Gluon saturation from DIS to AA collisions

III – AA collisions : gluon production

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General outline

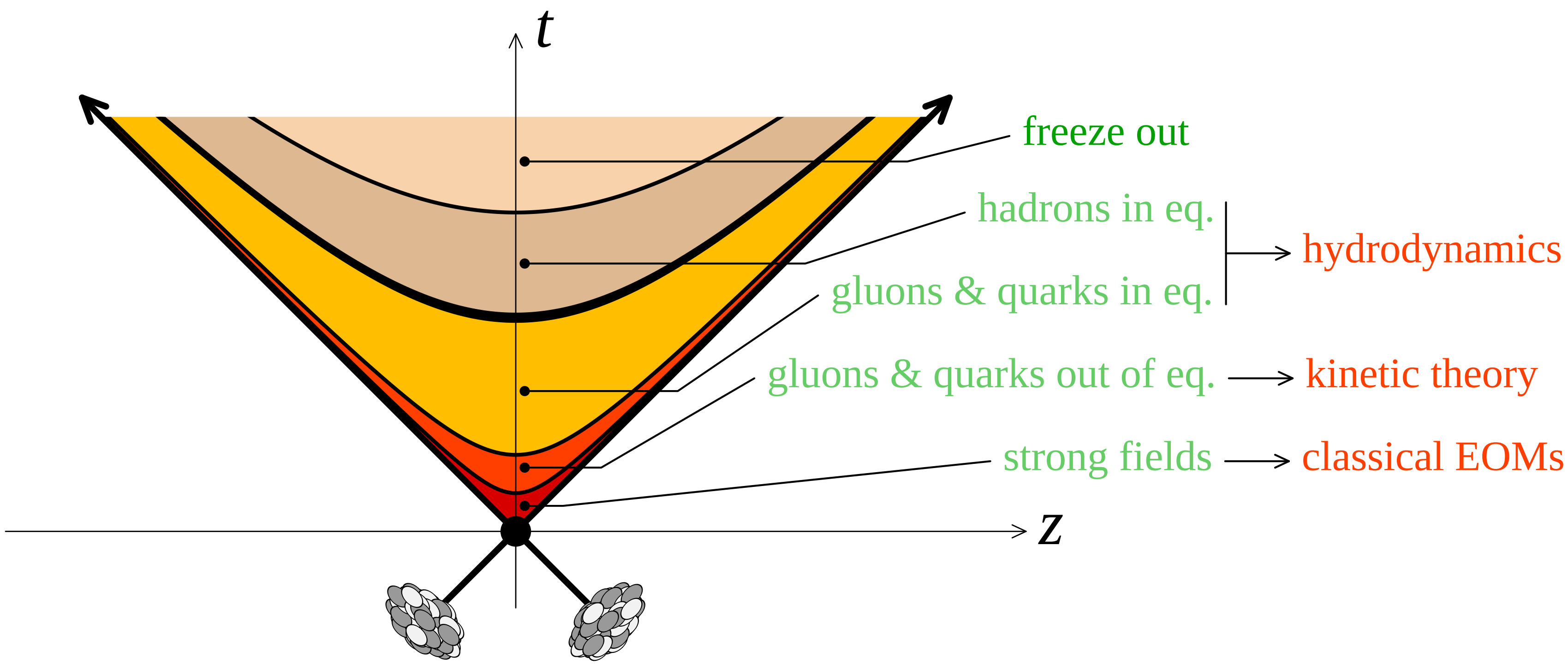
- Lecture I : Gluon saturation in DIS
- Lecture II : Proton-nucleus collisions
- Lecture III : AA collisions : gluon production
- Lecture IV : AA collisions : glasma instabilities

Lecture III : AA : gluon production

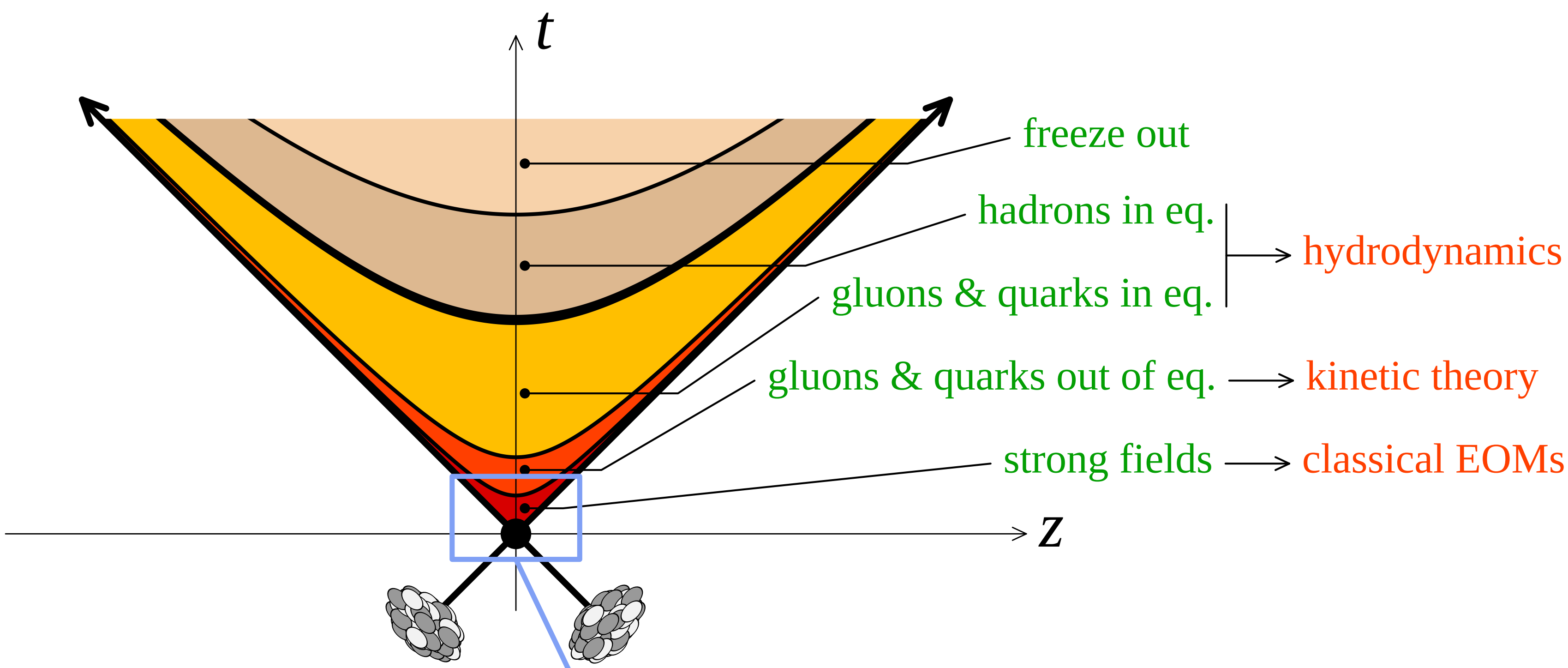
- Introduction to nucleus-nucleus collisions
- Power counting and bookeeping
- Classical fields, boundary conditions
- Factorization at small x

Introduction

Stages of a nucleus-nucleus collision



Stages of a nucleus-nucleus collision



- compute the initial production of semi-hard particles
- compute initial conditions for hydrodynamics

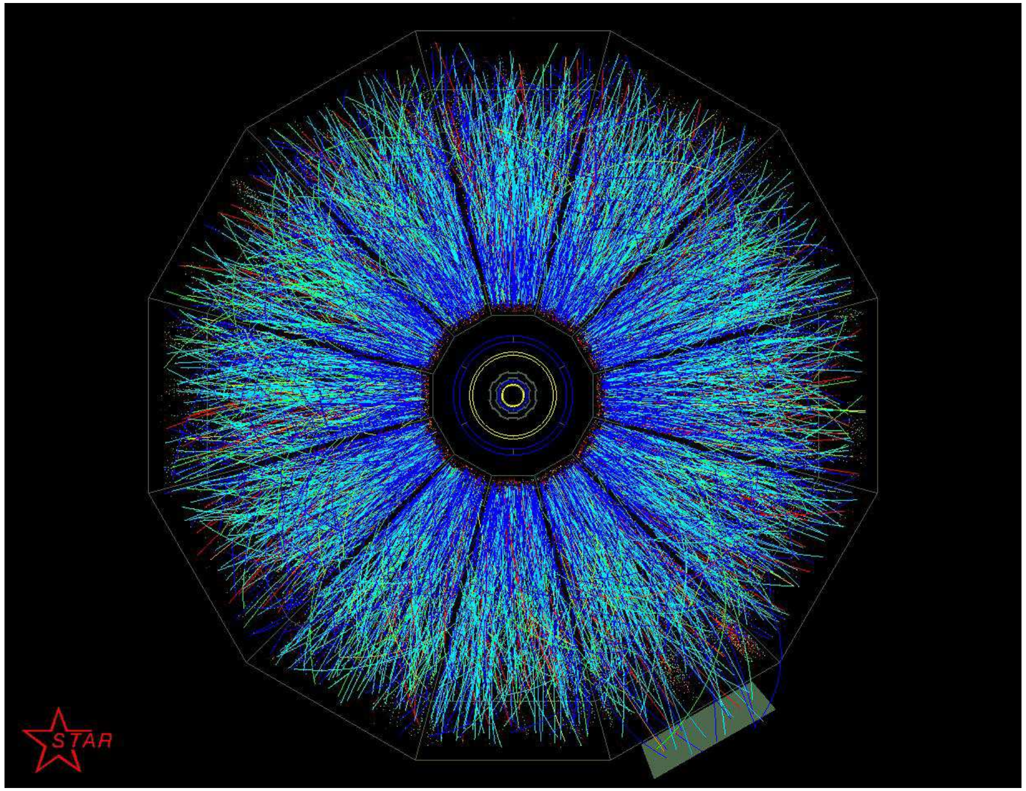
Small x QCD in AA collisions

- Saturation affects the early stages of heavy ion collisions, up to a time $\tau \sim Q_s^{-1}$
- The dynamics that takes place afterwards blurs the physics of saturation (for instance, if the system reaches thermalization, it does not remember the details of the dynamics at early times)

  ▷ Saturation affects only inclusive observables, like the overall multiplicity and its energy dependence

  ▷ Nucleus-nucleus collisions are a limited framework in order to probe saturation
- In AA collisions, the Color Glass Condensate provides a framework that can be used to compute an initial condition for the rest of the evolution

Small x QCD in AA collisions



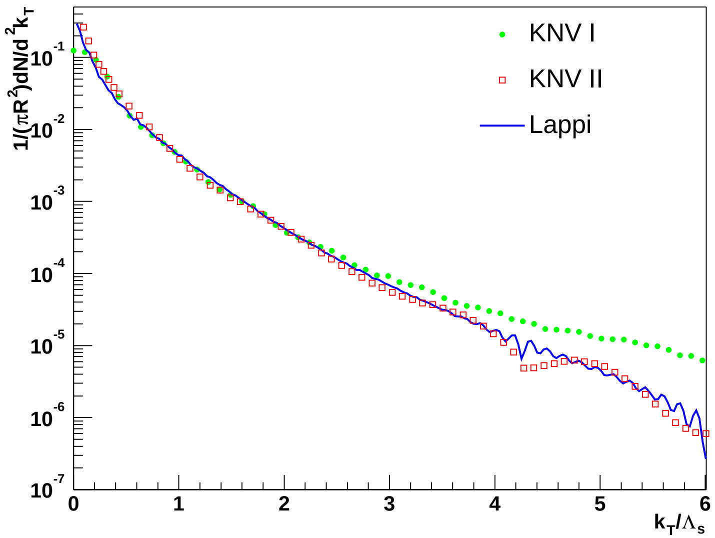
- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- the bulk of particle production comes from (very) low x
  - ▷ high gluon density (even more so in nuclei : $G_A/G_\mathrm{p} \approx A$)

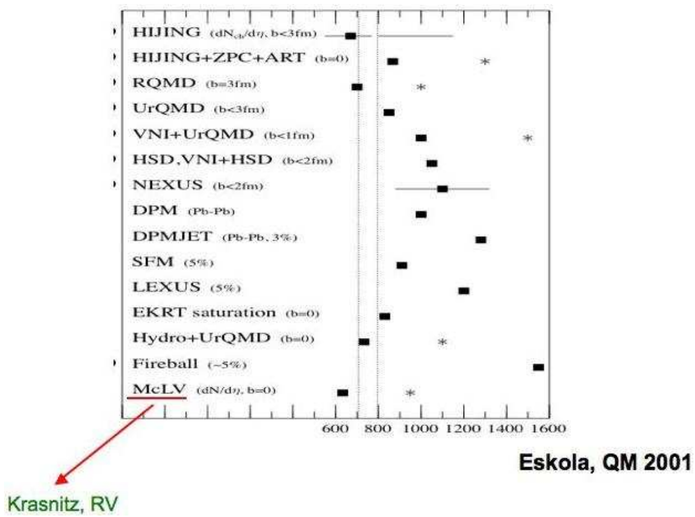
Krasnitz-Venugopalan computation

- Gluon spectrum from retarded classical solutions of Yang-Mills equations (Krasnitz, Venugopalan (1998); Lappi (2003)) :

$$\left\langle \frac{dN}{dY\,d^2\vec{\boldsymbol{p}}_\perp} \right\rangle_{\rm LLog} \propto \int_{x,y} e^{ip\cdot(x-y)} \;\cdots\; \langle \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)\rangle$$

$$\left[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}\right] = \delta^{\nu+}\delta(x^-)\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\nu-}\delta(x^+)\rho_2(\vec{\boldsymbol{x}}_\perp) \quad \text{with} \lim_{x_0\to-\infty} \mathcal{A}_\mu(x) = 0$$





Krasnitz-Venugopalan computation

- In nucleus-nucleus collisions, the two sources are equally strong, and should be treated on the same footing :

$$J^\mu \equiv \delta^{\mu +}\delta(x^-)\,\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu -}\delta(x^+)\,\rho_2(\vec{\boldsymbol{x}}_\perp)$$

- Average over the sources ρ_1, ρ_2

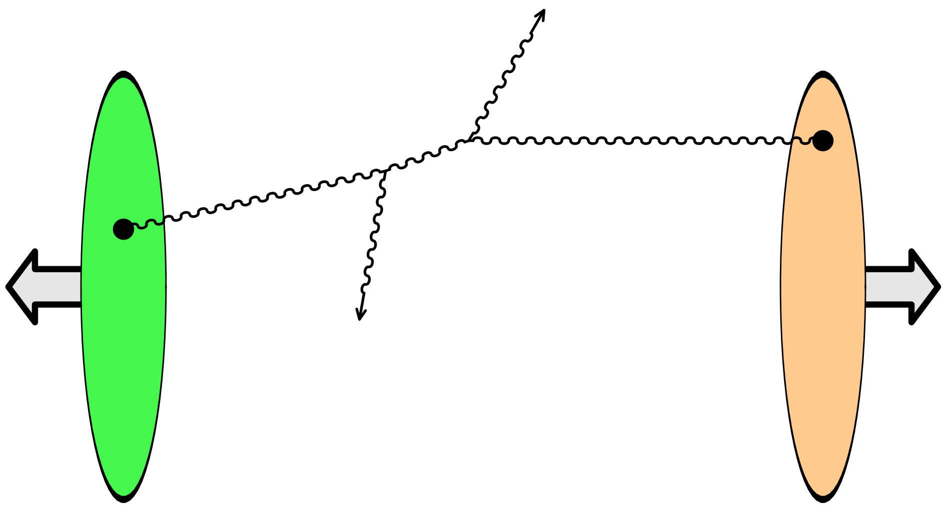
$$\langle \mathcal{O} \rangle_Y = \int [D\rho_1]\,[D\rho_2]\,W_{Y_{\rm beam}-Y}[\rho_1]\,W_{Y+Y_{\rm beam}}[\rho_2]\,\mathcal{O}[\rho_1,\rho_2]$$

- How to compute $\mathcal{O}[\rho_1,\rho_2]$ in the saturation regime ?
- What is the meaning of this factorization formula ?

Goals of this lecture

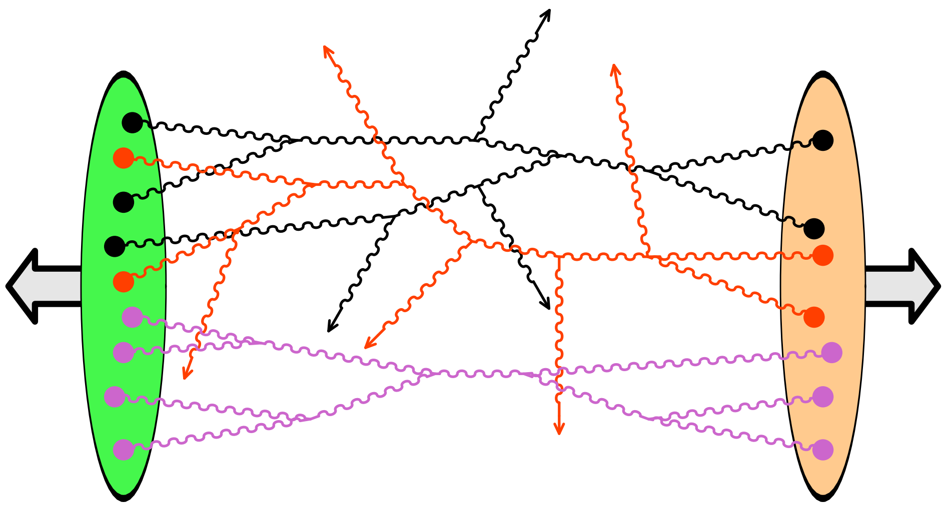
- Why can the gluon yield be obtained from classical solutions of Yang-Mills equations ?
- Why are the boundary conditions retarded ?
- Is this a controlled approximation, i.e. the first term in a more systematic expansion ?
- Is it possible to go beyond this computation, and study the 1-loop corrections ? Logs(1/x) and factorization ?

Initial particle production



- Dilute regime : one parton in each projectile interact

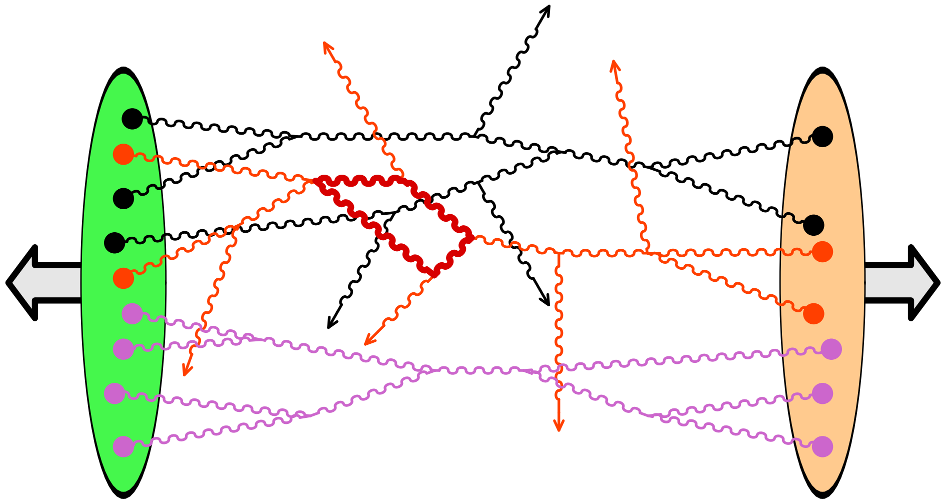
Initial particle production



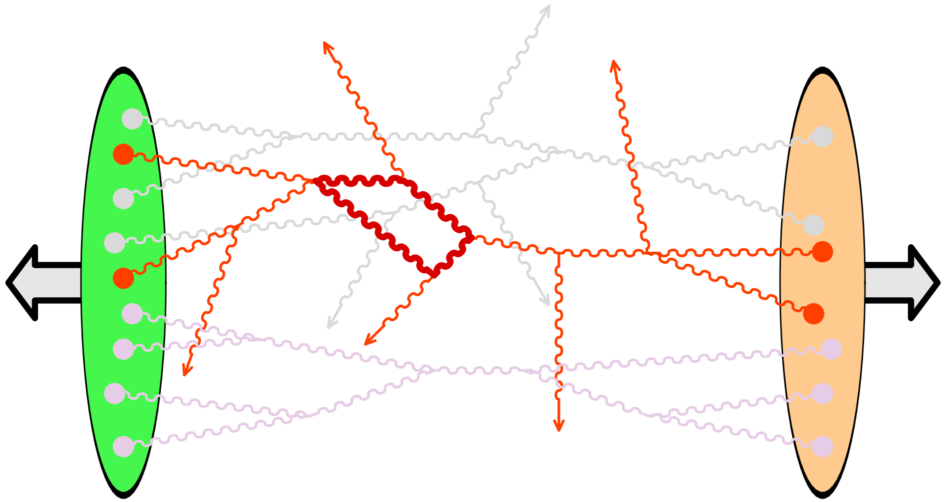
- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial (+ pileup of many simultaneous scatterings)

Power counting and Bookkeeping

Power counting



Power counting



- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle \sim$ occupation number $\sim 1/\alpha_s$)
- The order of each connected diagram is given by :

$$\frac{1}{g^2}\quad g^{\#\text{ produced gluons}}\quad g^{2(\#\text{ loops})}$$

- The total order of a graph is the product of the orders of its disconnected subdiagrams

Power counting

- Example : Inclusive gluon spectrum :

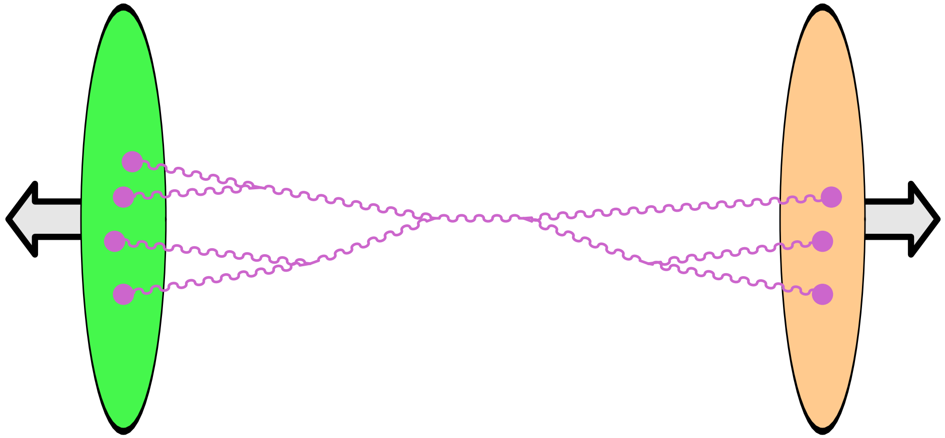
$$\frac{dN}{d^3\vec{\boldsymbol{p}}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

- The coefficients $c_{0,1,\cdots}$ are themselves series that resum all orders in $(g\rho_{1,2})^n$. For instance,

$$c_0 = \sum_{n=0}^{\infty} c_{0,n}\,(g\rho_{1,2})^n$$

- We want to calculate at least the entire c_0/g^2 contribution, and a subset of the higher order terms

Vacuum diagrams

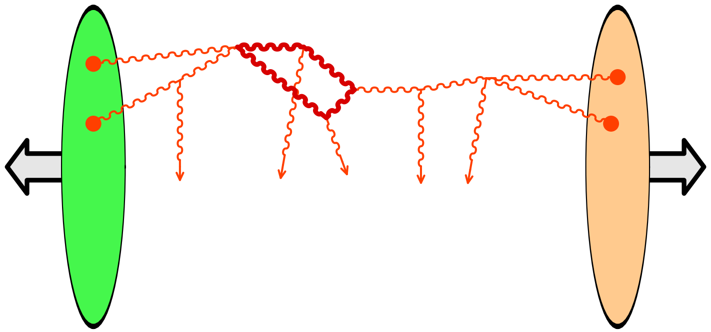


- Vacuum diagrams do not produce any gluon. They are contributions to the vacuum to vacuum amplitude $\langle 0_{\rm out} | 0_{\rm in} \rangle$
- The order of a connected vacuum diagram is given by :

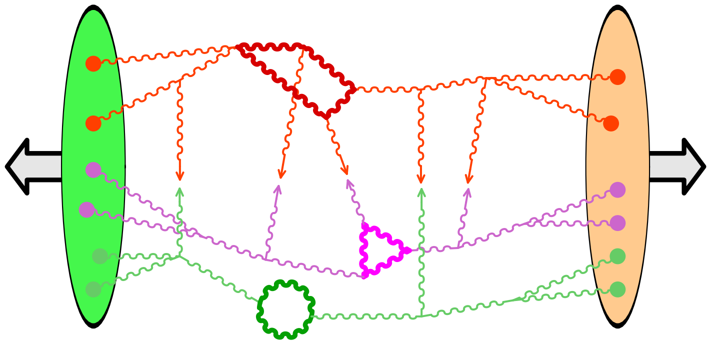
$$g^{-2} \quad g^{2(\#\ \text{loops})}$$

- Relation between connected and non connected vacuum diagrams :

$$\sum \begin{pmatrix} \text{all the vacuum} \\ \text{diagrams} \end{pmatrix} = \exp\left\{ \sum \begin{pmatrix} \text{simply connected} \\ \text{vacuum diagrams} \end{pmatrix} \right\} = e^{iV[j]}$$

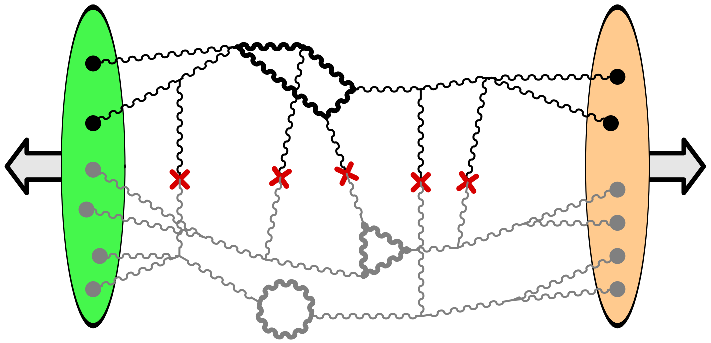


Bookkeeping



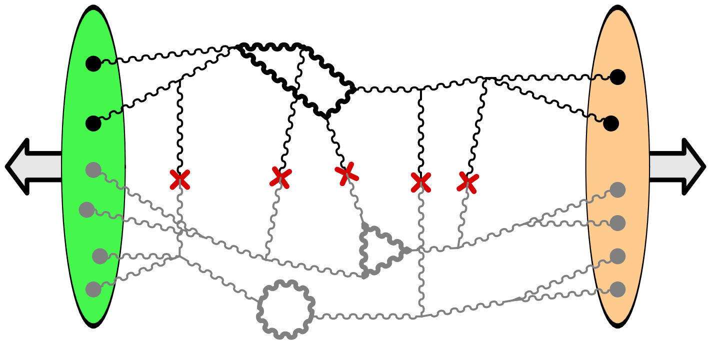
- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts through vacuum diagrams
  cut propagator : $2\pi\theta(-p^0)\delta(p^2)$
- The sum of the vacuum diagrams, $\exp(iV[j])$, is the generating functional for time-ordered products of fields :

$$\big\langle 0_{\rm out}\big|TA(x_1)\cdots A(x_n)\big|0_{\rm in}\big\rangle = \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\;e^{iV[j]}$$

Bookkeeping

- The probability of producing exactly n particles is :

$$P_n = \frac{1}{n!}\int \frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1}\cdots\frac{d^3\vec{\boldsymbol{p}}_n}{(2\pi)^3 2E_n}\left|\left\langle \vec{\boldsymbol{p}}_1\cdots\vec{\boldsymbol{p}}_{n\,\mathrm{out}}\middle|0_{\mathrm{in}}\right\rangle\right|^2$$

- Exercise. Show that :

$$P_n = \frac{1}{n!}\,\mathcal{C}^n\; e^{iV[j_+]}\; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

with

$$\begin{cases} \mathcal{C} \equiv \displaystyle\int_{x,y} G^0_{+-}(x,y)\,\square_x\square_y\,\frac{\delta}{\delta j_+(x)}\,\frac{\delta}{\delta j_-(y)} \\ G^0_{+-}(x,y) \equiv \displaystyle\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\; 2\pi\theta(-p^0)\delta(p^2) \end{cases}$$

Hint : start from the reduction formula for the transition amplitude, and use the fact that $\exp(iV[j])$ is the generating functional

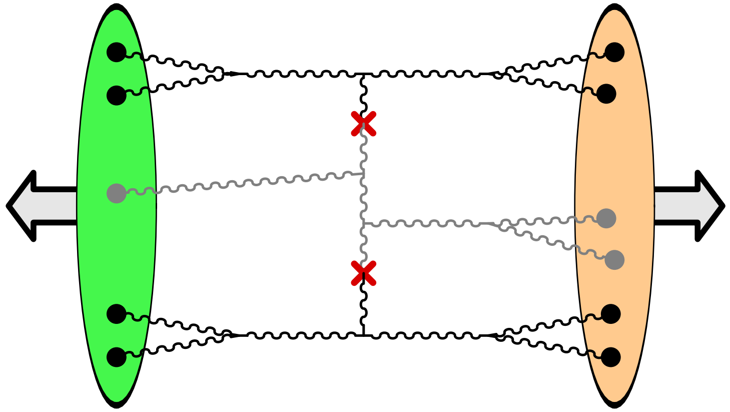
Note : the propagator $G^0_{+-}(x,y)$ is a cut propagator

Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y \, \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :



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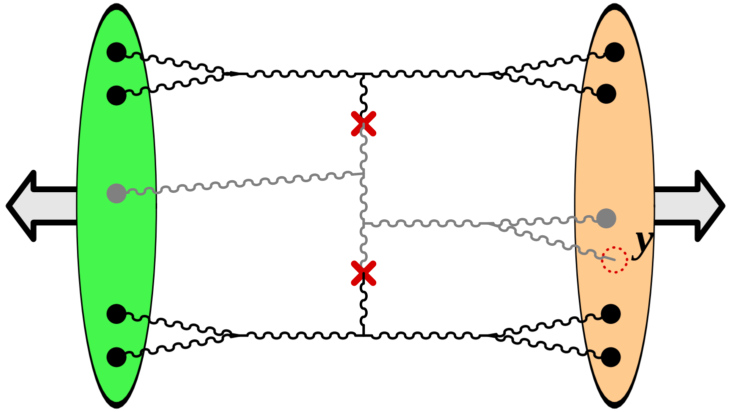
$$\frac{\delta}{\delta j_-(y)} \quad \longrightarrow$$



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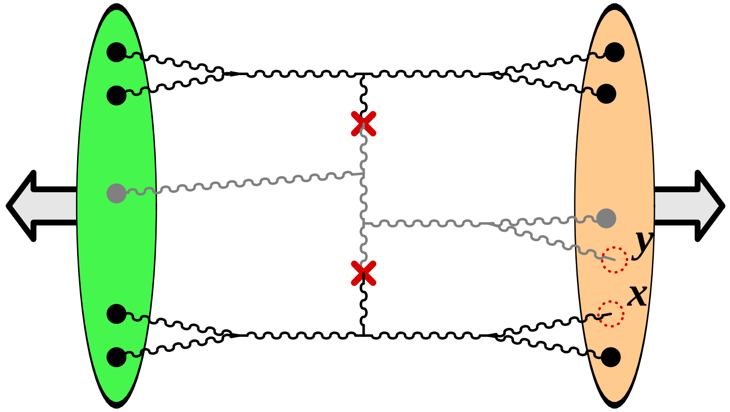

Bookkeeping

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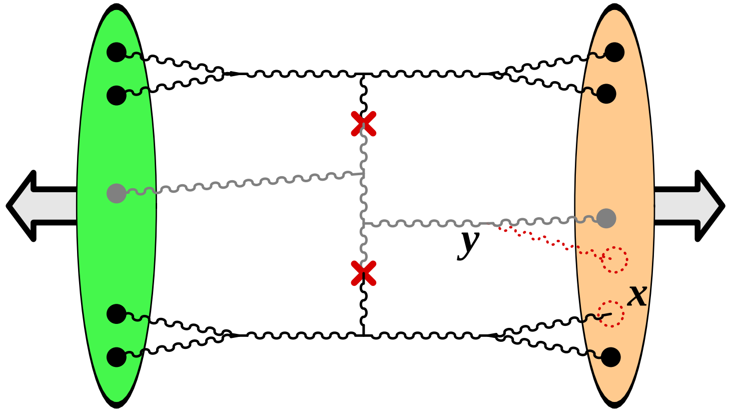
- Consider a generic cut vacuum diagram :

$$\square_y \quad \longrightarrow$$



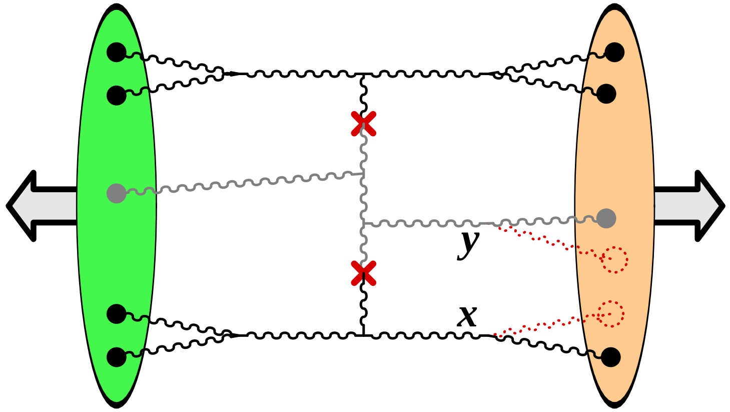
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\,\square_x \square_y\,\frac{\delta}{\delta j_+(x)}\frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$\square_x \quad \longrightarrow$$



Introduction
Bookkeeping
Power counting
Vacuum diagrams
Bookkeeping
Classical fields
Factorization
Summary

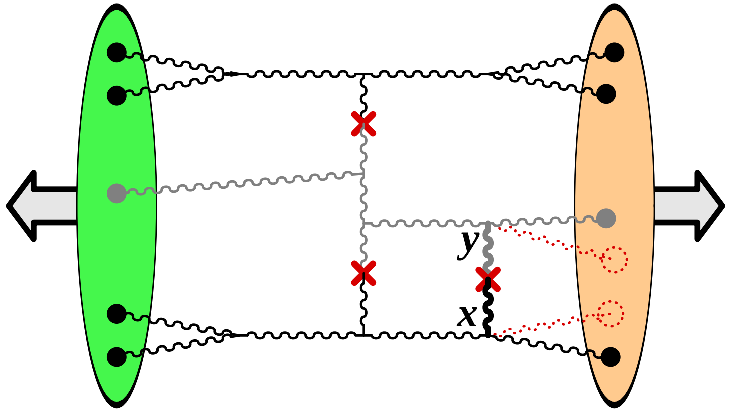
Bookkeeping

- Reminder :

$$\mathcal{C} \equiv \int_{x,y} G^0_{+-}(x,y)\, \square_x \square_y\, \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

- Consider a generic cut vacuum diagram :

$$G^0_{+-}(x,y) \quad \longrightarrow$$



▷ the operator $\mathcal{C}$ removes two sources (one in the amplitude and one in the complex conjugated amplitude), and creates a new cut propagator

Bookkeeping

- The sum of all the cut vacuum diagrams, with sources j_+ on one side of the cut and j_- on the other side, can be written as :

$$\sum \left(\begin{array}{c} \text{all the cut} \\ \text{vacuum diagrams} \end{array} \right) = e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}$$

- If we set $j_+ = j_- = j$, then we should get $\sum_n P_n = 1$

- Therefore, we have :

$$e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]} \Big|_{j_+=j_-} = 1$$

Note : the use of this identity renders automatic an important cancellation that would be hard to see at the level of diagrams (somewhat related to AGK)

Bookkeeping

- The operator $\mathcal{C}$ can be used to derive many useful formulas :

$$F(z) = \sum_{n=0}^{+\infty} z^n \, P_n = e^{z\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

▷ sum of all cut vacuum graphs, where each cut is weighted by z

$$\overline{N} = F'(1) = \mathcal{C} \; e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

$$\overline{N(N-1)} = F''(1) = \mathcal{C}^2 \; e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

- Main benefit :

The tracking of infinite sets of Feynman diagrams has been replaced by simple algebraic manipulations

Classical fields, Boundary conditions

Diagrammatic expansion

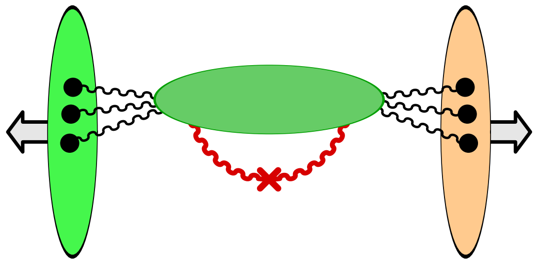
- It is easy to express the average multiplicity as :

$$\overline{N} = \sum_n n\, P_n = \mathcal{C}\left\{ \underbrace{e^{\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}} \right\}_{j_+=j_-=j}$$

sum of all the cut vacuum diagrams : $e^{iW[j_+,j_-]}$

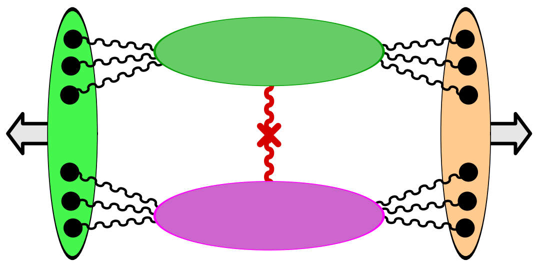
- There are two types of terms :
  - $\mathcal{C}$ picks two sources in the same connected cut diagram

$$\frac{\delta^2 iW}{\delta j_+(x)\delta j_-(y)} \quad \longrightarrow$$



  - $\mathcal{C}$ picks two sources in two distinct connected cut diagrams

$$\frac{\delta iW}{\delta j_+(x)}\;\frac{\delta iW}{\delta j_-(y)} \quad \longrightarrow$$

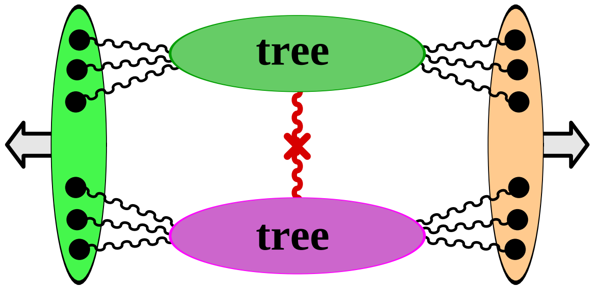


Diagrammatic expansion (LO)

- At LO, only tree diagrams contribute
  - ▷ the first type of topologies can be neglected (they have at least one loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- Note : at this point, the sources on both sides of the cut must be set equal :

$$j_+ = j_- = j$$

Retarded propagators

- In the previous diagrams, one must sum over all the possible ways of cutting lines inside the blobs
- This can be achieved via Cutkosky's cutting rules :
  - A vertex is $-ig$ on one side of the cut, and $+ig$ on the other side
  - A source ρ changes sign depending on the side of the cut
  - There are four propagators, depending on the location w.r.t. the cut of the vertices they connect :

$$
\begin{aligned}
G^0_{++}(p) &= i/(p^2 - m^2 + i\epsilon) && \text{(standard Feynman propagator)}\\
G^0_{--}(p) &= -i/(p^2 - m^2 - i\epsilon) && \text{(complex conjugate of } G^0_{++}(p))\\
G^0_{+-}(p) &= 2\pi\theta(-p^0)\delta(p^2 - m^2)
\end{aligned}
$$

  - At each vertex of a given diagram, sum over the types $+$ and $-$ (2^n terms for a diagram with n vertices)

Retarded propagators

- When summing over the cuts, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \frac{\boldsymbol{i}}{\boldsymbol{p^2 - m^2 + i\epsilon}} - 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

Retarded propagators

- When summing over the cuts, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \mathrm{PP}\left[\frac{i}{p^2-m^2}\right] + \underbrace{\pi\delta(p^2-m^2) - 2\pi\theta(-p^0)\delta(p^2-m^2)}_{\text{insert : } \boldsymbol{1=\theta(p^0)+\theta(-p^0)}}$$

Retarded propagators

- When summing over the cuts, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \mathrm{PP}\left[\frac{i}{p^2-m^2}\right] + \pi\left[\underbrace{\theta(p^0) - \theta(-p^0)}_{\textbf{sign}\,(p^0)}\right]\delta(p^2-m^2)$$

Retarded propagators

- When summing over the cuts, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \frac{i}{p^2 - m^2 + i\,\mathrm{sign}(p^0)\epsilon}$$

Retarded propagators

- When summing over the cuts, we only get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \boldsymbol{G^0_R(p)}$$

- Similarly : $G^0_{-+}(p) - G^0_{--}(p) = G^0_R(p)$

- Starting from the "leaves" of the trees, one can use these formulas in order to replace recursively all the $G^0_{\pm\pm}$ propagators by retarded propagators

  ▷ we have a sum of tree diagrams with retarded propagators

Classical fields

- The sum of all the tree diagrams constructed with retarded propagators is the solution of classical field equations with retarded boundary condition :

$$\lim_{t\to-\infty} \mathcal{A}(t,\vec{\boldsymbol{x}}) = 0$$

- Proof (for a scalar theory with a cubic self-interaction). The classical EOM reads

$$(\square + m^2)\,\varphi(x) + \frac{g}{2}\varphi^2(x) = j(x)$$

- Write the Green's formula for the retarded solution that obeys $\varphi(t,\vec{\boldsymbol{x}}) = 0$ at $t = -\infty$:

$$\varphi(x) = \int d^4y\; G^0_R(x-y)\left[-i\,\frac{g}{2}\varphi^2(y) + i\,j(y)\right]$$

Classical field

- One can construct the solution iteratively, by using in the r.h.s. the solution found in the previous orders

- Order g^0 :

$$\varphi_{(0)}(x) = \int d^4y\, G^0_R(x-y)\, i\, j(y)$$

- Order g^1 :

$$\varphi_{(0)}(x) + \varphi_{(1)}(x) = \int d^4y\, G^0_R(x-y)\left[-i\,\frac{g}{2}\varphi^2_{(0)}(y) + i\, j(y)\right]$$

i.e.

$$\varphi_{(1)}(x) = -i\,\frac{g}{2}\int d^4y\, G^0_R(x-y)\left[\int d^4z\, G^0_R(y-z)\, i\, j(z)\right]^2$$

Retarded classical solution

- The diagrammatic expansion of this classical solution is :

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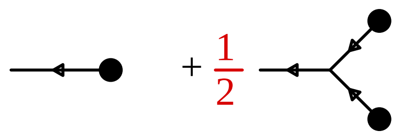

Retarded classical solution

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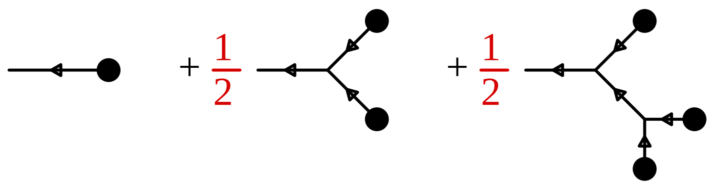

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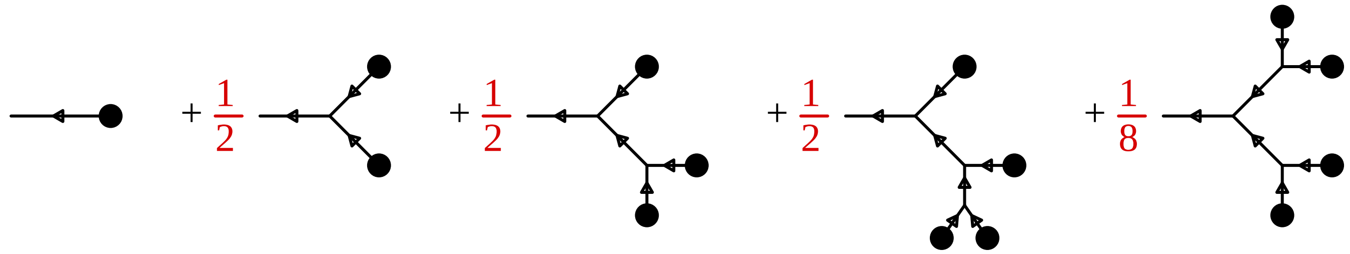

Retarded classical solution

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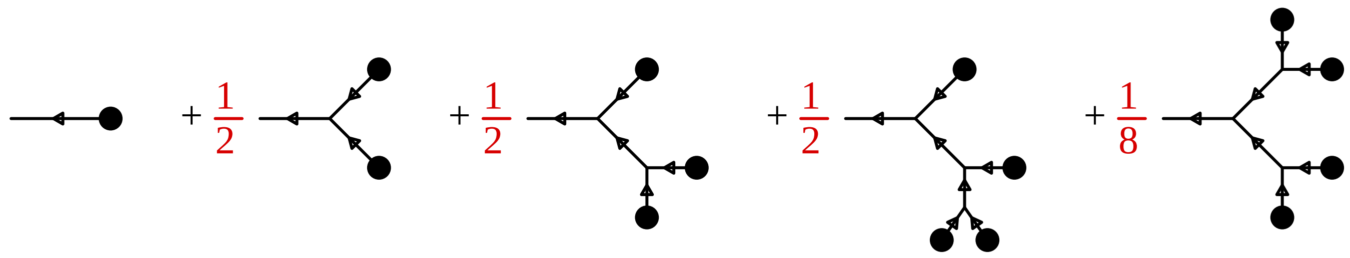


- The classical solution is given by the sum of all the tree diagrams with retarded propagators

Gluon spectrum at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)

- The gluon spectrum at LO is given by :

$$\left.\frac{dN}{dY\,d^2\vec{\boldsymbol{p}}_\perp}\right|_{\rm LO} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y \sum_\lambda \epsilon_\lambda^\mu\epsilon_\lambda^\nu\;\mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

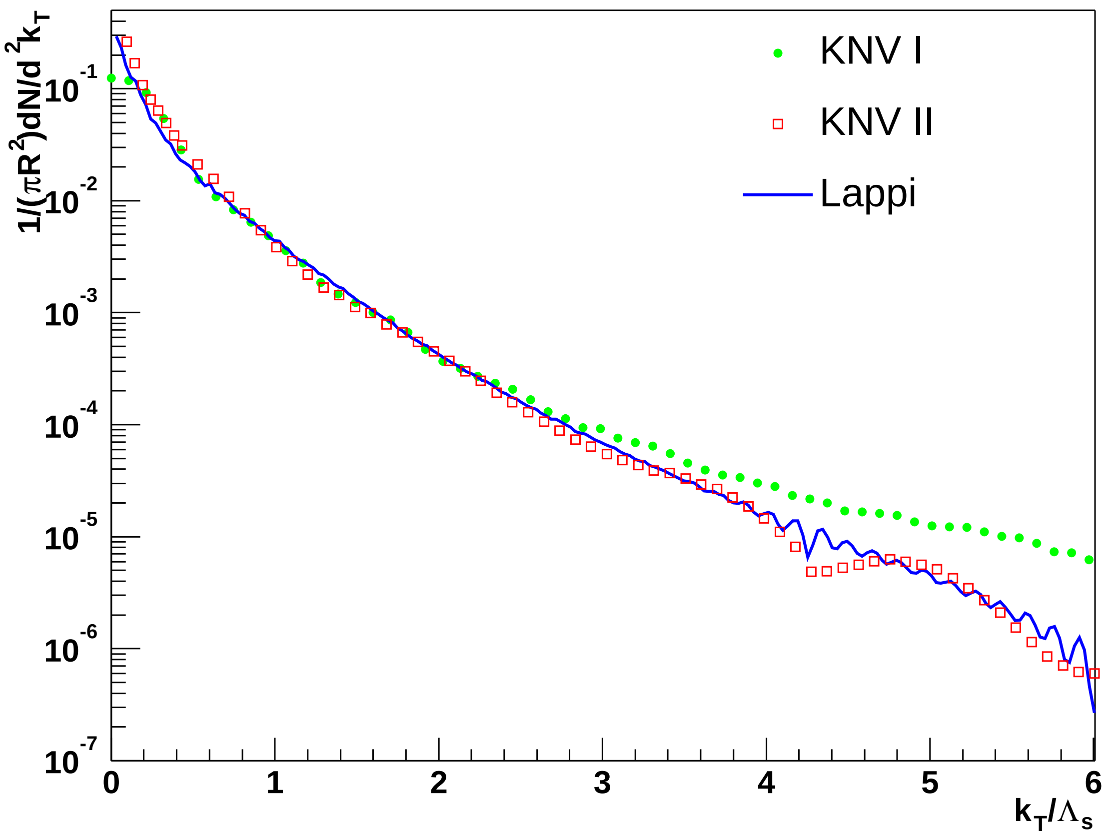
where $\mathcal{A}_\mu(x)$ is the solution of Yang-Mills equations,

$$[\mathcal{D}_\mu, \mathcal{F}^{\mu\nu}] = J^\nu$$

such that

$$\lim_{x^0\to-\infty} \mathcal{A}_\mu(x) = 0$$

Gluon spectrum at LO



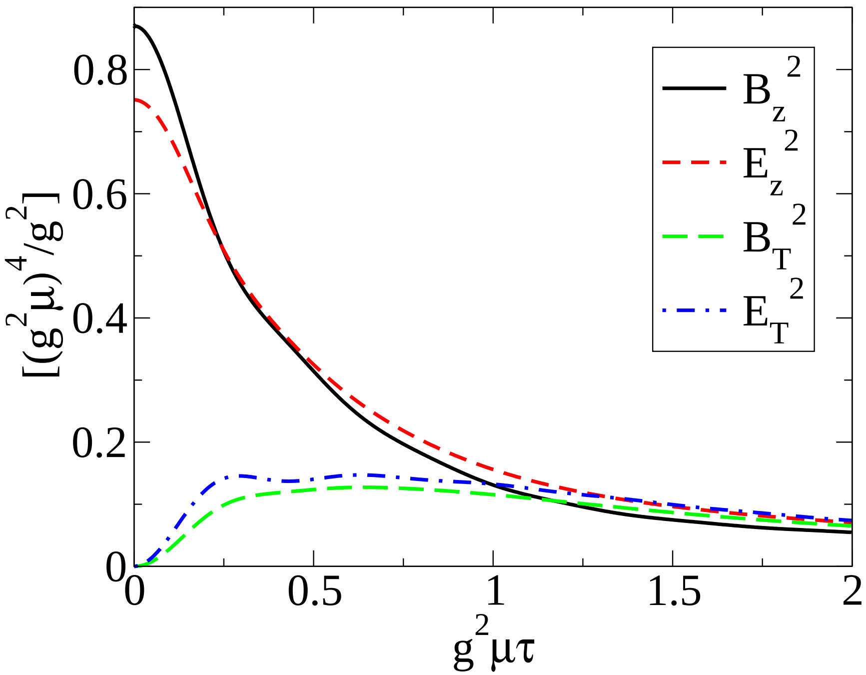
- Lattice artifacts at large momentum
  (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)

Initial Glasma fields

Lappi, McLerran (2006) (Semantics : Glasma ≡ Glas[s - plas]ma)

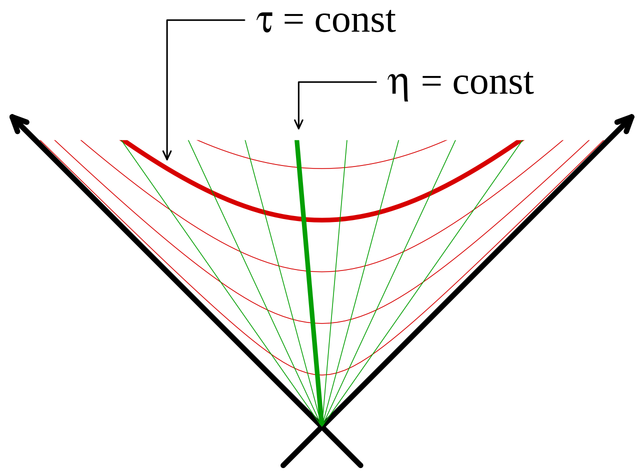
- Before the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields are localized in two sheets transverse to the beam axis
- Immediately after the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields have become longitudinal :

$$\boldsymbol{E}^z = ig\left[\mathcal{A}_1^i, \mathcal{A}_2^i\right] \quad , \qquad \boldsymbol{B}^z = ig\epsilon^{ij}\left[\mathcal{A}_1^i, \mathcal{A}_2^j\right]$$



Boost invariance

- Gauge condition : $x^+\mathcal{A}^- + x^-\mathcal{A}^+ = 0$



$$\Rightarrow \quad \mathcal{A}^\pm(x) \quad = \quad \pm\, x^\pm\, \beta(\tau,\eta,\vec{\boldsymbol{x}}_\perp)$$

- Initial values at $\tau = 0^+$: $\mathcal{A}^i(0^+,\eta,\vec{\boldsymbol{x}}_\perp)$ and $\beta(0^+,\eta,\vec{\boldsymbol{x}}_\perp)$ do not depend on the rapidity η

  ▷ $\mathcal{A}^i$ and β remain independent of η at all times

Exercise : Generating functional

- Consider a function $z(\vec{\boldsymbol{p}})$, and define the functional

$$F[z] \equiv \frac{1}{n!}\sum_{n=0}^{+\infty}\int d\Phi_1 \cdot d\Phi_n \; z(\vec{\boldsymbol{p}}_1)\cdots z(\vec{\boldsymbol{p}}_n) \; \left|\left\langle \vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_{n\,\mathrm{out}} \middle| 0_{\mathrm{in}}\right\rangle\right|^2$$

- At LO, one can write it in terms of two classical fields $\mathcal{A}_\pm(x)$:

$$\left.\frac{\delta \ln F[z]}{\delta z(\vec{\boldsymbol{p}})}\right|_{\mathrm{LO}} = \int_{x,y} e^{ip\cdot(x-y)} \cdots \mathcal{A}_+^\mu(x)\mathcal{A}_-^\nu(y)$$

- Non retarded boundary conditions unless $z(\vec{\boldsymbol{p}}) \equiv 1$:

$$a_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) = a_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0$$

$$a_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) = z(\vec{\boldsymbol{p}})\; a_+^{(+)}(+\infty,\vec{\boldsymbol{p}})$$

$$a_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) = z(\vec{\boldsymbol{p}})\; a_-^{(-)}(+\infty,\vec{\boldsymbol{p}})$$

where : $\mathcal{A}_\epsilon(x) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[a_\epsilon^{(+)}(x_0,\vec{\boldsymbol{p}})\, e^{-ip\cdot x} + a_\epsilon^{(-)}(x_0,\vec{\boldsymbol{p}})\, e^{+ip\cdot x}\right]$

Factorization at small x

What is the problem ?

- Naive perturbative expansion :

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} = \frac{1}{g^2}\left[c_0 + c_1\, g^2 + c_2\, g^4 + \cdots\right]$$

Note : so far, we have seen how to compute c_0 given $\rho_{1,2}$

- Problem : $c_{1,2,\cdots}$ contain logarithms of $1/x_{1,2}$:

$$\begin{aligned} c_1 &= & & & c_{10} &+ c_{11}\ln\left(\frac{1}{x_{1,2}}\right) \\ c_2 &= c_{20} &+& c_{21}\ln\left(\frac{1}{x_{1,2}}\right) &&+ \underbrace{c_{22}\ln^2\left(\frac{1}{x_{1,2}}\right)}_{\text{Leading Log terms}} \end{aligned}$$

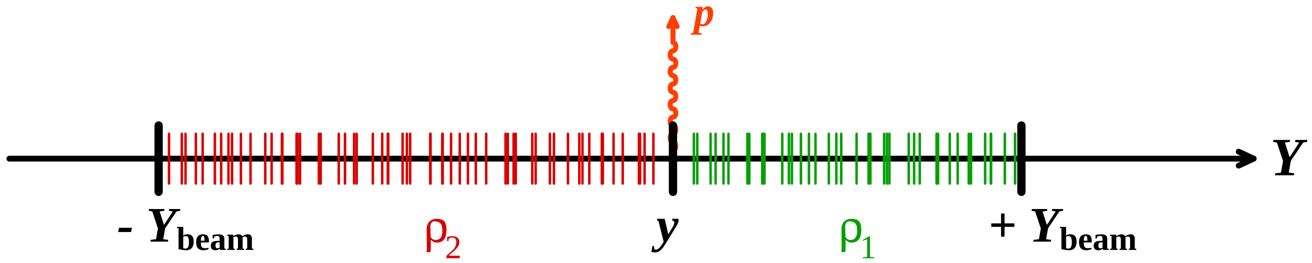
- At small $x_{1,2}$, these logs are large, and we would like to resum all the terms that have as many logs as powers of g^2

What is the problem ?

- For the single gluon spectrum in AA collisions, one would like to establish a formula such as :

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle \underset{\mathrm{LLog}}{=} \int \big[D\rho_1\, D\rho_2\big]\; W_{Y_{\mathrm{beam}}-y}[\rho_1]\; W_{y+Y_{\mathrm{beam}}}[\rho_2]\; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}}$$

with $\frac{\partial}{\partial Y} W_Y = \mathcal{H}\, W$



- All the leading logs of $1/x_{1,2}$ are absorbed in the $W's$
- The $W's$ obey the JIMWLK evolution equation

Factorization in four easy steps

- I : Express the single gluon spectrum at LO and NLO in terms of classical fields and small field fluctuations. Check that their boundary conditions are retarded
- II : Write the NLO terms as a perturbation of the initial value of the classical fields on the light-cone :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \left[\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

- III : For $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}$ on the same branch of the light-cone, one has :

$$\frac{1}{2}\int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}} = \log\left(\frac{\Lambda^+}{p^+}\right)\times\mathcal{H} + \text{finite terms}$$

- IV : There are no other logs. Factorization follows trivially

Single gluon spectrum at LO

- LO results for the single gluon spectrum :
  - At LO, the single gluon spectrum can be expressed in terms of classical solutions of the field equation of motion
  - These classical fields obey retarded boundary conditions

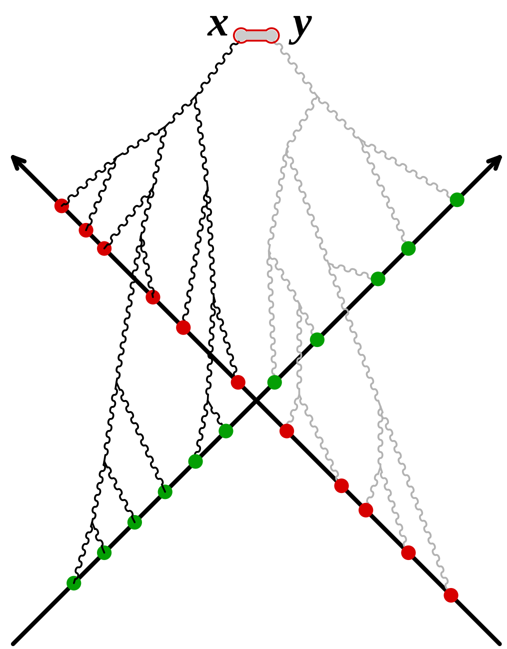
$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO} = \lim_{t\to+\infty} \int d^3\vec{\boldsymbol{x}}d^3\vec{\boldsymbol{y}}\; e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\;\cdots\mathcal{A}^\mu(t,\vec{\boldsymbol{x}})\;\mathcal{A}^\nu(t,\vec{\boldsymbol{y}})$$

$$\left[\mathcal{D}_\mu,\mathcal{F}^{\mu\nu}\right] = J^\nu$$

$$\lim_{t\to-\infty} \mathcal{A}^\mu(t,\vec{\boldsymbol{x}}) = 0$$

Single gluon spectrum at LO

- Retarded classical fields are sums of tree diagrams :



Single gluon spectrum at LO

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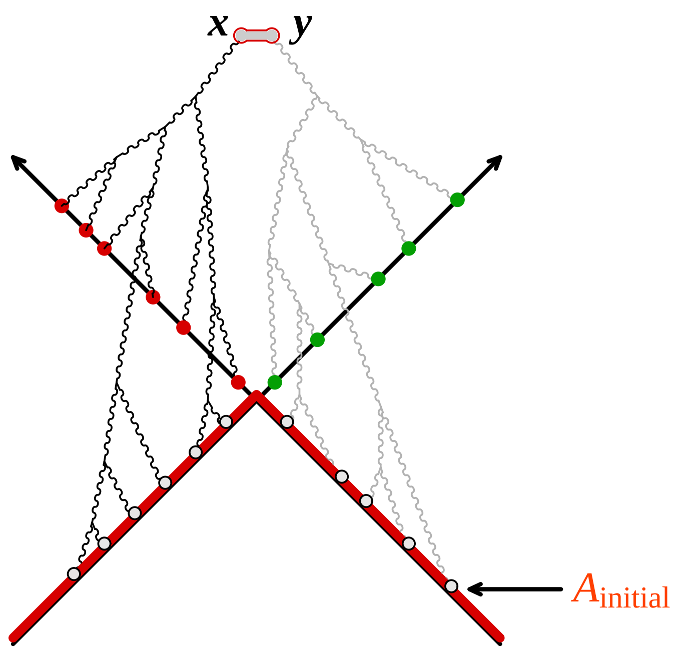
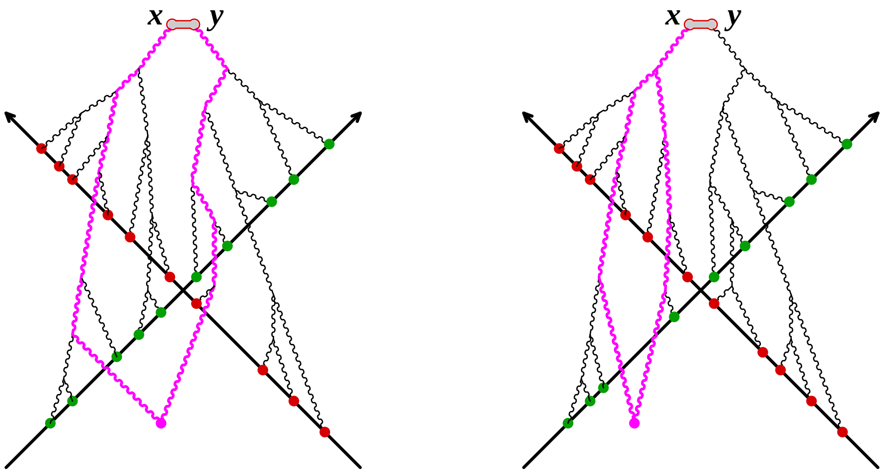


- Note : the gluon spectrum is a functional of the value of the classical field just above the backward light-cone :

$$\frac{dN}{d^3\vec{\boldsymbol{p}}} = \mathcal{F}[\mathcal{A}_{\rm initial}]$$

Single gluon spectrum at NLO

- 1-loop graphs contributing to the gluon spectrum at NLO :



$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \lim_{t\to+\infty}\int d^3\vec{\boldsymbol{x}}d^3\vec{\boldsymbol{y}}\; e^{i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\;\cdots\Big[\mathcal{G}^{\mu\nu}(x,y) + \beta^\mu(t,\vec{\boldsymbol{x}})\;\mathcal{A}^\nu(t,\vec{\boldsymbol{y}}) + \mathcal{A}^\mu(t,\vec{\boldsymbol{x}})\;\beta^\nu(t,\vec{\boldsymbol{y}})\Big]$$

- $\mathcal{G}^{\mu\nu}$ is a 2-point function on top of the classical field
- β^μ is a small field fluctuation driven by a 1-loop source

Single gluon spectrum at NLO

- The 2-point function $\mathcal{G}^{\mu\nu}$ can be written as

$$\mathcal{G}(x,y) = \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\; a_{-\boldsymbol{k}}(x)\; a_{+\boldsymbol{k}}(y)$$

with
$$\begin{cases} \dfrac{\delta^2 \mathcal{S}_{YM}}{\delta \mathcal{A}^2} \cdot a_{\pm\boldsymbol{k}} = 0 \\ \lim\limits_{t\to-\infty} a_{\pm\boldsymbol{k}}(t,\vec{\boldsymbol{x}}) = \boldsymbol{\epsilon}(\boldsymbol{k})\; e^{\pm ik\cdot x} \end{cases}$$

- The equation of motion for β^μ reads

$$\frac{\delta^2 \mathcal{S}_{YM}}{\delta \mathcal{A}^2} \cdot \beta = \underbrace{\frac{\partial^3 \mathcal{S}_{YM}(\mathcal{A})}{\partial \mathcal{A}^3}}_{\text{3-gluon vertex in the background A}} \underbrace{\frac{1}{2}\int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\; a_{-\boldsymbol{k}}(x)\; a_{+\boldsymbol{k}}(x)}_{\text{value of the loop}}$$

$$\lim_{t\to-\infty} \beta(t,\vec{\boldsymbol{x}}) = 0$$

Single gluon spectrum at NLO

- The retarded nature of the field fluctuations allows a factorization between the initial condition (calculable analytically) and the evolution on top of $\mathcal{A}^\mu$ (complicated) :

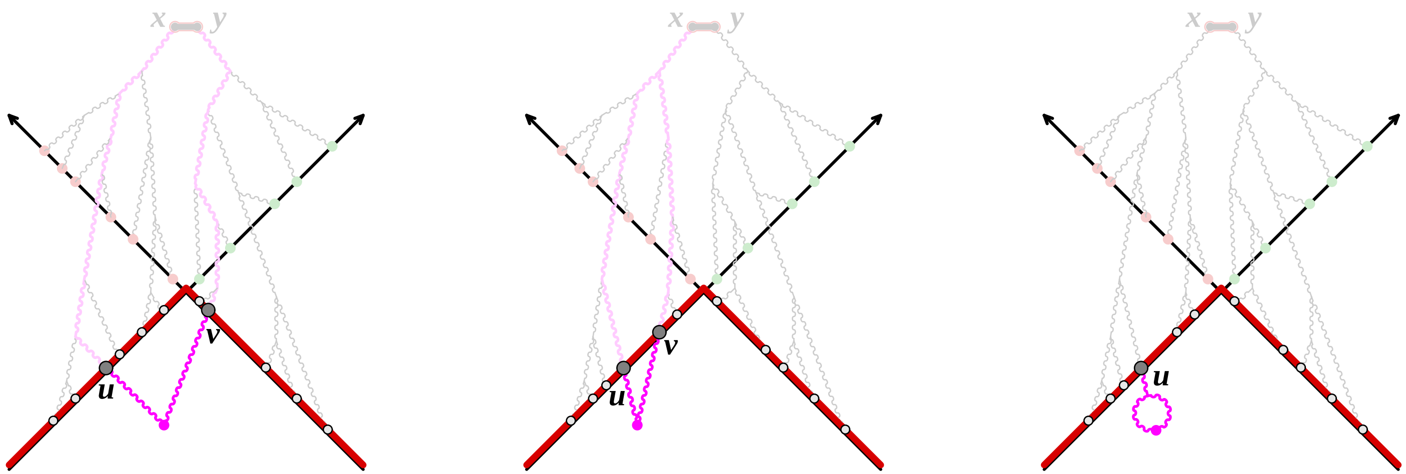
$$a^\mu(x) = \underbrace{\Big[\int_{\vec{\boldsymbol{u}}\in \mathrm{LC}} a(u)\cdot \mathbb{T}_{\boldsymbol{u}}\Big]}_{\text{initial condition}} \; \mathcal{A}^\mu(x)$$

  - 'LC' is a surface just above the backward light-cone
  - $\mathbb{T}_{\boldsymbol{u}}$ is the generator of shifts of the initial value of the fields on this surface :

$$\mathcal{F}[\mathcal{A}_{\mathrm{initial}} + a] \equiv \exp\Big[\int_{\vec{\boldsymbol{u}}\in \mathrm{LC}} a(u)\cdot \mathbb{T}_{\boldsymbol{u}}\Big] \; \mathcal{F}[\mathcal{A}_{\mathrm{initial}}]$$

Single gluon spectrum at NLO

- 1-loop graphs contributing to the gluon spectrum at NLO :



- The NLO corrections can be written as :

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm NLO} = \left[\frac{1}{2}\int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in {\rm LC}} \mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int_{\vec{\boldsymbol{u}}\in{\rm LC}} \boldsymbol{\beta}(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

▷ the functions $\mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})$ and $\beta(\vec{\boldsymbol{u}})$ can be evaluated analytically

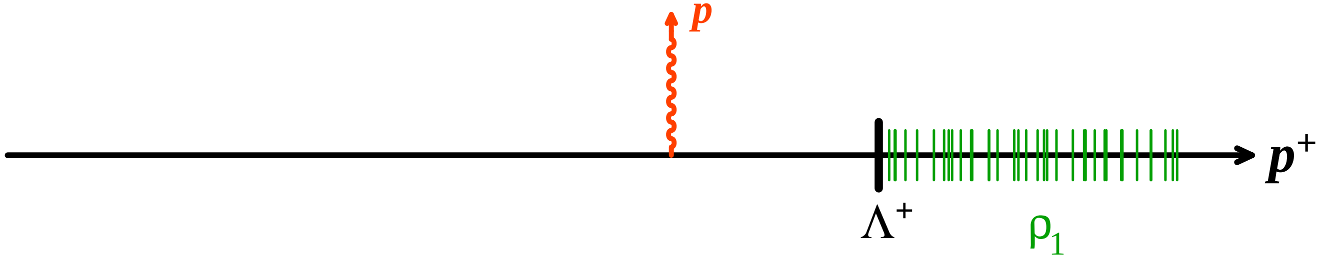
Divergences

- If $\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}$ belong to the same branch of the LC (e.g. $u^- = v^- = \epsilon$), the function $\mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})$ contains

$$\mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \sim \int_0^{+\infty} \frac{dk^+}{k^+} \cdots e^{ik^-(u^+-v^+)} \qquad \text{with} \quad k^- \equiv \frac{\boldsymbol{k}_\perp^2}{2k^+}$$

▷ the integral converges at $k^+ = 0$ but not when $k^+ \to +\infty$

Note : the log is a $\log(\Lambda^+/p^+)$, where Λ^+ is the boundary between the hard color sources and the fields, and p^+ the longitudinal momentum of the produced gluon



JIMWLK Hamiltonian

- When $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the same branch of the LC, we have

$$\Big[\frac{1}{2}\int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in \mathrm{LC}} \mathcal{G}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\,\mathbb{T}_{\boldsymbol{u}}\mathbb{T}_{\boldsymbol{v}} + \int_{\vec{\boldsymbol{u}}\in \mathrm{LC}} \beta(\vec{\boldsymbol{u}})\,\mathbb{T}_{\boldsymbol{u}}\Big]$$

$$\underset{\mathrm{LLog}}{=} \log\left(\frac{\Lambda^+}{p^+}\right) \times \Big[\text{JIMWLK } \mathcal{H}\Big]$$

- The configuration where $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the first branch of the LC can be rewritten as

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{NLO}} \underset{\mathrm{LLog}}{=} \log\left(\frac{\Lambda^+}{p^+}\right)\mathcal{H}_1 \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}}$$

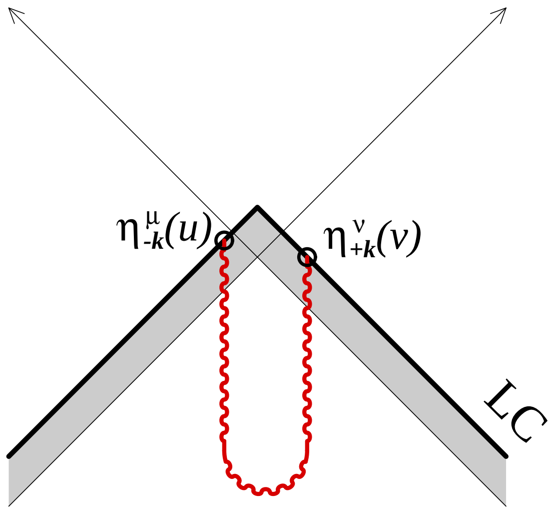
with $\mathcal{H}_1$ the JIMWLK Hamiltonian for the first nucleus

- Including also the configuration where both $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}}$ are on the second branch of the LC, we get

$$\left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{NLO}} \underset{\mathrm{LLog}}{=} \left[\log\left(\frac{\Lambda^+}{p^+}\right)\mathcal{H}_1 + \log\left(\frac{\Lambda^-}{p^-}\right)\mathcal{H}_2\right] \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\mathrm{LO}}$$

Leading Log divergences

- The only remaining possibility is to have $\vec{u}$ and $\vec{v}$ on different branches of the LC



However, there is no log divergence in this case, since the k^+ integral is of the form :

$$\int \frac{dk^+}{k^+} \cdots e^{ik^+(u^- - v^-)} \; e^{ik^-(u^+ - v^+)}$$

▷ no mixing of the divergences of the two nuclei

- Therefore, one gets the expected factorization formula :

$$\left\langle \frac{dN}{d^3\vec{\boldsymbol{p}}} \right\rangle_{\rm LLog} = \int \left[D\rho_1 \; D\rho_2\right] \; W_{Y_1}[\rho_1] \, W_{Y_2}[\rho_2] \; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}}\right|_{\rm LO}$$

with $Y_1 = \log(\sqrt{s}/p^+) \quad , \quad Y_2 = \log(\sqrt{s}/p^-)$

Extensions

- One can prove similar factorization results for the inclusive two-gluon spectrum,

$$\left\langle \frac{d^2N}{d^3\vec{\boldsymbol{p}}_1 d^3\vec{\boldsymbol{p}}_2} \right\rangle_{\rm LLog} = \int \left[D\rho_1\; D\rho_2\right] W_{Y_1}[\rho_1]\, W_{Y_2}[\rho_2]\; \left.\frac{dN}{d^3\vec{\boldsymbol{p}}_1}\right|_{\rm LO} \times \left.\frac{dN}{d^3\vec{\boldsymbol{p}}_2}\right|_{\rm LO}$$

  (valid provided the two gluons are nearby in rapidity)

- Obvious extensions of this result hold for the n-gluon spectrum

- When there is a large rapidity separation between the measured gluons, additional large logs that are not resummed by this formula can exist

Summary

Summary

- Nucleus-nucleus collisions are not a good framework in order to probe saturation, but the physics of saturation is crucial in order to correctly assess what happens in the early stages of AA collisions
  - Leading order $\triangleright$ classical fields (retarded in the case of inclusive observables)
  - The resummation of Leading Logs of $1/x_{1,2}$ can be factorized in the evolved distribution of color sources

- Next lecture : among the higher order corrections, there are other terms that may become large due to an instability

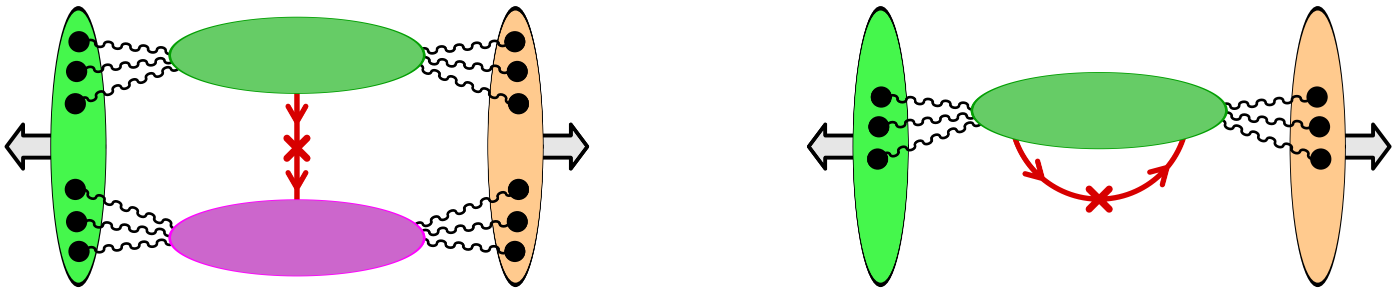
  $\triangleright$ these terms must also be resummed

Extra bits

Inclusive quark spectrum

FG, Kajantie, Lappi (2004, 2005)

- One can construct for quarks an operator $\mathcal{C}_{\rm q}$ that plays the same role as $\mathcal{C}$ for the gluons
- By repeating the same arguments, we find two generic topologies contributing to the inclusive quark spectrum :



(the blobs are sums of cut diagrams)

- The first topology cannot exist because the quark line is not closed on itself
  - ▷ the quark spectrum starts at one loop

Quark production at one loop

- At lowest order (one loop), the quark spectrum reads :

$$\frac{d\overline{N}_{\rm q}}{dY\, d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{ip\cdot x}\, \overline{u}(\vec{\boldsymbol{p}})\, (i\, \overrightarrow{\not{\partial}}_x - m)\, S_{+-}(x,y)\, (i\, \overleftarrow{\not{\partial}}_y + m)\, u(\vec{\boldsymbol{p}})\, e^{-ip\cdot y}$$

where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

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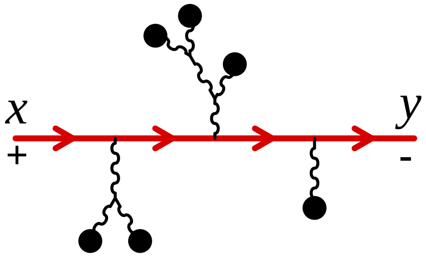
  where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

- We need to calculate the sum of the following tree diagrams :



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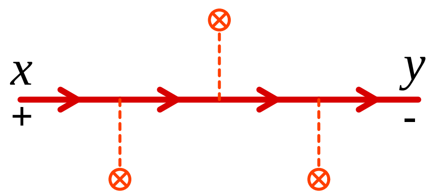
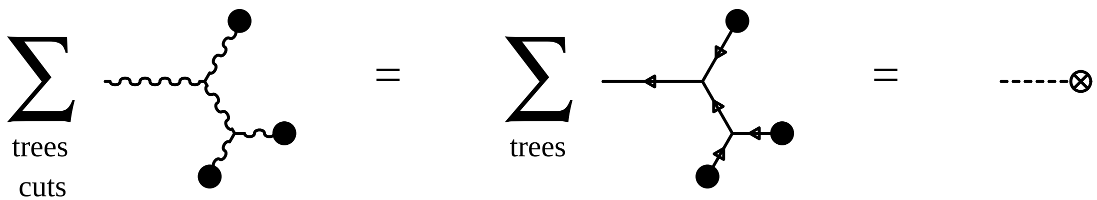


- Perform a resummation of all the sub-diagrams that correspond to the retarded classical solution :



Quark propagator

- The summation of all the classical field insertions can be done by solving a Lippmann-Schwinger equation :

$$S_{\epsilon\epsilon'}(x,y) = S^0_{\epsilon\epsilon'}(x,y) - ig \sum_{\eta=\pm} (-1)^\eta \int d^4z\, S^0_{\epsilon\eta}(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_{\eta\epsilon'}(z,y)$$

- This equation is rather non-trivial to solve in this form, because of the mixing of the 4 components of the propagator. Perform a rotation on the $\pm$ indices :

$$S_{\epsilon\epsilon'} \quad \longrightarrow \quad \boldsymbol{S}_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} U_{\alpha\epsilon} U_{\beta\epsilon'} S_{\epsilon\epsilon'}$$

$$(-1)^\epsilon \delta_{\epsilon\epsilon'} \quad \longrightarrow \quad \boldsymbol{\Sigma}_{\alpha\beta} \equiv \sum_{\epsilon=\pm} U_{\alpha\epsilon} U_{\beta\epsilon} (-1)^\epsilon$$

- A useful choice for the rotation matrix U is $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Quark propagator

- Under this rotation, the matrix propagator and field insertion become :

$$\boldsymbol{S}_{\alpha\beta} = \begin{pmatrix} 0 & S_A \\ S_R & S_D \end{pmatrix} \quad , \qquad \boldsymbol{\Sigma}_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $S^0_D(p) = 2\pi(\not{p} + m)\delta(p^2 - m^2)$

- The main simplification comes from the fact that $\boldsymbol{S}^0\boldsymbol{\Sigma}$ is the sum of a diagonal matrix and a nilpotent matrix
- One finds that S_R and S_A do not mix, i.e. they obey equations such as :

$$S_R(x,y) = S^0_R(x,y) - i\,g\int d^4z\, S^0_R(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_R(z,y)$$

- One can solve S_D in terms of S_R and S_A :

$$S_D = S_R * S^{0\ -1}_R * S^0_D * S^{0\ -1}_A * S_A$$

Quark propagator

- In order to go back to S_{+-}, invert the rotation :

$$S_{+-} = \frac{1}{2}\left[S_{A} - S_{R} - S_{D}\right]$$

- At this point, we can rewrite the quark spectrum in terms of retarded and advanced quark propagators in the classical background

- Finally, one can rewrite it in terms of retarded solutions of the Dirac equation on top of the background $\mathcal{A}_\mu(x)$

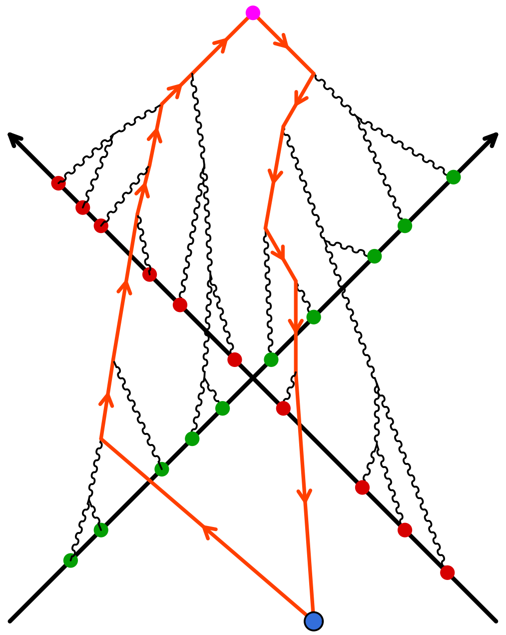
$$\frac{d\overline{N}_{\rm q}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int\frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\left|\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}})\right|^2$$

with

$$\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}}) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\; u^\dagger(\vec{\boldsymbol{p}})\psi_{\boldsymbol{q}}(x)$$

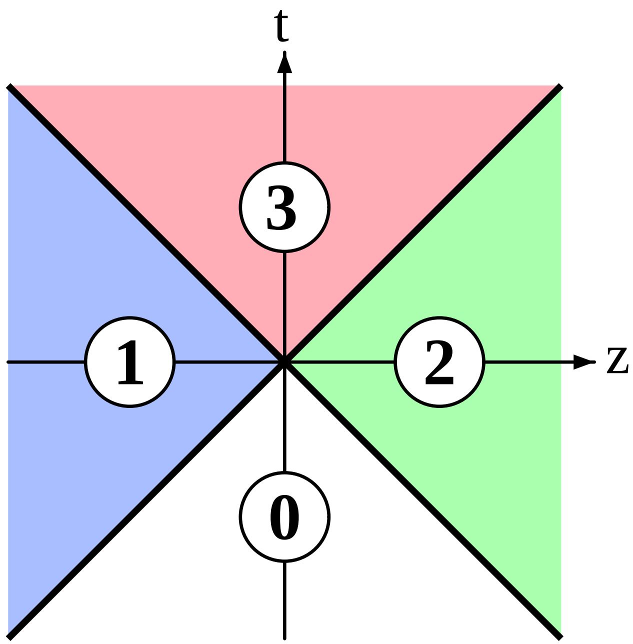
$$(i\partial\!\!\!/_x - g\mathcal{A}\!\!\!/(x) - m)\psi_{\boldsymbol{q}}(x) = 0\;,\quad \psi_{\boldsymbol{q}}(x^0,\vec{\boldsymbol{x}}) \underset{x^0\to-\infty}{=} v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

- This calculation amounts to summing the following diagrams :



Background field

- Space-time structure of the classical color field:

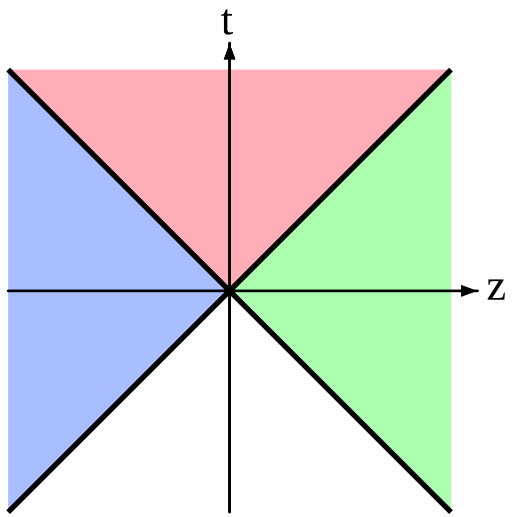


- Region 0: $\mathcal{A}^\mu = 0$
- Region 1: $\mathcal{A}^\pm = 0$,
  $\mathcal{A}^i = \frac{i}{g} U_1 \nabla_\perp^i U_1^\dagger$
- Region 2: $\mathcal{A}^\pm = 0$,
  $\mathcal{A}^i = \frac{i}{g} U_2 \nabla_\perp^i U_2^\dagger$
- Region 3: $\mathcal{A}^\mu \neq 0$

- **Notes:**
  - In the region 3, $\mathcal{A}^\mu$ is known only numerically
  - We must solve the Dirac equation numerically as well

Quark propagation

- Propagation through region 0:

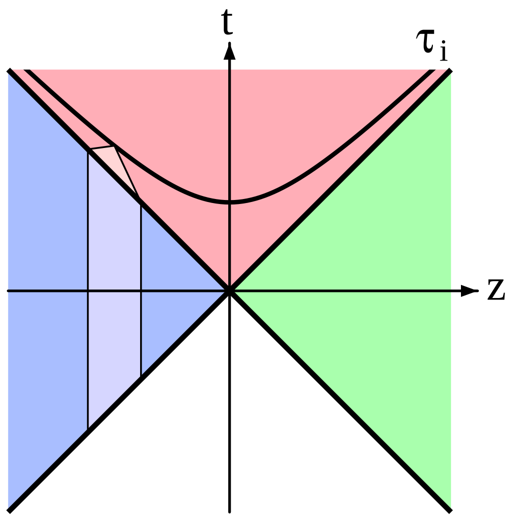


▷ trivial because there is no background field

$$\psi_{\boldsymbol{q}}(x) = v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

Quark propagation

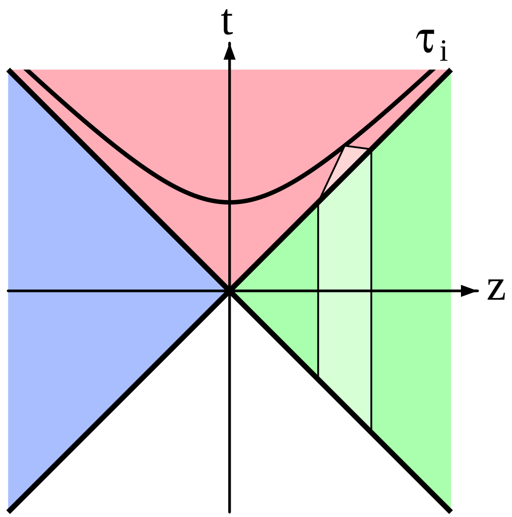
- Propagation through region 1:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},1}(\tau_i)$ can be obtained analytically

Quark propagation

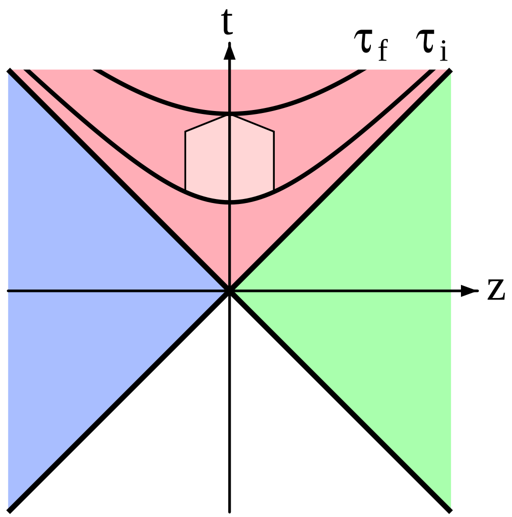
- Propagation through region 2:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},2}(\tau_i)$ can be obtained analytically

Quark propagation

- Propagation through region 3:



▷ One must solve the Dirac equation :

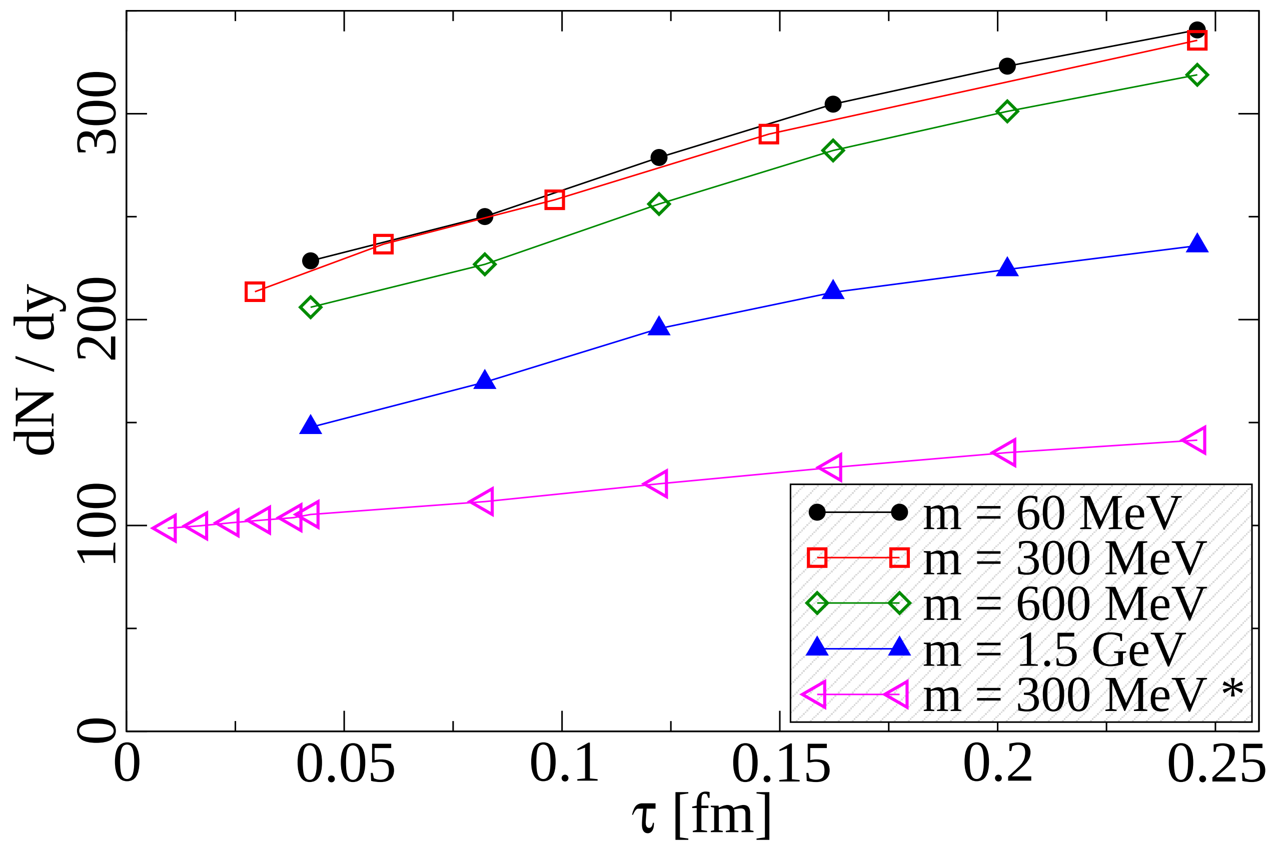
$$\left[i\partial\!\!\!/ - g A\!\!\!/ - m\right] \psi_{\boldsymbol{q}}(\tau, \eta, \vec{\boldsymbol{x}}_\perp) = 0$$

▷ initial condition: $\psi_{\boldsymbol{q}}(\tau_i) = \psi_{\boldsymbol{q},1}(\tau_i) + \psi_{\boldsymbol{q},2}(\tau_i)$

($\tau_i = 0^+$ in practice)

Time dependence

- $g^2\mu = 2$ GeV , (*) $g^2\mu = 1$ GeV :



Spectra for various quark masses

- $g^2\mu = 2$ GeV , $\tau = 0.25$ fm :