How quantum fields start to flow in Heavy Ion Collisions

University of Cape Town, April 2013

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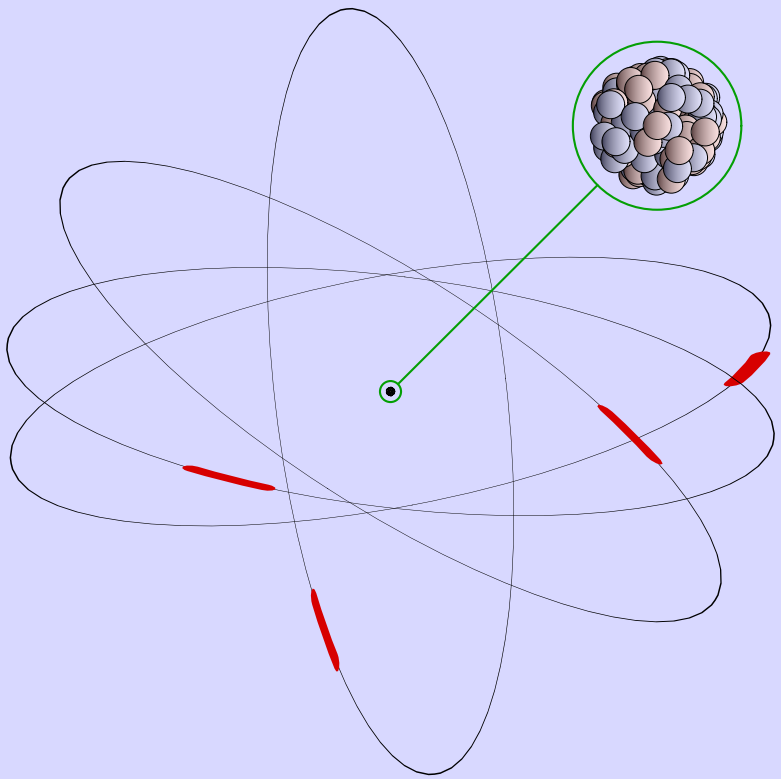
Outline

Heavy Ion Collisions

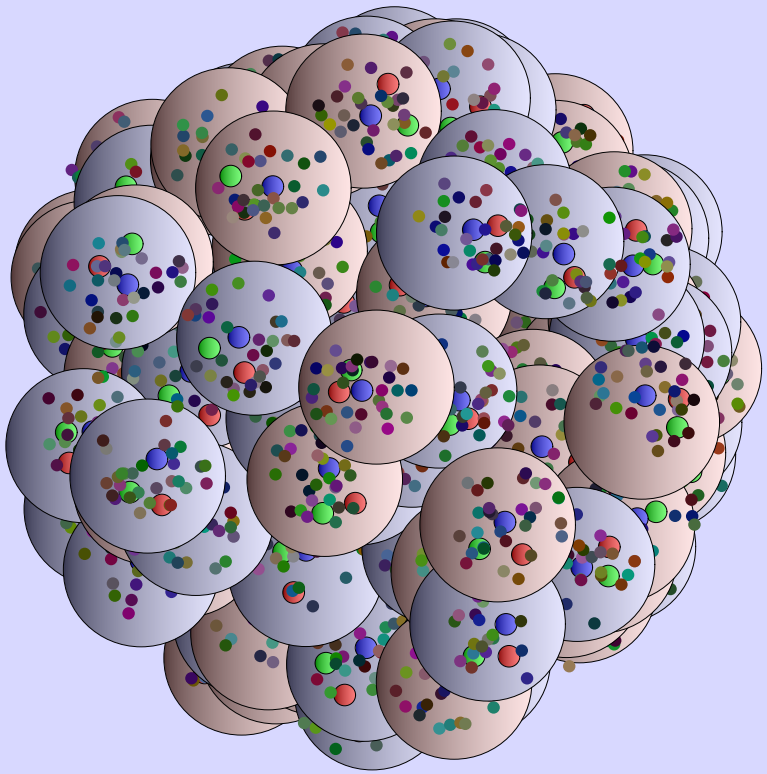
From atoms to nuclei, to quarks and gluons

10^{-10} m : atom (99.98% of the mass is in the nucleus)



From atoms to nuclei, to quarks and gluons

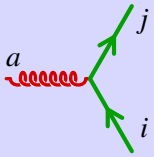
$< 10^{-15}$ m : quarks + gluons



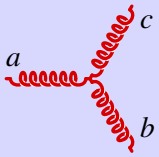
Quarks and gluons

Strong interactions : Quantum Chromo-Dynamics

- Matter : quarks ; Interaction carriers : gluons



$\sim \; g\,(t^a)_{ij}$



$\sim \; g\,(T^a)_{bc}$

- i, j : quark colors ; a, b, c : gluon colors
- $(t^a)_{ij}$: 3×3 SU(3) matrix ; $(T^a)_{bc}$: 8×8 SU(3) matrix

Lagrangian

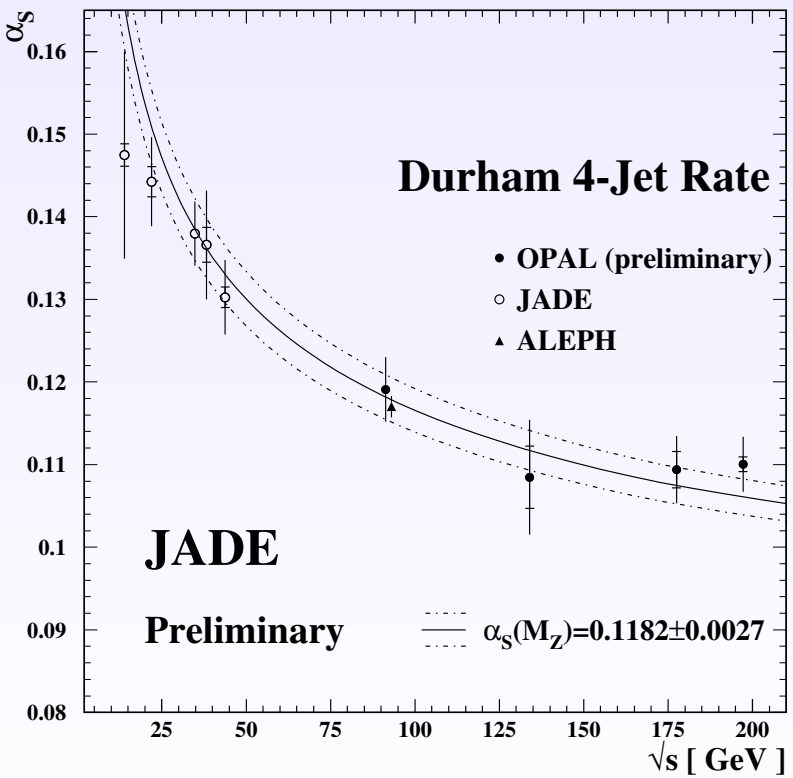
$$\mathcal{L} = -\frac{1}{4}F^2 + \sum_f \overline{\psi}_f (i\not{D} - m_f)\psi_f$$

- Free parameters : quark masses m_f, scale Λ_{QCD}

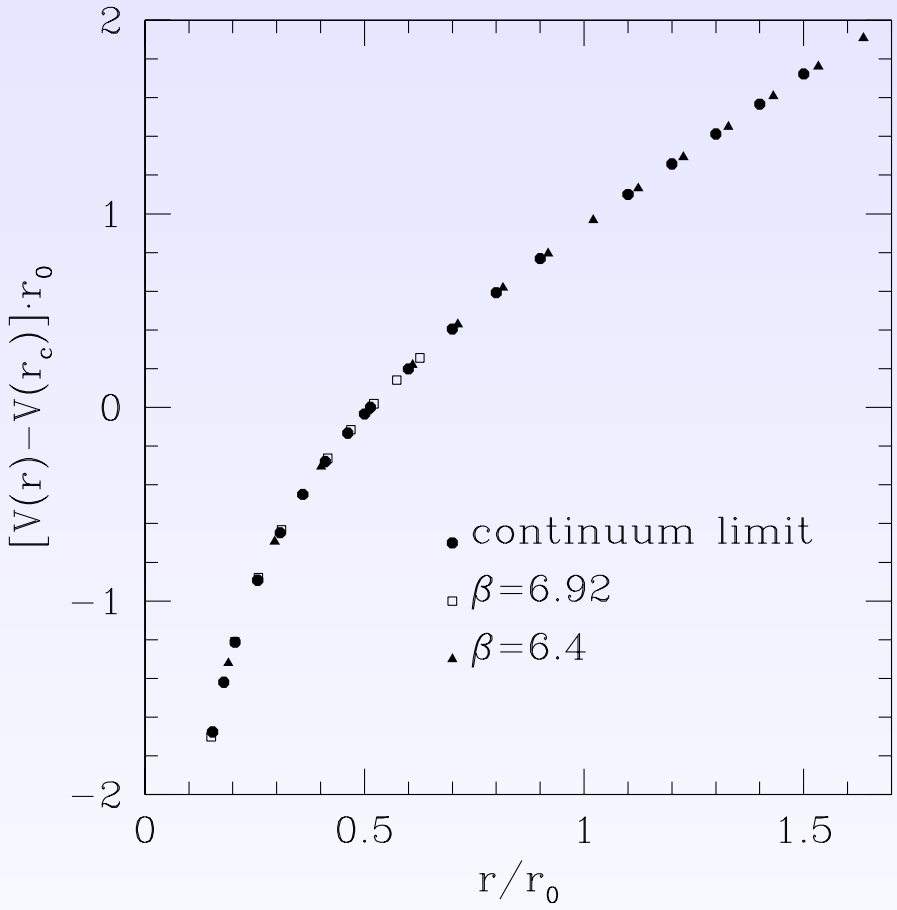
Asymptotic freedom

Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(E) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(E/\Lambda_{QCD})}$$



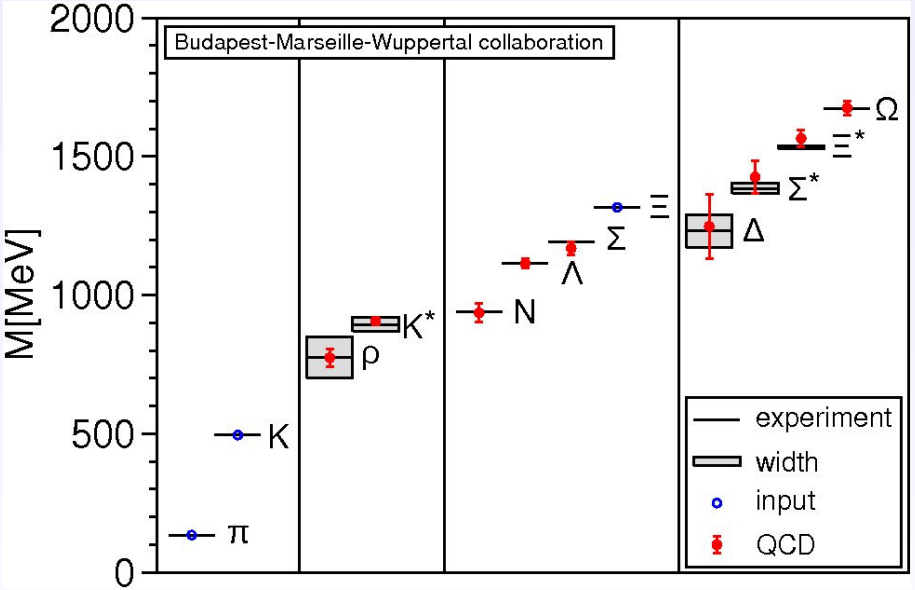
Color confinement



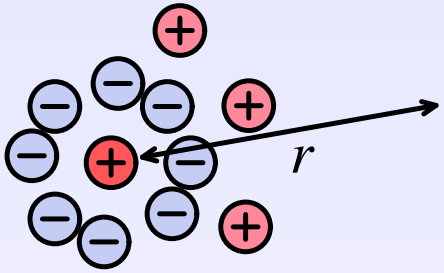
- The quark-antiquark potential increases linearly with distance

Color confinement

- In nature, we do not see free quarks and gluons (the closest we have to actual quarks and gluons are jets)
- Instead, we see hadrons (quark-gluon bound states):



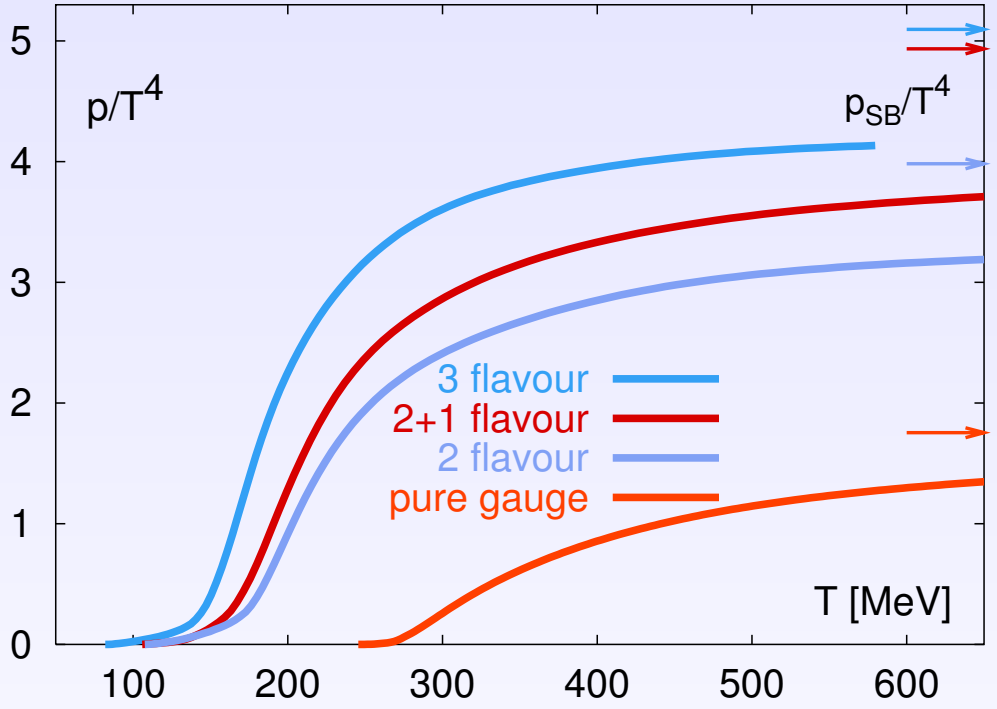
Debye screening at high density



$$V(r) = \frac{\exp(-r/r_{debye})}{r}$$

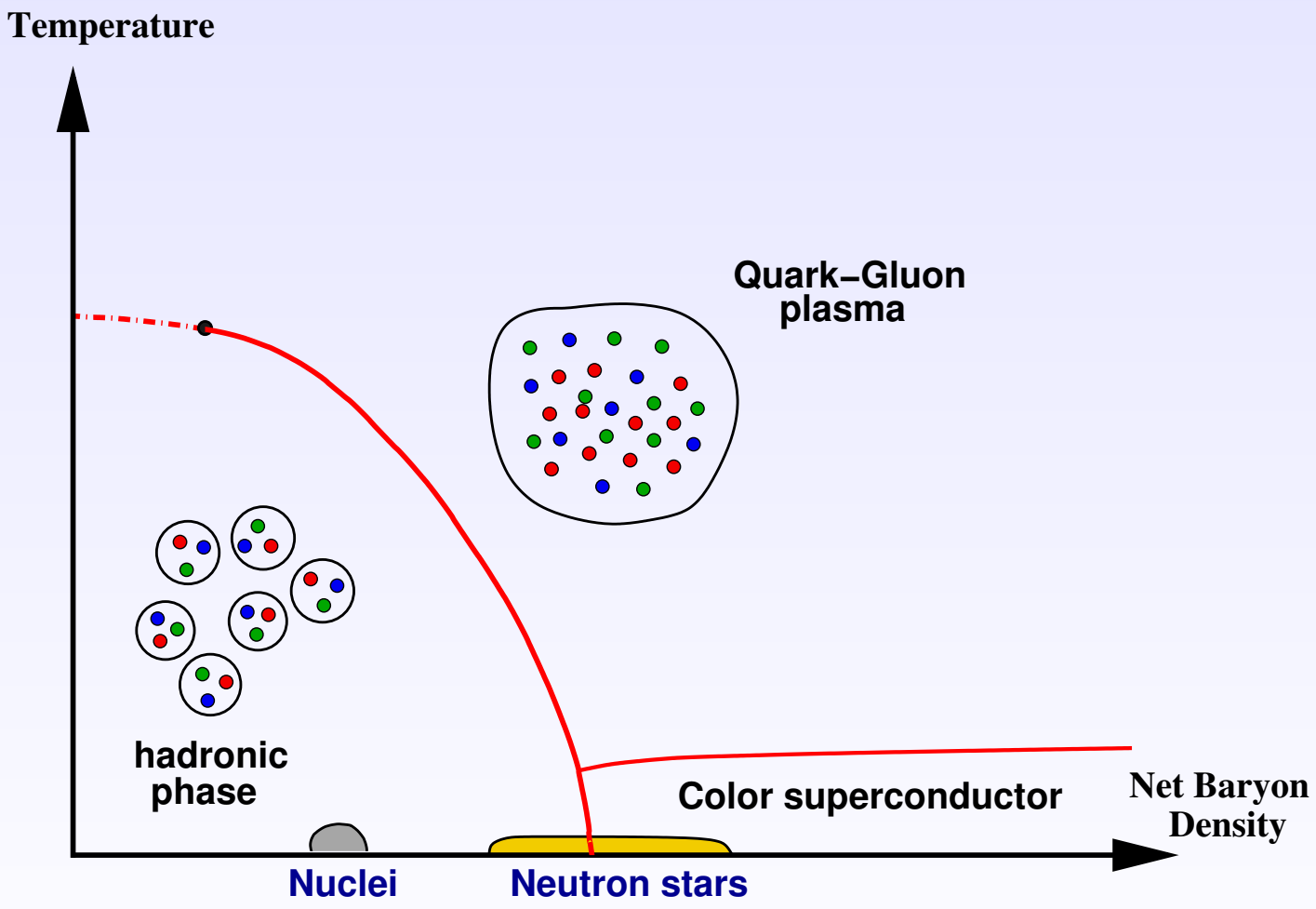
- In a dense environment, color charges are screened by their neighbours
- The Coulomb potential decreases exponentially beyond the Debye radius r_{debye}
- Bound states larger than r_{debye} cannot survive

Deconfinement transition

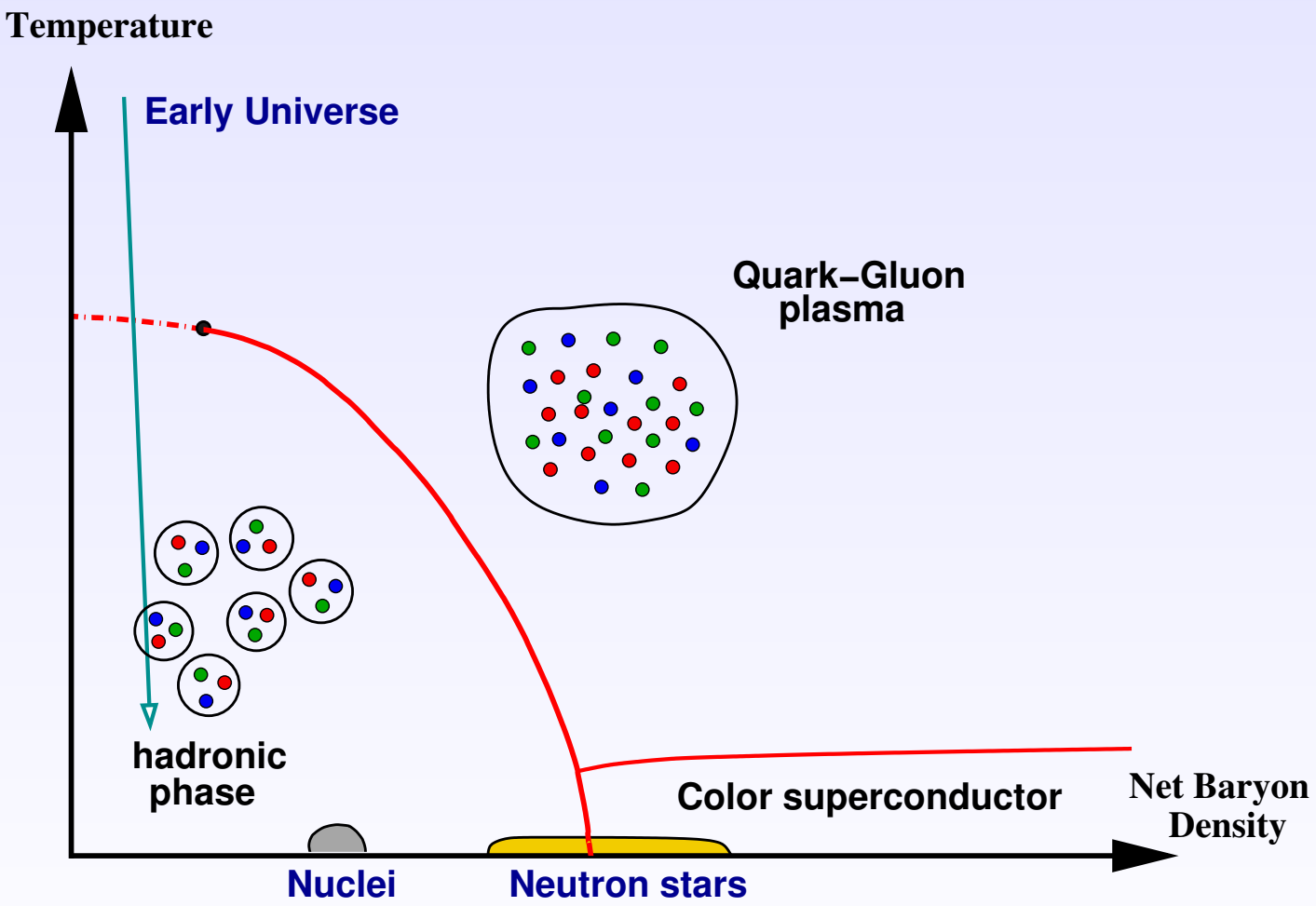


- Fast increase of the pressure :
  - at $T \sim 270$ MeV, if there are only gluons
  - at $T \sim 150$–170 MeV, depending on the number of light quarks

QCD phase diagram



QGP in the early universe



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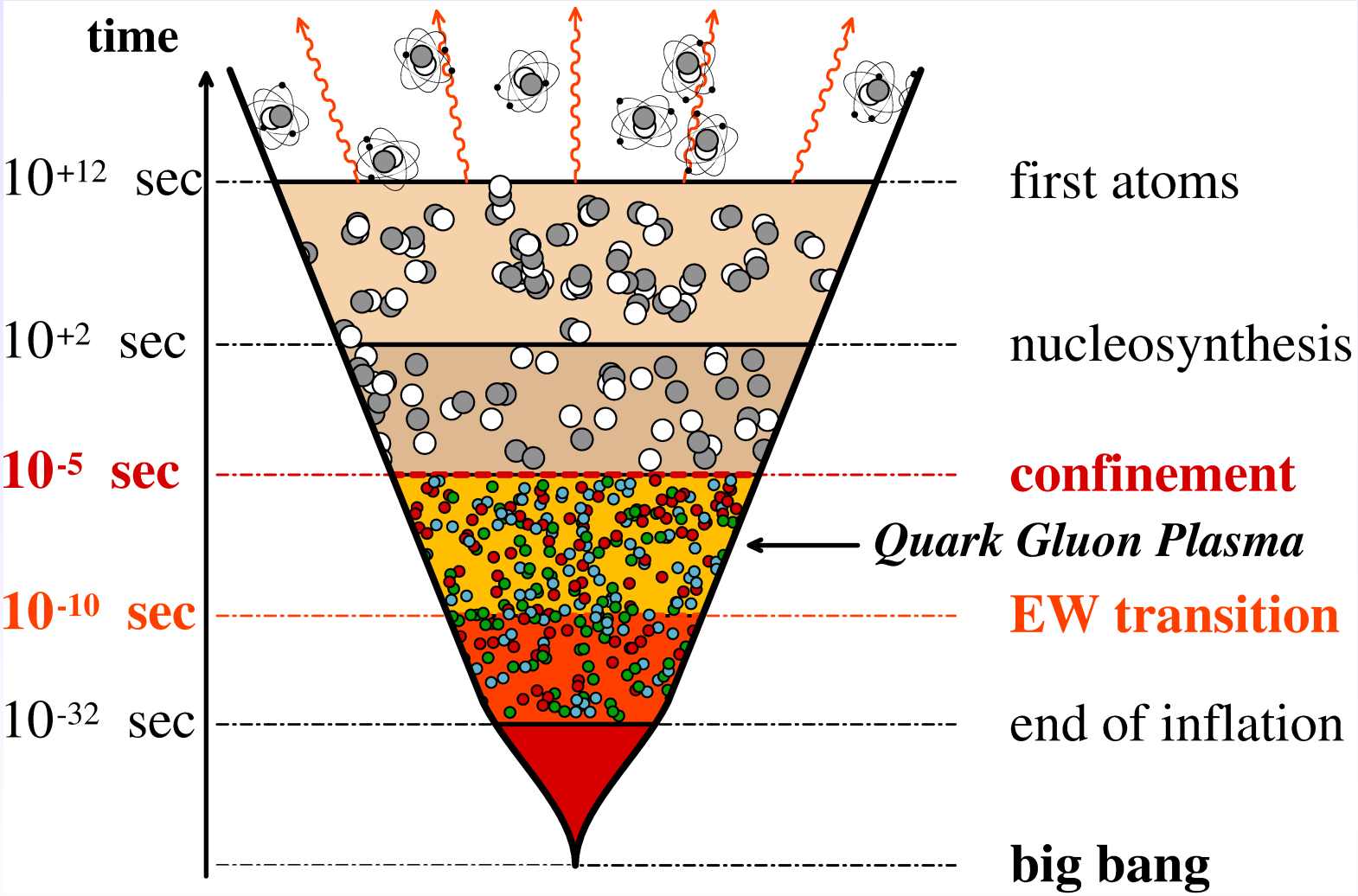
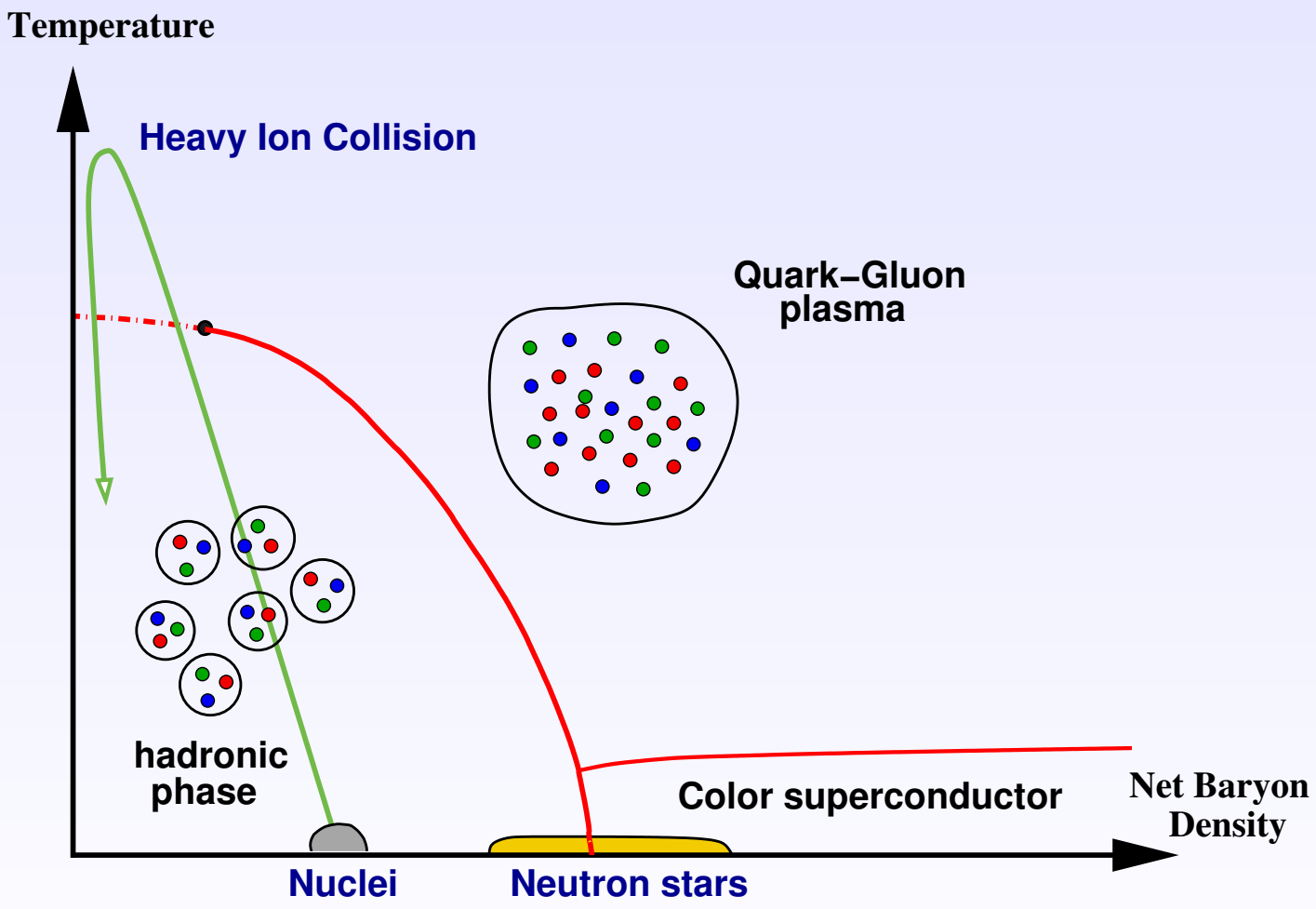
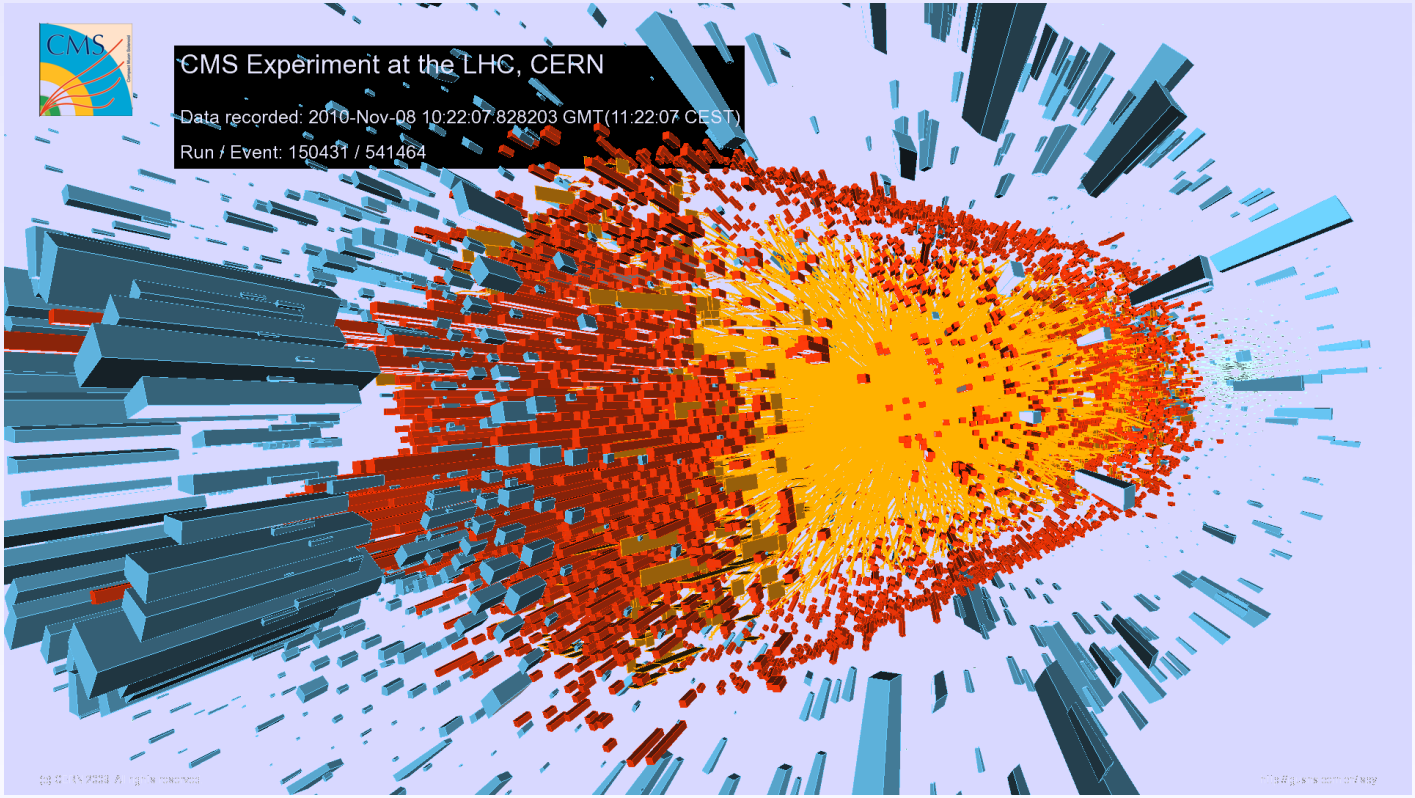


Heavy ion collisions

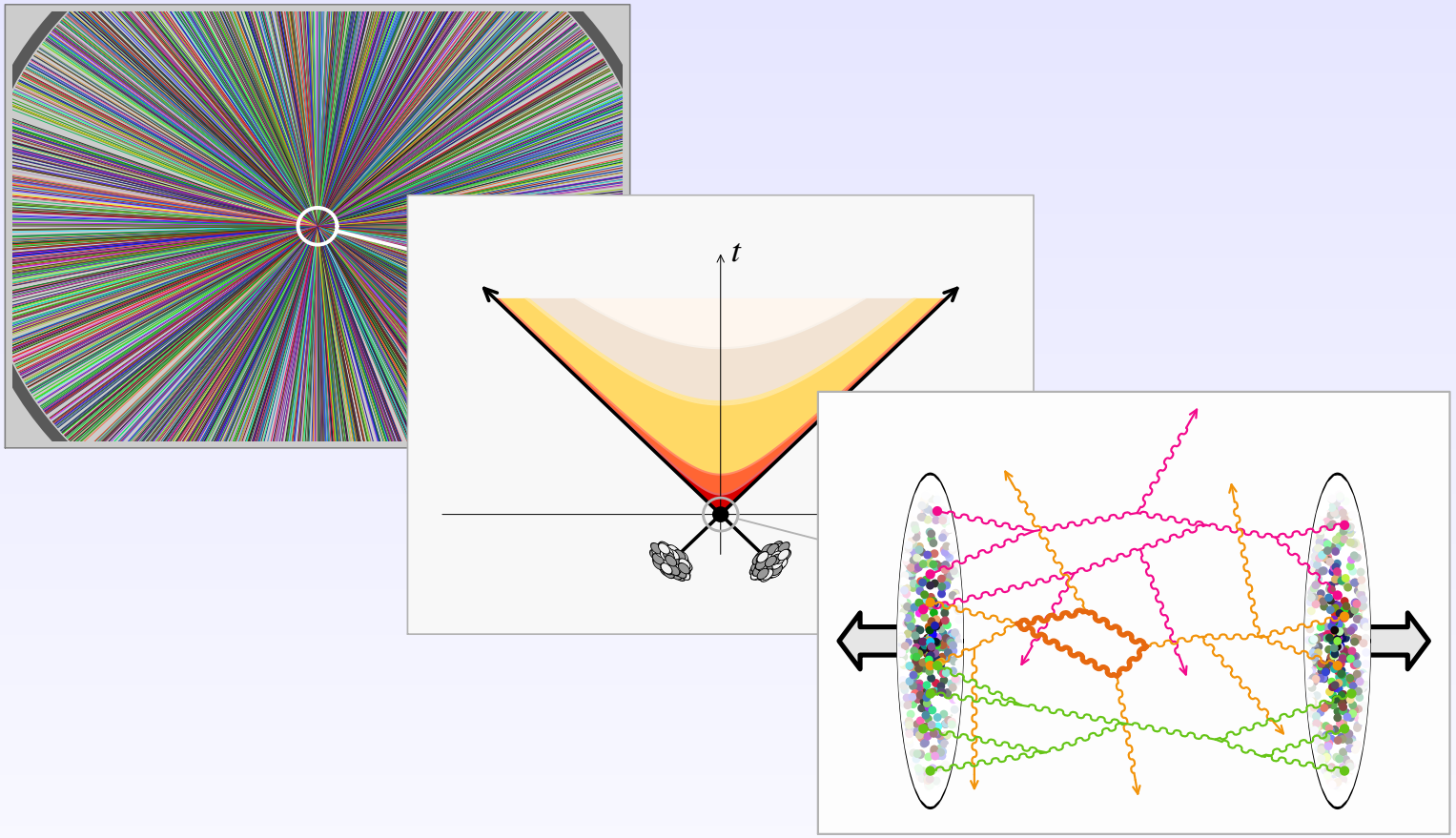


Experimental facilities : RHIC and LHC

Heavy ion collision at the LHC

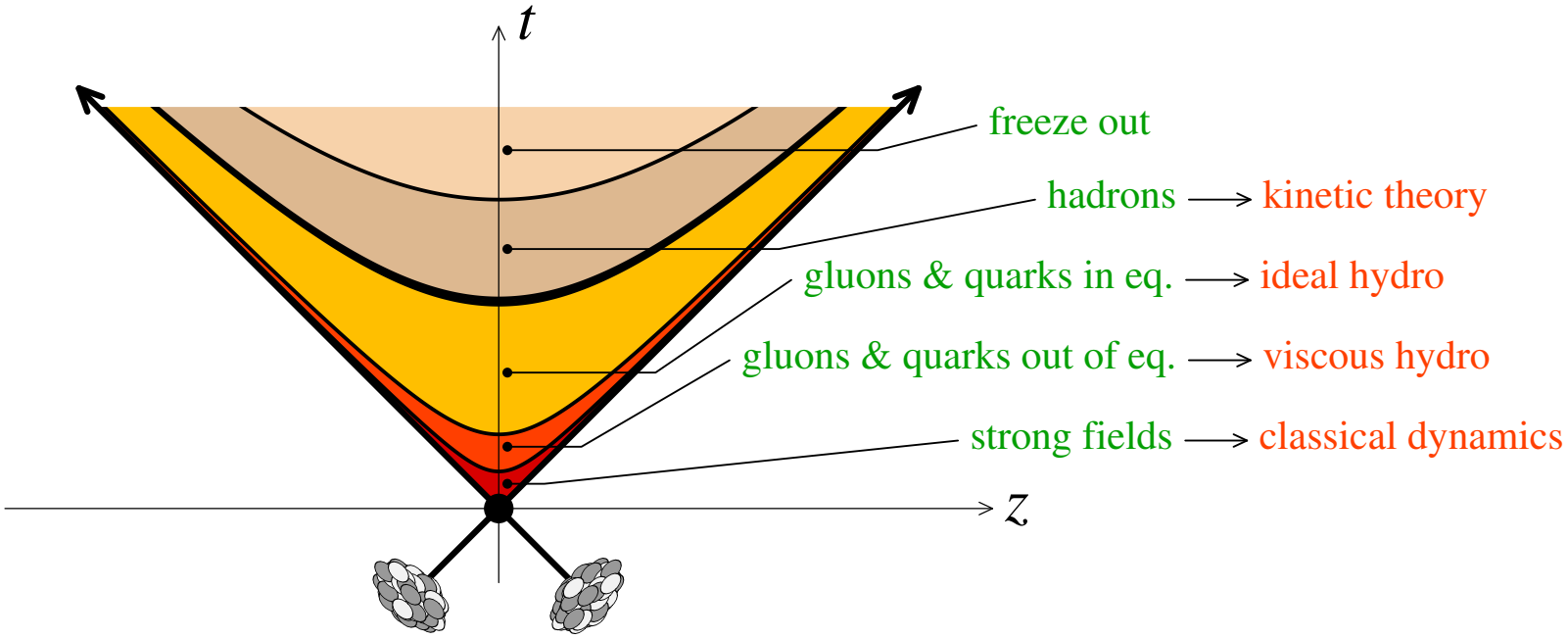


From measured hadrons back to QCD...



Goal : from the final state particles (hadrons), understand the microscopic dynamics of the quarks and gluons

Stages of a nucleus-nucleus collision



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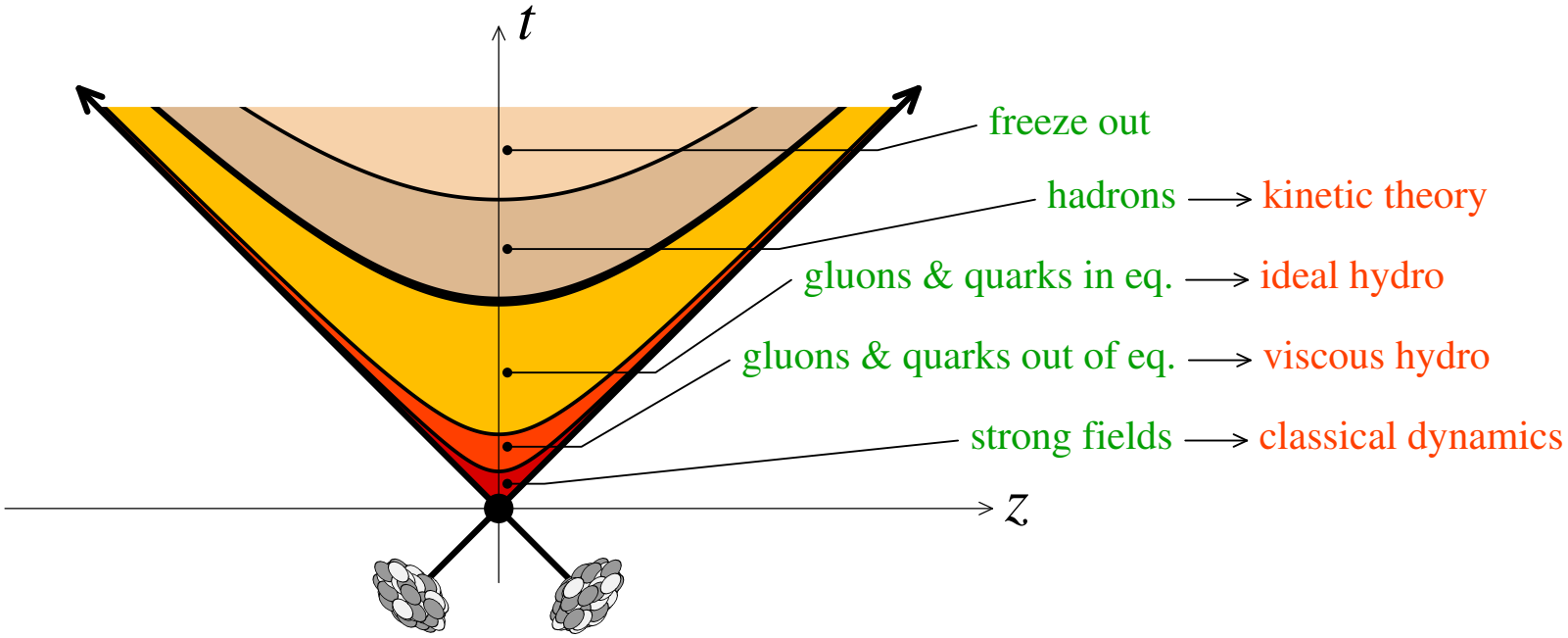
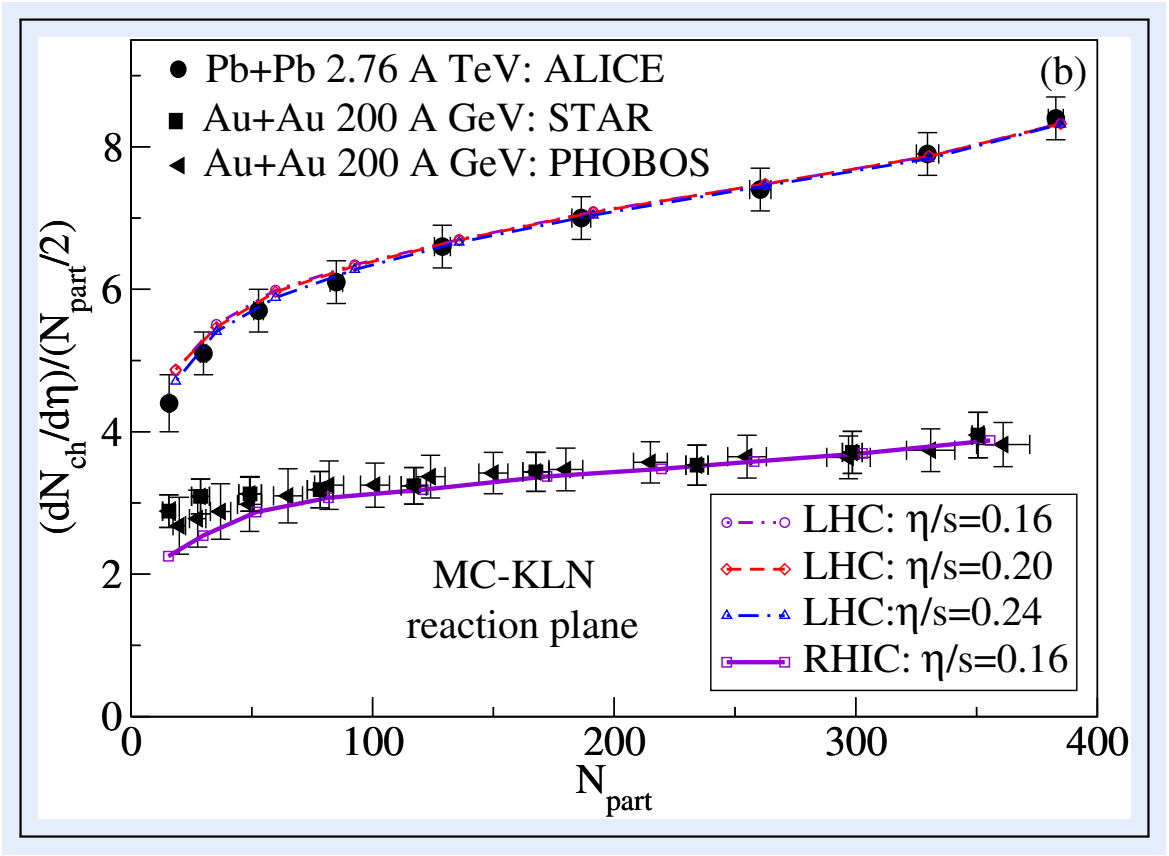


- Well described as a fluid expanding into vacuum according to relativistic hydrodynamics

Evidence for hydrodynamical behavior



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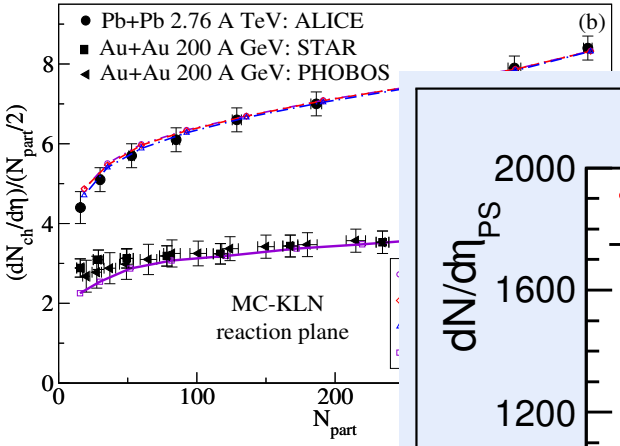

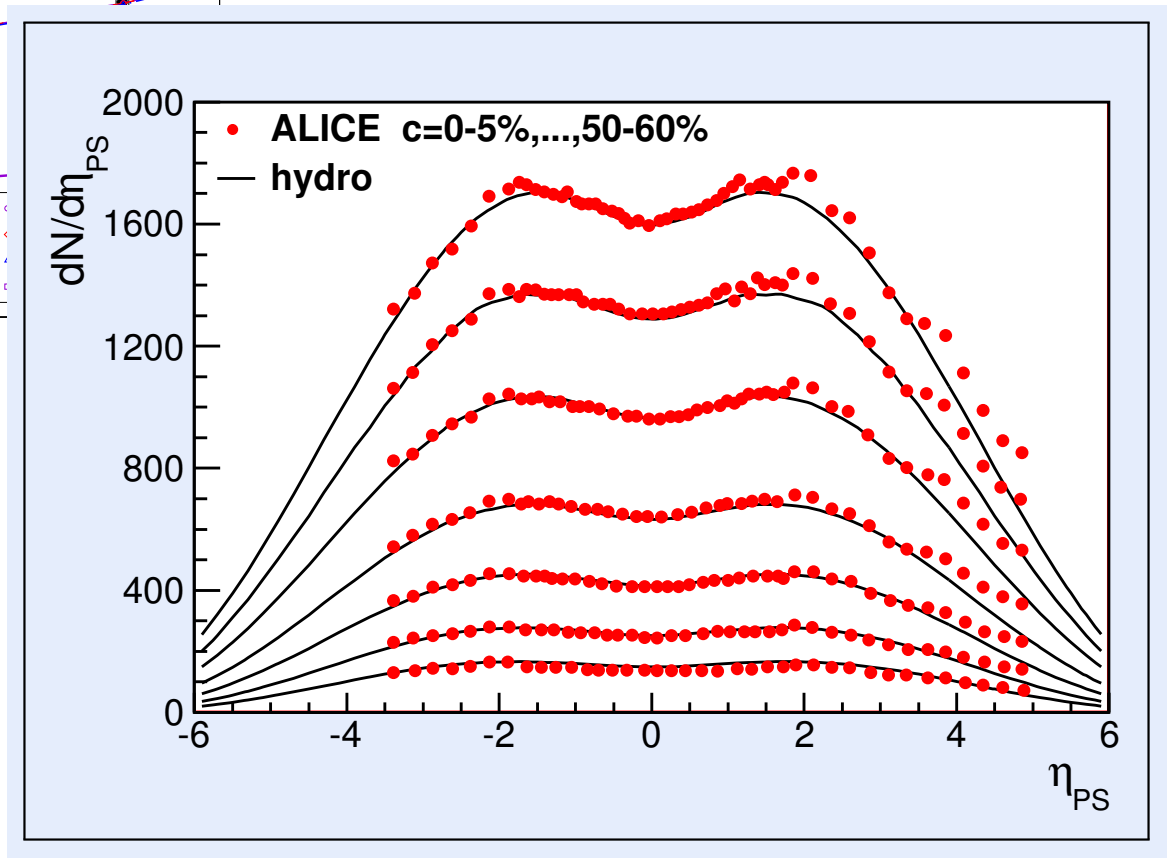

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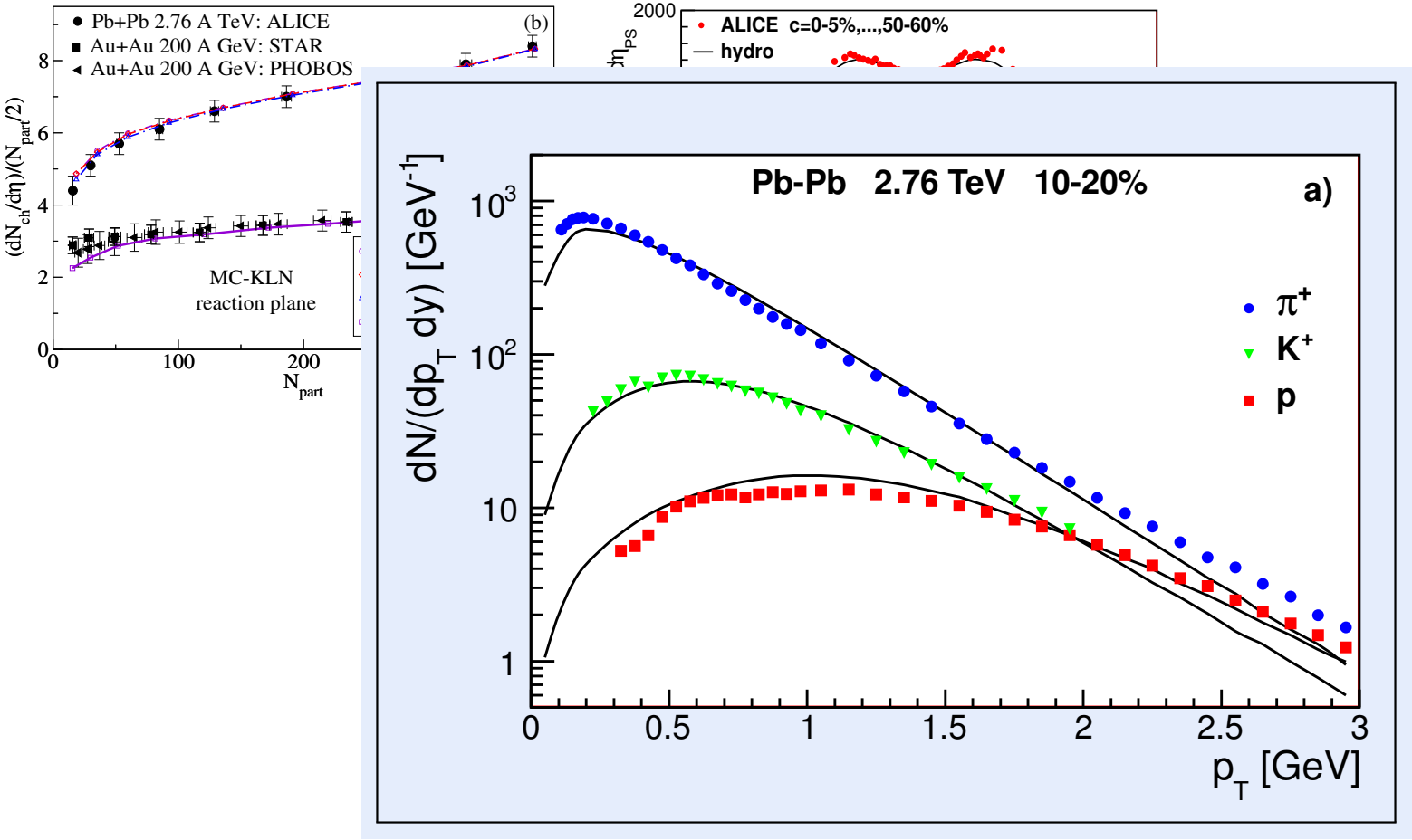

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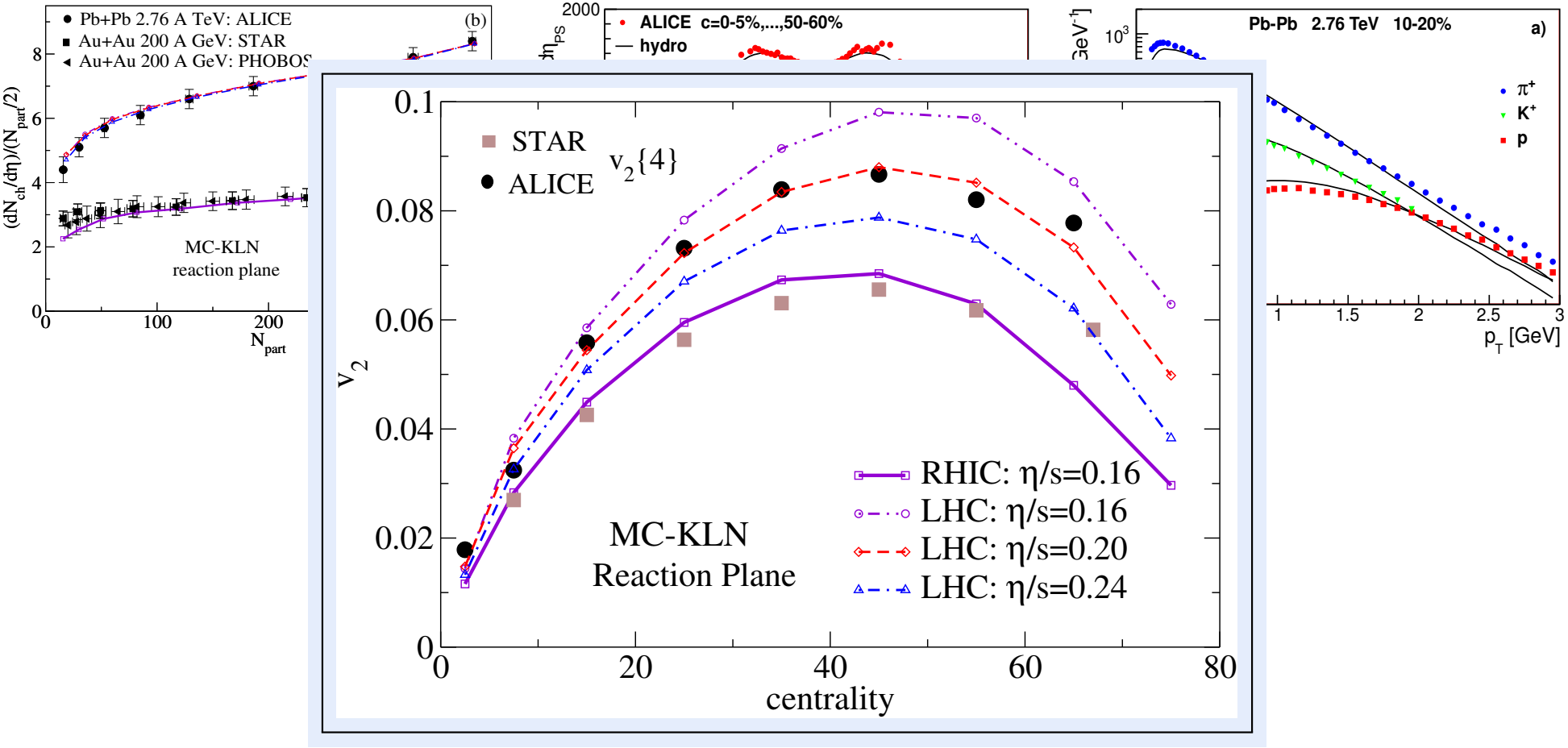

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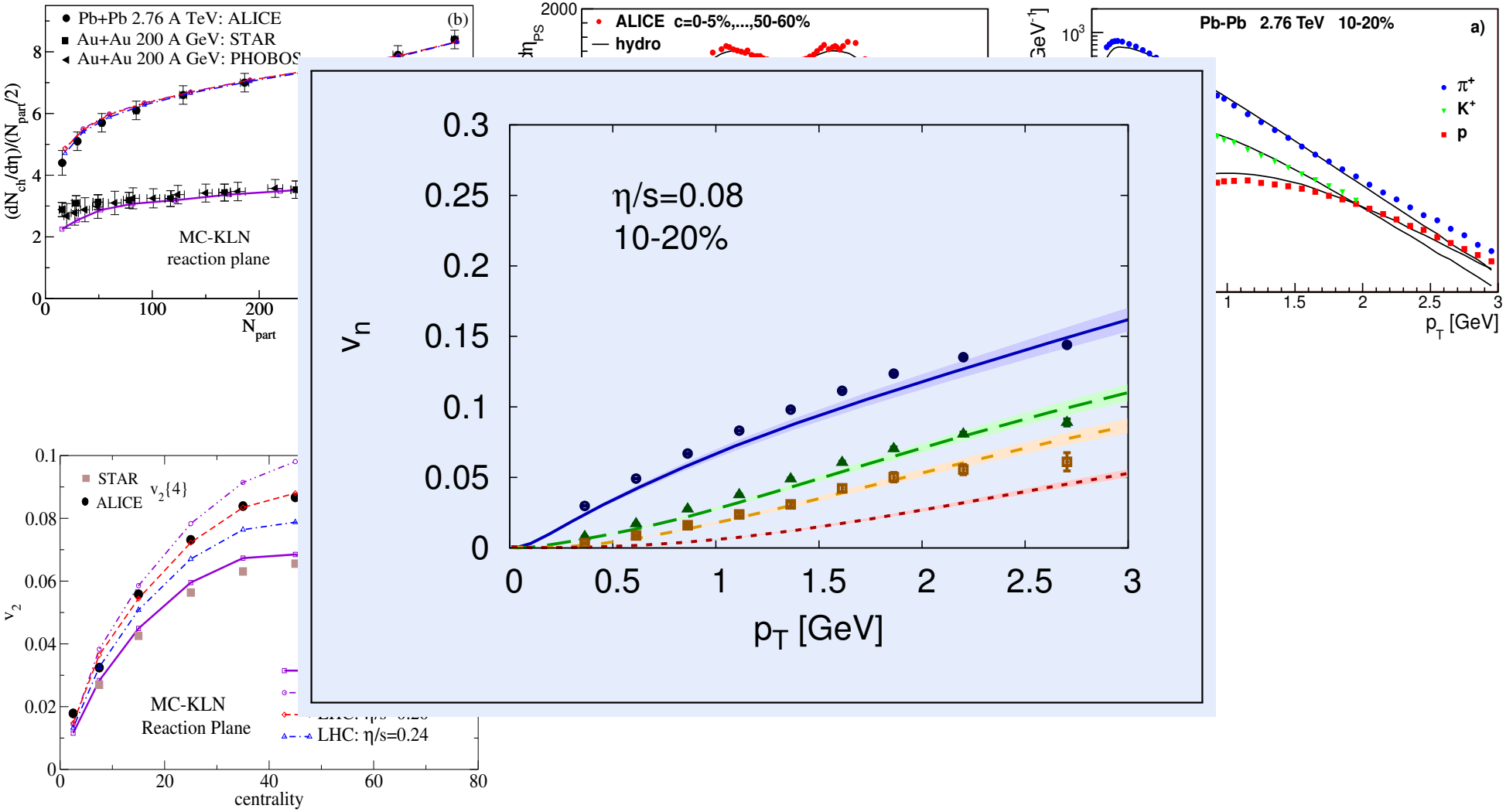

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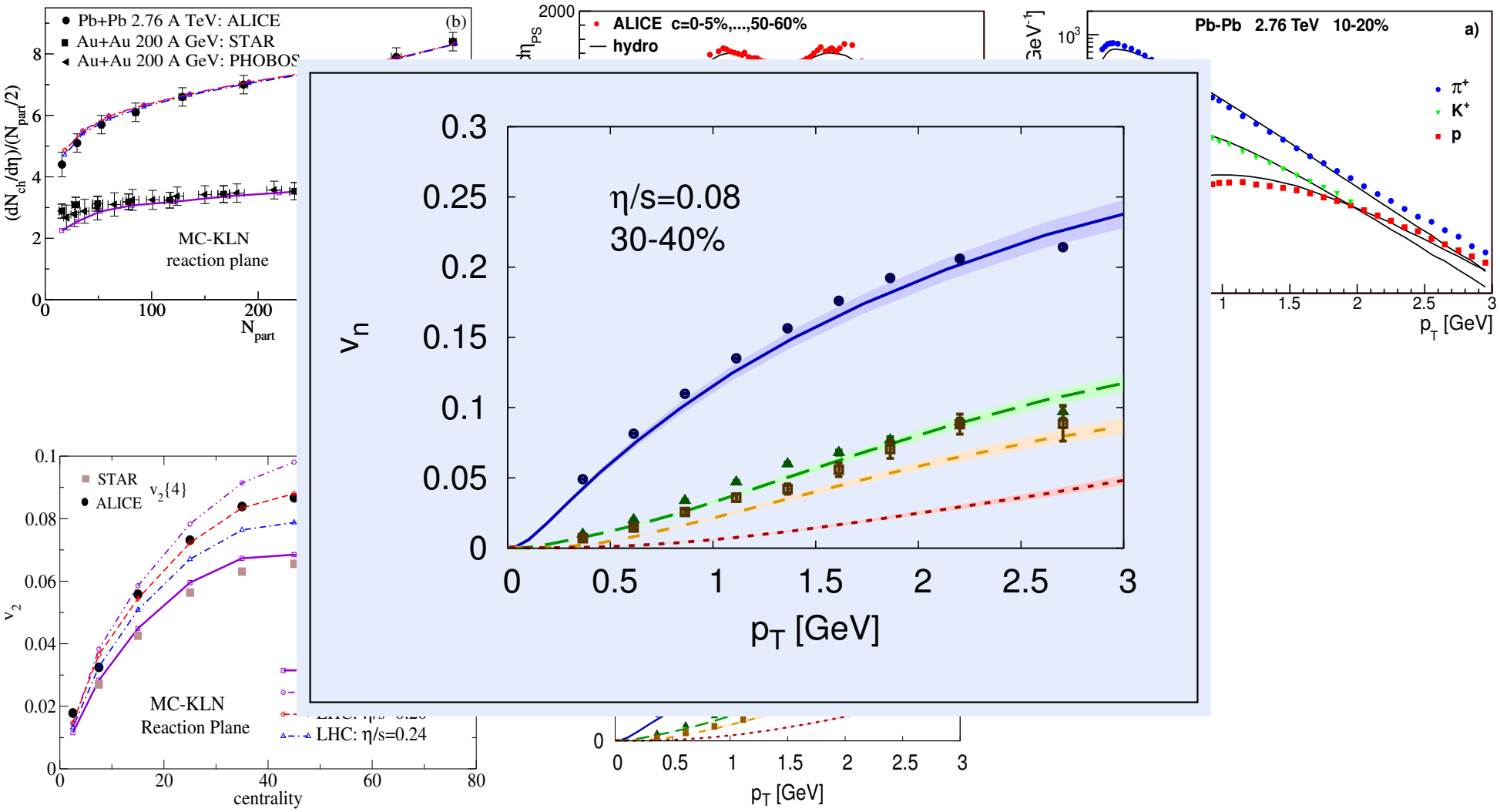

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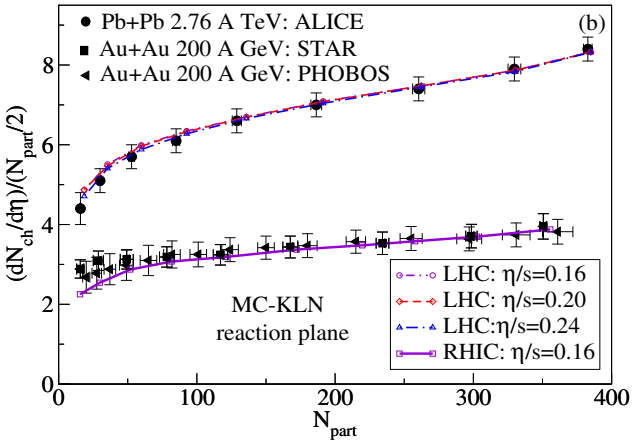

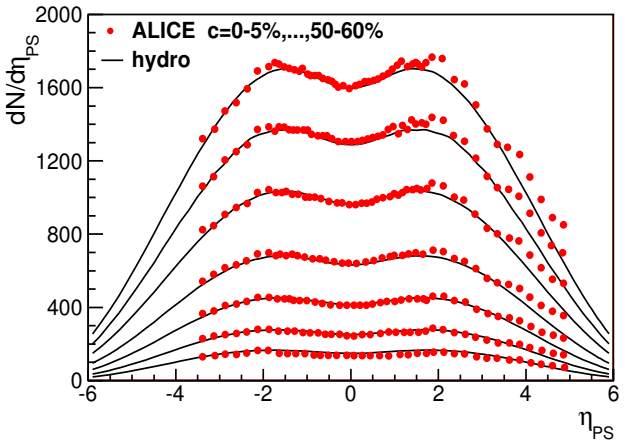

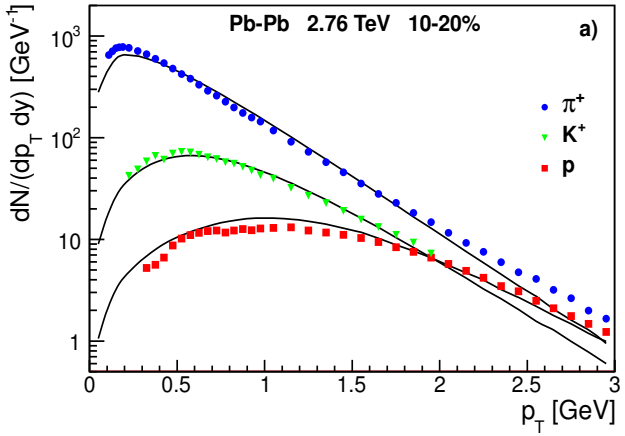

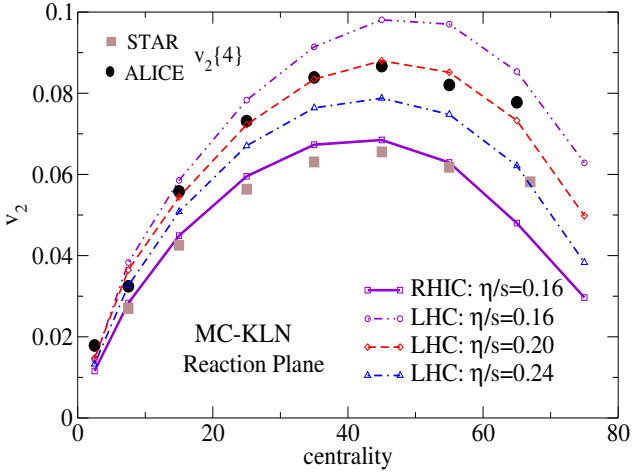

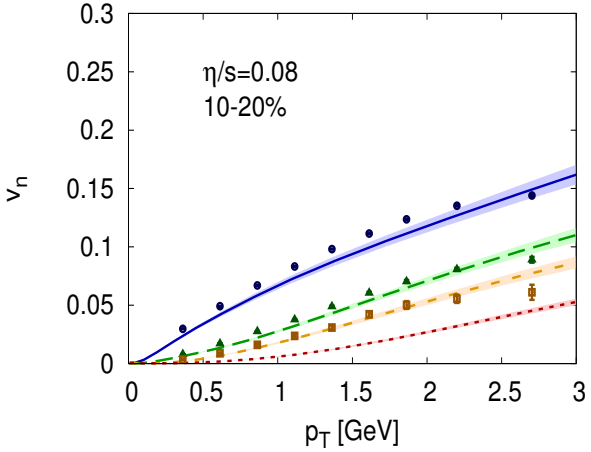

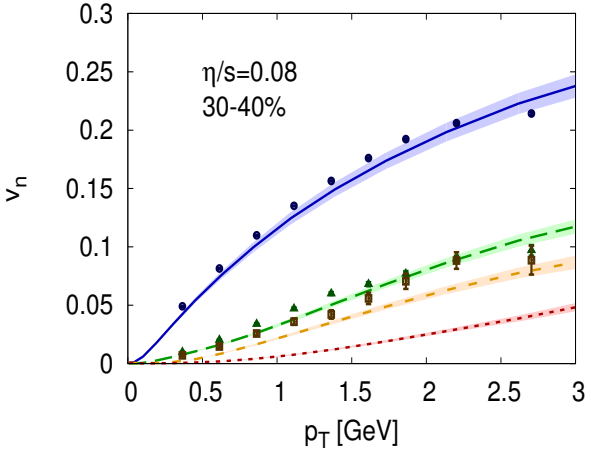

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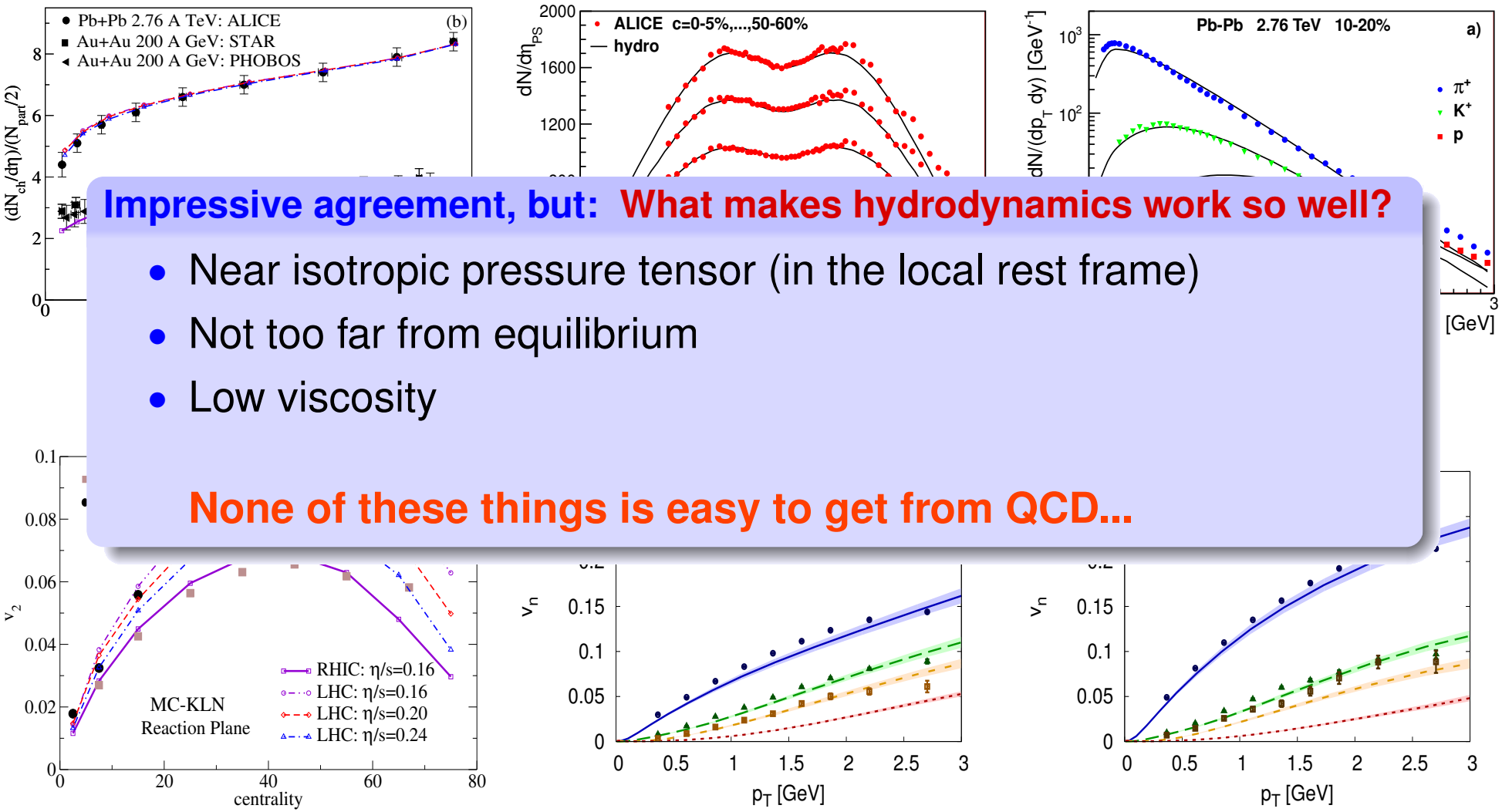


Impressive agreement, but: What makes hydrodynamics work so well?

- Near isotropic pressure tensor (in the local rest frame)
- Not too far from equilibrium
- Low viscosity

None of these things is easy to get from QCD...

Thermalization :
A brief history

The second law of thermodynamics

- Thermodynamics is a big success of 19th century physics
- The 1st law is easy to understand from the underlying dynamics, since it just states that energy is conserved, provided one does the bookkeeping correctly
- But the 2nd law, about the increase of entropy in closed systems, had remained rather elusive from the point of view of mechanics

  Main issue : Newtonian mechanics is time-reversible and does not impose a preferred direction to the flow of time

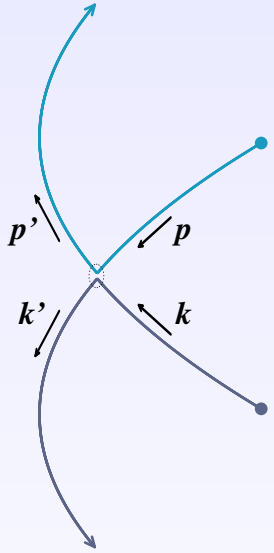
Various concepts of equilibrium

- Kinetic equilibration : the single particle distribution in a closed system with many constituents is the Boltzmann distribution
- Micro-canonical equilibration : all the micro-states that have the same energy are equiprobable (assuming no other knowledge about the system)
- Ergodicity : for a measurement that lasts long enough, another interesting question is whether a generic phase-space trajectory covers uniformly the entire energy surface

Boltzmann equation, H theorem
Poincaré recurrence theorem

Boltzmann equation (1872)

- The Boltzmann equation describes the evolution of the 1-particle distribution of point-like objects that collide at short distance :



$$\left[\partial_t + \vec{v}_p \cdot \vec{\nabla}_x\right] f(t, \vec{x}, \vec{p}) = \mathcal{C}_p[f]$$

▷ the functional $\mathcal{C}_p[f]$ is the collision term :

$$\mathcal{C}_p[f] = \frac{1}{2E_p} \int \frac{d^3\vec{p}'}{(2\pi)^3 2E_{p'}} \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} \int \frac{d^3\vec{k}'}{(2\pi)^3 2E_{k'}} (2\pi)^4 \delta(p+k-p'-k')$$
$$\times \left[f(X, \vec{p}')f(X, \vec{k}') - f(X, \vec{p})f(X, \vec{k})\right] \left|\mathcal{M}\right|^2$$

- Elementary collisions are reversible ($A \to B$ and $B \to A$ have the same cross-section)

H theorem (1872)

- Entropy density and flux :

$$
\begin{aligned}
s(X) &\equiv -\int \frac{d^3\vec{p}}{(2\pi)^3}\, f(X,\vec{p})\, \log f(X,\vec{p}) \\
\vec{J}_s(X) &\equiv -\int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{v}_p\, f(X,\vec{p})\, \log f(X,\vec{p})
\end{aligned}
$$

H theorem

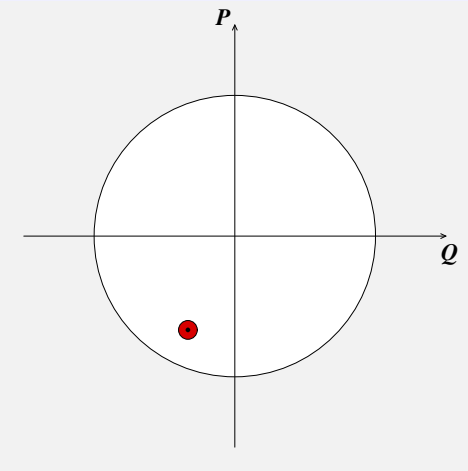
$$\partial_t s + \vec{\nabla}_x \cdot \vec{J}_s = \sigma$$

- $\sigma \geq 0$ for any distribution $f(X,\vec{p})$
- $\sigma = 0$ when $f(X,\vec{p}) = \exp(-\beta E_p)$

Poincaré recurrence theorem (1880)

- Liouville theorem : the volume in phase-space is conserved under a conservative Hamiltonian flow
- Phase-space trajectories do not intersect

- Poincaré recurrence theorem : any dynamical system whose conserved energy surface has a finite measure will return arbitrarily close to its initial condition



- Surround the phase-space point by a small sphere that sweeps through phase-space

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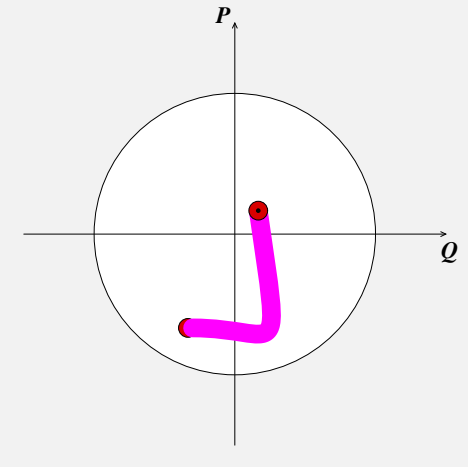

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- The phase-space tube cannot shrink nor cross itself. Eventually it will fill all the available volume
- It must connect with its starting point in a finite time

Note : the accessible phase-space volume of a system with many degrees of freedom is LARGE $\Rightarrow$ very large recurrence time

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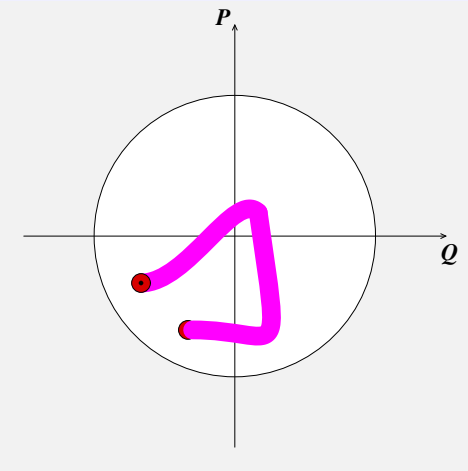

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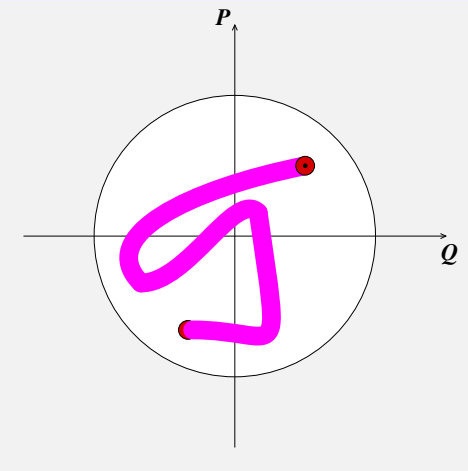

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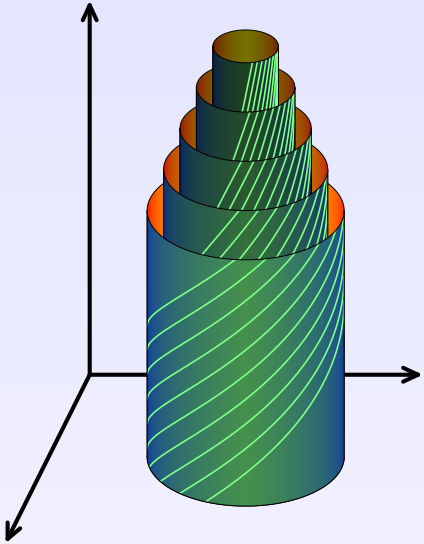
Chaos, KAM theorem

Generic dynamical system

- Generic system with N degrees of freedom:
  - Phase-space : $2N$ dimensional
  - Constant energy surface : $2N - 1$ dimensional

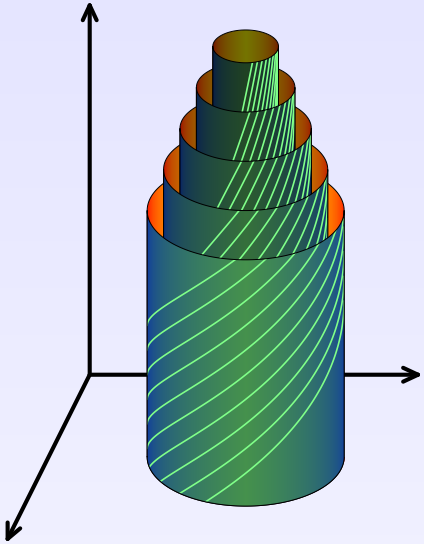
- Does a typical trajectory span most of the allowed domain?

Integrable systems : N independent conserved quantities



- Accessible phase-space : N dimensional torus
- The constant energy surface is foliated into invariant tori, and the motion is quasi-periodic on these tori
- classical integrable systems never thermalize

Integrable systems : N independent conserved quantities



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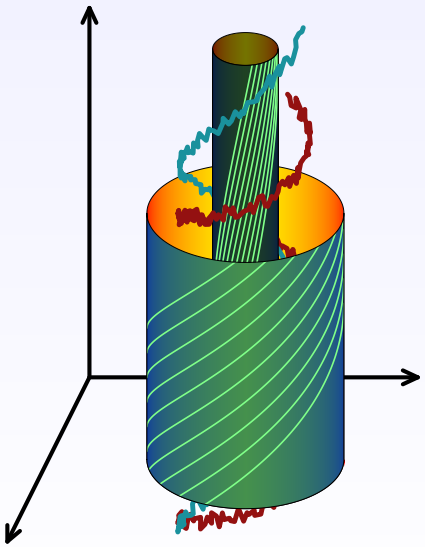
- What happens if we perturb an integrable system? Does the accessible phase-space become immediately $2\mathrm{N}-1$ dimensional?

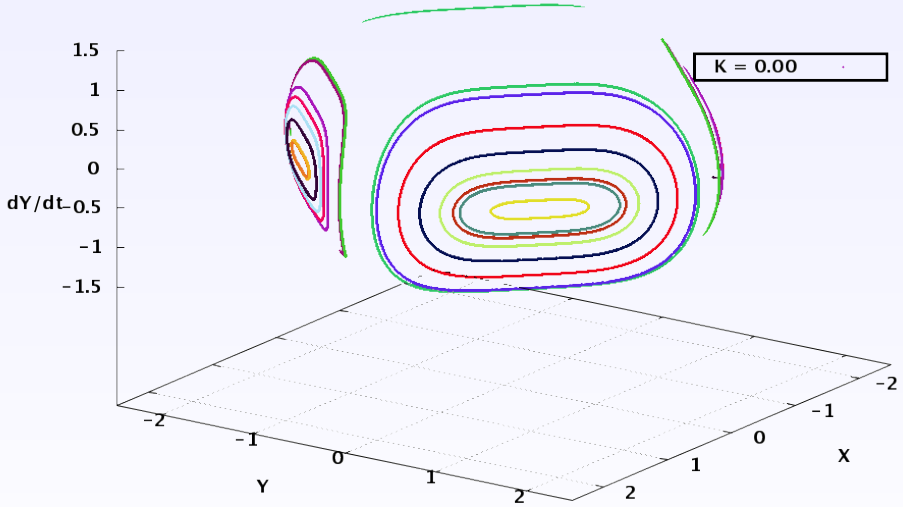
  Kolmogorov-Arnold-Moser theorem (1954,1963) : Not completely : many invariant tori survive, with chaotic domains in between ($2\mathrm{N}-1$ dimensional)



Toy example of two coupled oscillators

$$H \equiv \frac{1}{2}(\dot{X}^2 + \dot{Y}^2) + \frac{1}{24}(X^4 + Y^4 + 6\,X^2\,Y^2) + \frac{1}{2}K^2\,Y^2$$

- Integrable for $K = 0$ (H separable in the variables $X \pm Y$)
- Chaotic at moderate $K > 0$ (and regular again at large K)
- Phase-space section $\dot{X} \equiv 0$, at fixed H :



QM, Quantum chaos
Berry's conjecture

Formulation of QM in the classical phase-space

- Quantum Mechanics introduces a natural smearing due to the uncertainty principle. To make the connection with classical mechanics, it is useful to use Moyal's formulation of QM in terms of classical variables

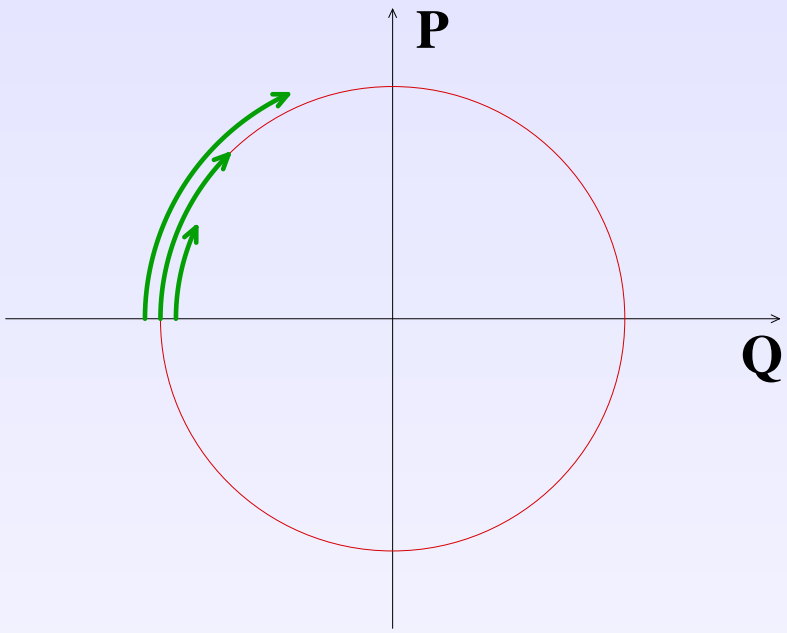
- Dual formulation of QM :

Density	$\hat{\rho}$		$W(Q,P)$
Evolution	$\partial_t \hat{\rho} + i[\widehat{H}, \hat{\rho}] = 0$	$\xleftrightarrow[\text{trans.}]{\text{Weyl-Wigner}}$	$\partial_t W + \{\{W, H\}\} = 0$
Initial condition	coherent state		Gaussian in Q, P

- Moyal bracket : $\{\{\cdot,\cdot\}\} = \underbrace{\{\cdot,\cdot\}}_{\text{Poisson bracket}} + \mathcal{O}(\hbar^2)$

  The Moyal equation becomes the Liouville equation in the classical limit $\hbar \to 0$

Microcanonical equilibration of an anharmonic oscillator



- The oscillation frequency depends on the initial condition

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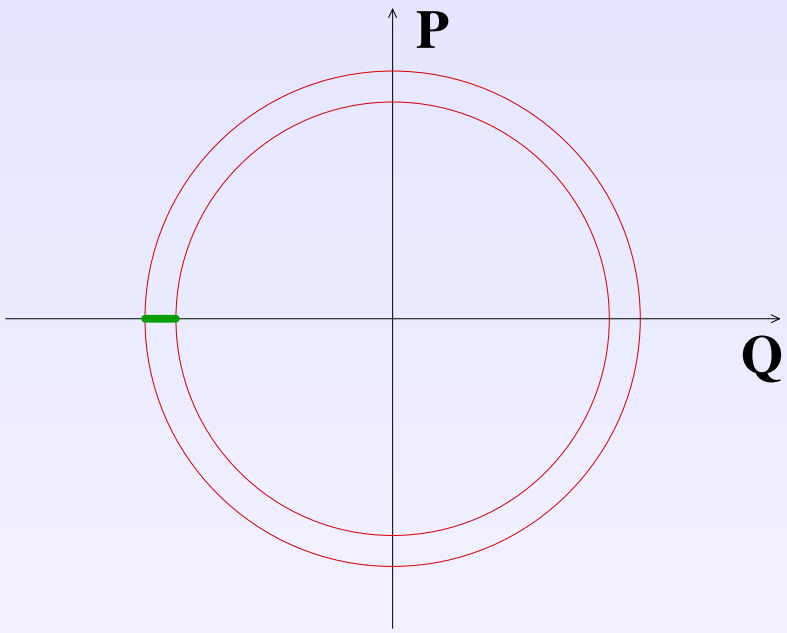
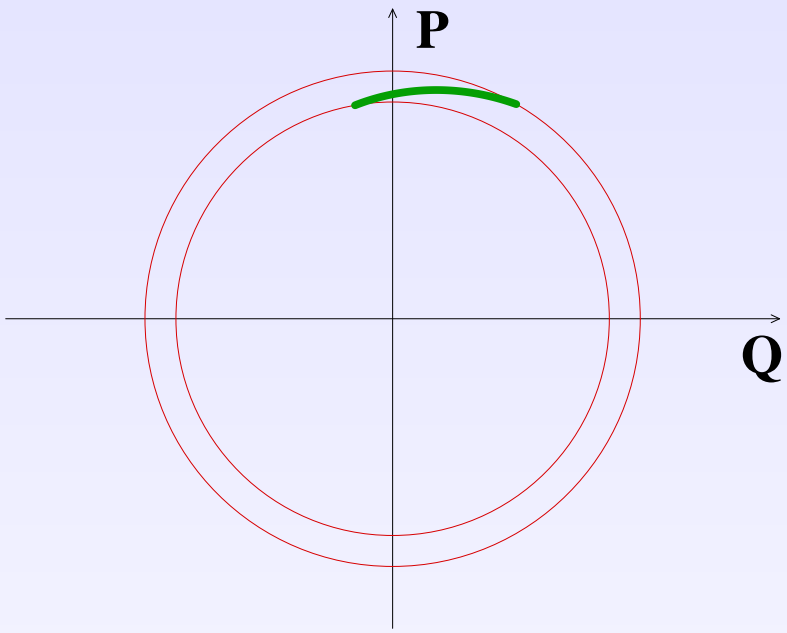


- The oscillation frequency depends on the initial condition
- Because of QM, the initial ensemble is a set of width $\sim \hbar$

Microcanonical equilibration of an anharmonic oscillator



- The oscillation frequency depends on the initial condition
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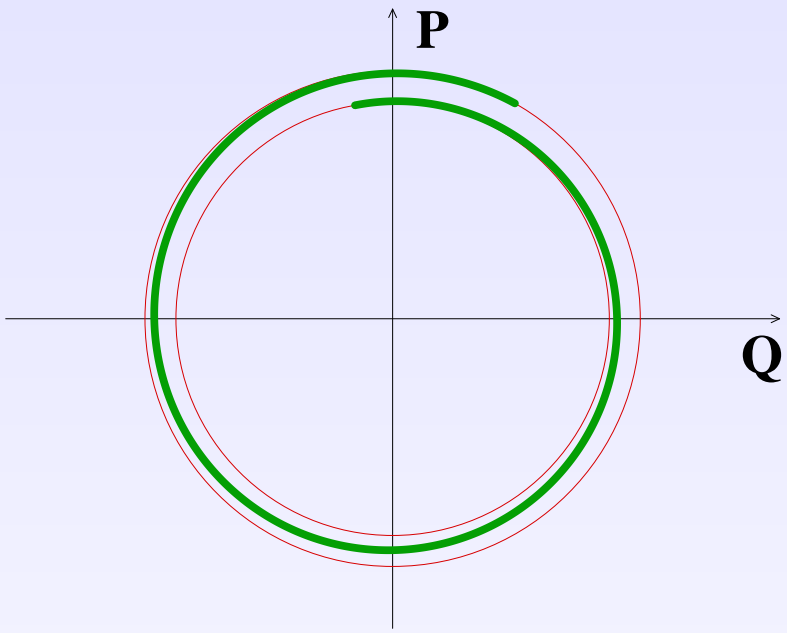

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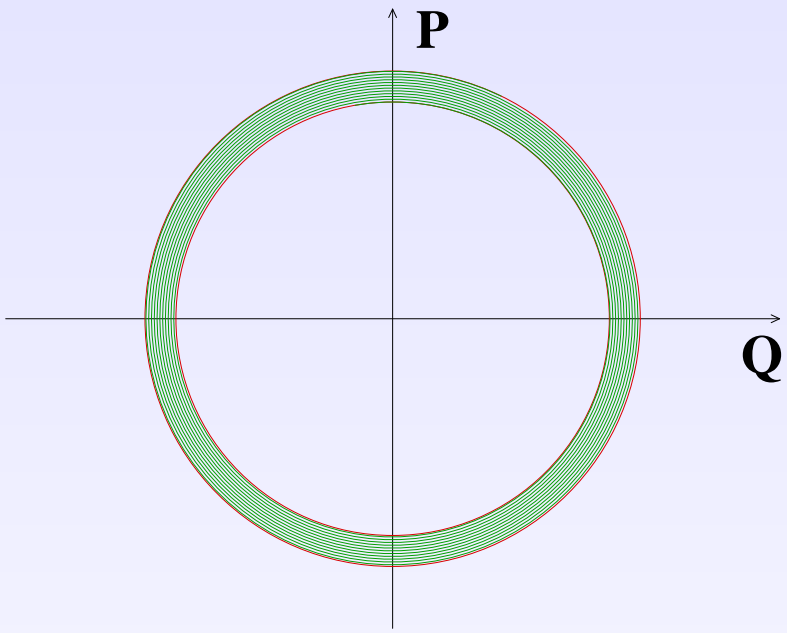


- The oscillation frequency depends on the initial condition
- Because of QM, the initial ensemble is a set of width $\sim \hbar$
- This ensemble of initial configurations spreads in time
- At large times, the ensemble fills densely all the region allowed by energy conservation $\Rightarrow$ microcanonical equilibrium even without a time average

Quantum chaos

- Central issue : consider a Hamiltonian that leads to chaotic classical behavior; What happens when this system is quantized?

- Schrodinger's equation is linear :

$$i\partial_t \Psi = \widehat{H}\,\Psi$$

- Once we know the spectrum of the Hamiltonian $\{E_n, \Psi_n\}$, any wavefunction evolves as :

$$\Psi(t) = \sum_n c_n \; e^{iE_n t}\,\Psi_n$$

$E_n \in \mathbb{R} \;\Rightarrow\;$ nothing is unstable. Where is the chaos?

Berry's conjecture (1977)

- The complexity of the classical dynamics translates in the complexity of the high lying eigenfunctions

- Berry's conjecture : for most practical purposes, high lying eigenfunctions of classically chaotic systems behave as Gaussian random functions with 2-point correlations given by

$$\left\langle \Psi^*(\mathbf{X}-\frac{\mathbf{s}}{2})\Psi(\mathbf{X}+\frac{\mathbf{s}}{2})\right\rangle = \int d\mathbf{P}\; e^{i\mathbf{P}\cdot\mathbf{s}/\hbar}\;\delta\big[E-H(\mathbf{X},\mathbf{P})\big]$$

- Then, the Wigner distribution associated with the eigenfunction Ψ_E is

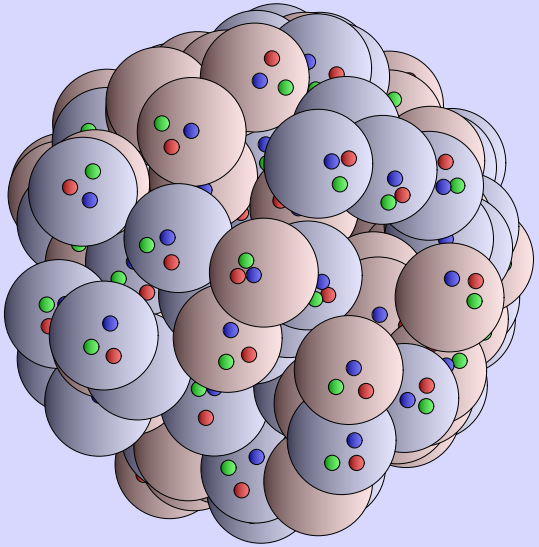
$$\begin{aligned} W(\mathbf{X},\mathbf{P}) &= \int d\mathbf{s}\; e^{-i\mathbf{P}\cdot\mathbf{s}/\hbar}\;\Psi_E^*(\mathbf{X}-\frac{\mathbf{s}}{2})\Psi_E(\mathbf{X}+\frac{\mathbf{s}}{2}) \\ &\sim \delta\big[E-H(\mathbf{X},\mathbf{P})\big] \end{aligned}$$

$\Rightarrow$ micro-canonical equilibrium for a single eigenstate

Semi-Classical Methods in High Energy Collisions

What do we need to know about nuclei?

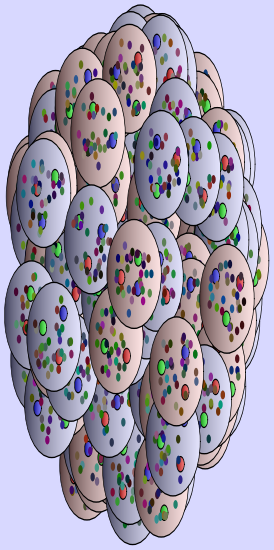
Nucleus at rest



- At low energy : valence quarks

What do we need to know about nuclei?

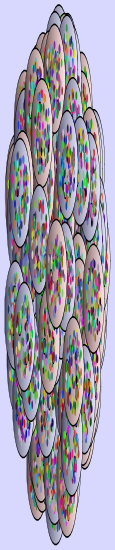
Slightly boosted nucleus



- At low energy : valence quarks
- At higher energy :
  - Lorenz contraction of longitudinal sizes
  - Time dilation $\triangleright$ slowing down of the internal dynamics
  - Gluons start becoming important

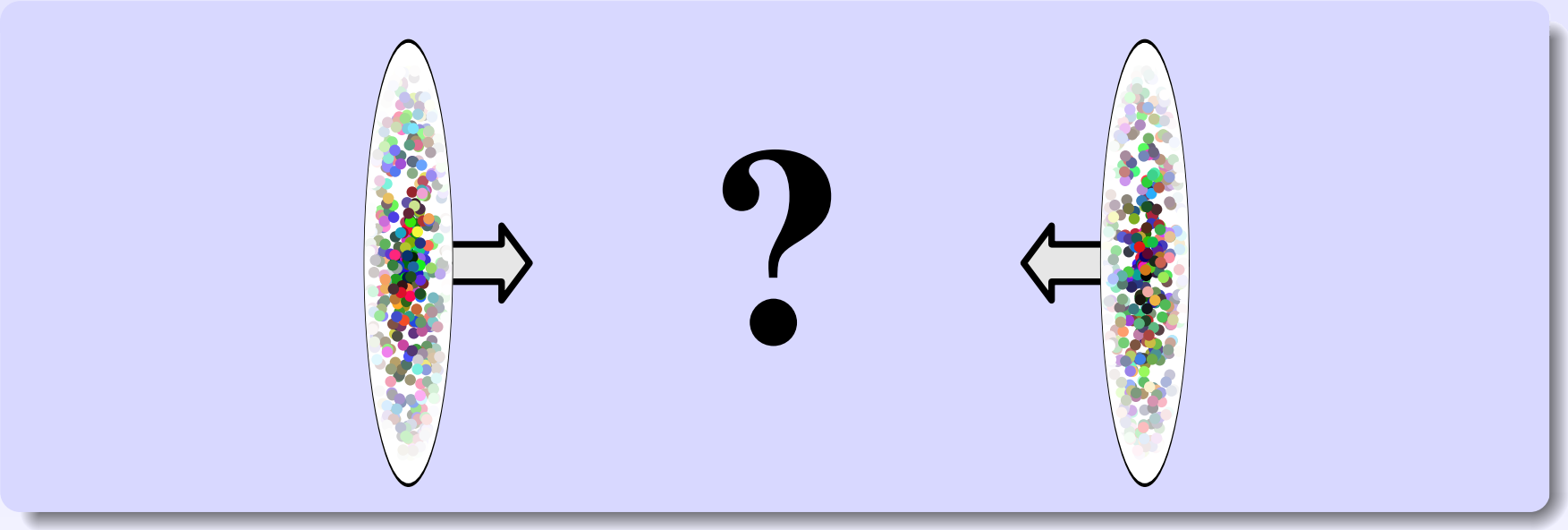
What do we need to know about nuclei?

High energy nucleus



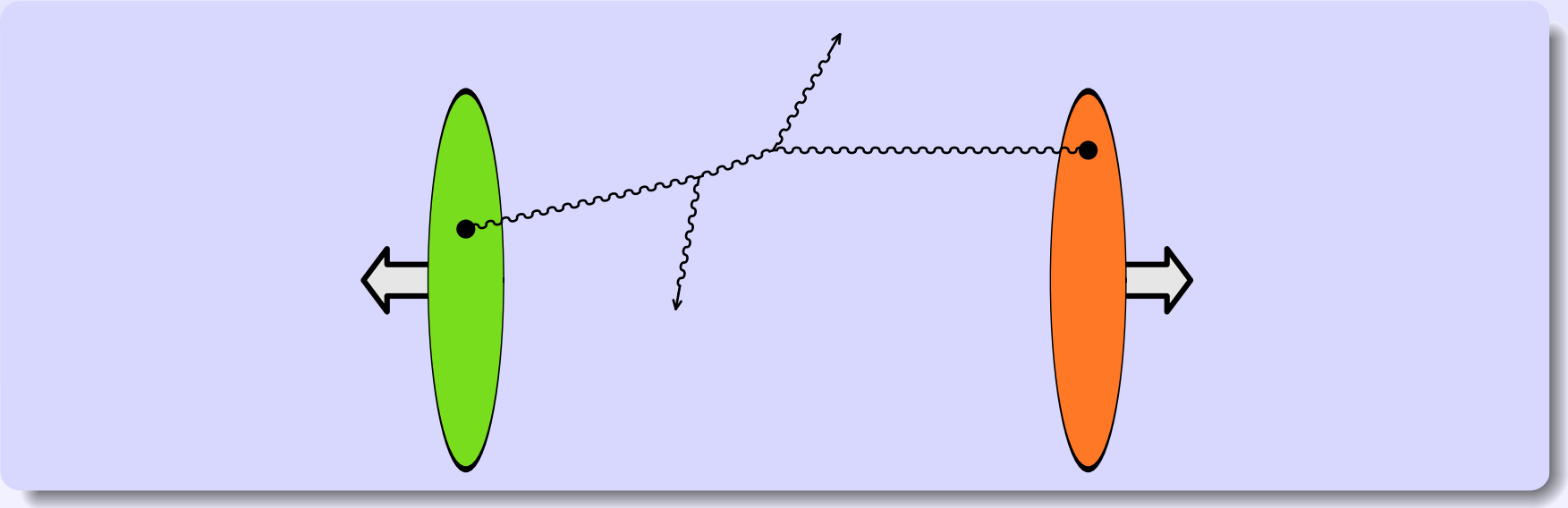
- At low energy : valence quarks
- At higher energy :
  - Lorenz contraction of longitudinal sizes
  - Time dilation $\triangleright$ slowing down of the internal dynamics
  - Gluons start becoming important
- At very high energy : gluons dominate

Multiple scatterings and gluon recombination



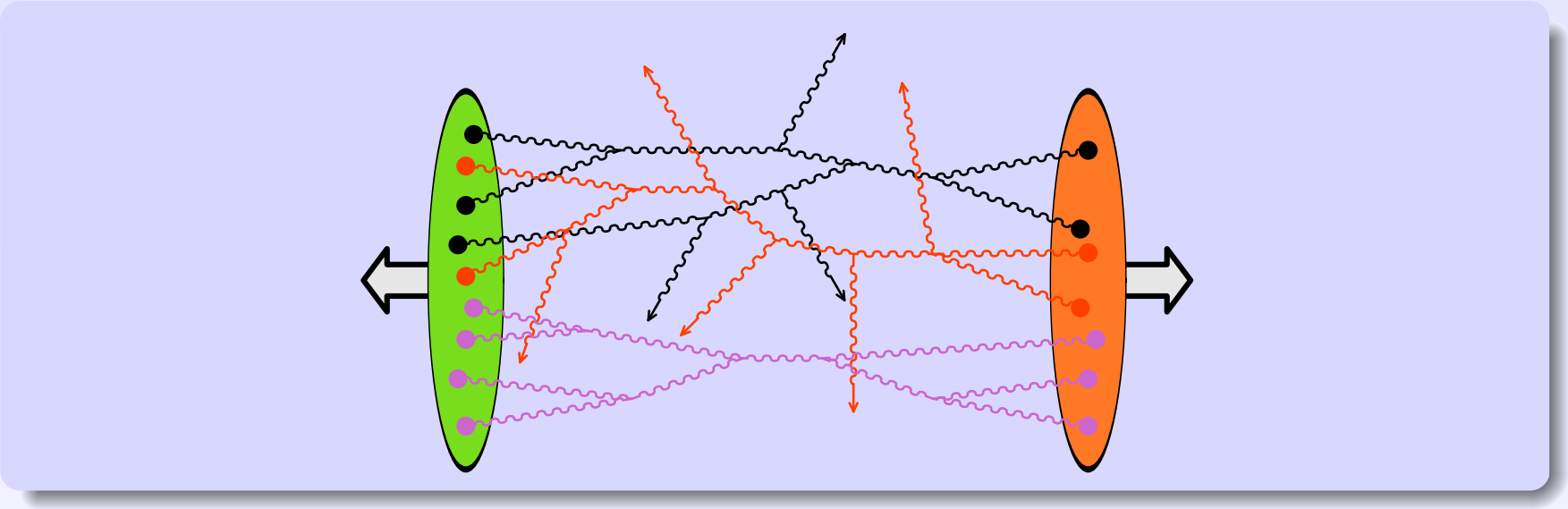
- Main difficulty: How to treat collisions involving a large number of partons?

Multiple scatterings and gluon recombination



- Dilute regime : one parton in each projectile interact
  - ▷ single parton distributions, standard perturbation theory

Multiple scatterings and gluon recombination



- Dense regime : multiparton processes become crucial
  - ▷ gluon recombinations are important (saturation)
  - ▷ multi-parton distributions
  - ▷ alternative approach : treat the gluons in the projectiles as external currents

$$\mathcal{L} = -\frac{1}{4}F^2 + A \cdot (J_1 + J_2)$$

(gluons only, field A for $k^+ < \Lambda$, classical source J for $k^+ > \Lambda$)

Color Glass Condensate

CGC = effective theory of small x gluons

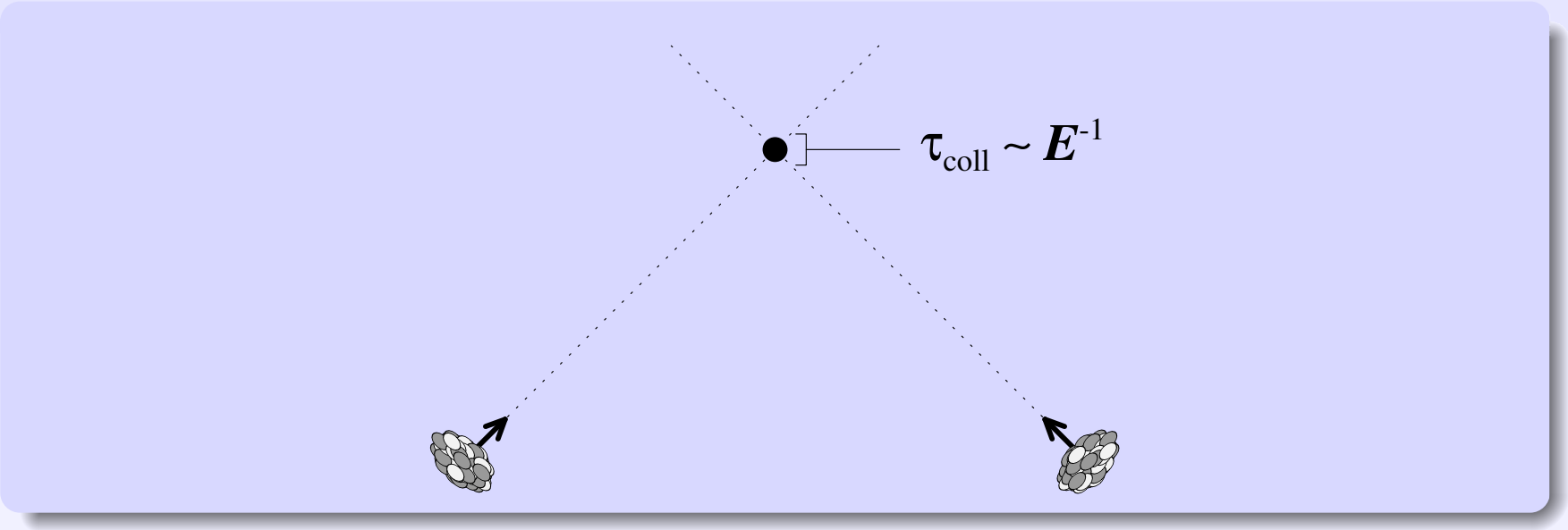
- The fast partons ($k^+ > \Lambda^+$) are frozen by time dilation
  ▷ described as static color sources on the light-cone :

$$J^\mu = \delta^{\mu+}\rho(x^-, \vec{x}_\perp) \qquad (0 < x^- < 1/\Lambda^+)$$

- The color sources ρ are random, and described by a probability distribution $W_{\Lambda^+}[\rho]$

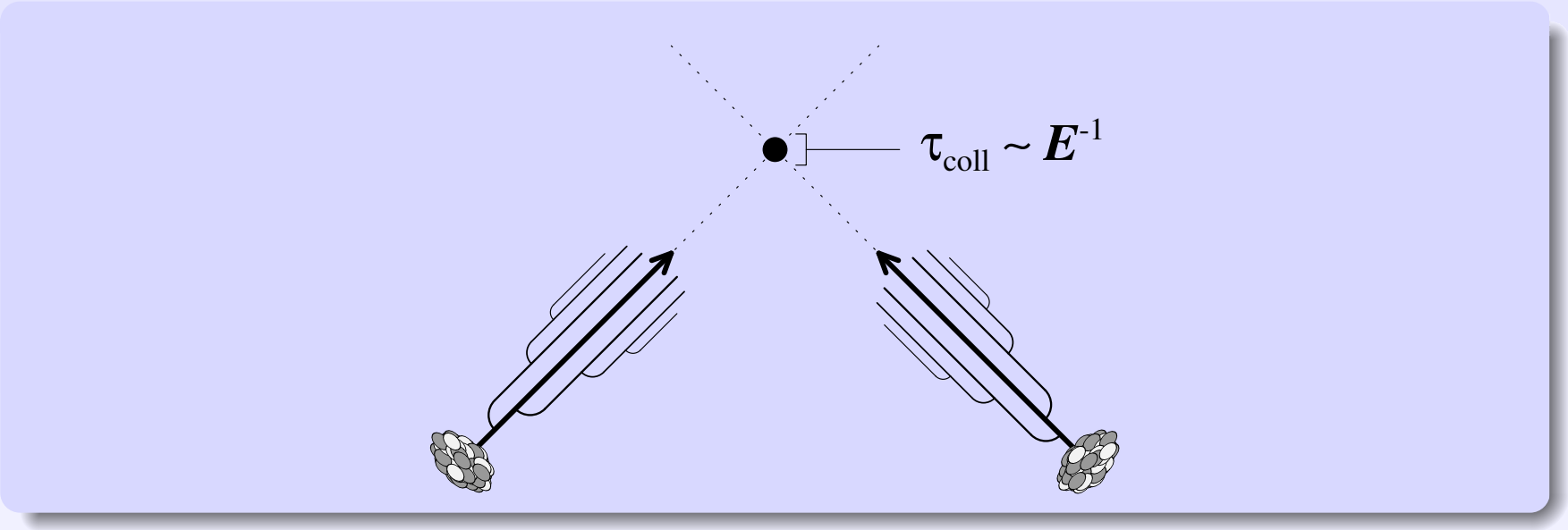
- Slow partons ($k^+ < \Lambda^+$) cannot be considered static over the time-scales of the collision process
  ▷ must be treated as standard gauge fields
  ▷ eikonal coupling to the current J^μ : $A_\mu J^\mu$

Universality of the distribution $W[\rho]$



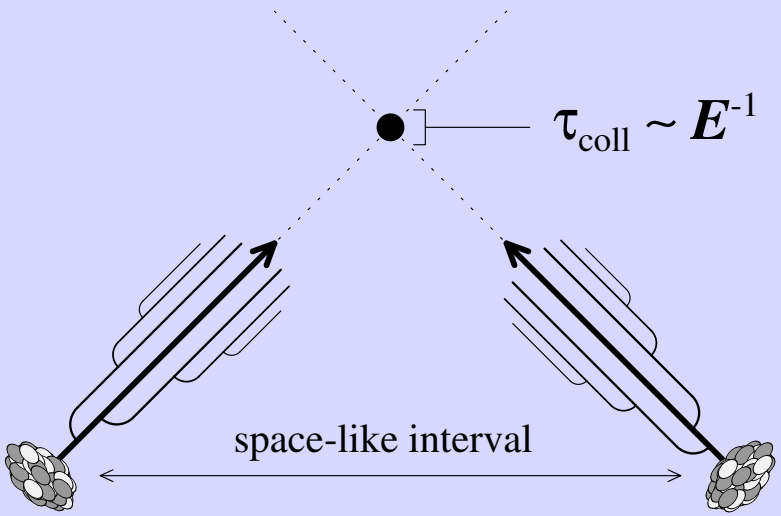
- The duration of the collision is very short: $\tau_{coll} \sim E^{-1}$

Universality of the distribution $W[\rho]$



- The duration of the collision is very short: $\tau_{coll} \sim E^{-1}$
- The evolution of the distribution $W[\rho]$ the radiation of soft gluons, which takes a long time
  $\triangleright$ it must happen (long) before the collision

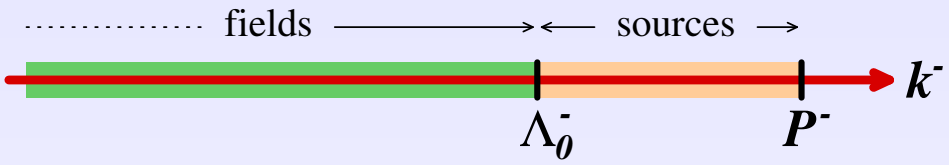
Universality of the distribution $W[\rho]$



- The duration of the collision is very short: $\tau_{coll} \sim E^{-1}$
- The evolution of the distribution $W[\rho]$ the radiation of soft gluons, which takes a long time
  $\triangleright$ it must happen (long) before the collision
- The projectiles are not in causal contact before the impact
  $\triangleright$ the distributions are intrinsic properties of the projectiles, independent of the measured observable

Power counting

- CGC effective theory with cutoff at the scale Λ_0 :



$$\mathcal{S} = \underbrace{-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}}_{\mathcal{S}_{\rm YM}} + \int \underbrace{(J_1^\mu + J_2^\mu)}_{\text{fast partons}} \; A_\mu$$

- Expansion in g^2 in the saturated regime:

$$T^{\mu\nu} \sim \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

Leading Order in g^2 : tree diagrams

- The saturated regime corresponds to sources of order $J \sim \mathcal{O}(g^{-1})$
- The Leading Order is the sum of all the tree diagrams

  Observables can be expressed in terms of classical solutions of Yang-Mills equations (QCD analogue of Maxwell's equations) :

$$\mathcal{D}_\mu \mathcal{F}^{\mu\nu} = J_1^\nu + J_2^\nu$$

- Boundary conditions for inclusive observables :

$$\lim_{x^0 \to -\infty} \mathcal{A}^\mu(x) = 0$$

Example : 00 component of the energy-momentum tensor

$$T^{00}_{\text{LO}} = \frac{1}{2}\Big[\underbrace{\mathcal{E}^2 + \mathcal{B}^2}_{\text{class. fields}}\Big]$$

Next to Leading Order in g^2 : 1-loop diagrams

Getting loops from trees...

$$\mathcal{O}_{\text{NLO}} = \left[\frac{1}{2} \int_{u,v} \mathbf{G}_{uv} \, \mathbb{T}_u \mathbb{T}_v \right] \mathcal{O}_{\text{LO}}$$

- $\mathbb{T}$ is the generator of the shifts of the initial value of the field :

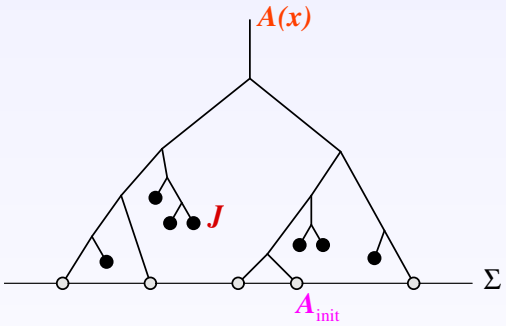
$$\mathbb{T}_u \sim \frac{\partial}{\partial \mathcal{A}_{\text{init}}}$$

$$\exp\left[\int_u \alpha_u \, \mathbb{T}_u \right] \mathcal{O}\Big[\overbrace{\mathcal{A}_\tau(\underbrace{\mathcal{A}_{\text{init}}}_{\text{init. value}})}^{\text{class. field at } \tau} \Big] = \mathcal{O}\Big[\mathcal{A}_\tau(\underbrace{\mathcal{A}_{\text{init}} + \alpha}_{\text{shifted init. value}}) \Big]$$

Shift operator $\mathbb{T}$ – Definition

Equations of motion for a field $\mathcal{A}$ and a small perturbation α

$$\begin{aligned} \Box\mathcal{A} + V'(\mathcal{A}) &= J \\ [\Box + V''(\mathcal{A})]\ \alpha &= 0 \end{aligned}$$



- Getting the perturbation by shifting the initial condition of $\mathcal{A}$ at one point :

$$\alpha(x) = \int_u \alpha_u\, \mathbb{T}_u\ \mathcal{A}(x)$$

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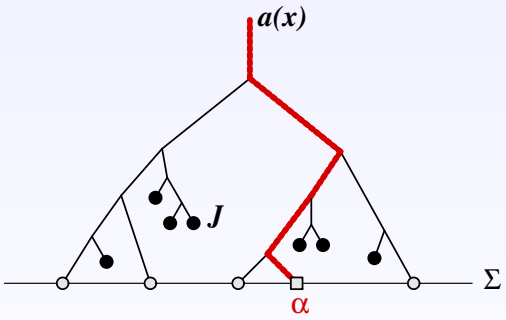

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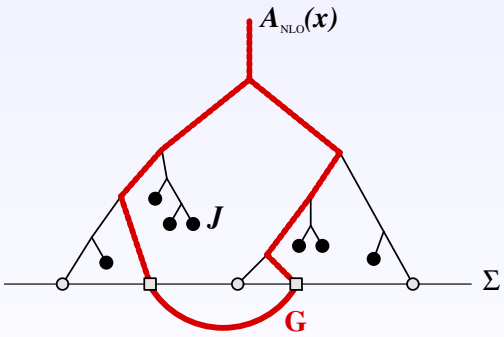


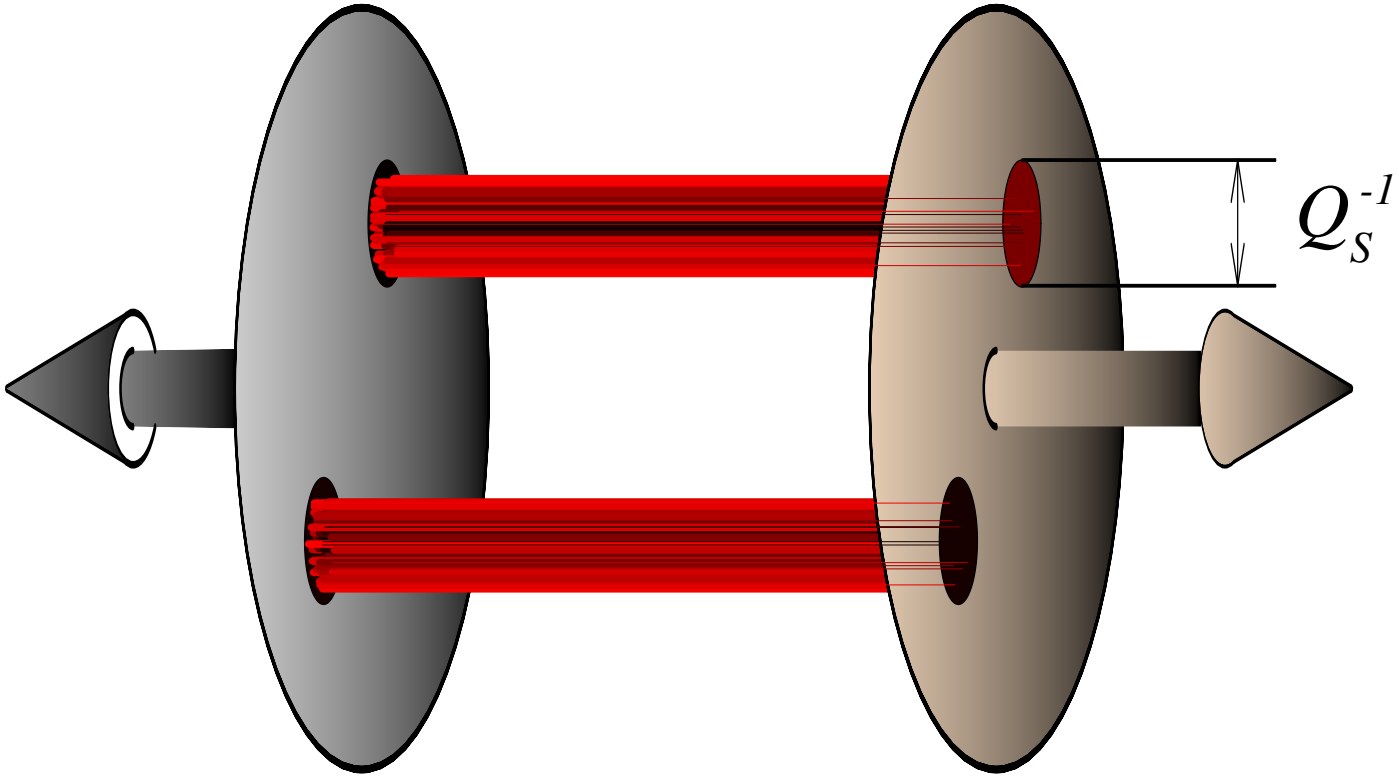
- Getting the perturbation by shifting the initial condition of $\mathcal{A}$ at one point :

$$\alpha(x) = \int_u \alpha_u\, \mathbb{T}_u\ \mathcal{A}(x)$$

- A loop is obtained by shifting the initial condition of $\mathcal{A}$ at two points

Thermalization in QFT

Energy momentum tensor of the initial classical field



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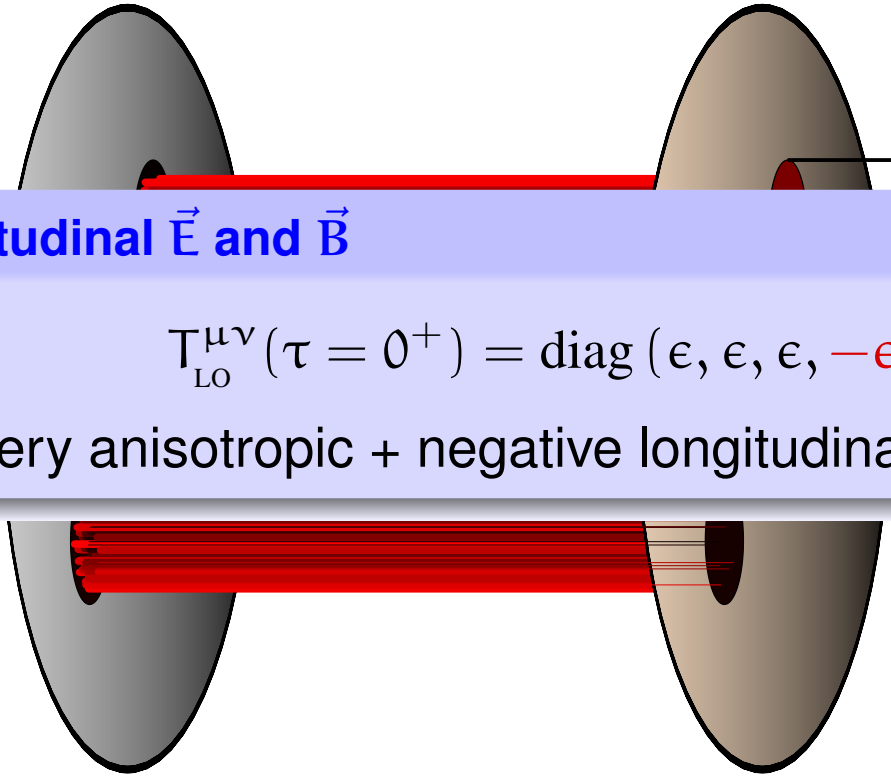
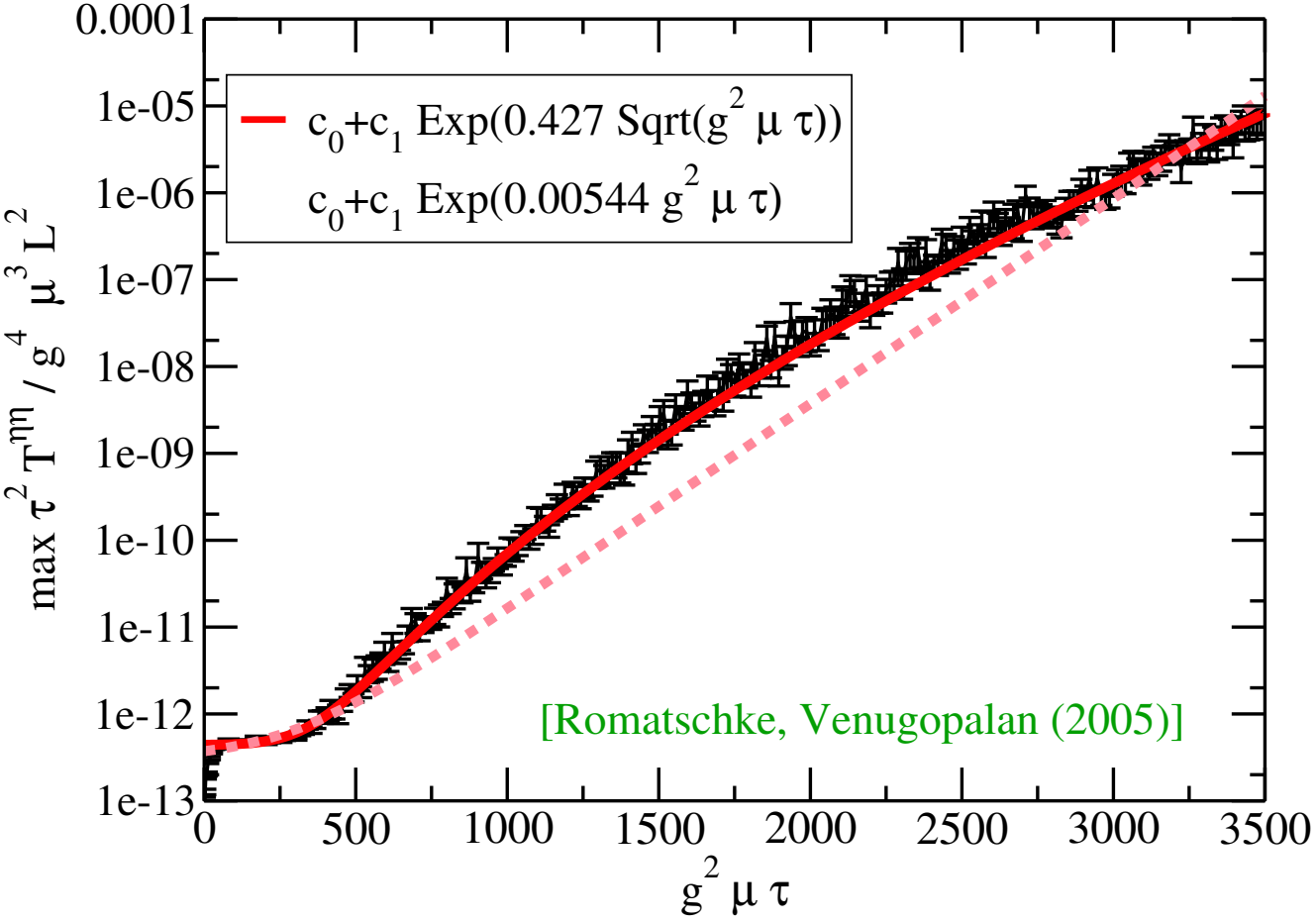


$T^{\mu\nu}$ for longitudinal $\vec{E}$ and $\vec{B}$

$$T^{\mu\nu}_{\text{LO}}(\tau = 0^+) = \text{diag}\,(\epsilon, \epsilon, \epsilon, -\epsilon)$$

▷ very anisotropic + negative longitudinal pressure!

Weibel instabilities for small perturbations



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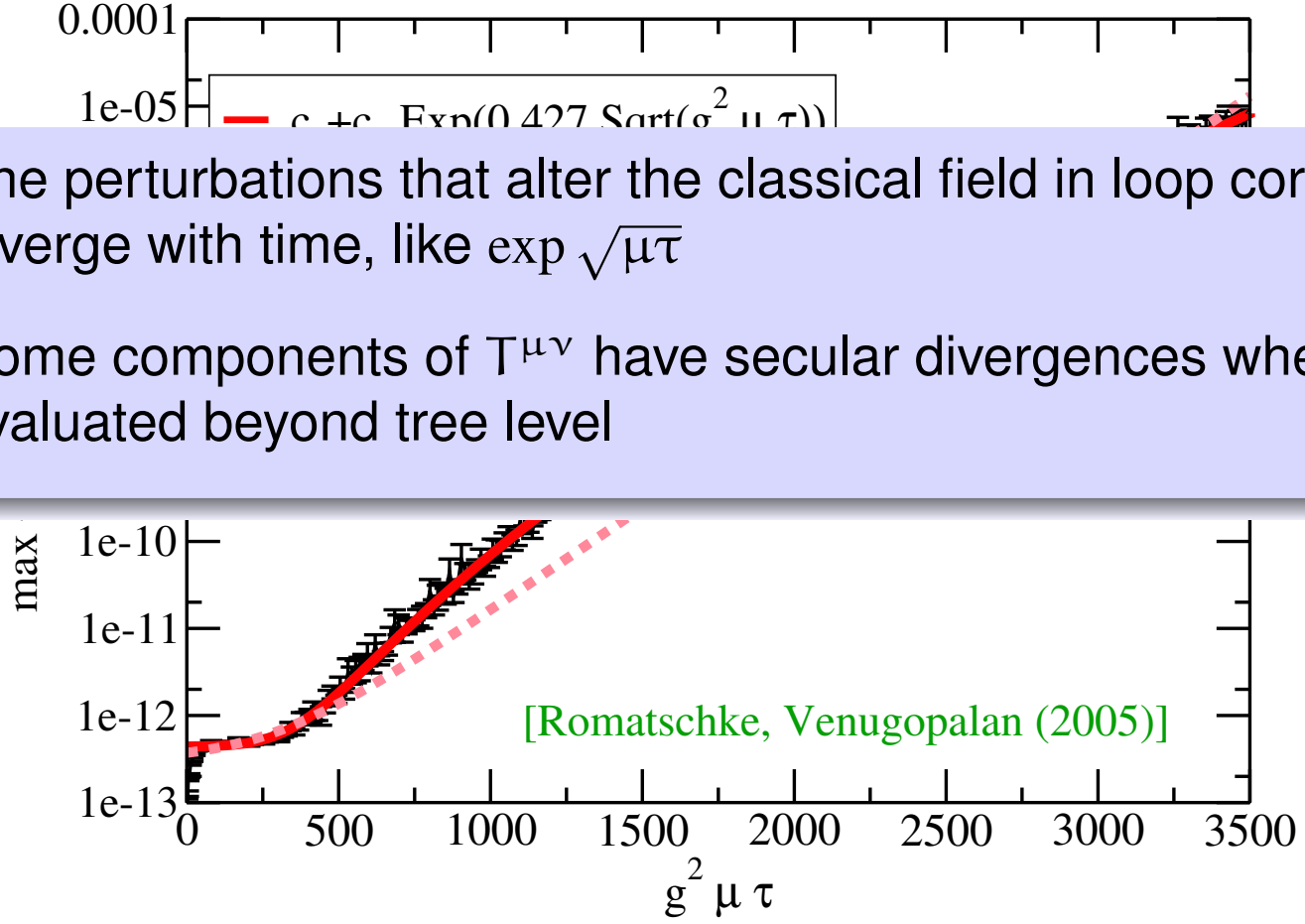
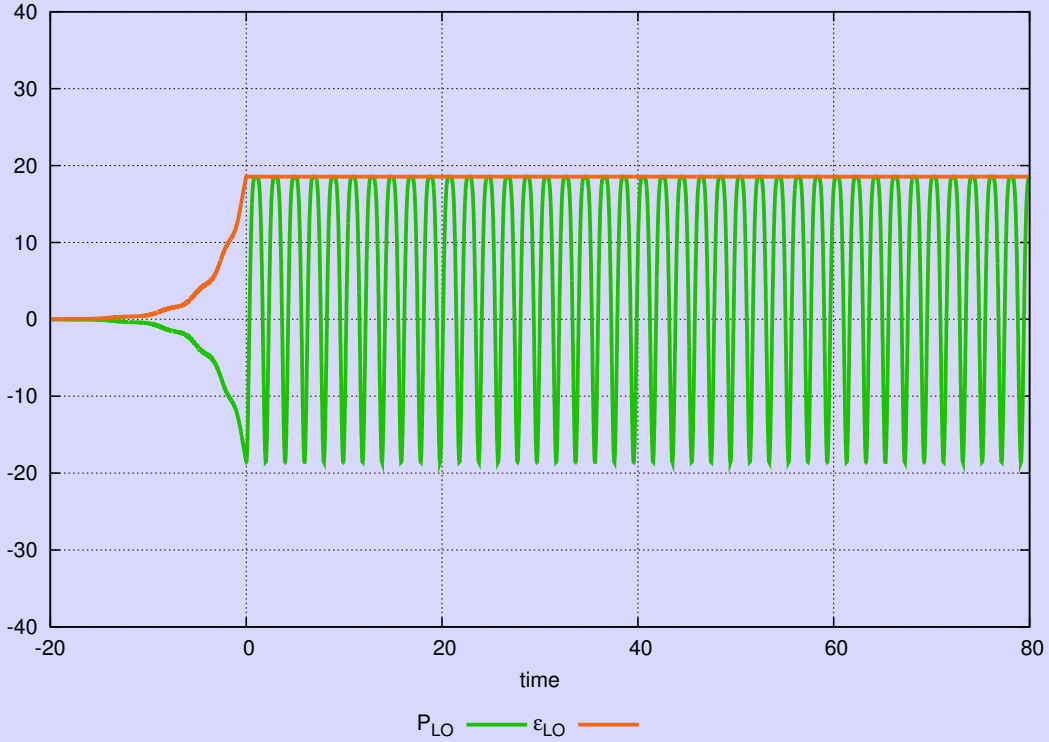


- The perturbations that alter the classical field in loop corrections diverge with time, like $\exp\sqrt{\mu\tau}$
- Some components of $T^{\mu\nu}$ have secular divergences when evaluated beyond tree level

Example of pathologies in fixed order calculations

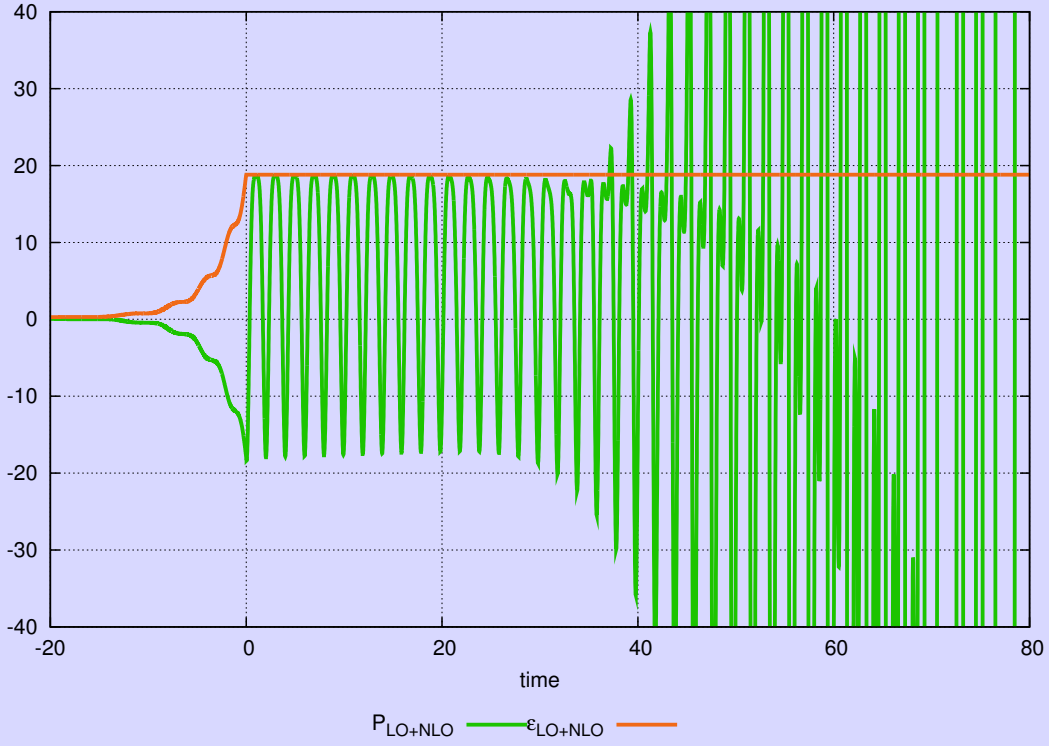
Leading Order



- Oscillating pressure at LO : no equation of state

Example of pathologies in fixed order calculations

Leading + Next-to-Leading Orders



- Oscillating pressure at LO : no equation of state
- Small correction to the energy density
  (protected by energy conservation)
- Secular divergence in the pressure

Resummation of the leading secular terms

$$
\begin{aligned}
T^{\mu\nu}_{\text{resummed}} &= \exp\left[\frac{1}{2}\int_{u,v} G_{uv}\,\mathbb{T}_u\,\mathbb{T}_v\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}] \\
&= \underbrace{T^{\mu\nu}_{\text{LO}} + T^{\mu\nu}_{\text{NLO}}}_{\text{in full}} + \underbrace{T^{\mu\nu}_{\text{NNLO}} + \cdots}_{\text{partially}}
\end{aligned}
$$

- The exponentiation of the 1-loop result collects all the terms with the worst time behavior

Resummation of the leading secular terms

$$
\begin{aligned}
T^{\mu\nu}_{\text{resummed}} &= \exp\left[\frac{1}{2}\int_{u,v} G_{uv}\,\mathbb{T}_u\,\mathbb{T}_v\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}] \\
&= \int [D\chi]\,\exp\left[-\frac{1}{2}\int_{u,v}\chi(u)\,G^{-1}_{uv}\,\chi(v)\right] T^{\mu\nu}_{\text{LO}}[\mathcal{A}_{\text{init}}+\chi]
\end{aligned}
$$

- The exponentiation of the 1-loop result collects all the terms with the worst time behavior
- The evolution remains classical, with a Gaussian average over initial conditions

Classical field + Fluctuations = Coherent state

- This Gaussian distribution of initial fields is the Wigner representation of a **coherent state** $\left|\mathcal{A}\right\rangle$

  Coherent states are the "most classical quantum states" (their wavefunction has the minimal extent permitted by the uncertainty principle, shared equally in $\mathbf{X}$ and $\mathbf{P}$)

- $\left|\mathcal{A}\right\rangle$ is not an eigenstate of the Hamiltonian
  $\triangleright$ decoherence via interactions

Analogous (but simpler) scalar toy model

ϕ^4 field theory coupled to a source

$$\mathcal{L} = \frac{1}{2}(\partial_\alpha \phi)^2 - \frac{g^2}{4!}\phi^4 + J\phi$$

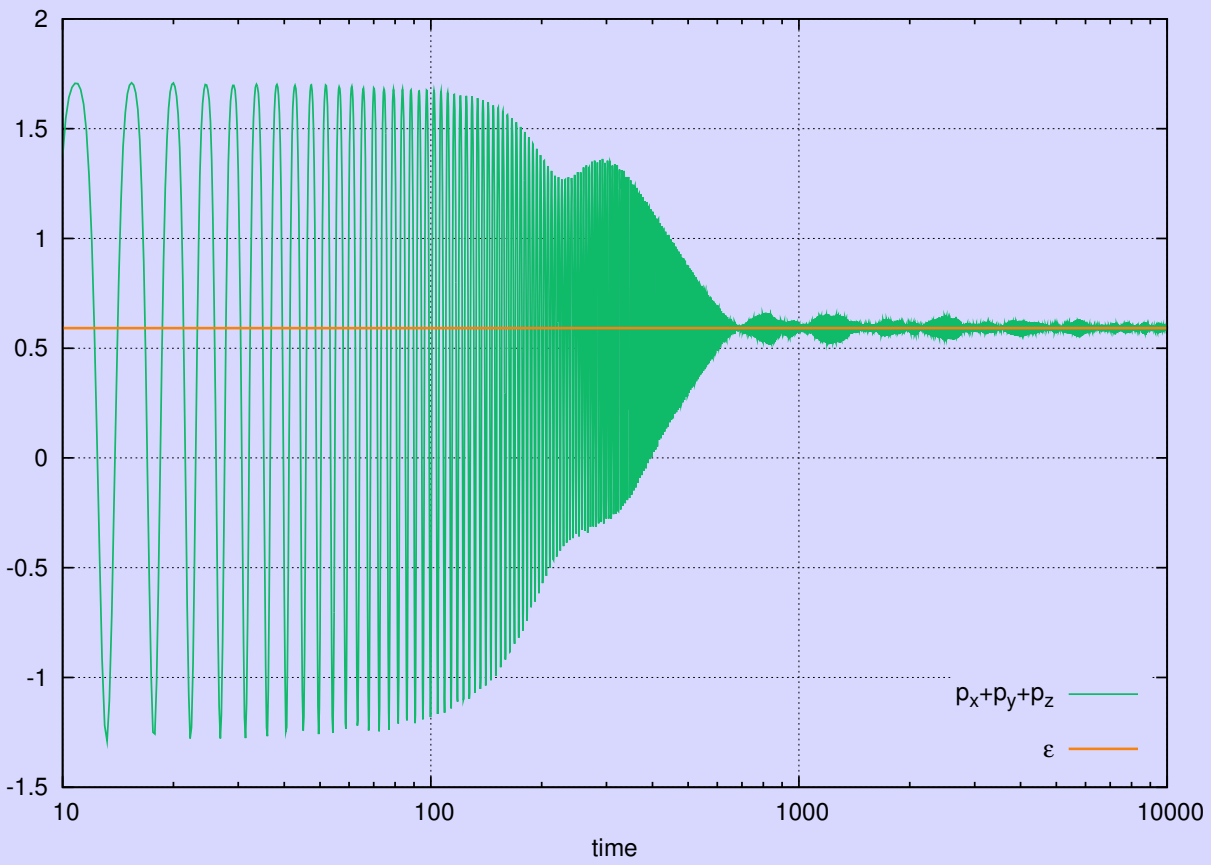
Strong external source: $J \quad \propto \quad \frac{Q^3}{g}$

- In 3+1-dim, g is dimensionless, and the only scale is Q
- This theory has unstable modes (parametric resonance)
- Two setups have been studied :
  - Fixed volume system (equation of state, thermalization)
  - Longitudinally expanding system (isotropization)

Equation of State

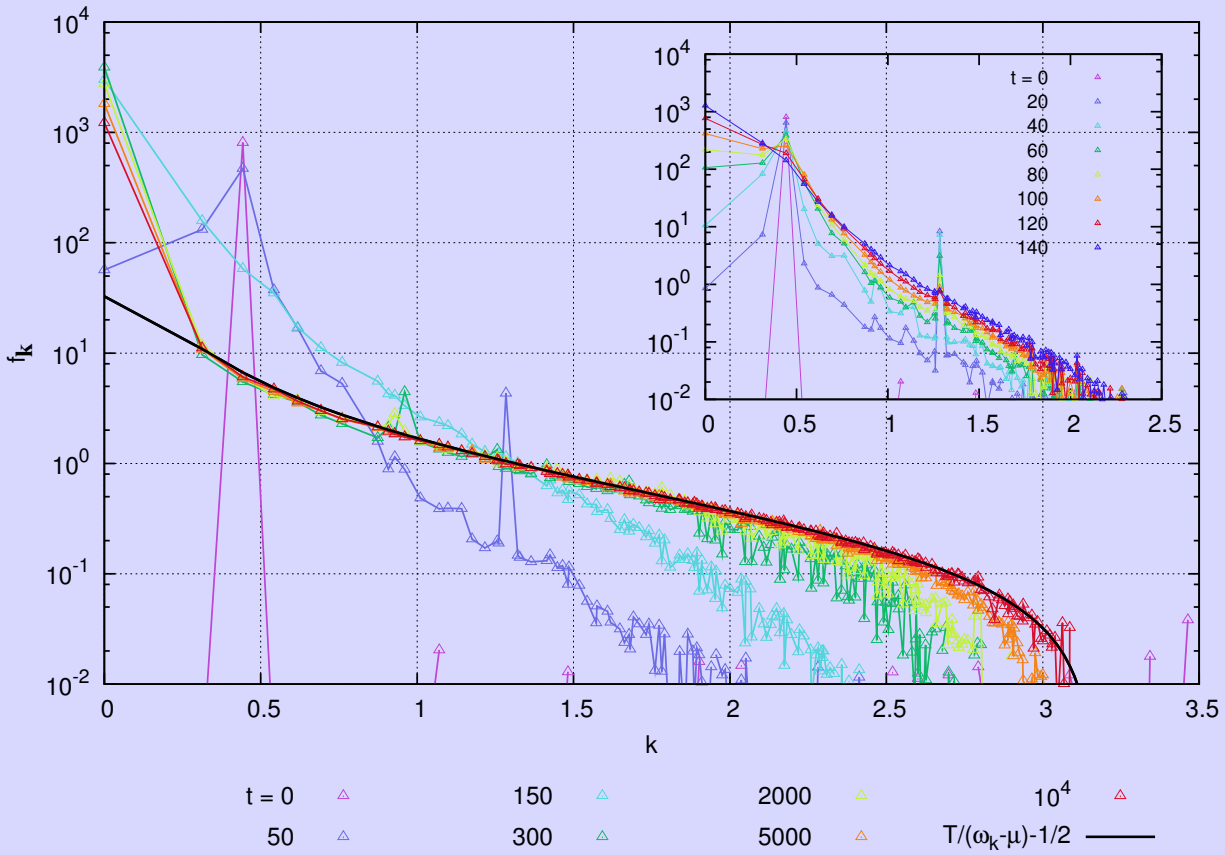
Thermalization

Resummed energy momentum tensor



- No secular divergence in the resummed pressure
- The pressure relaxes to the equilibrium equation of state

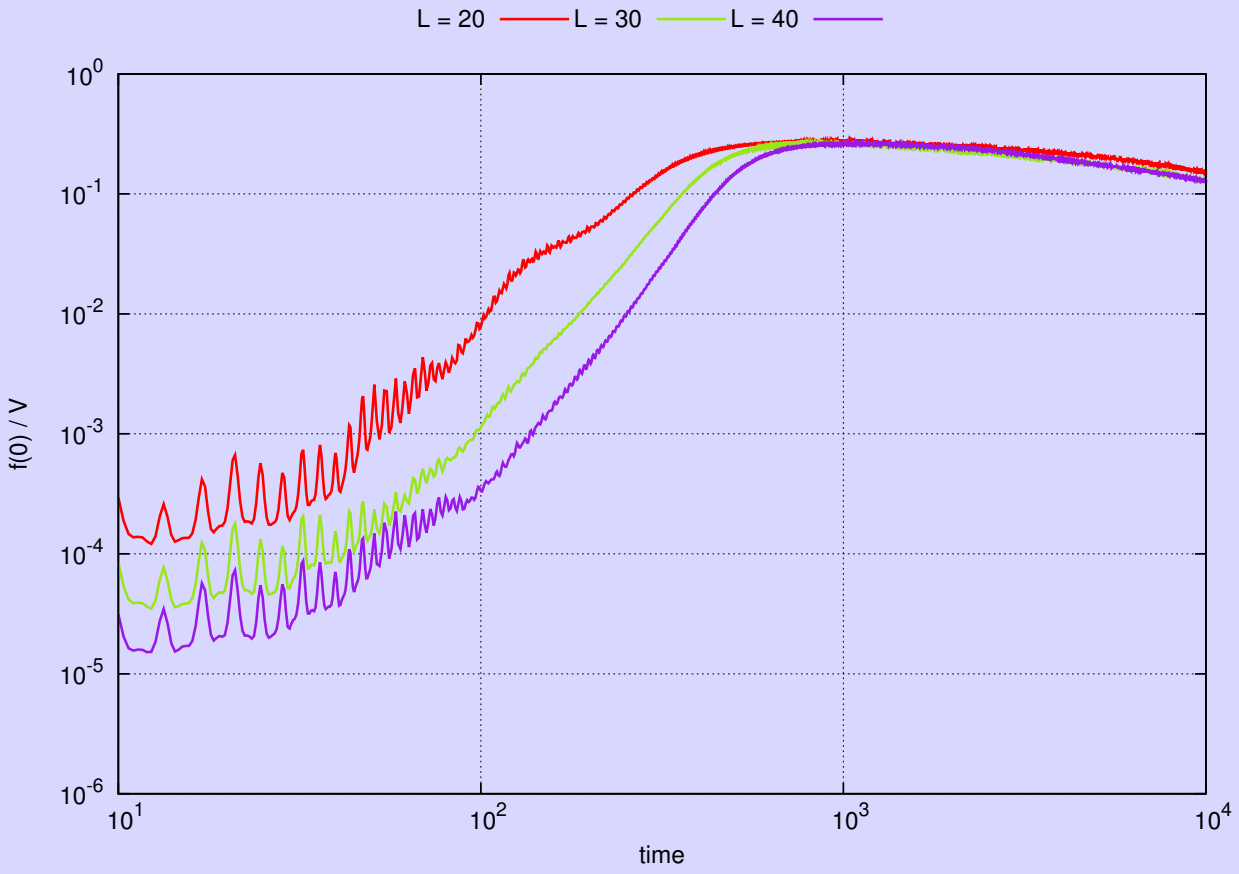
Occupation number



- Resonant peak at early times
- Late times : classical equilibrium with a chemical potential
- $\mu \approx m$ + excess at $k = 0$: Bose-Einstein condensation?

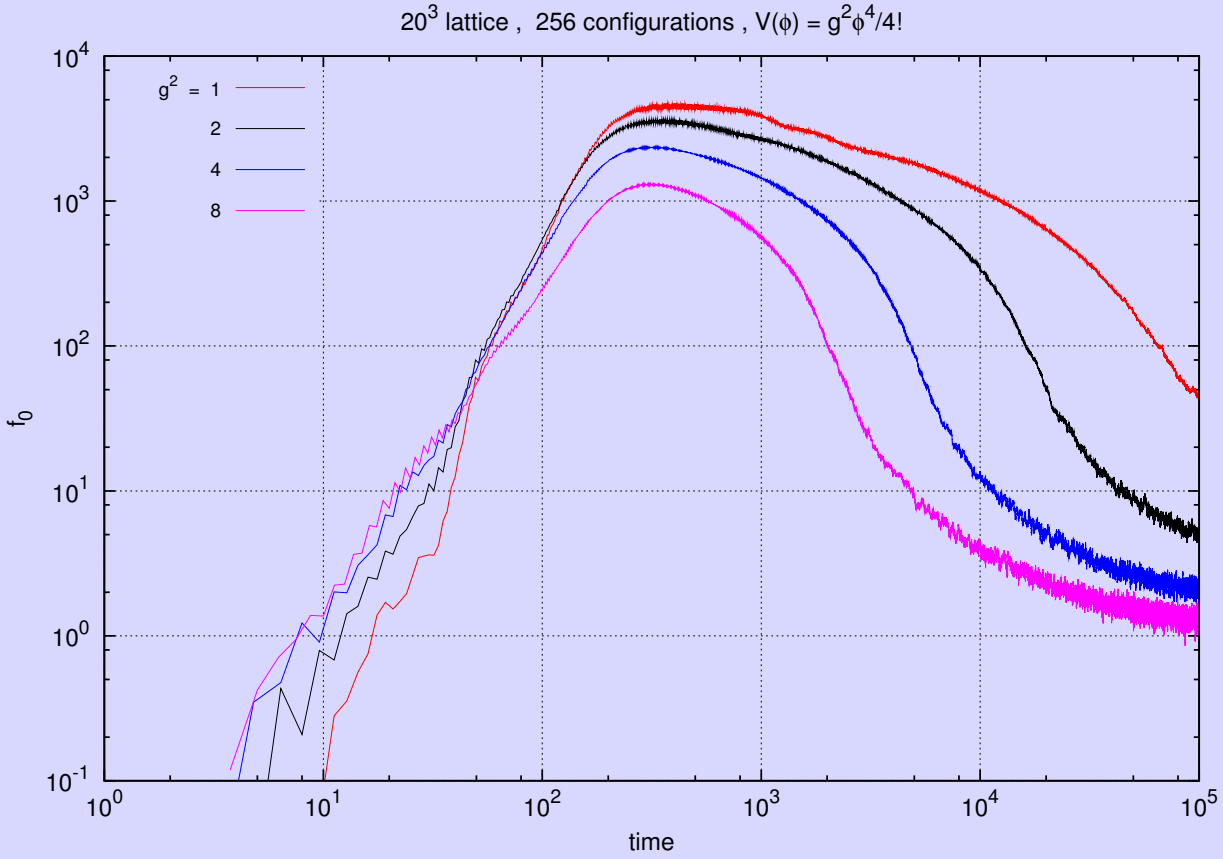
Bose-Einstein Condensation?

Volume dependence of the zero mode



$$f(\mathbf{k}) = \frac{1}{e^{\beta(\omega_k - \mu)} - 1} + n_0 \delta(\mathbf{k}) \quad \Longrightarrow f(0) \propto V = L^3$$

Evolution of the condensate

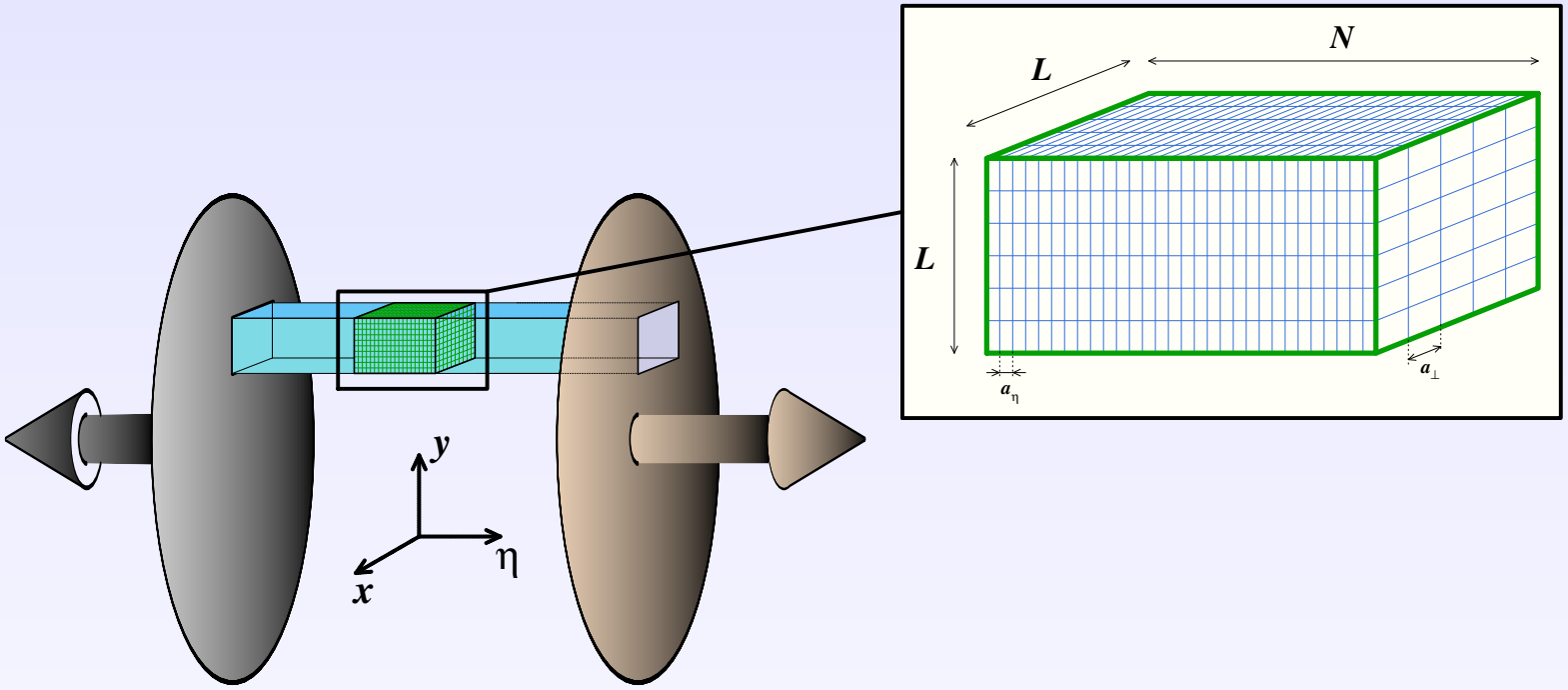


- Formation time almost independent of the coupling
- Condensate lifetime much longer than its formation time

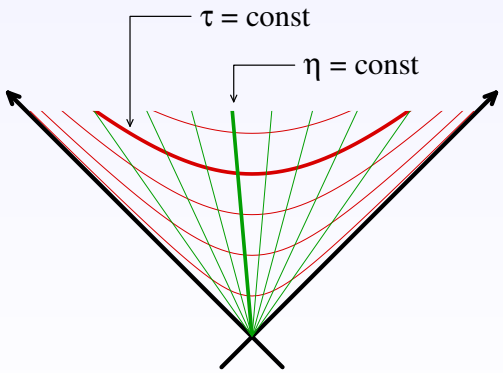
Isotropization

(in an expanding system)

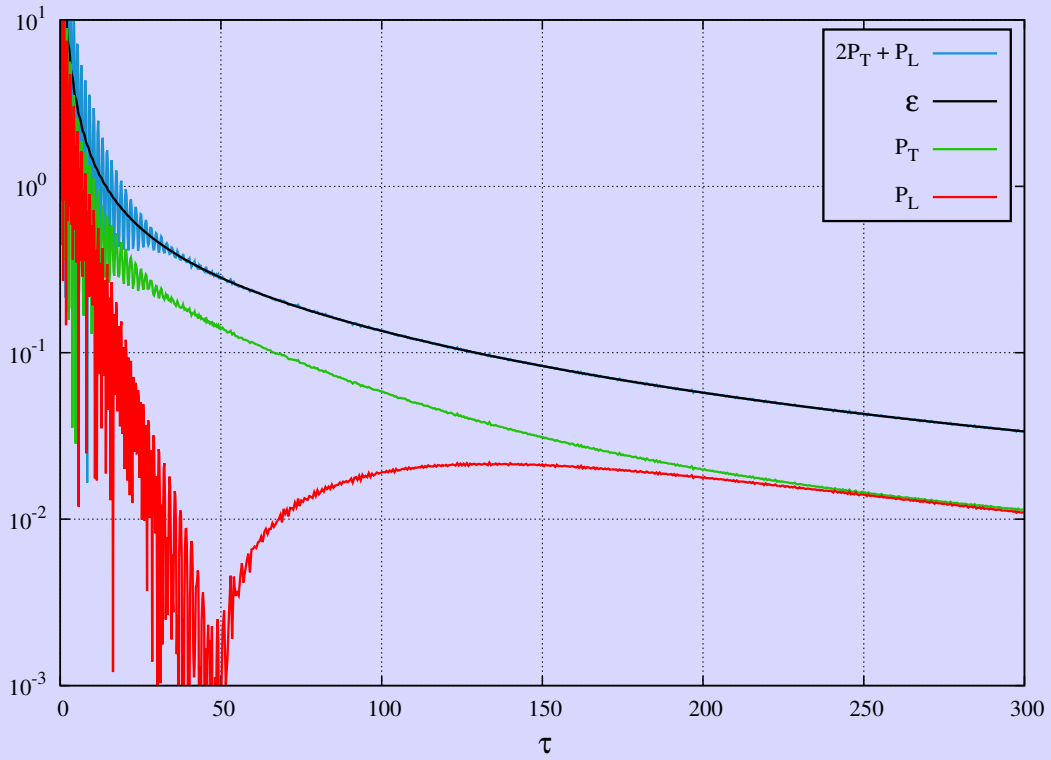
Discretization of the expanding volume



- Comoving coordinates : $\tau, \eta, \mathbf{x}_\perp$
- Only a small volume is simulated
  + periodic boundary conditions
- $L^2 \times N$ lattice with $L \sim 30 - 50$, $N \sim 300 - 600$

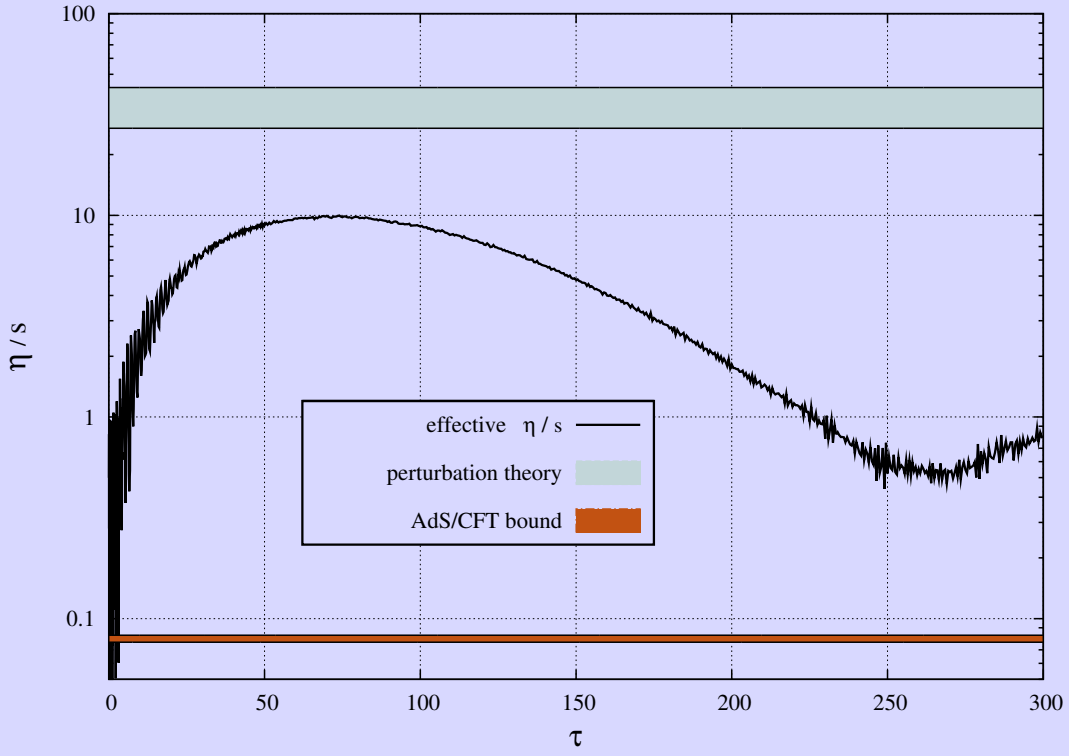


Isotropization



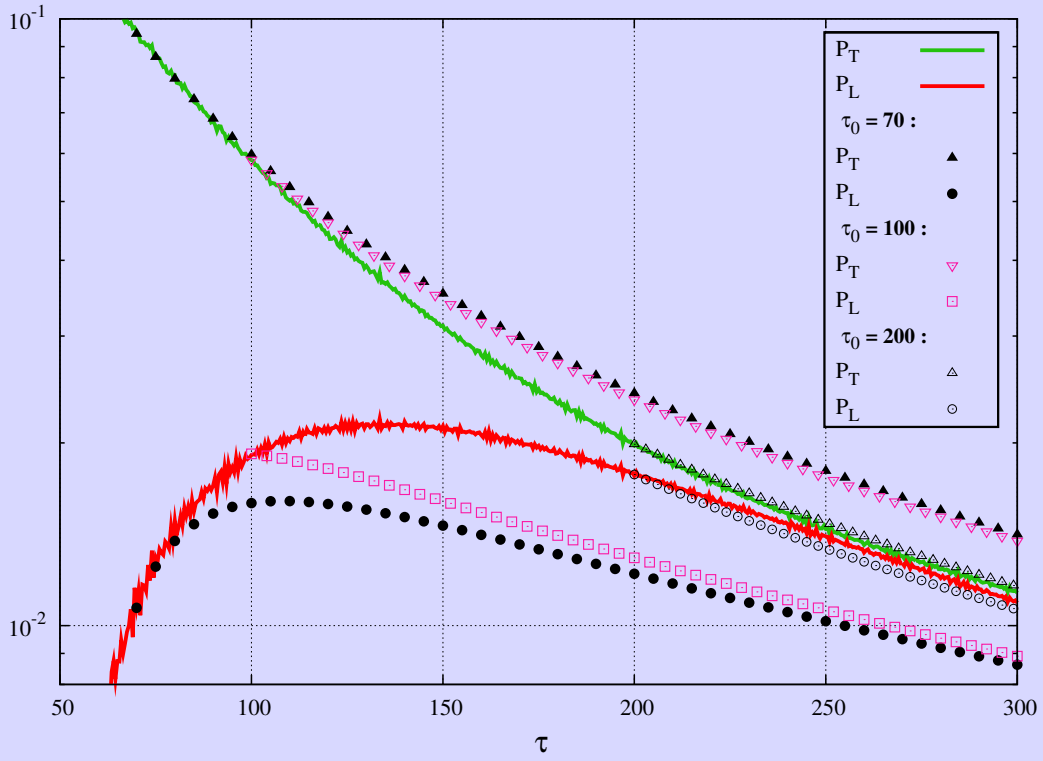
- At early times, P_L drops much faster than P_T (redshifting of the longitudinal momenta due to the expansion)
- Drastic change of behavior when the expansion rate becomes smaller than the growth rate of the unstability
- Eventually, isotropic pressure tensor : $P_L \approx P_T$

Effective shear viscosity



$$P_T = \frac{\epsilon}{3} + \frac{2}{3}\frac{\eta}{s}\frac{s}{\tau} \quad , \qquad P_L = \frac{\epsilon}{3} - \frac{4}{3}\frac{\eta}{s}\frac{s}{\tau} \quad , \qquad s \approx \epsilon^{3/4}$$

Comparison with 1st order hydrodynamics



- Faster relaxation than in hydrodynamics
- Hydrodynamics works well once $P_L \approx P_T$

Summary

Summary and outlook

Coherent state ($\tau = 0^+$)

$\Downarrow$

Equation of State

$\Downarrow$

Isotropic stress tensor

$\Downarrow$

Kinetic equilibrium

$\Downarrow$

Chemical equilibrium

BEC ?

Hydrodynamics

BUT : so far, all numerical studies done for a toy scalar model

What's next?

- generalizable to QCD
- gauge invariant formulation
- computationally expensive ($\sim$ [scalar case] $\times 3 \times (N_c^2 - 1)$)