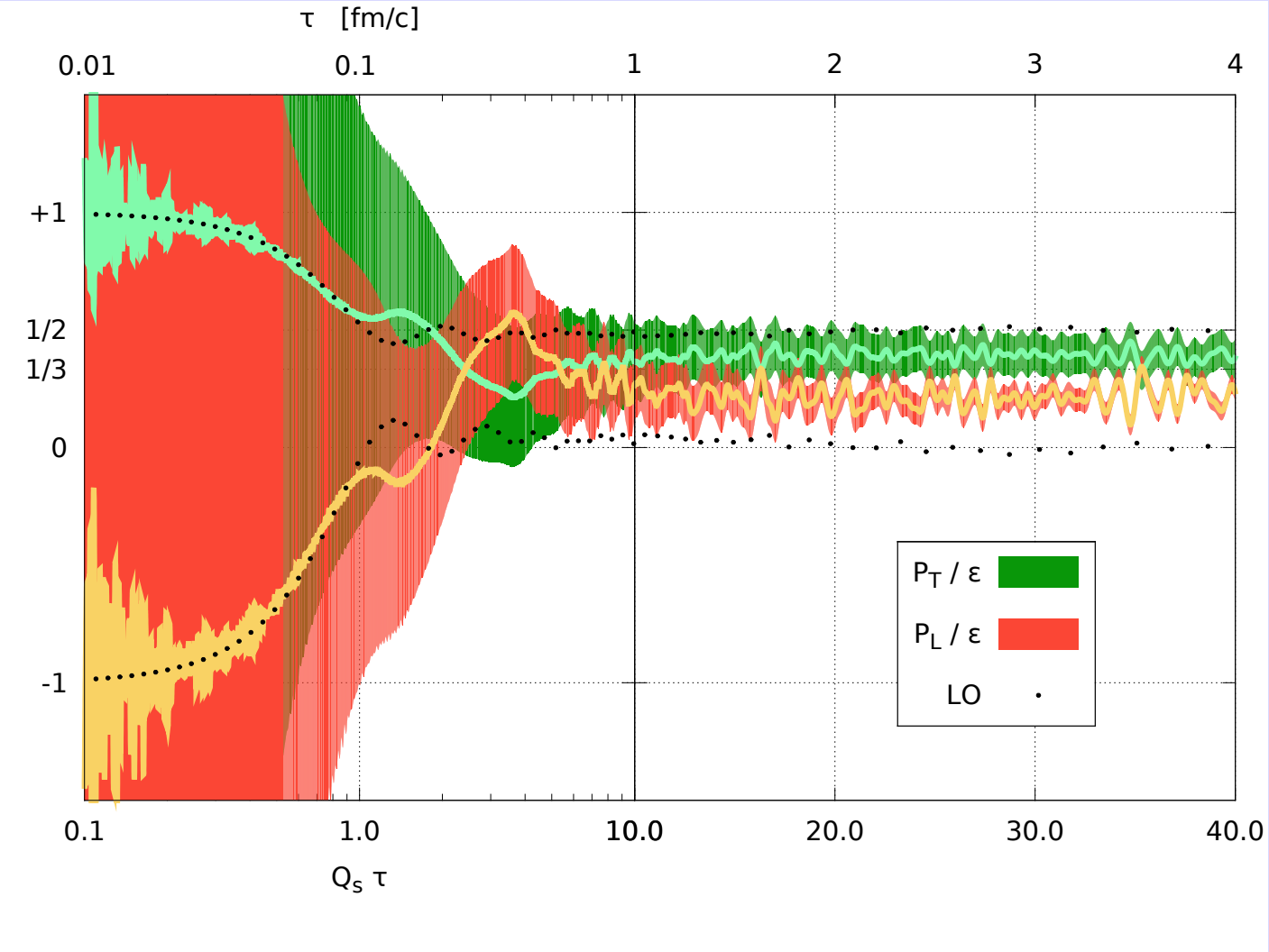
Time evolution of P_T/ϵ and P_L/ϵ ($64 \times 64 \times 128$ lattice)

$g = 0.1$ ($N_{\text{confs}} = 200$)



Time evolution of P_T/ϵ and P_L/ϵ ($64 \times 64 \times 128$ lattice)

$g = 0.5$ ($N_{\text{confs}} = 2000$)



Non Renormalizability

Schwinger-Keldysh formalism in the R/A basis

Schwinger-Keldysh propagators

$$G^0_{++}(p) = \frac{i}{p^2 + i\epsilon}\,, \qquad G^0_{--}(p) = \frac{-i}{p^2 - i\epsilon}$$

$$G^0_{+-}(p) = 2\pi\,\theta(-p^0)\delta(p^2)\,, \qquad G^0_{-+}(p) = 2\pi\,\theta(p^0)\delta(p^2)$$

- Define a new basis by "rotating" the propagators :

$$\mathbb{G}^0_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} \Omega_{\alpha\epsilon}\Omega_{\beta\epsilon'}G^0_{\epsilon\epsilon'} \qquad \text{with} \quad \Omega_{\alpha\epsilon} \equiv \begin{pmatrix} 1 & -1 \\ 1/2 & 1/2 \end{pmatrix}$$

New propagators

$$\mathbb{G}^0_{\alpha\beta} = \begin{pmatrix} 0 & G^0_A \\ G^0_R & G^0_S \equiv \pi\,\delta(p^2) \end{pmatrix}$$

Classical Statistical Approximation

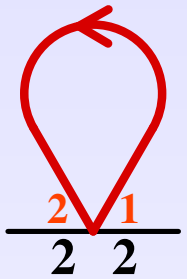
New vertices

$$\Gamma_{1111} = \Gamma_{1122} = \Gamma_{2222} = 0$$
$$\Gamma_{1222} = -ig^2 \ , \quad \Gamma_{1112} = -ig^2/4$$

- Classical statistical approximation : drop the Γ_{1112} vertex
- Only a subset of the graphs of the full theory
- Same ultraviolet superficial power counting

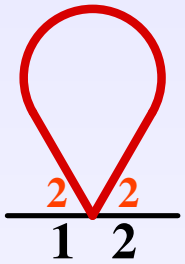
Ultraviolet divergences

$$\left[\Sigma_{11}\right]_{\text{CSA}}^{1\ \text{loop}} = 0 \quad , \qquad \left[\Sigma_{22}\right]_{\text{CSA}}^{1\ \text{loop}} =$$



$$= 0$$

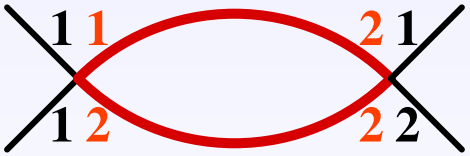
$$\left[\Sigma_{12}\right]_{\text{CSA}}^{1\ \text{loop}} =$$



$$= \frac{g^2 \Lambda_{\text{UV}}^2}{16\pi^2}$$

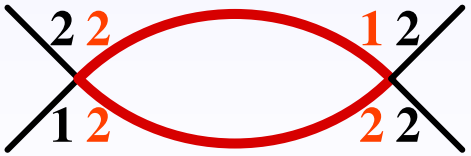
(usual mass renormalization)

$$\left[\Gamma_{1112}\right]_{\text{CSA}}^{1\ \text{loop}} =$$



$$= 0 \, , \qquad \left[\Gamma_{1111}\right]_{\text{CSA}}^{1\ \text{loop}} = \left[\Gamma_{2222}\right]_{\text{CSA}}^{1\ \text{loop}} = 0$$

$$\left[\Gamma_{1222}\right]_{\text{CSA}}^{1\ \text{loop}} =$$



$$\sim g^4 \log(\Lambda_{\text{UV}})$$

Ultraviolet divergences, the weird stuff...

$$\left[\Gamma_{1122}\right]^{1\ \mathrm{loop}}_{\mathrm{CSA}} = \frac{ig^4}{64\pi}\left[\mathrm{sign}(t) + \mathrm{sign}(u) + 2\,\Lambda_{\mathrm{UV}}\left(\frac{\theta(-t)}{|\mathbf{p}_1+\mathbf{p}_3|} + \frac{\theta(-u)}{|\mathbf{p}_1+\mathbf{p}_4|}\right)\right]$$

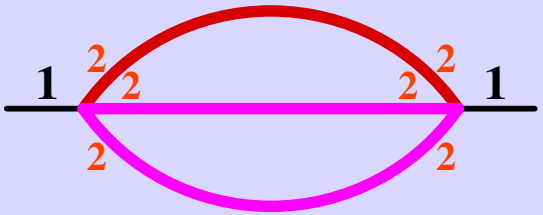
with $t \equiv (p_1+p_3)^2 \quad , \quad u \equiv (p_1+p_4)^2$

- No divergence if we keep the 1112 bare vertex
- UV divergence with no corresponding operator in the Lagrangian
- Stronger than the superficial power counting suggests
- Non-polynomial in the external momenta (i.e. non-local)

$\Longrightarrow$ the classical statistical approximation violates Weinberg's theorem, and is not renormalizable

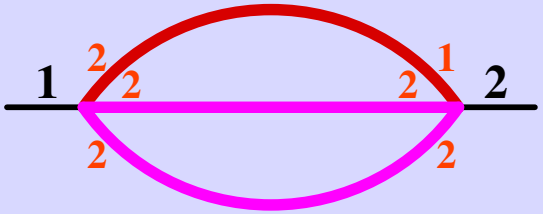
Unusual UV divergences in self-energies

$$\left[\Sigma_{11}(P)\right]^{2\text{ loop}}_{\text{CSA}} =$$



$$= -\frac{ig^4}{1024\pi^3}\left(\Lambda^2_{\text{UV}} - \frac{2}{3}p^2\right)$$

$$\text{Im}\left[\Sigma_{12}(P)\right]^{2\text{ loop}}_{\text{CSA}} =$$



$$= -\frac{g^4}{1024\pi^3}\left(\Lambda^2_{\text{UV}} - \frac{2}{3}p^2\right)$$

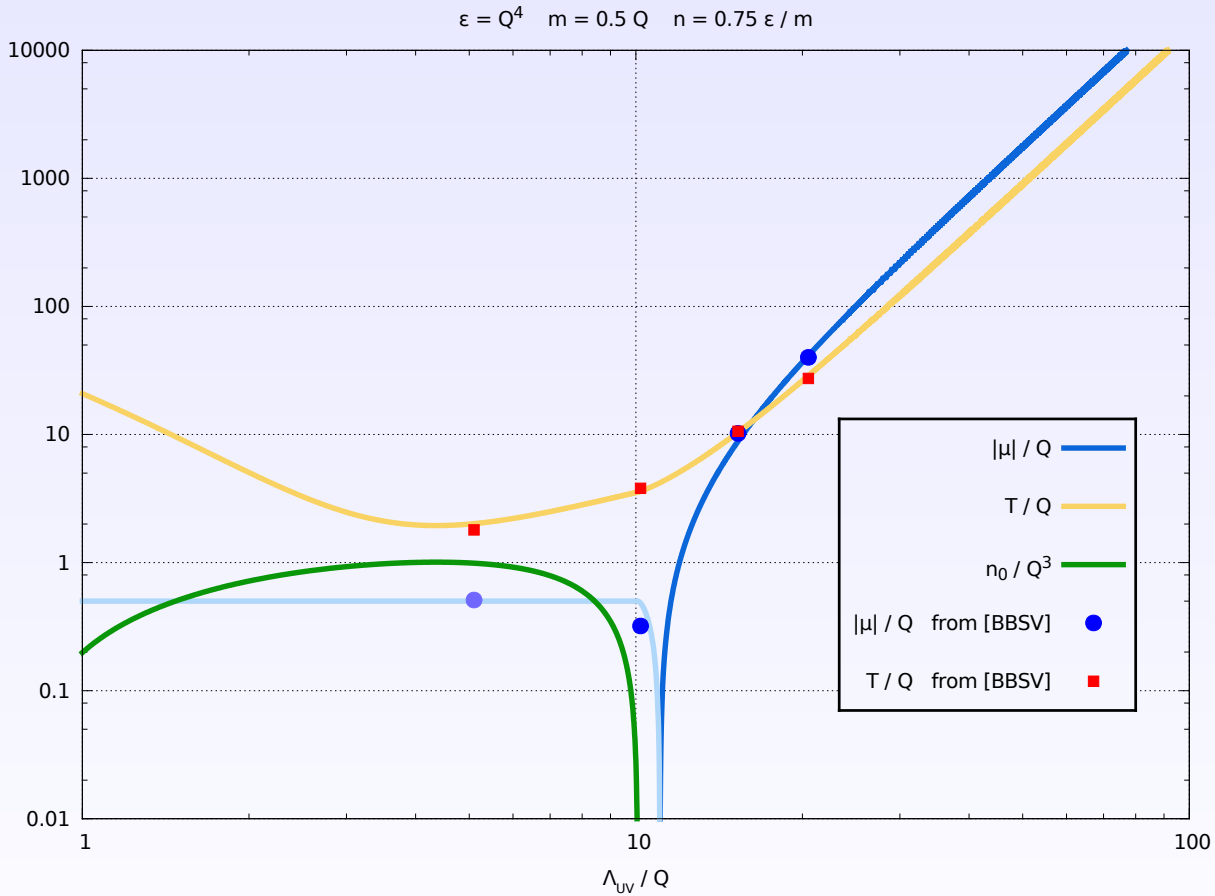
- Add a particle distribution $f(\mathbf{p})$ in the game, and compute the collision term for the Boltzmann equation :

$$\mathcal{C}_{\mathbf{p}}[f] = -\frac{i}{2\omega_{\mathbf{p}}}\left[\Sigma_{11}(P) + \left(f(\mathbf{p}) + \frac{1}{2}\right)\left(\Sigma_{21}(P) - \Sigma_{12}(P)\right)\right]$$

$\Longrightarrow$ fake scattering term $\sim g^4 f(\mathbf{p})\Lambda^2_{\text{UV}}/\omega_{\mathbf{p}}$

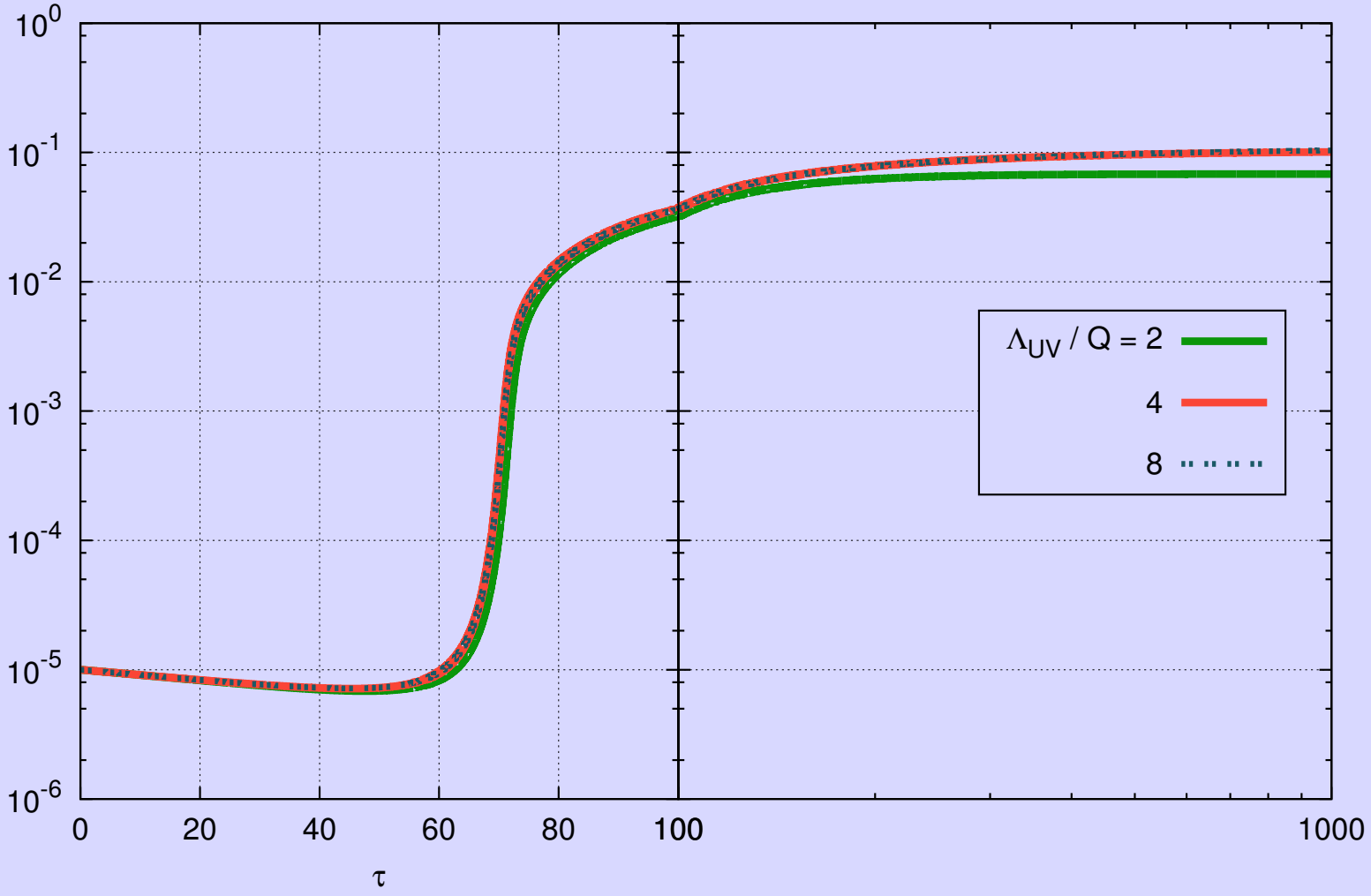
UV cutoff dependence of the asymptotic distribution

- At late times, $f(\mathbf{p}) \approx \dfrac{T}{\omega_{\mathbf{p}} - \mu} - \dfrac{1}{2}$, but T and μ depend on Λ_{UV}

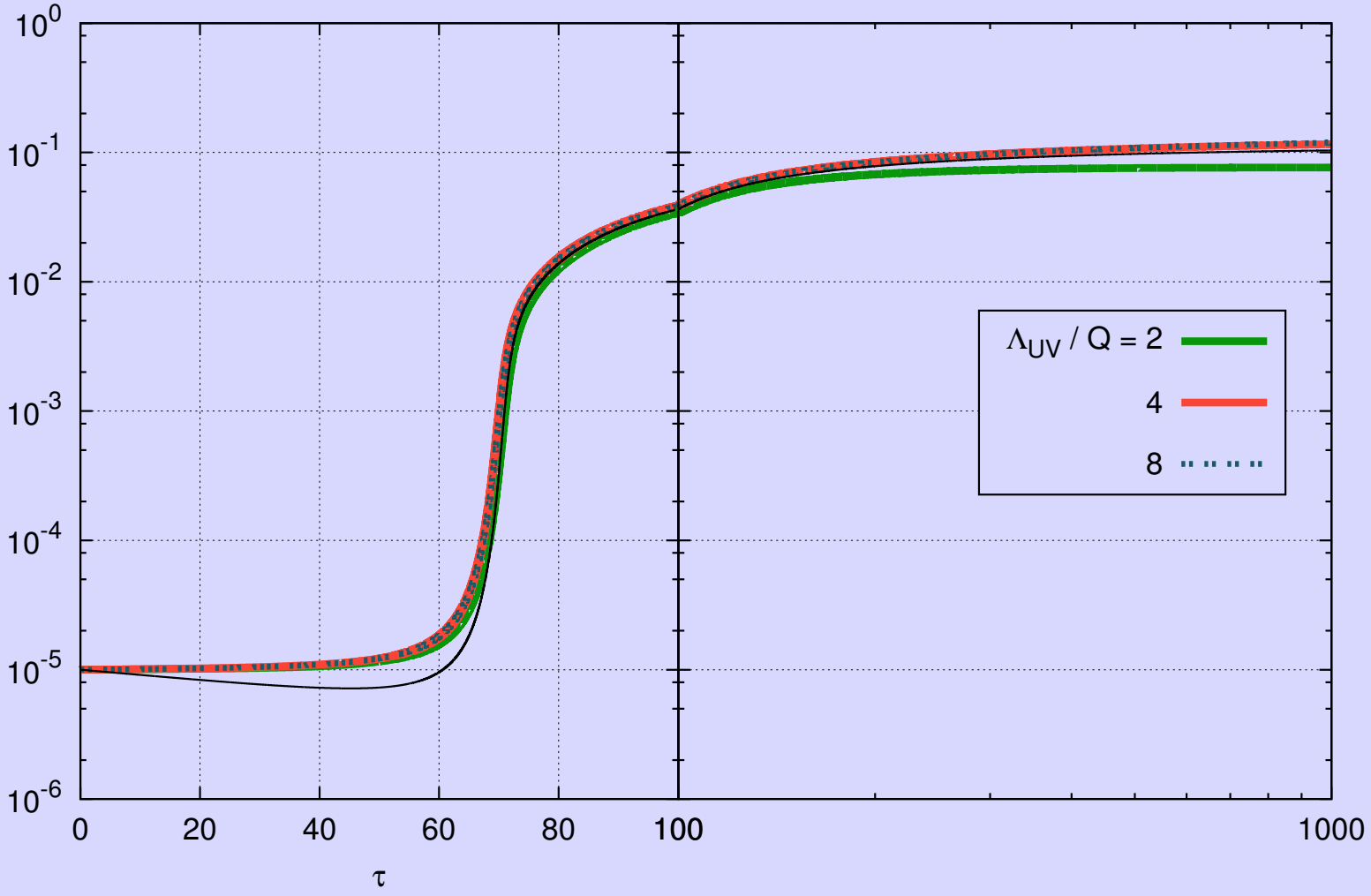


(points : classical statistical simulations, curves : Boltzmann eq.)

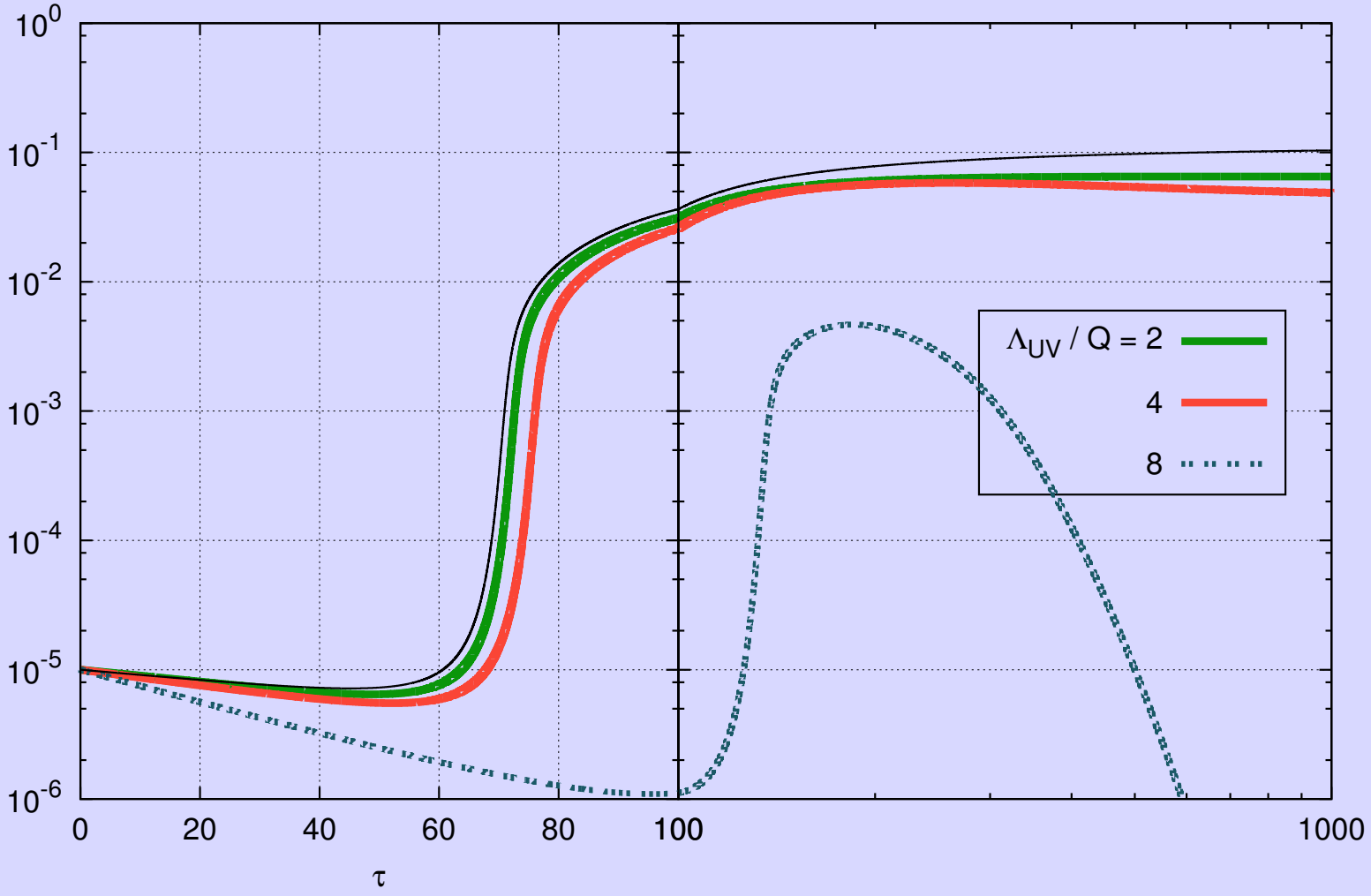
Full Quantum : $f_1 f_2 (1+f_3)(1+f_4) - (1,2 \leftrightarrow 3,4)$



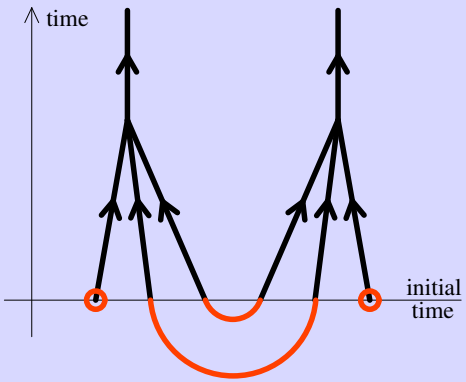
Strict classical : $f_1 f_2 (f_3 + f_4) - (1,2 \leftrightarrow 3,4)$

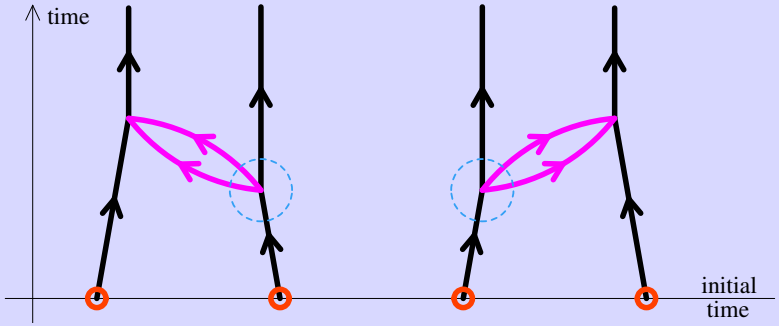


Classical + vac. fluct. : $(f_1 + 1/2)(f_2 + 1/2)(1 + f_3 + f_4) - (1,2 \leftrightarrow 3,4)$

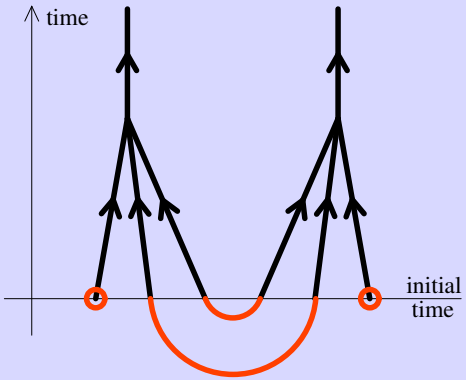


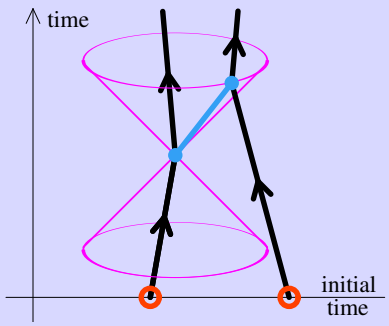
Classical + missing graphs





Classical + divergent part of the missing graphs





- In coordinate space :

$$L(x,y) \equiv \frac{g^4}{64\pi^3}\frac{\Lambda_{\rm UV}}{|\mathbf{x}-\mathbf{y}|}\,\delta((x^0-y^0)^2-(\mathbf{x}-\mathbf{y})^2)$$

- Can be generated by a Gaussian multiplicative noise term :

$$\Box\varphi + \frac{g^2}{6}\varphi^3 + i\xi\varphi = 0$$
$$\langle\xi(x)\rangle = 0\,, \qquad \langle\xi(x)\xi(y)\rangle = L(x,y)$$

Summary

Summary

- CGC calculation of the energy momentum tensor in a nucleus-nucleus collision, up to $Q_s\tau \lesssim 20$
- Method :
  - Classical statistical method
  - Initial Gaussian fluctuations : analytical, from a 1-loop calculation
  - Time evolution : numerical, 3+1d Yang-Mills equations on a lattice
- Accuracy :

  $\langle 0_{\rm in} | T^{\mu\nu}(\tau, x) | 0_{\rm in} \rangle$ at LO + NLO + leading secular terms
- Results :
  - Sizable longitudinal pressure ($P_L/P_T \sim 60\%$ for $g = 0.5$)
  - Typical timescale : $Q_s\tau \sim 2 - 3$

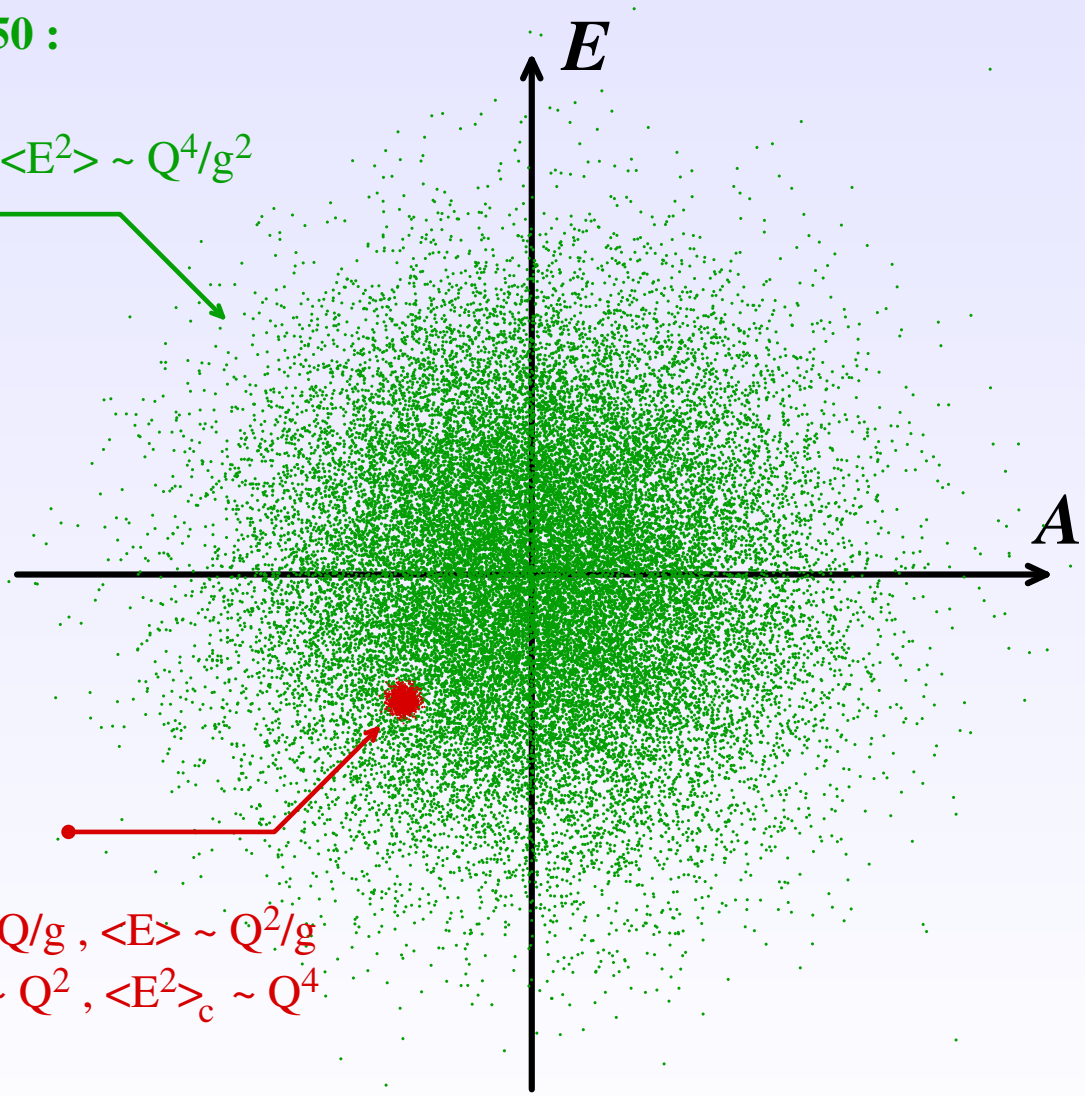
Extra Bits

Coherent vs Incoherent initial conditions

arXiv:1303.5650 :

$\langle A \rangle$, $\langle E \rangle = 0$

$\langle A^2 \rangle \sim Q^2/g^2$, $\langle E^2 \rangle \sim Q^4/g^2$

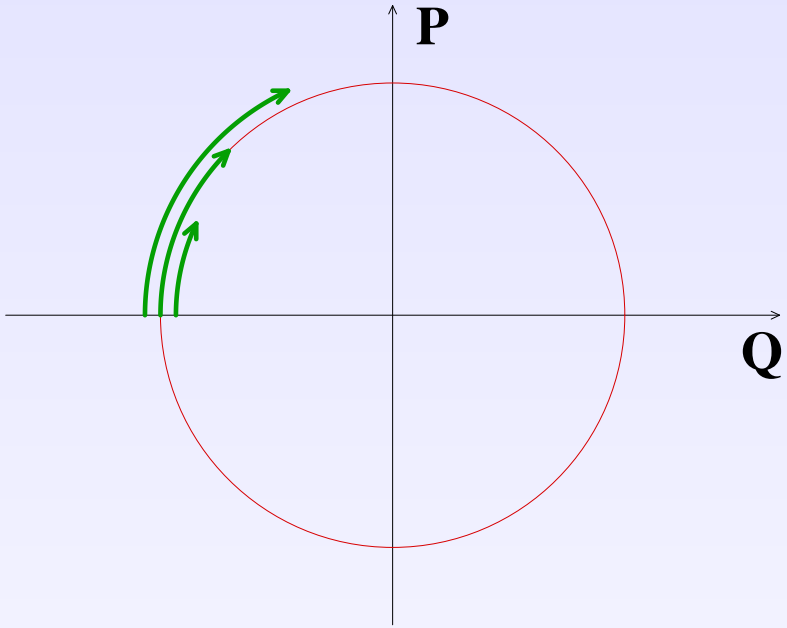


CGC :

$\langle A \rangle \sim Q/g$, $\langle E \rangle \sim Q^2/g$

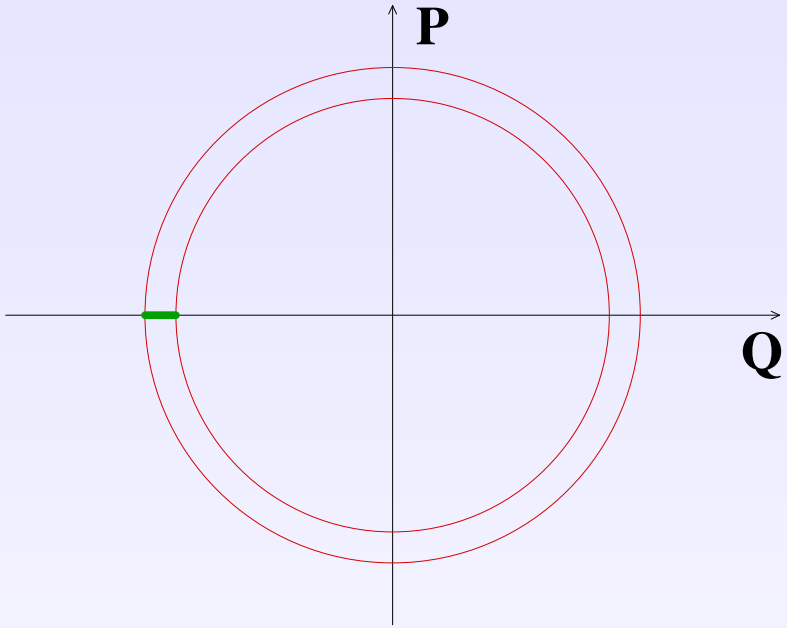
$\langle A^2 \rangle_c \sim Q^2$, $\langle E^2 \rangle_c \sim Q^4$

Microcanonical equilibration of an anharmonic oscillator



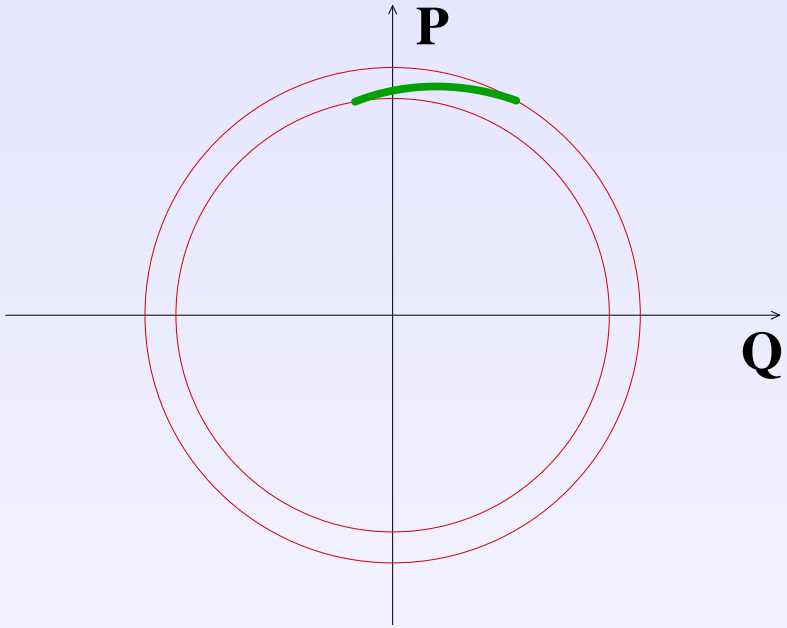
- The oscillation frequency depends on the initial condition

Microcanonical equilibration of an anharmonic oscillator



- The oscillation frequency depends on the initial condition
- Because of QM, the initial ensemble is a set of width $\sim \hbar$

Microcanonical equilibration of an anharmonic oscillator



- The oscillation frequency depends on the initial condition
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- This ensemble of initial configurations spreads in time

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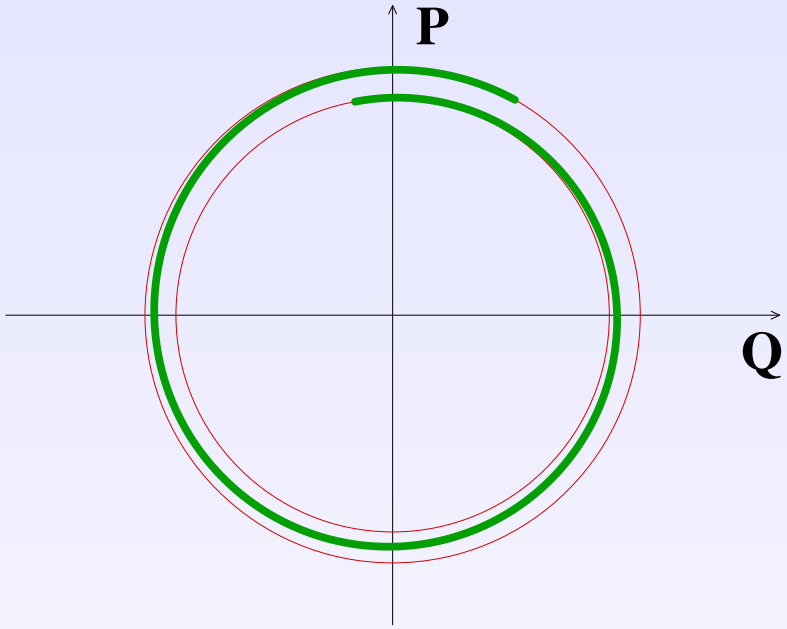

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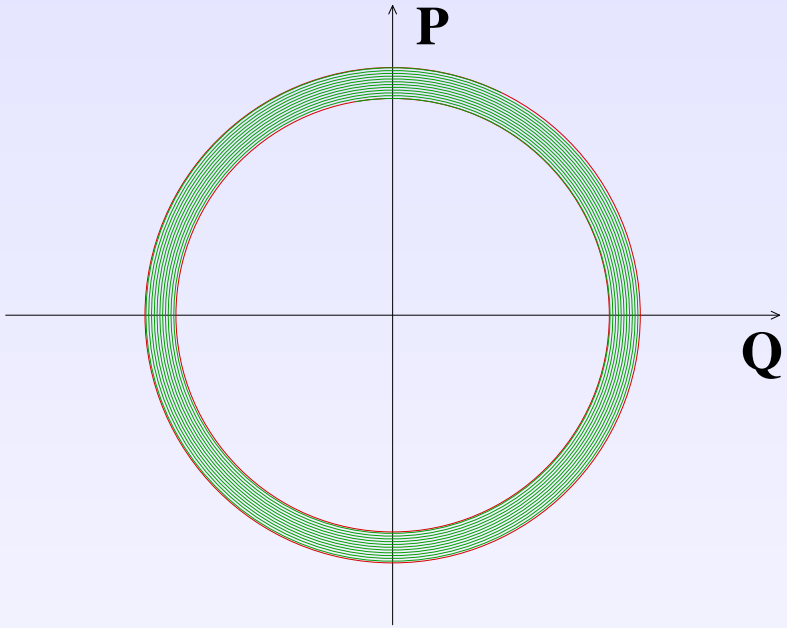

- The oscillation frequency depends on the initial condition
- Because of QM, the initial ensemble is a set of width $\sim \hbar$
- This ensemble of initial configurations spreads in time
- At large times, the ensemble fills densely all the region allowed by energy conservation $\Rightarrow$ microcanonical equilibrium

Quantum chaos

- Central issue : consider a Hamiltonian that leads to chaotic classical behavior; What happens when this system is quantized?

- Schrodinger's equation is linear, and $\widehat{H}$ Hermitian:

$$i\partial_t \Psi = \widehat{H}\,\Psi$$

- Once we know the spectrum of the Hamiltonian $\{E_n, \Psi_n\}$, any wavefunction evolves as :

$$\Psi(t) = \sum_n c_n \; e^{iE_n t}\,\Psi_n$$

$E_n \in \mathbb{R} \;\Rightarrow\;$ nothing is unstable. Where is the chaos in QM ?

Berry's conjecture (1977)

- The complexity of the classical dynamics is hidden in the complexity of the high lying eigenfunctions

- Berry's conjecture : for most practical purposes, high lying eigenfunctions of classically chaotic systems behave as Gaussian random functions with 2-point correlations given by

$$\left\langle \Psi^*(\mathbf{X}-\frac{\mathbf{s}}{2})\Psi(\mathbf{X}+\frac{\mathbf{s}}{2})\right\rangle = \int d\mathbf{P}\; e^{i\mathbf{P}\cdot\mathbf{s}/\hbar}\;\delta\big[E-H(\mathbf{X},\mathbf{P})\big]$$

- If this hypothesis is true, the Wigner distribution associated with the eigenfunction Ψ_E is

$$\begin{aligned} W(\mathbf{X},\mathbf{P}) &= \int d\mathbf{s}\; e^{-i\mathbf{P}\cdot\mathbf{s}/\hbar}\;\Psi^*_E(\mathbf{X}-\frac{\mathbf{s}}{2})\Psi_E(\mathbf{X}+\frac{\mathbf{s}}{2}) \\ &\sim \delta\big[E-H(\mathbf{X},\mathbf{P})\big] \end{aligned}$$

$\Rightarrow$ micro-canonical equilibrium for a single eigenstate

Eigenstate thermalization hypothesis (Srednicki, 1994)

- If an energy eigenstate obeys Berry's conjecture, then a measurement performed on that state will lead to the Bose-Einstein (or Fermi-Dirac) distribution for the single particle distribution

- Generic states approach equilibrium via decoherence of their energy eigenstate components