Quark-Gluon Plasma and Heavy Ion Collisions

III – Hydrodynamical behavior

François Gelis

CEA/Saclay

General outline

- I : Introduction to Heavy Ion Collisions
- II : QCD at finite temperature
- III : Hydrodynamical behavior
- IV : Kinetic theory

Lecture III

- Relativistic hydrodynamics
- Phenomenological results
- Viscous corrections

Relativistic hydrodynamics

Thermodynamics

- Differential of the internal energy U:

$$dU = -PdV + TdS + \mu_B dN_B$$

  - P, V = pressure, volume
  - T, S = temperature, entropy
  - μ_B, N_B = baryonic chemical potential, baryon number

  Note: the rest energy (mc^2) is included in the internal energy

- The energy U is an extensive function of V, S, N_B:

$$U(\lambda V, \lambda S, \lambda N_B) = \lambda U(V, S, N_B)$$

- Taking the derivative with respect to λ at $\lambda = 1$:

$$U = -PV + TS + \mu_B N_B$$

- One also gets the Gibbs-Duhem relation:

$$VdP = SdT + N_B d\mu_B$$

Thermodynamics

- In hydrodynamics, the relevant quantities are the densities $\epsilon = U/V$, $s = S/V$ and $n_B = N_B/V$ (intensive quantities)

- By dividing the previous thermodynamical relations by V, one gets:

$$\begin{aligned} \epsilon &= -P + Ts + \mu_B n_B \\ dP &= sdT + n_B d\mu_B \end{aligned}$$

- Differentiating again, one obtains:

$$d\epsilon = Tds + \mu_B dn_B$$

Thermodynamics: isentropic evolution

- The entropy is conserved in the evolution of an ideal fluid: S and N_B are conserved, and only V can change
- The variations of the densities are controlled by:

$$\frac{ds}{s} = \frac{dn_B}{n_B} = -\frac{dV}{V}$$

- The variation of energy density is given by:

$$\epsilon dV + V d\epsilon = -P dV$$

- Therefore,

$$\frac{d\epsilon}{\epsilon + P} = -\frac{dV}{V} = \frac{ds}{s} = \frac{dn_B}{n_B}$$

Thermodynamics

- Global thermal equilibrium: all the thermodynamical quantities are uniform in the system, and the system is at rest ($\vec{v} = 0$)

- In hydrodynamics, the system is not in global equilibrium. However, one requests that it is in local equilibrium:

$$P, T, \mu_B \to P(t, \vec{x}), T(t, \vec{x}), \mu_B(t, \vec{x})$$

- One assumes that the thermodynamical relations involving densities and intensive quantities remain valid locally

  ▷ this requires that $P, T, \cdots$ vary very slowly

  ▷ this is satisfied if the mean free path of particles between successive collisions is much smaller than all the other lengths in the problem

Energy-momentum tensor

- Noether's theorem states that for each continuous symmetry of the Lagrangian, there is an associated conserved current J^μ, such that $\partial_\mu J^\mu = 0$

- As a consequence, the quantity

$$Q(t) \equiv \int d^3\vec{\boldsymbol{x}}\; J^0(t,\vec{\boldsymbol{x}})$$

is time independent. Proof :

$$\begin{aligned}\partial_t Q(t) &= \int d^3\vec{\boldsymbol{x}}\; \partial_t J^0(t,\vec{\boldsymbol{x}}) = -\int d^3\vec{\boldsymbol{x}}\; \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} \cdot \vec{\boldsymbol{J}}(t,\vec{\boldsymbol{x}}) \\ &= -\oint d^2\vec{\boldsymbol{S}} \cdot \vec{\boldsymbol{J}}(t,\vec{\boldsymbol{x}}) = 0\end{aligned}$$

- Note : the spatial vector $\vec{\boldsymbol{J}}$ describes the flow of the quantity Q across a surface

Energy-momentum tensor

- In a theory invariant under translations in time and position, the energy and the momentum are conserved quantities
- For each direction ν, there is a conserved current, denoted $T^{\mu\nu}$, called the energy-momentum tensor, that obeys

$$\partial_\mu T^{\mu\nu} = 0$$

- The integral over space of the zero component gives the 4-momentum of the system

$$P^\nu = \int d^3\vec{\boldsymbol{x}}\, T^{0\nu}(t, \vec{\boldsymbol{x}})$$

- The vector $T^{i\nu}$ (i=1,2,3) represents the flow of the component ν of momentum. For $\nu = 0$, this is an energy flow. For $\nu = 1, 2, 3$, this is a 3-momentum flow and it is thus related to pressure

Energy-momentum tensor

- Consider a fluid cell at rest, of volume δV. It has an energy $\delta P^0 = \epsilon\, \delta V$ and a 3-momentum $\delta \vec{\boldsymbol{P}} = 0$. This can be achieved if the energy momentum tensor has the following components :

$$T^{00} = \epsilon \quad , \quad T^{0i} = 0$$

- The flow of momentum P^i across an element of surface $d\vec{\boldsymbol{S}}$ is $dP^i = dS^j T^{ji}$. From the definition of the pressure p, this must be equal to $p dS^i$. Hence $T^{ij} = p\delta^{ij}$.
- Therefore, in the local rest frame of the fluid :

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Energy-momentum tensor

- In the laboratory frame where the fluid 4-velocity is v^μ, the energy-momentum tensor can only be built from the symmetric tensors $g^{\mu\nu}$ and $v^\mu v^\nu$. In the local rest frame ($v^\mu = (1, 0, 0, 0)$), we must recover the previous expression. Therefore :

$$T^{\mu\nu} = (p + \epsilon)\, v^\mu\, v^\nu - p\, g^{\mu\nu}$$

- Note : this expression is valid only for an ideal fluid, with no dissipative phenomena. In a viscous fluid, there can be a transport of momentum due to the friction of fluid layers that move at different velocities. This is taken into account by additional terms in $T^{\mu\nu}$ that are proportional to the derivatives $\partial_i v^j$, multiplied by the viscosity η.

Ideal hydrodynamics

- The fundamental equation of hydrodynamics is simply the conservation of energy-momentum,

$$\partial_\mu T^{\mu\nu} = 0$$

and of the baryon number,

$$\partial_\mu (n_B v^\mu) = 0$$

- The unknown functions are :
  - $p(t, \vec{x})$, $\epsilon(t, \vec{x})$, $n_B(t, \vec{x})$
  - $v^\mu(t, \vec{x})$ (3 unknowns only, since $v_\mu v^\mu = 1$)

- $\partial_\mu T^{\mu\nu} = 0$ and $\partial_\mu (n_B v^\mu) = 0$ give only 5 equations

- An additional constraint comes from the equation of state of the matter under consideration, as a relation between the local pressure p and energy density ϵ

Ideal hydrodynamics

- Non-relativistic limit,
  - $v^\mu \approx (1, \vec{\boldsymbol{v}})$
  - ϵ becomes the mass density ρ
  - the pressure p is much smaller than the energy density ϵ

It is easy to check that the above equation is equivalent to the continuity equation for mass and to Euler's equation :

$$\nu = 0 \ : \qquad \partial_t \rho + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} \cdot (\rho \vec{\boldsymbol{v}}) = 0$$

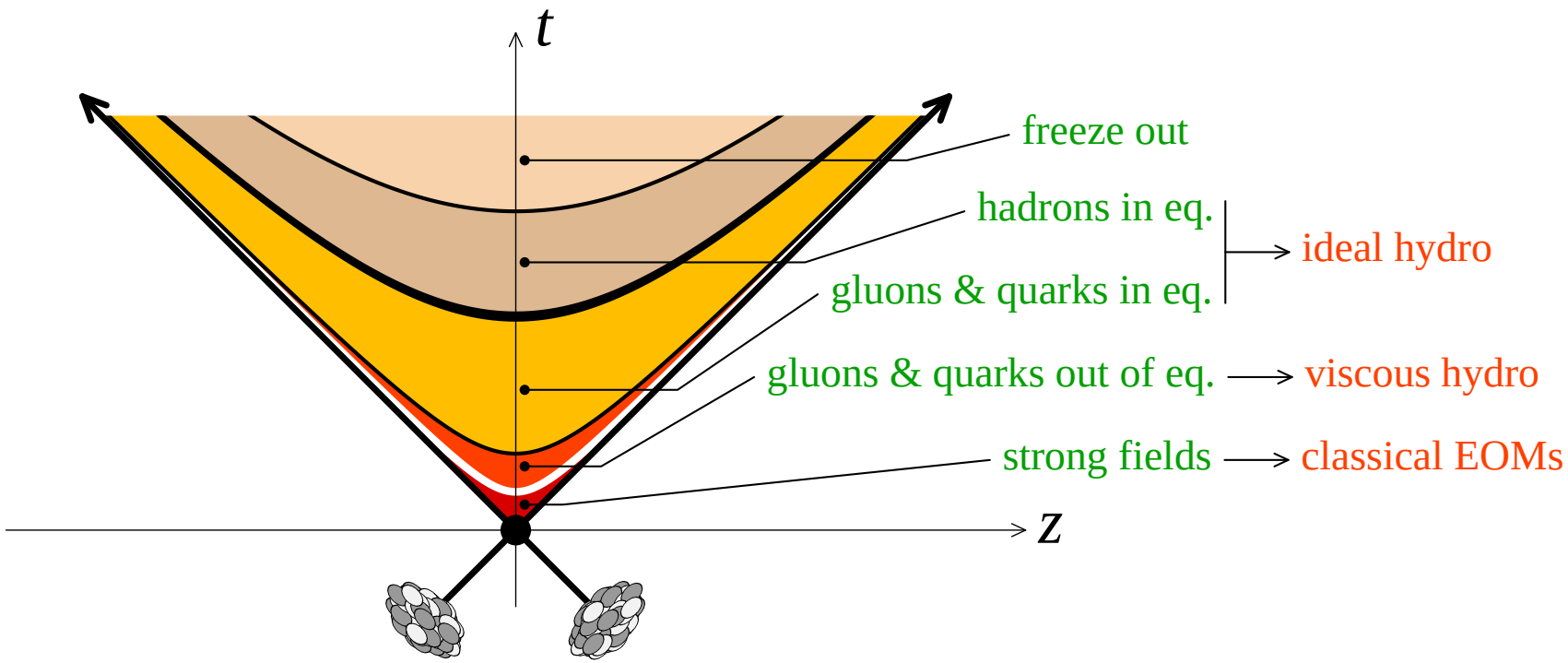
$$\nu = i \ : \qquad \partial_t (\rho v^i) + \partial_j (\rho v^i v^j) + \partial_i p = 0$$

Note : the second equation can be cast into the more familiar form

$$\rho \left[\partial_t + \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} \right] \vec{\boldsymbol{v}} + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} p = 0$$

Initial conditions

- An initial condition $p_0(\vec{x}), \epsilon_0(\vec{x}), \vec{v}_0(\vec{x}), n_{B0}(\vec{x})$ must be specified at a certain time τ_0. Since the relativistic Euler equation contains only first derivatives in time, this is sufficient to obtain the solution at any time $\tau > \tau_0$.



Initial conditions

- All the particles are produced in a very short interval at $t = z = 0$

- For a uniform longitudinal motion, the longitudinal velocity is

$$v_z = \frac{z}{t}$$

(Note: this condition is invariant under boosts in the z direction)

- One uses usually proper time and rapidity:

$$\begin{aligned} t &= \tau \cosh \eta \\ z &= \tau \sinh \eta \\ v_z &= \tanh Y \end{aligned}$$

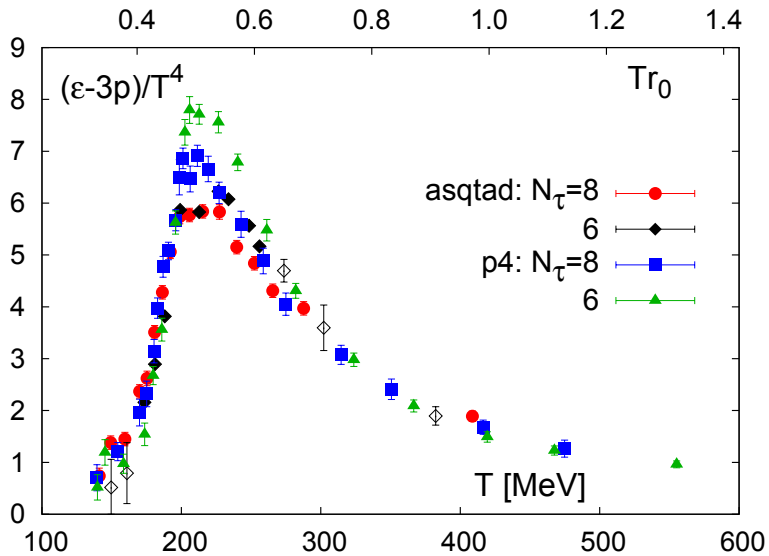
(Note: $v_z = z/t$ becomes $Y = \eta$)

Equation of state

- For a free gas of massless particles: $\epsilon = 3p$ This is easily checked from

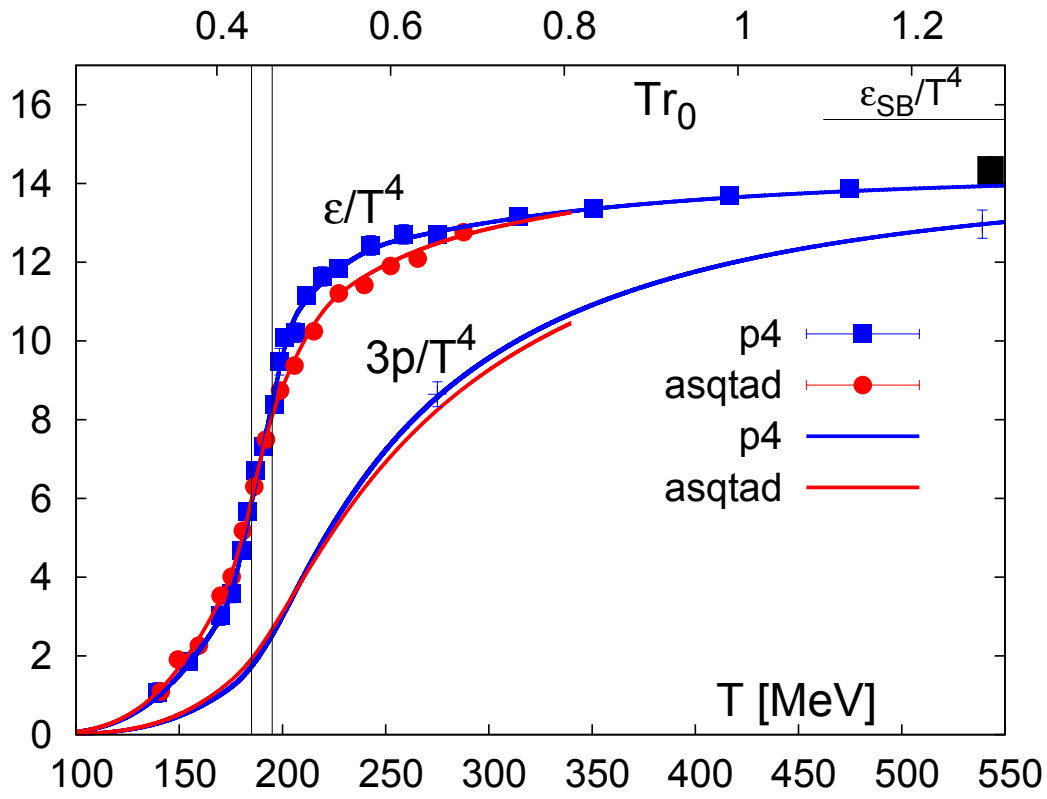
$$T^{\mu\nu} = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3} \frac{p^\mu p^\nu}{|\vec{\boldsymbol{p}}|} f(|\vec{\boldsymbol{p}}|)$$

- This relation is approximately true at high T (where interactions become weak thanks to asymptotic freedom), but deviations are important near the phase transition:



Equation of state

- In practical simulations, one can use an equation of state obtained from lattice calculations:



Sound propagation

- Consider a small perturbation on top of a static fluid :

$$p = p_0 + p'$$
$$\epsilon = \epsilon_0 + \epsilon'$$

- The Euler equation, linearized in the perturbations, reads :

$$\partial_t \epsilon' + (p_0 + \epsilon_0)\vec{\nabla}_{\boldsymbol{x}} \cdot \vec{\boldsymbol{v}}' = 0$$
$$(p_0 + \epsilon_0)\partial_t \vec{\boldsymbol{v}}' + \vec{\nabla}_{\boldsymbol{x}}\, p' = 0$$

- Differentiate the 1st equation with respect to time, and eliminate the velocity $\vec{\boldsymbol{v}}'$. We get :

$$\partial_t^2\, \epsilon' = \vec{\nabla}_{\boldsymbol{x}}^2\, p'$$

- For small perturbations, write $\epsilon' = (\partial\epsilon/\partial p)_0\, p'$. Therefore,

$$\frac{1}{c_s^2}\partial_t^2\, p' = \vec{\nabla}_{\boldsymbol{x}}^2\, p' \qquad \text{with} \quad c_s^2 \equiv \left(\frac{\partial p}{\partial \epsilon}\right)_0$$

Phenomenological results

Longitudinal cooling

- Assume that the transverse velocity and its gradients are small, and that $v_z = z/t$
- At $z = 0$, the equation of conservation of baryon number is

$$\frac{\partial n_B}{\partial t} + \frac{n_B}{t} = 0$$

Therefore: $n_B t = \text{constant}$

- Similarly, for the energy density:

$$\frac{\partial \epsilon}{\partial t} + \frac{\epsilon + P}{t} = 0$$

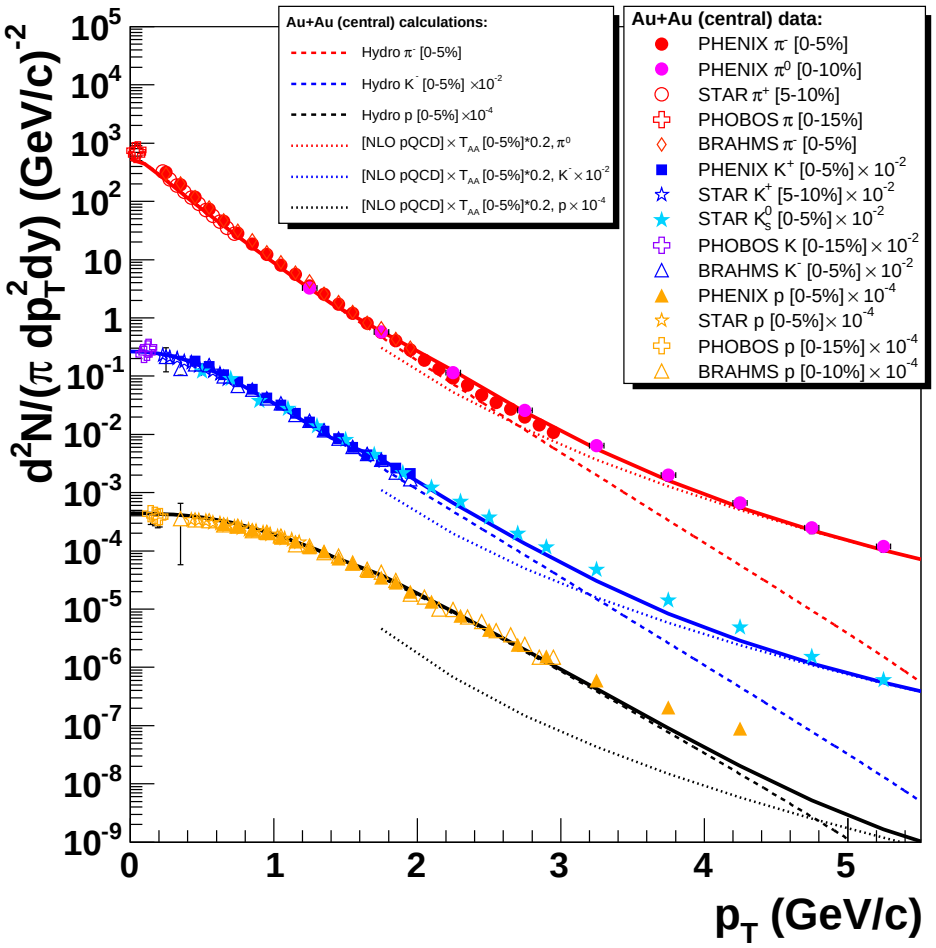
Note: now, we have: $d(\epsilon t) = -P dt \neq 0$

- For an ideal gas: $p = \epsilon/3$, and

$$\epsilon \propto \frac{1}{\tau^{4/3}}$$

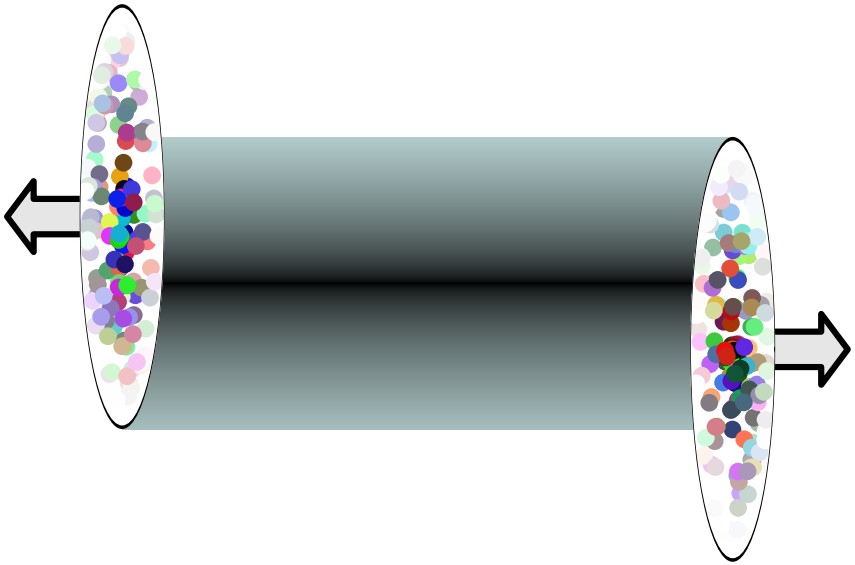
Particle spectra

- Hydrodynamics reproduces the hadron spectra at low $p_\perp$



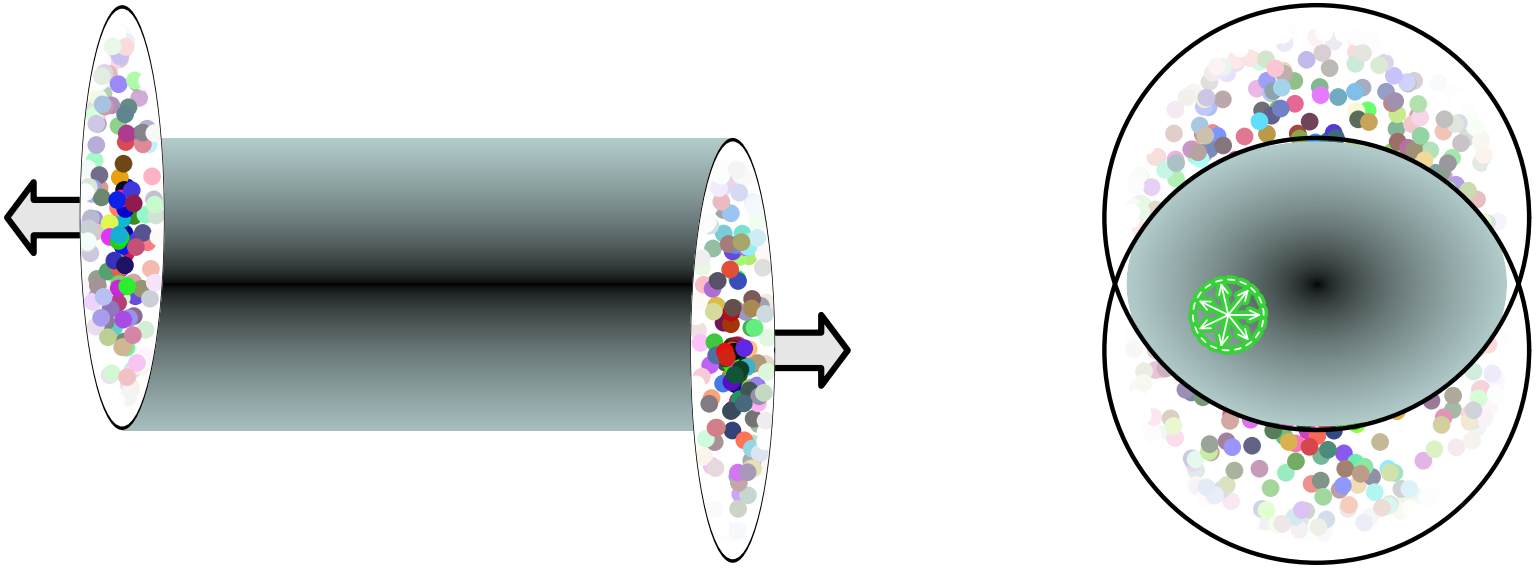
Elliptic flow

- Consider a non-central collision :



Elliptic flow

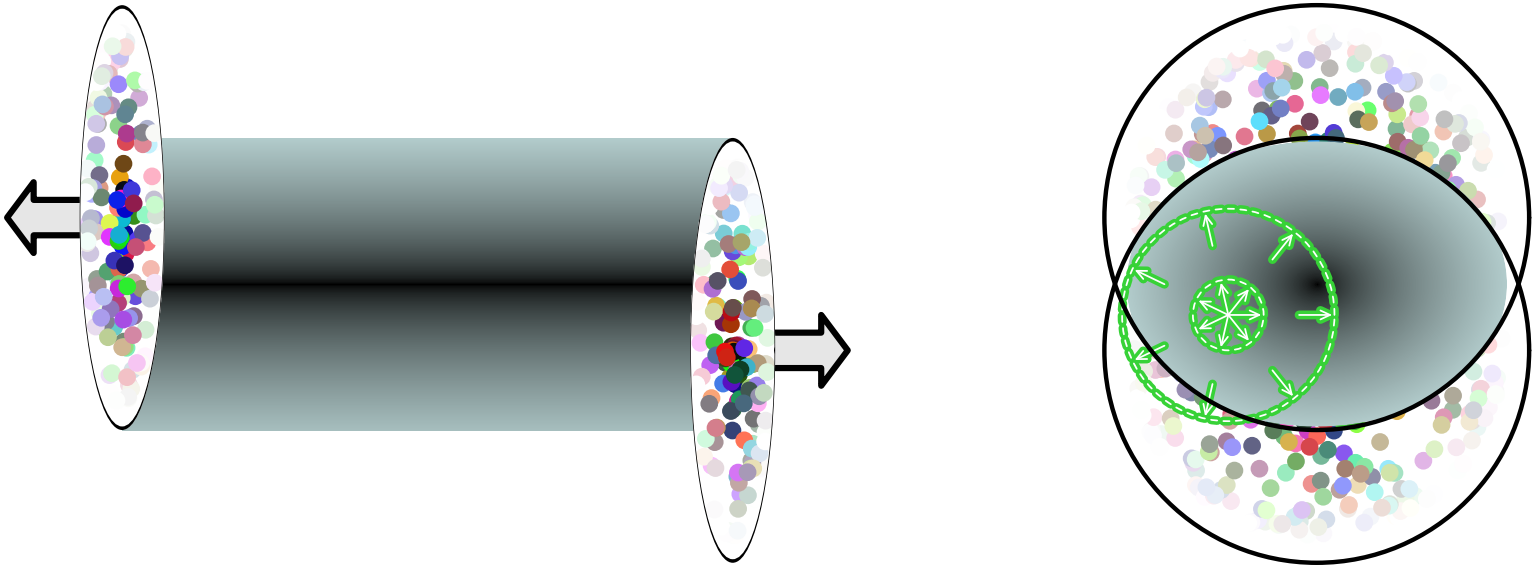
- Consider a non-central collision :



  - Initially, the momentum distribution of particles is isotropic in the transverse plane, because their production comes from local partonic interactions

Elliptic flow

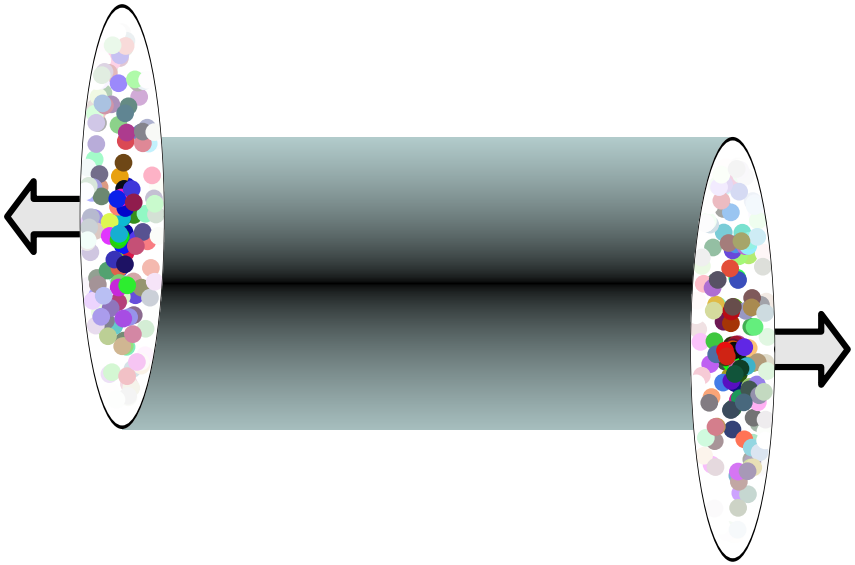
- Consider a non-central collision :

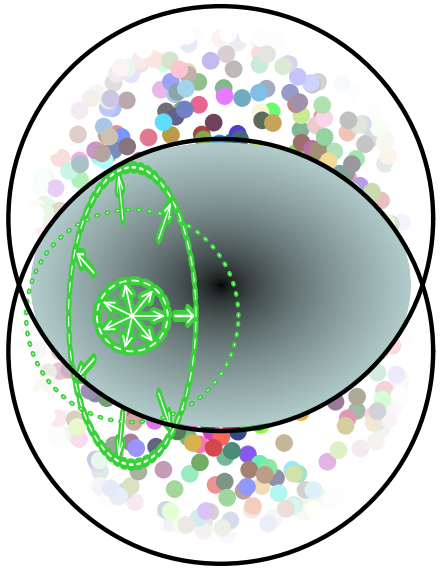


  - Initially, the momentum distribution of particles is isotropic in the transverse plane, because their production comes from local partonic interactions
  - If these particles were escaping freely, the distribution would remain isotropic at all times

Elliptic flow

- Consider a non-central collision :





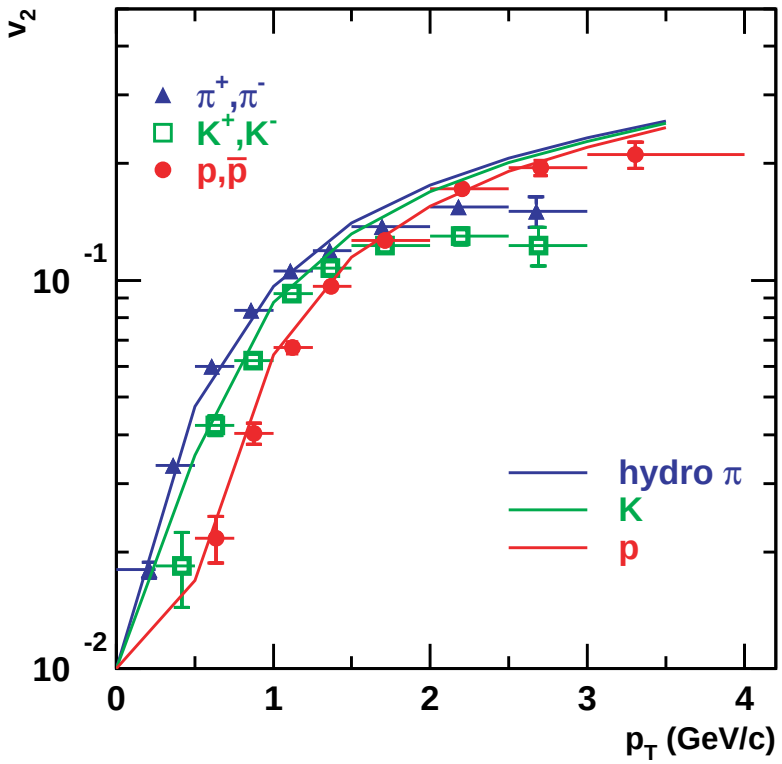
  - Initially, the momentum distribution of particles is isotropic in the transverse plane, because their production comes from local partonic interactions
  - If these particles were escaping freely, the distribution would remain isotropic at all times
  - If the system has a small mean free path, pressure gradients are anisotropic and induce an anisotropy of the distribution

Elliptic flow

- Observable: 2nd harmonic of the azimuthal distribution

$$\frac{dN}{d\varphi} \sim 1 + 2v_1 \cos(\varphi) + 2v_2 \cos(2\varphi) + \cdots$$

▷ v_2 measures the ellipticity of the momentum distribution



Viscous corrections

Is the QGP a perfect fluid?

- Note: a perfect fluid is a fluid with a very small viscosity, that can be described with Euler equations (ideal hydrodynamics)
- The elliptic flow coefficient v_2 measured at RHIC is well reproduced by ideal hydrodynamics, that has no viscosity
  - In hydrodynamics, the relevant parameter is the dimensionless ratio η/s of the shear viscosity to the entropy density
  - It has been concluded from there that the QGP must have a very small ratio η/s

Flow parameters

- Knudsen number : $K \equiv \lambda / R$
  - λ = mean free path between two collisions
  - R = size of the system
  - ▷ hydrodynamics is applicable if $K \ll 1$
- Mach number : $M \equiv v / c_s$
  - v = typical flow velocity
  - R = sound velocity
  - ▷ the flow is incompressible if $M \ll 1$
- Reynolds number : $R_e \equiv Rv/(\eta/\rho)$
  - η = shear viscosity
  - ρ = mass density
  - ▷ the flow is non-viscous if $R_e \gg 1$

Flow parameters

- Kinetic theory indicates that $\eta/\rho \sim \lambda c_s$. Therefore, the flow parameters are related by:

$$R_e\, K \sim M$$

- In heavy ion collisions, the flow velocity is comparable to the speed of sound, and $M \sim 1$. Therefore, one has

$$K \sim R_e^{-1}$$

$\triangleright$ departures from the applicability of hydrodynamics and viscous effects are related

Viscous hydrodynamics

- One can study corrections to ideal hydrodynamics by an expansion in powers of K:
  - Order zero in K: Euler equations
  - Order one in K: Navier-Stokes equations

- The Navier-Stokes equations can be obtained by adding a correction to the spatial part of the energy-momentum tensor:

$$\delta T^{ij} = \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3} \delta^{ij} (\partial \cdot \vec{\boldsymbol{v}}) \right) + \zeta \, \delta^{ij} (\partial \cdot \vec{\boldsymbol{v}})$$

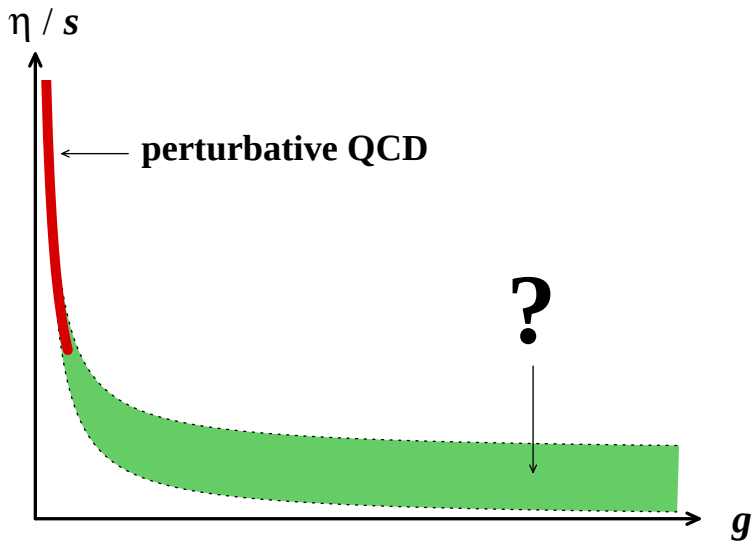
  - η = shear viscosity
  - ζ = bulk viscosity

  Note: things are more complicated in the relativistic case, due to the necessity to preserve Lorentz invariance and causality

Shear viscosity

- The shear viscosity has been calculated in QCD at weak coupling ($g \to 0$), and it is quite large :

$$\frac{\eta}{s} = \frac{5.12}{g^4 \ln\left(\frac{2.42}{g}\right)}$$



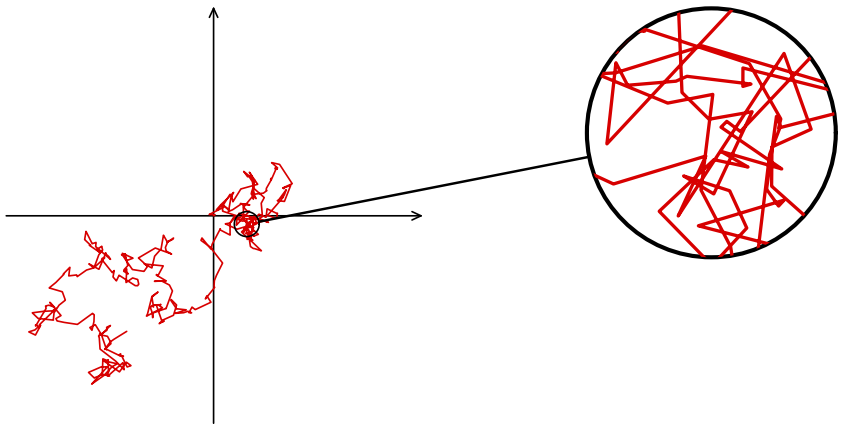
- However, η/s decreases quickly when the coupling increases $\triangleright$ one way to have a small viscosity is to have a large coupling. Problem : how to calculate it?

Lower bound for η/s

- $\eta \sim \lambda\epsilon$ ($\lambda =$ mean free path, $\epsilon =$ energy density). Thus,

$$\frac{\eta}{s} \sim \lambda \underbrace{\frac{\epsilon}{s}}_{\text{energy per particle}}$$

- Heisenberg inequalities forbid the mean free path to be smaller than the De Broglie wavelength of the particles. Scatterings by an $\mathcal{O}(1)$ angle can occur only every $\lambda_{\mathrm{Broglie}}$ at most :

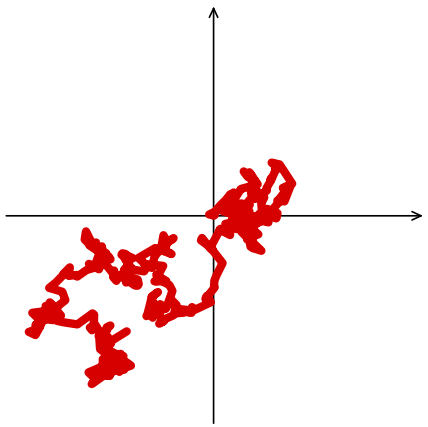


Lower bound for η/s

- $\eta \sim \lambda\epsilon$ ($\lambda =$ mean free path, $\epsilon =$ energy density). Thus,

$$\frac{\eta}{s} \sim \lambda \underbrace{\frac{\epsilon}{s}}_{\text{energy per particle}}$$

- Heisenberg inequalities forbid the mean free path to be smaller than the De Broglie wavelength of the particles. Scatterings by an $\mathcal{O}(1)$ angle can occur only every $\lambda_{\mathrm{Broglie}}$ at most :

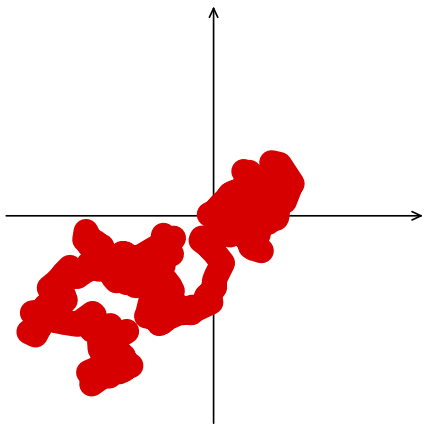


Lower bound for η/s

- $\eta \sim \lambda\epsilon$ ($\lambda =$ mean free path, $\epsilon =$ energy density). Thus,

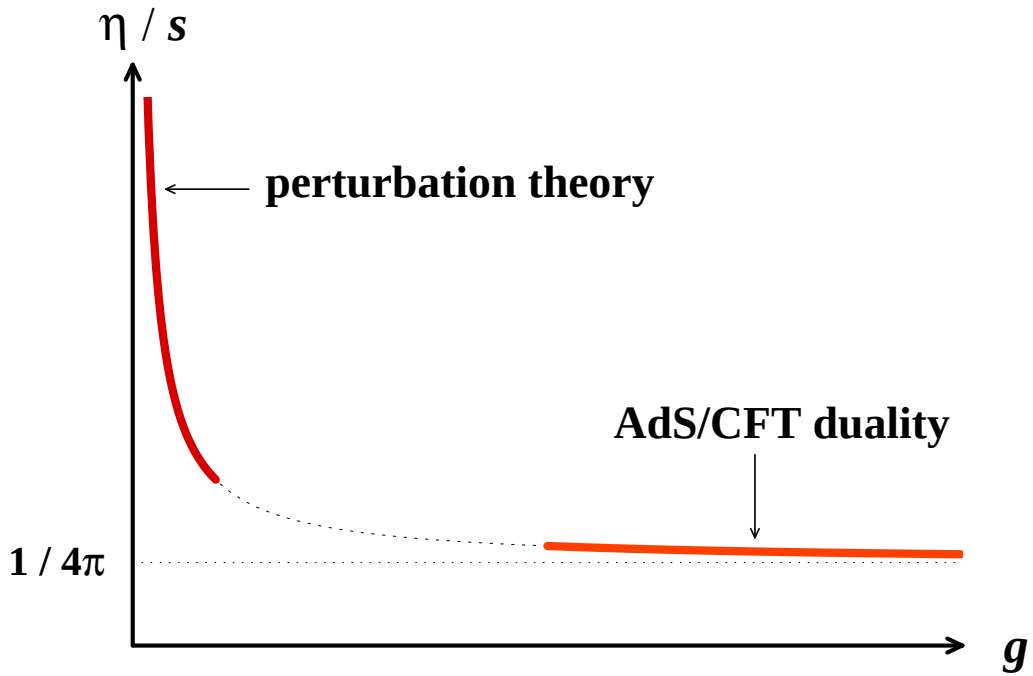
$$\frac{\eta}{s} \sim \lambda \underbrace{\frac{\epsilon}{s}}_{\text{energy per particle}}$$

- Heisenberg inequalities forbid the mean free path to be smaller than the De Broglie wavelength of the particles. Scatterings by an $\mathcal{O}(1)$ angle can occur only every $\lambda_{\mathrm{Broglie}}$ at most :



- Hence, $\dfrac{\eta}{s} \geq \mathcal{O}(1)$

Shear viscosity at strong coupling



- Conjecture : $1/4\pi$ is the lowest possible value for η/s
- The study of viscous effects on the elliptic flow v_2 suggests that at RHIC:

$$\eta/s \leq (2-3) \times \frac{1}{4\pi}$$

Lecture IV

- Collisionless transport
- Boltzmann equation
- Transport coefficients