
Quark-Gluon Plasma and Heavy Ion Collisions

IV – Kinetic theory

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General outline

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[Boltzmann equation](#)

[Transport coefficients](#)

- I : Introduction to Heavy Ion Collisions
- II : QCD at finite temperature
- III : Hydrodynamical behavior
- IV : Kinetic theory

Lecture IV

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[Boltzmann equation](#)

[Transport coefficients](#)

- Collisionless transport
- Boltzmann equation
- Transport coefficients

Collisionless kinetic equations

- Free transport
- Vlasov equation

Boltzmann equation

Transport coefficients

Collisionless kinetic equations

Kinetic theory

Collisionless kinetic equations

- Free transport
- Vlasov equation

Boltzmann equation

Transport coefficients

- In kinetic theory, the system is described by distributions of particles

$$f(t, \vec{x}, \vec{p}) \equiv \frac{dN}{d^3\vec{x} d^3\vec{p}}$$

- Implicit hypothesis :
 - ◆ The particle distributions vary slowly with t and \vec{x} (gradients in t, \vec{x} are much smaller than the typical momentum \vec{p})
 - ◆ On-shell particles propagate freely between two collisions
- Kinetic equations describe the time evolution of these distributions under the influence of **external forces**, or of the **mutual interactions of the particles**
- Kinetic equations :
 - ◆ Free transport equation
 - ◆ Vlasov equation
 - ◆ Boltzmann equation

Free transport

Collisionless kinetic equations

- Free transport

- Vlasov equation

Boltzmann equation

Transport coefficients

- Free transport is a regime in which the particles do not interact. Given an initial $f(t_0, \vec{x}, \vec{p})$, the particles propagate on straight lines, at constant velocity
- The kinetic equation that describes this regime reads :

$$\vec{p} \cdot \partial_{\vec{x}} f(t, \vec{x}, \vec{p}) = 0$$

or, equivalently :

$$\left[\partial_t + \vec{v}_p \cdot \vec{\nabla}_{\vec{x}} \right] f(t, \vec{x}, \vec{p}) = 0 \quad \text{with } \vec{v}_p \equiv \frac{\vec{p}}{E_p}$$

- This equation can be solved trivially from its initial condition :

$$f(t, \vec{x}, \vec{p}) = f(t_0, \vec{x} - \vec{v}_p(t - t_0), \vec{p})$$

Interpretation :

- ◆ The momentum \vec{p} of the particles does not change
- ◆ If a particle of momentum \vec{p} is at the position \vec{x} at time t , it comes from the position $\vec{x} - \vec{v}_p(t - t_0)$ at the time t_0

Free transport

Collisionless kinetic equations

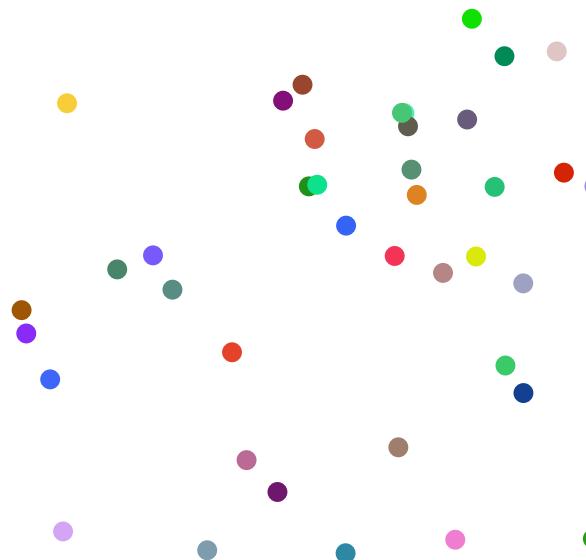
● Free transport

● Vlasov equation

Boltzmann equation

Transport coefficients

■ At the time t_0 :



Free transport

Collisionless kinetic equations

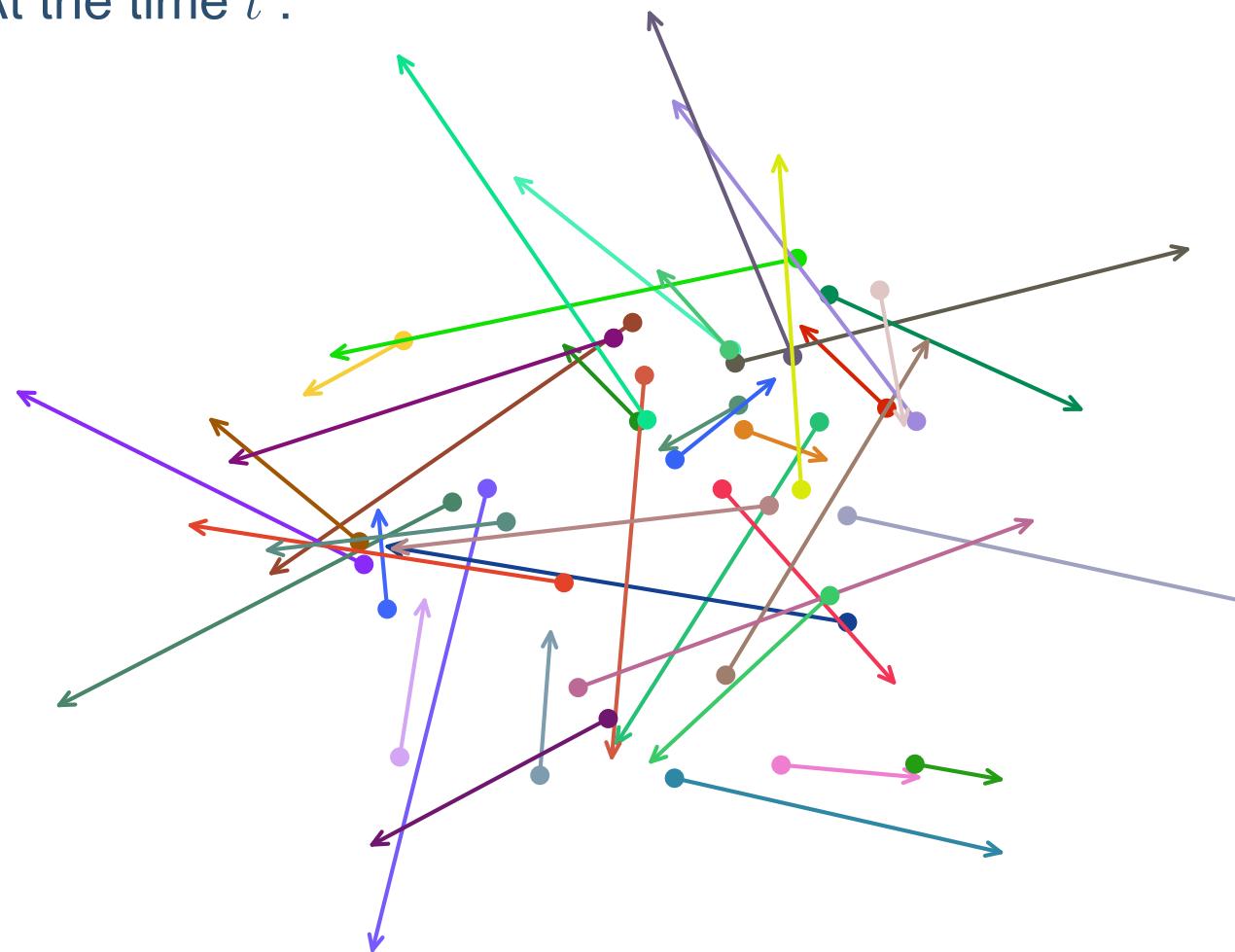
● Free transport

● Vlasov equation

Boltzmann equation

Transport coefficients

■ At the time t :



Vlasov equation

Collisionless kinetic equations

- Free transport
- Vlasov equation

Boltzmann equation

Transport coefficients

- The Vlasov equation describes the time evolution of a distribution of particles under the influence of a force \vec{F}

- The Vlasov equation reads :

$$\left[\partial_t + \vec{v}_p \cdot \vec{\nabla}_x \right] f(t, \vec{x}, \vec{p}) + \underline{\vec{F} \cdot \vec{\nabla}_p f(t, \vec{x}, \vec{p})} = 0$$

- When the force is externally applied, it can be solved formally by :

$$f(t, \vec{x}, \vec{p}) = f(t_0, \vec{x}_0, \vec{p}_0)$$

where (\vec{x}_0, \vec{p}_0) is the position in phase space at time t_0 that leads to (\vec{x}, \vec{p}) at time t under the effect of the force \vec{F} . If $(\vec{x}(\tau), \vec{p}(\tau))$ denotes the trajectory between t_0 and t , one has

$$\vec{x} = \vec{x}_0 + \int_{t_0}^t d\tau \frac{\vec{p}(\tau)}{E_p(\tau)} \quad , \quad \vec{p} = \vec{p}_0 + \int_{t_0}^t d\tau \vec{F}(\tau, \vec{x}(\tau))$$

Vlasov equation

Collisionless kinetic equations

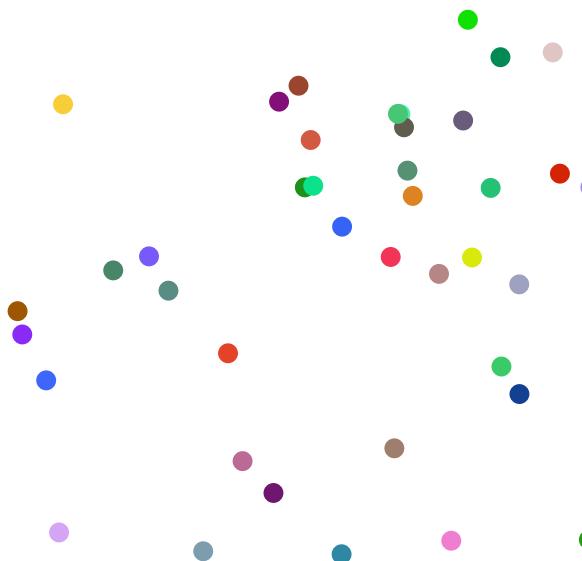
● Free transport

● Vlasov equation

Boltzmann equation

Transport coefficients

■ At the time t_0 :



Vlasov equation

Collisionless kinetic equations

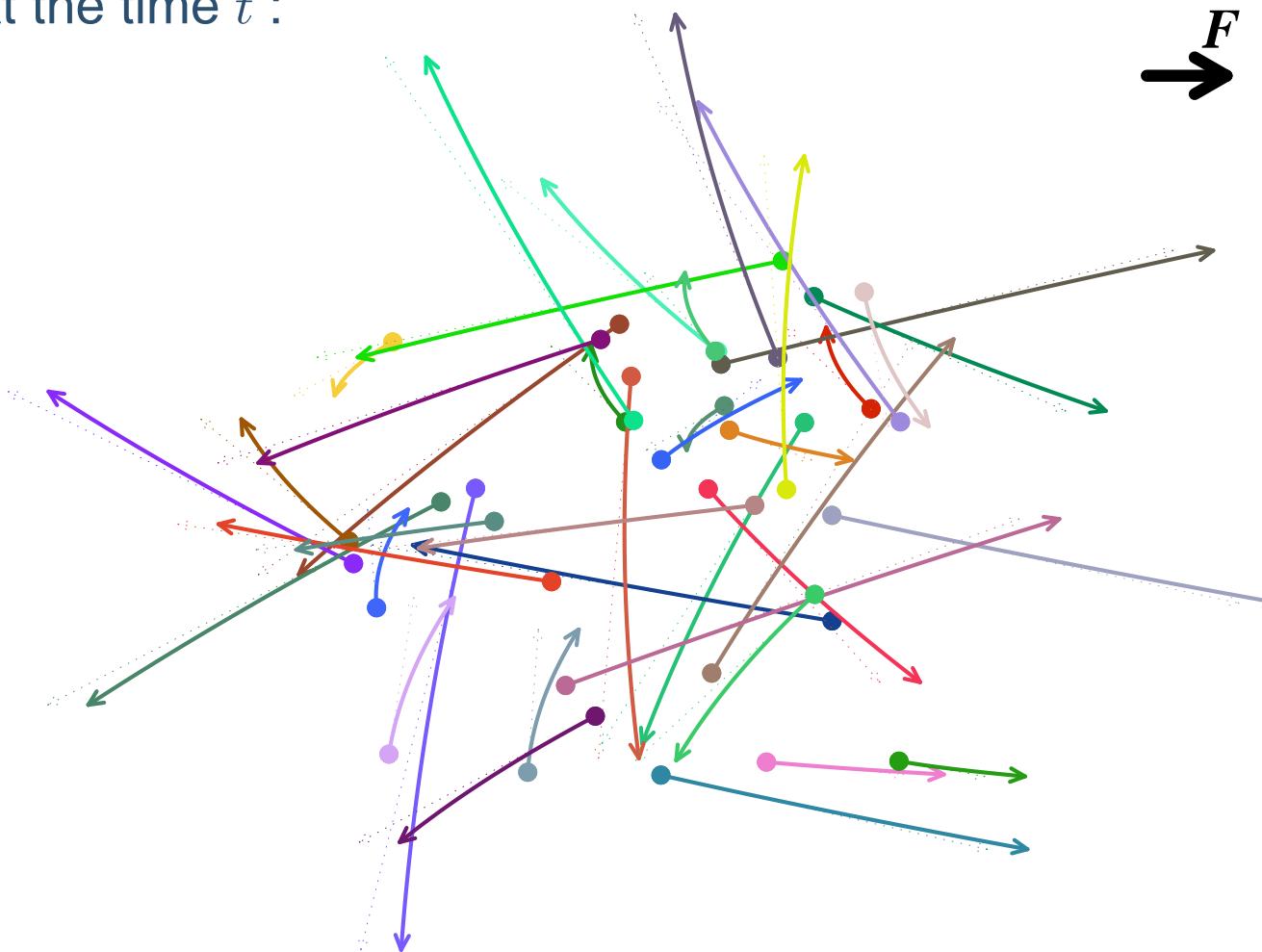
● Free transport

● Vlasov equation

Boltzmann equation

Transport coefficients

■ At the time t :



Vlasov equation

Collisionless kinetic equations

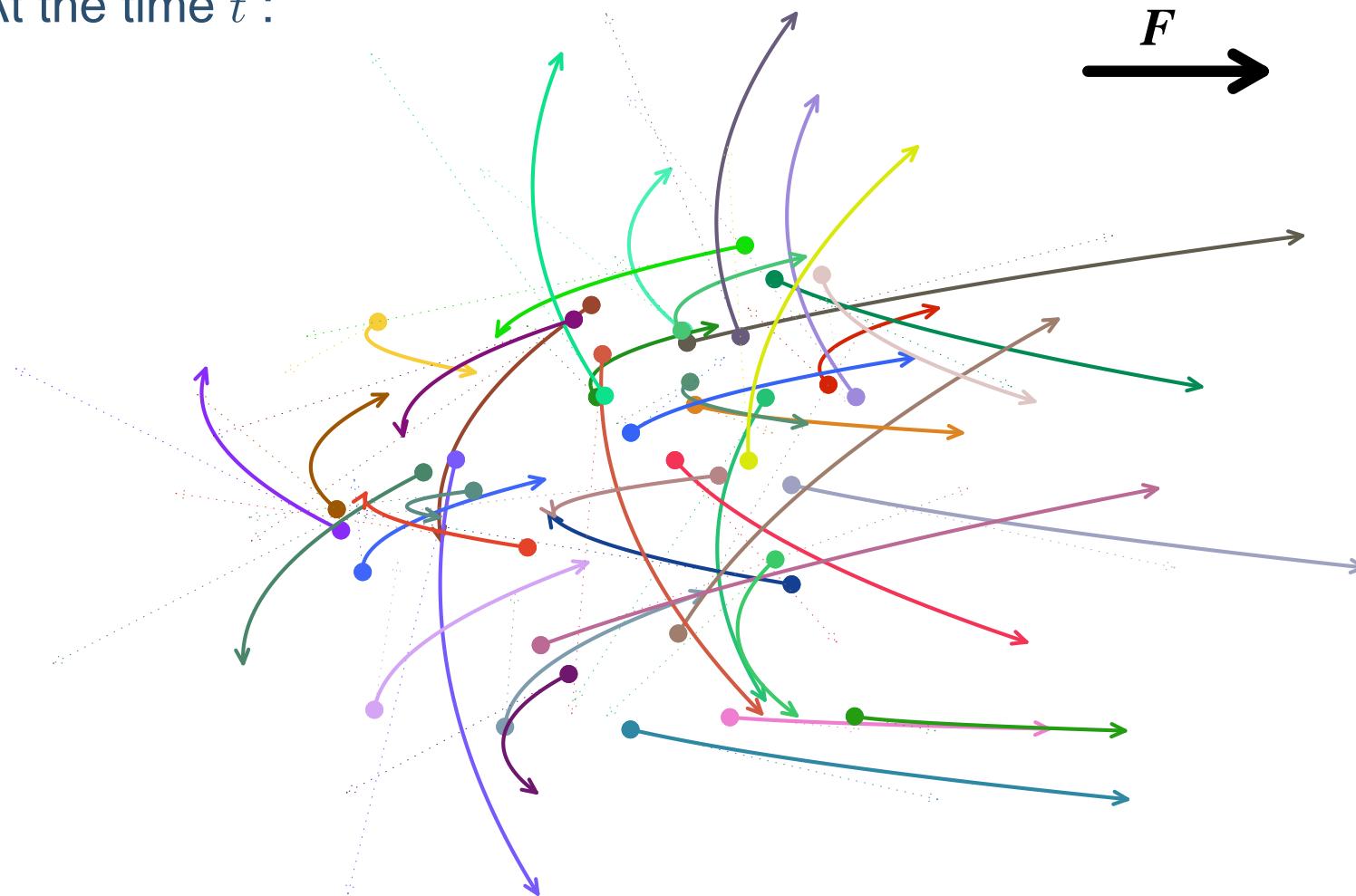
● Free transport

● Vlasov equation

Boltzmann equation

Transport coefficients

■ At the time t :



Vlasov equation

Collisionless kinetic equations

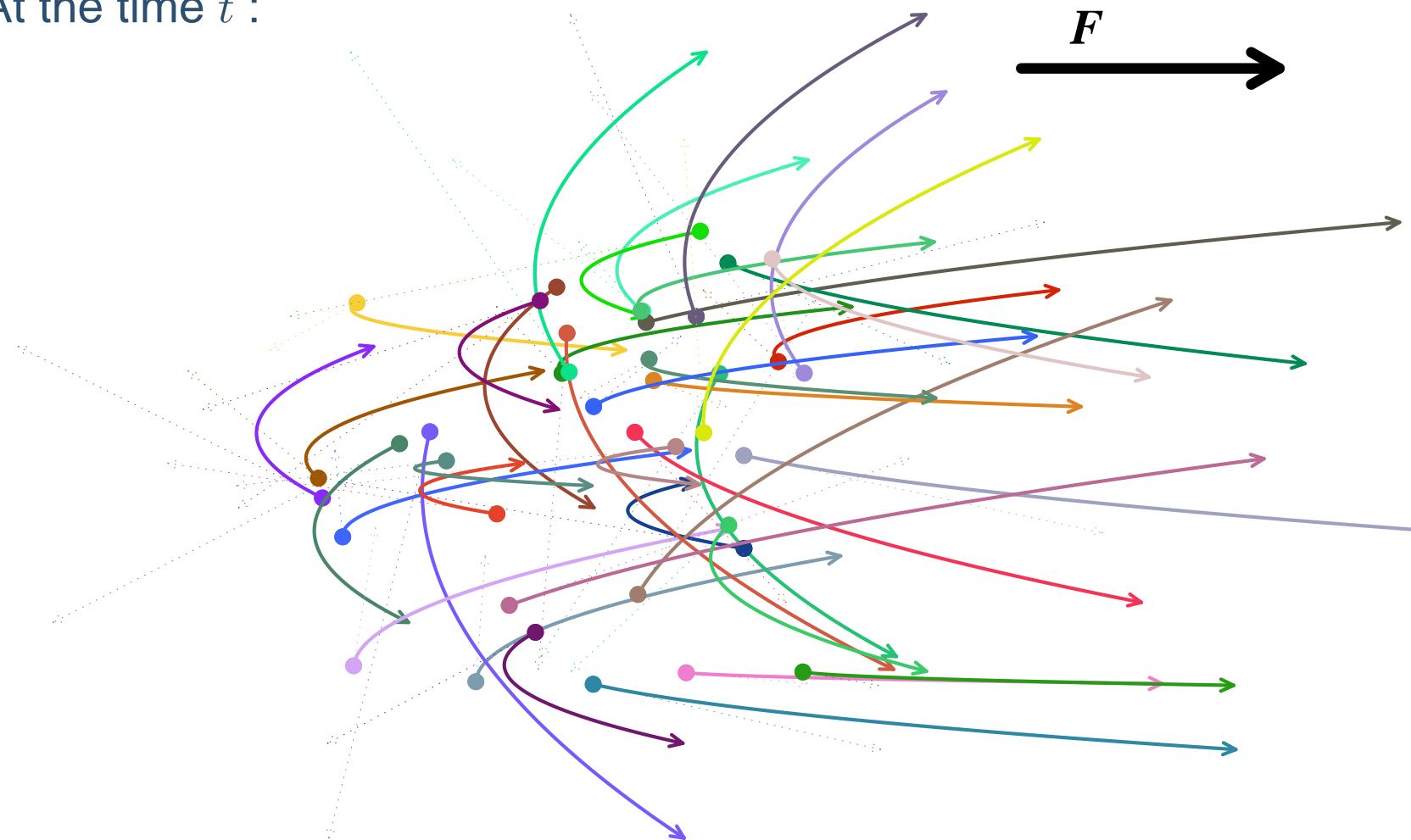
● Free transport

● Vlasov equation

Boltzmann equation

Transport coefficients

■ At the time t :



Vlasov equation + mean field

Collisionless kinetic equations

- Free transport
- Vlasov equation

Boltzmann equation

Transport coefficients

- In many applications, the force \vec{F} is not externally applied, but results from the action of all the other particles
- Example : for electro-magnetic interactions among the particles in the system, the force term in the Vlasov equation reads

$$\underbrace{e v_p^\mu F_{\mu\nu}} \partial_p^\nu f(t, \vec{x}, \vec{p})$$

Lorentz force in covariant form

with

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\partial_\mu F^{\mu\nu}(x) = \underbrace{e \int \frac{d^3 \vec{p}}{(2\pi)^3} v_p^\nu f(t, \vec{x}, \vec{p})}_{\text{EM current created by the particles}} \quad (\text{Maxwell's equation})$$

EM current created by the particles

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Boltzmann equation

- Collision term
- Implicit assumptions
- Collisions or mean field ?
- Collisional invariants
- H theorem
- Equilibrium state

[Transport coefficients](#)

Boltzmann equation

Collision term

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Boltzmann equation

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- The Boltzmann equation takes into account the **collisions** among particles. It is valid when these collisions are sufficiently local (i.e. no long range interactions among pairs of particles). Thanks to the **Debye screening**, this is a valid assumption for a neutral plasma
- The Boltzmann equation reads :

$$\left[\partial_t + \vec{v}_p \cdot \vec{\nabla}_x \right] f(t, \vec{x}, \vec{p}) + \vec{F} \cdot \vec{\nabla}_p f(t, \vec{x}, \vec{p}) = \mathcal{C}_p[f]$$

▷ the functional $\mathcal{C}_p[f]$ is the **collision term**. For $2 \rightarrow 2$ collisions, it can be written as :

$$\begin{aligned} \mathcal{C}_p[f] = & \frac{1}{2E_p} \int \frac{d^3 \vec{p}'}{(2\pi)^3 2E_{p'}} \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_k} \int \frac{d^3 \vec{k}'}{(2\pi)^3 2E_{k'}} (2\pi)^4 \delta(p + k - p' - k') \\ & \times \left[f(X, \vec{p}') f(X, \vec{k}') (1 + f(X, \vec{p})) (1 + f(X, \vec{k})) \right. \\ & \left. - f(X, \vec{p}) f(X, \vec{k}) (1 + f(X, \vec{k}')) (1 + f(X, \vec{p}')) \right] |\mathcal{M}|^2 \end{aligned}$$

Collision term

Collisionless kinetic equations

Boltzmann equation



Collision term

Implicit assumptions

Collisions or mean field ?

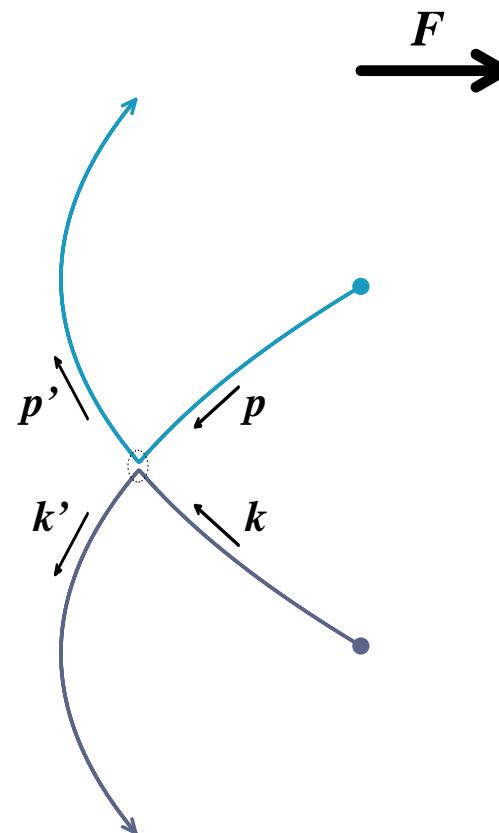
Collisional invariants

H theorem

Equilibrium state

Transport coefficients

■ Elementary 2-body collision :



Note : microscopic collisions are **reversible**

Diluteness assumption

Collisionless kinetic equations

Boltzmann equation

● Collision term

● Implicit assumptions

● Collisions or mean field ?

● Collisional invariants

● H theorem

● Equilibrium state

Transport coefficients

- In order to be able to neglect 3-body collisions and higher, the system under study must be sufficiently dilute
- For a system of N hard spheres of radius r , the Boltzmann equation is valid in the limit :

$$\left\{ \begin{array}{l} Nr^2 = \text{const} \\ Nr^3 \rightarrow 0 \end{array} \right.$$

(Boltzmann-Grad limit)

- The first condition means that the mean free path is fixed ($\lambda = 1/n\sigma$, $n = N/V$, $\sigma = 2\pi r^2$)
- The second condition means that the volume occupied by the particles tend to zero

Molecular chaos assumption

Collisionless kinetic equations

Boltzmann equation

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Transport coefficients

- Strictly speaking, the collision term should contain the probability to find a pair of particles of momenta \vec{p}, \vec{k} at the point (t, \vec{x}) before the collision
 - ▷ one should have used the 2-particle phase-space distribution :

$$f_2(X, \vec{p}; X, \vec{k})$$

that contains information about the 2-particle correlations

- By writing :

$$f_2(X, \vec{p}; X, \vec{k}) = f(X, \vec{p})f(X, \vec{k})$$

one assumes that the two colliding particles have uncorrelated momenta before the collision

- Although the microscopic processes are reversible, the Boltzmann equation is not, because the two momenta become correlated after the collision

Collisions or mean field ?

Collisionless kinetic equations

Boltzmann equation

- Collision term
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Transport coefficients

- Given a two-body interaction between particles, should we treat it as part of the mean field force term, or as part of the collision term?
- Bobylev, Illner : for inverse power forces in r^{-s}
 - ◆ the collision term prevails if $s > 3$
 - ◆ the mean-field term prevails if $s < 3$
- This indicates that short-range interactions should be treated as collisions, while long range interactions go in the mean-field term
- Examples :

Debye screened forces	→	collisions
Hard sphere interactions	→	collisions
Gravitational forces	→	mean-field

Collisional invariants

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Boltzmann equation

- Collision term
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● Collisional invariants

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Transport coefficients

- Consider a quantity $I(\vec{p})$, and the integral

$$\mathcal{I}[f] \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3} \mathcal{C}_{\vec{p}}[f] I(\vec{p})$$

- By symmetry under the exchange $(\vec{p}, \vec{p}') \leftrightarrow (\vec{k}, \vec{k}')$ and antisymmetry under $(\vec{p}, \vec{k}) \leftrightarrow (\vec{p}', \vec{k}')$, we can write

$$\mathcal{I}[f] = \frac{1}{4} \int \frac{d^3 \vec{p}}{(2\pi)^3} \mathcal{C}_{\vec{p}}[f] \left[I(\vec{p}) + I(\vec{k}) - I(\vec{p}') - I(\vec{k}') \right]$$

- A quantity $I(\vec{p})$ for which the bracket $[\dots]$ vanishes is called a collisional invariant
- Collisional invariants :
 - ◆ $I(\vec{p}) = 1$ (elastic collisions conserve the number of particles)
 - ◆ $I(\vec{p}) = p^\mu$ (energy-momentum conservation)

Local conservation laws

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[Transport coefficients](#)

- Define the density and current at point X for the quantity I :

$$I(X) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3} I(\vec{p}) f(X, \vec{p})$$

$$\vec{J}_I(X) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3} I(\vec{p}) \vec{v}_p f(X, \vec{p})$$

- Multiply the Boltzmann equation by $I(\vec{p})$ and integrate it over all the momenta p :
 - ◆ The collision term gives zero for a collisional invariant
 - ◆ If there is no force term, then one obtains

$$\partial_t I(X) + \vec{\nabla}_x \cdot \vec{J}_I(X) = 0$$

▷ continuity equation for the local conservation of the quantity I

- Note : if there is a force term, the number of particles is locally conserved, but not their momentum

H theorem

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● H theorem

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■ Define the quantities

$$h(X, \vec{p}) \equiv (1 + f(X, \vec{p})) \ln(1 + f(X, \vec{p})) - f(X, \vec{p}) \ln(f(X, \vec{p}))$$

$$H(X) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3} h(X, \vec{p}) \quad , \quad \vec{J}_H(X) \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3} h(X, \vec{p}) \vec{v}_p$$

■ From the Boltzmann equation, we get

$$\left[\partial_t + \vec{v}_p \cdot \vec{\nabla}_x \right] h + \vec{F}(X) \cdot \vec{\nabla}_p h = \mathcal{C}_p[f] \ln \left(\frac{1 + f(X, \vec{p})}{f(X, \vec{p})} \right)$$

$$\partial_t H + \vec{\nabla}_x \cdot \vec{J}_H = \sigma_H$$

with $\sigma_H \equiv \int \frac{d^3 \vec{p}}{(2\pi)^3} \mathcal{C}_p[f] \ln \left(\frac{1 + f(X, \vec{p})}{f(X, \vec{p})} \right)$

H theorem

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Transport coefficients

- Using the symmetry properties of the collision term, we can rewrite σ_H as

$$\sigma_H = \frac{1}{4} \int \frac{d^3 \vec{p}}{(2\pi)^3} \mathcal{C}_{\mathbf{p}}[f] \left[\ln \left(\frac{1 + f(X, \vec{p})}{f(X, \vec{p})} \right) + \ln \left(\frac{1 + f(X, \vec{k})}{f(X, \vec{k})} \right) - \ln \left(\frac{1 + f(X, \vec{p}')}{f(X, \vec{p}')} \right) - \ln \left(\frac{1 + f(X, \vec{k}')}{f(X, \vec{k}')} \right) \right]$$

- In the right hand side, one can rewrite the factors that depend on f as follows :

$$f(X, \vec{p}) f(X, \vec{k}) (1 + f(X, \vec{p}')) (1 + f(X, \vec{k}')) \left[\frac{\alpha_{\mathbf{p}} \alpha_{\mathbf{k}}}{\alpha_{\mathbf{p}'} \alpha_{\mathbf{k}'}} - 1 \right] \ln \left(\frac{\alpha_{\mathbf{p}} \alpha_{\mathbf{k}}}{\alpha_{\mathbf{p}'} \alpha_{\mathbf{k}'}} \right)$$

with $\alpha_{\mathbf{p}} \equiv (1 + f(X, \vec{p}))/f(X, \vec{p})$

- Since $(X - 1) \ln(X) \geq 0$, we have $\sigma_H \geq 0$
 - ▷ the quantity H has a positive source term

H theorem

Collisionless kinetic equations

Boltzmann equation

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Transport coefficients

■ Interpretation :

- ◆ $H(X)$ is the entropy density, and $\vec{J}_H(X)$ its current
- ◆ Because the continuity equation for H has a right hand side σ_H , it is not a conserved quantity
- ◆ Because $\sigma_H \geq 0$, the total amount of H in the system can only increase

■ Remarks :

- ◆ This seems to contradict Poincaré's recurrence theorem :
“Any system with a finite volume phase-space will return arbitrarily close to its initial conditions in a finite time”
 ▷ where does the irreversibility come from in the Boltzmann eq.?
- ◆ Molecular chaos assumption : the Boltzmann equation is an approximation of the full dynamical evolution of the system, in which one neglects correlations among particles prior to collisions. By dropping these correlations, one loses the information necessary to reverse the time evolution of the system

Equilibrium state

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- When the equilibrium is reached, $\sigma_H = 0$

- ▷ $\ln((1 + f_{\text{eq}})/f_{\text{eq}})$ is a collisional invariant
- ▷ it is a linear combination of 1 and p^μ :

$$\ln \left(\frac{1 + f_{\text{eq}}(X, \vec{p})}{f_{\text{eq}}(X, \vec{p})} \right) = \alpha + \beta_\mu p^\mu \quad \Rightarrow \quad f_{\text{eq}}(X, \vec{p}) = \frac{1}{e^{\alpha + \beta_\mu p^\mu} - 1}$$

(Bose-Einstein distribution)

- $\beta_\mu p^\mu$ is the Lorentz covariant form of p^0/T ($\beta = 1/T$)
- α is a chemical potential associated to the conservation of the number of particles

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Transport coefficients

Transport coefficients

- The Boltzmann equation is a powerful tool for calculating transport coefficients such as **conductivity**, **viscosity**, **diffusion constants**
- These transport coefficients can also be calculated in **quantum field at finite temperature**. Example for the **electric conductivity** :
 - ◆ σ_{el} is the coefficient of proportionality between the induced electric current and the applied electric field :

$$\vec{j}_{\text{el}} = \sigma_{\text{el}} \vec{E}$$

- ◆ It is given by a current-current correlator (**Kubo's formula**) :

$$\sigma_{\text{el}} = \frac{1}{6} \lim_{\omega \rightarrow 0} \int d^4x e^{i\omega t} \left\langle j_{\text{el}}^i(t, \vec{x}) j_{\text{el}}^i(0, \vec{0}) \right\rangle_T$$

- ◆ This correlation function can be evaluated from Feynman diagrams at finite temperature, but one needs to **sum an infinite series of graphs** ▷ quite difficult

Transport coefficients

- In the evaluation of σ_{el} from the Boltzmann equation, one perturbs a system at equilibrium by a small electric field
 - ▷ it enters in the Boltzmann equation via the force $\vec{F} \equiv e\vec{E}$

- This force induces a departure of f away from f_{eq} . It is convenient to parameterize it by

$$f(X, \vec{p}) \equiv f_{\text{eq}}(X, \vec{p}) + f_{\text{eq}}(X, \vec{p})(1 + f_1(X, \vec{p})) f_1(X, \vec{p})$$

- Since the applied field is small, the deviation f_1 is also small
 - ▷ linearize the collision term in f_1 :

$$\begin{aligned} \mathcal{C}_{\vec{p}}[f] &= \mathbf{L}_{\vec{p}} \cdot f_1 + \mathcal{O}(f_1^2) \\ &\equiv \frac{1}{2E_{\vec{p}}} \int \frac{d^3 \vec{p}'}{(2\pi)^3 2E_{\vec{p}'}} \int \frac{d^3 \vec{k}}{(2\pi)^3 2E_{\vec{k}}} \int \frac{d^3 \vec{k}'}{(2\pi)^3 2E_{\vec{k}'}} (2\pi)^4 \delta(p + k - p' - k') \\ &\quad \times f_{\text{eq}}(X, \vec{p}) f_{\text{eq}}(X, \vec{k}) (1 + f_{\text{eq}}(X, \vec{k}')) (1 + f_{\text{eq}}(X, \vec{p}')) \\ &\quad \times \left[f_1(X, \vec{p}) + f_1(X, \vec{k}) - f_1(X, \vec{p}') - f_1(X, \vec{k}') \right] |\mathcal{M}|^2 \end{aligned}$$

Transport coefficients

- We apply an uniform electric field, hence $\vec{\nabla}_x f(X, \vec{p}) = 0$
- In order to have $\vec{j}_{\text{el}} = \sigma_{\text{el}} \vec{E}$, we must reach the stationary regime. Therefore $\partial_t f(X, \vec{p}) = 0$
- Since the applied field is small, it is legitimate to replace f by f_{eq} in the force term. Thus, the linearized Boltzmann equation reads :

$$\mathbf{L}_p \cdot f_1 = e \vec{E} \cdot \vec{\nabla}_p f_{\text{eq}}(X, \vec{p})$$

- Solve this equation (not easy, but doable numerically). Since it is a linear equation, the solution f_1 is linear in \vec{E}
- Then, one calculates the current induced by this perturbation of the particle distribution,

$$\vec{j}_{\text{el}} = e \int \frac{d^3 \vec{p}}{(2\pi)^3} \vec{v}_p f_{\text{eq}}(X, \vec{p}) (1 + f_{\text{eq}}(X, \vec{p})) f_1(X, \vec{p})$$

▷ read σ_{el} as the coefficient of proportionality between \vec{j}_{el} and \vec{E}