Gluon saturation from DIS to AA collisions

III – AA collisions: gluon production

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General outline

- Lecture I : Gluon saturation in DIS
- Lecture II : Proton-nucleus collisions
- Lecture III : AA collisions : gluon production
- Lecture IV : AA collisions : glasma instabilities
Introduction to nucleus-nucleus collisions

Power counting and bookkeeping

Classical fields, boundary conditions

Factorization at small $x$
Introduction
Stages of a nucleus-nucleus collision

- freeze out
- hadrons in eq. → hydrodynamics
- gluons & quarks in eq. → kinetic theory
- gluons & quarks out of eq.
- strong fields → classical EOMs
Stages of a nucleus-nucleus collision

- compute the initial production of semi-hard particles
- compute initial conditions for hydrodynamics
Saturation affects the early stages of heavy ion collisions, up to a time $\tau \sim Q_s^{-1}$.

The dynamics that takes place afterwards blurs the physics of saturation (for instance, if the system reaches thermalization, it does not remember the details of the dynamics at early times).

- Saturation affects only inclusive observables, like the overall multiplicity and its energy dependence.
- Nucleus-nucleus collisions are a limited framework in order to probe saturation.

In AA collisions, the Color Glass Condensate provides a framework that can be used to compute an initial condition for the rest of the evolution.
Small x QCD in AA collisions

- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- the bulk of particle production comes from (very) low $x$
  - high gluon density (even more so in nuclei: $G_A/G_P \approx A$)
Krasnitz-Venugopalan computation

- Gluon spectrum from retarded classical solutions of Yang-Mills equations (Krasnitz, Venugopalan (1998); Lappi (2003)):

\[
\left\langle \frac{dN}{dY d^2\mathbf{p}_\perp} \right\rangle_{L\text{Log}} \propto \int_{x,y} e^{i\mathbf{p} \cdot (x-y)} \cdots \langle A_\mu(x) A_\nu(y) \rangle
\]

\[
[D_\mu, F^{\mu\nu}] = \delta^{\nu+} \delta(x^-) \rho_1(\vec{x}_\perp) + \delta^{\nu-} \delta(x^+) \rho_2(\vec{x}_\perp) \quad \text{with} \quad \lim_{x_0 \to -\infty} A_\mu(x) = 0
\]
Krasnitz-Venugopalan computation

In nucleus-nucleus collisions, the two sources are equally strong, and should be treated on the same footing:

\[ J^\mu \equiv \delta^{\mu+}(x^-) \rho_1(\vec{x}_\perp) + \delta^{\mu-}(x^+) \rho_2(\vec{x}_\perp) \]

Average over the sources \( \rho_1, \rho_2 \)

\[ \langle \mathcal{O}\rangle_Y = \int [D\rho_1][D\rho_2] W_{Y_{\text{beam}}-Y}[\rho_1] W_{Y+Y_{\text{beam}}}[\rho_2] \mathcal{O}[\rho_1, \rho_2] \]

How to compute \( \mathcal{O}[\rho_1, \rho_2] \) in the saturation regime?

What is the meaning of this factorization formula?
Goals of this lecture

- Why can the gluon yield be obtained from classical solutions of Yang-Mills equations?

- Why are the boundary conditions retarded?

- Is this a controlled approximation, i.e. the first term in a more systematic expansion?

- Is it possible to go beyond this computation, and study the 1-loop corrections? Logs(1/x) and factorization?
Initial particle production

- **Dilute regime**: one parton in each projectile interact
Initial particle production

- **Dilute regime**: one parton in each projectile interact
- **Dense regime**: multiparton processes become crucial (+ pileup of many simultaneous scatterings)
Power counting and Bookkeeping
Power counting
In the saturated regime, the sources are of order $1/g$ (because $\langle \rho \rho \rangle \sim$ occupation number $\sim 1/\alpha_s$).

The order of each connected diagram is given by:

$$\frac{1}{g^2} g^{\# \text{ produced gluons}} g^{2(\# \text{ loops})}$$

The total order of a graph is the product of the orders of its disconnected subdiagrams.
**Power counting**

- **Example**: Inclusive gluon spectrum:

\[
\frac{dN}{d^3\vec{p}} = \frac{1}{g^2} \left[ c_0 + c_1 g^2 + c_2 g^4 + \cdots \right]
\]

- The coefficients \(c_0, 1, \ldots\) are themselves series that resum all orders in \((g\rho_{1,2})^n\). For instance,

\[
c_0 = \sum_{n=0}^{\infty} c_{0,n} (g\rho_{1,2})^n
\]

- We want to calculate at least the entire \(c_0/g^2\) contribution, and a subset of the higher order terms.
Vacuum diagrams do not produce any gluon. They are contributions to the vacuum to vacuum amplitude $\langle 0_{\text{out}} | 0_{\text{in}} \rangle$.

The order of a connected vacuum diagram is given by:

$$g^{-2} g^2 (\text{# loops})$$

Relation between connected and non connected vacuum diagrams:

$$\sum \left( \text{all the vacuum diagrams} \right) = \exp \left\{ \sum \left( \text{simply connected vacuum diagrams} \right) \right\} = e^{iV[j]}$$
Bookkeeping
Consider **squared amplitudes** (including interference terms) rather than the amplitudes themselves
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See them as cuts through vacuum diagrams.

cut propagator: \(2\pi \theta(-p^0)\delta(p^2)\)
Consider **squared amplitudes** (including interference terms) rather than the amplitudes themselves

See them as **cuts through vacuum diagrams**

*cut propagator: \(2\pi \theta(-p^0)\delta(p^2)\)*

The sum of the vacuum diagrams, \(\exp(iV[j])\), is the generating functional for time-ordered products of fields:

\[
\langle 0_{\text{out}} | T A(x_1) \cdots A(x_n) | 0_{\text{in}} \rangle = \frac{\delta}{\delta j(x_1)} \cdots \frac{\delta}{\delta j(x_n)} e^{iV[j]}
\]
The probability of producing exactly $n$ particles is:

$$P_n = \frac{1}{n!} \int \frac{d^3 \vec{p}_1}{(2\pi)^3 2E_1} \cdots \frac{d^3 \vec{p}_n}{(2\pi)^3 2E_n} \left| \langle \vec{p}_1 \cdots \vec{p}_n_{\text{out}} | 0_{\text{in}} \rangle \right|^2$$

Exercise. Show that:

$$P_n = \frac{1}{n!} \mathcal{C}^n e^{iV[j+]} e^{-iV^*[j-]} \bigg|_{j+ = j- = j}$$

with

$$\mathcal{C} = \int_{x,y} G^0_{+-}(x,y) \square_x \square_y \frac{\delta}{\delta j_(x)} \frac{\delta}{\delta j_-(y)}$$

$$G^0_{+-}(x,y) \equiv \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} 2\pi \theta(-p^0) \delta(p^2)$$

Hint: start from the reduction formula for the transition amplitude, and use the fact that $\exp(iV[j])$ is the generating functional.

Note: the propagator $G^0_{+-}(x,y)$ is a cut propagator.
Reminder:

$$C \equiv \int_{x,y} G^0_{+-}(x, y) \delta_{x} \delta_{y} \frac{\delta}{\delta j^+(x)} \frac{\delta}{\delta j^-(y)}$$

Consider a generic cut vacuum diagram:
Bookkeeping

- Reminder:

\[ C \equiv \int_{x,y} G^{0}_{+-}(x, y) \frac{\delta}{\delta j^+(x)} \frac{\delta}{\delta j^-(y)} \]

- Consider a generic cut vacuum diagram:

\[ \frac{\delta}{\delta j^-(y)} \rightarrow y \]
Reminder:

\[
C \equiv \int_{x,y} G_{+-}^0(x, y) \begin{pmatrix} x & y \end{pmatrix} \frac{\delta}{\delta j^+(x)} \frac{\delta}{\delta j^-(y)}
\]

Consider a generic cut vacuum diagram:

\[
\frac{\delta}{\delta j^+(x)} \rightarrow \begin{array}{c}
\end{array}
\]
Bookkeeping

- Reminder:

\[ C \equiv \int_{x,y} G^0_{+-}(x,y) \delta_{x} \delta_{y} \frac{\delta}{\delta j^+ (x)} \frac{\delta}{\delta j^- (y)} \]

- Consider a generic cut vacuum diagram:

\[ \square \rightarrow \square \]
Reminder:

\[ C \equiv \int_{x,y} G^0_{+-}(x, y) \delta x \delta y \frac{\delta j_-(y)}{\delta j_+(x)} \]

Consider a generic cut vacuum diagram:

\[ \square_x \rightarrow y \]
Reminder:

\[ C \equiv \int_{x,y} G^0_{+-}(x, y) \Delta x \Delta y \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)} \]

Consider a generic cut vacuum diagram:

\[ G^0_{+-}(x, y) \rightarrow \]

- the operator \( C \) removes two sources (one in the amplitude and one in the complex conjugated amplitude), and creates a new cut propagator.
The sum of all the cut vacuum diagrams, with sources $j_+$ on one side of the cut and $j_-$ on the other side, can be written as:

$$\sum \text{(all the cut vacuum diagrams)} = e^C e^{iV[j_+]} e^{-iV^*[j_-]}$$

If we set $j_+ = j_- = j$, then we should get $\sum_n P_n = 1$

Therefore, we have:

$$e^C e^{iV[j_+]} e^{-iV^*[j_-]} \bigg|_{j_+ = j_-} = 1$$

Note: the use of this identity renders automatic an important cancellation that would be hard to see at the level of diagrams (somewhat related to AGK)
Bookkeeping

- The operator $C$ can be used to derive many useful formulas:

$$F(z) = \sum_{n=0}^{+\infty} z^n \ P_n = e^{zC} \ e^{iV[j_+]} \ e^{-iV^*[j_-]} \bigg|_{j_+ = j_- = j}$$

- sum of all cut vacuum graphs, where each cut is weighted by $z$

$$\overline{N} = F'(1) = C \ e^{C} \ e^{iV[j_+]} \ e^{-iV^*[j_-]} \bigg|_{j_+ = j_- = j}$$

$$\overline{N}(\overline{N} - 1) = F''(1) = C^2 \ e^{C} \ e^{iV[j_+]} \ e^{-iV^*[j_-]} \bigg|_{j_+ = j_- = j}$$

- Main benefit:

The tracking of infinite sets of Feynman diagrams has been replaced by simple algebraic manipulations.
Classical fields, Boundary conditions
Diagrammatic expansion

- It is easy to express the average multiplicity as:

\[
\overline{N} = \sum_n n P_n = C \left\{ e^{iV[j_+]} e^{-iV^*[j_-]} \right\}_{j_+ = j_- = j}
\]

sum of all the cut vacuum diagrams: \( e^{iW[j_+, j_-]} \)

- There are two types of terms:
  - \( C \) picks two sources in the same connected cut diagram

\[
\frac{\delta^2 iW}{\delta j_+(x) \delta j_-(y)} \rightarrow
\]

- \( C \) picks two sources in two distinct connected cut diagrams

\[
\frac{\delta iW}{\delta j_+(x)} \frac{\delta iW}{\delta j_-(y)} \rightarrow
\]
Diagrammatic expansion (LO)

- At LO, only tree diagrams contribute
  - the first type of topologies can be neglected (they have at least one loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts:

\[ \overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}} \]

- Note: at this point, the sources on both sides of the cut must be set equal:
  \[ j_+ = j_- = j \]
Retarded propagators

In the previous diagrams, one must sum over all the possible ways of cutting lines inside the blobs

This can be achieved via Cutkosky’s cutting rules:

- A vertex is $-ig$ on one side of the cut, and $+ig$ on the other side
- A source $\rho$ changes sign depending on the side of the cut
- There are four propagators, depending on the location w.r.t. the cut of the vertices they connect:

\[
G^{0+}(p) = i/(p^2 - m^2 + i\epsilon) \quad \text{(standard Feynman propagator)}
\]
\[
G^{0-}(p) = -i/(p^2 - m^2 - i\epsilon) \quad \text{(complex conjugate of } G^{0+}(p))
\]
\[
G^{00}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)
\]

- At each vertex of a given diagram, sum over the types $+$ and $-$ ($2^n$ terms for a diagram with $n$ vertices)
Retarded propagators

When summing over the cuts, we only get combinations of propagators such as:

\[ G_{++}^0(p) - G_{+-}^0(p) = \frac{i}{p^2 - m^2 + i\epsilon} - 2\pi\theta(-p^0)\delta(p^2 - m^2) \]
Retarded propagators

When summing over the cuts, we only get combinations of propagators such as:

\[ G_{++}^0(p) - G_{+-}^0(p) = \text{PP} \left[ \frac{i}{p^2 - m^2} \right] + \pi \delta(p^2 - m^2) - 2\pi \theta(-p^0) \delta(p^2 - m^2) \]

\[ \text{insert : } 1 = \theta(p^0) + \theta(-p^0) \]
Retarded propagators

When summing over the cuts, we only get combinations of propagators such as:

\[
G_{++}^0(p) - G_{+-}^0(p) = \text{PP} \left[ \frac{i}{p^2 - m^2} \right] + \pi [ \theta(p^0) - \theta(-p^0) ] \delta(p^2 - m^2) \sign(p^0)
\]
Retarded propagators

When summing over the cuts, we only get combinations of propagators such as:

\[ G^0_{++}(p) - G^0_{+-}(p) = \frac{i}{p^2 - m^2 + i\text{ sign}(p^0)\epsilon} \]
Retarded propagators

- When summing over the cuts, we only get combinations of propagators such as:

\[ G_{++}^0(p) - G_{+-}^0(p) = G_R^0(p) \]

- Similarly:

\[ G_{-+}^0(p) - G_{--}^0(p) = G_R^0(p) \]

- Starting from the “leaves” of the trees, one can use these formulas in order to replace recursively all the \( G_{\pm\pm}^0 \) propagators by retarded propagators

▷ we have a sum of tree diagrams with retarded propagators
Classical fields

- The sum of all the tree diagrams constructed with retarded propagators is the solution of classical field equations with retarded boundary condition:

\[
\lim_{t \to -\infty} A(t, \vec{x}) = 0
\]

- Proof (for a scalar theory with a cubic self-interaction). The classical EOM reads

\[
(\Box + m^2) \varphi(x) + \frac{g}{2} \varphi^2(x) = j(x)
\]

- Write the Green’s formula for the retarded solution that obeys \( \varphi(t, \vec{x}) = 0 \) at \( t = -\infty \):

\[
\varphi(x) = \int d^4y \ G^0_R(x - y) \left[ -i \frac{g}{2} \varphi^2(y) + i j(y) \right]
\]
Classical field

One can construct the solution iteratively, by using in the r.h.s. the solution found in the previous orders

Order $g^0$:

$$\varphi_{(0)}(x) = \int d^4 y \ G_R^0 (x - y) \ i j(y)$$

Order $g^1$:

$$\varphi_{(0)}(x) + \varphi_{(1)}(x) = \int d^4 y \ G_R^0 (x - y) \left[-i \frac{g}{2} \varphi_{(0)}(y) + i j(y)\right]$$

i.e.

$$\varphi_{(1)}(x) = -i \frac{g}{2} \int d^4 y \ G_R^0 (x - y) \left[\int d^4 z \ G_R^0 (y - z) \ i j(z)\right]^2$$
The diagrammatic expansion of this classical solution is:
Retarded classical solution

The diagrammatic expansion of this classical solution is:

\[ + \frac{1}{2} \]
Retarded classical solution

The diagrammatic expansion of this classical solution is:

\[ \begin{align*}
    & + \frac{1}{2} \quad + \frac{1}{2} \\
    \end{align*} \]
The diagrammatic expansion of this classical solution is:

\[ \text{Diagrammatic expansion} \]

\[ + \frac{1}{2} \quad + \frac{1}{2} \quad + \frac{1}{2} \quad + \frac{1}{8} \]
The diagrammatic expansion of this classical solution is:

\[ + \frac{1}{2} \quad + \frac{1}{2} \quad + \frac{1}{2} \quad + \frac{1}{8} \]

The classical solution is given by the sum of all the tree diagrams with retarded propagators.
Gluon spectrum at LO


The gluon spectrum at LO is given by:

$$\left. \frac{dN}{dY d^2 \vec{p}_\perp} \right|_{LO} = \frac{1}{16 \pi^3} \int_{x,y} e^{i \vec{p} \cdot (x-y)} \Box x \Box y \sum_\lambda \epsilon^\mu_\lambda \epsilon^\nu_\lambda A_\mu(x) A_\nu(y)$$

where $A_\mu(x)$ is the solution of Yang-Mills equations,

$$[D_\mu, F^{\mu\nu}] = J^\nu$$

such that

$$\lim_{x^0 \to -\infty} A_\mu(x) = 0$$
Gluon spectrum at LO

- Lattice artifacts at large momentum (they do not affect much the overall number of gluons)
- Important softening at small $k_\perp$ compared to pQCD (saturation)
Initial Glasma fields

Lappi, McLerran (2006)  (Semantics: Glasma $\equiv$ Glas[s - plas]ma)

- Before the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields are localized in two sheets transverse to the beam axis.
- Immediately after the collision, the chromo-$\vec{E}$ and $\vec{B}$ fields have become longitudinal:

$$E^z = ig \left[ A^i_1, A^i_2 \right], \quad B^z = ig \epsilon^{ij} \left[ A^i_1, A^j_2 \right]$$

![Graph](image-url)
Boost invariance

- **Gauge condition**: \( x^+ A^- + x^- A^+ = 0 \)

\[ A^\pm(x) = \pm x^\pm \beta(\tau, \eta, \vec{x}_\perp) \]

- Initial values at \( \tau = 0^+ \): \( A^i(0^+, \eta, \vec{x}_\perp) \) and \( \beta(0^+, \eta, \vec{x}_\perp) \) do not depend on the rapidity \( \eta \)

\( A^i \) and \( \beta \) remain independent of \( \eta \) at all times
Exercise : Generating functional

Consider a function $z(\vec{p})$, and define the functional

$$F[z] \equiv \frac{1}{n!} \sum_{n=0}^{+\infty} \int d\Phi_1 \cdot d\Phi_n \ z(\vec{p}_1) \cdots z(\vec{p}_n) \ |\langle \vec{p}_1 \cdots \vec{p}_n\text{out}\ |0\text{in}\rangle|^2$$

At LO, one can write it in terms of two classical fields $A_\pm(x)$:

$$\frac{\delta \ln F[z]}{\delta z(\vec{p})}_{\text{LO}} = \int_{x,y} e^{i\vec{p} \cdot (x-y)} \cdots A_+^\mu(x) A_-^\nu(y)$$

Non retarded boundary conditions unless $z(\vec{p}) \equiv 1$:

$$a_+^{(\pm)}(-\infty, \vec{p}) = a_-^{(\pm)}(-\infty, \vec{p}) = 0$$

$$a_-^{(\pm)}(+\infty, \vec{p}) = z(\vec{p}) \ a_+^{(\pm)}(+\infty, \vec{p})$$

$$a_+^{(-)}(+\infty, \vec{p}) = z(\vec{p}) \ a_-^{(-)}(+\infty, \vec{p})$$

where: $A_\epsilon(x) \equiv \int \frac{d^3\vec{p}}{(2\pi)^3 E_p} \left[a_\epsilon^{(\pm)}(x_0, \vec{p}) \ e^{-i\vec{p} \cdot x} + a_\epsilon^{(\pm)}(x_0, \vec{p}) \ e^{+i\vec{p} \cdot x}\right]$
Factorization at small $x$
What is the problem?

- Naive perturbative expansion:

\[
\frac{dN}{d^3\vec{p}} = \frac{1}{g^2} \left[ c_0 + c_1 g^2 + c_2 g^4 + \cdots \right]
\]

Note: so far, we have seen how to compute \( c_0 \) given \( \rho_{1,2} \).

- Problem: \( c_{1,2,\ldots} \) contain logarithms of \( 1/x_{1,2} \):

\[
\begin{align*}
c_1 &= c_{10} + c_{11} \ln \left( \frac{1}{x_{1,2}} \right) \\
c_2 &= c_{20} + c_{21} \ln \left( \frac{1}{x_{1,2}} \right) + c_{22} \ln^2 \left( \frac{1}{x_{1,2}} \right)
\end{align*}
\]

Leading Log terms

- At small \( x_{1,2} \), these logs are large, and we would like to resum all the terms that have as many logs as powers of \( g^2 \).
What is the problem?

For the single gluon spectrum in AA collisions, one would like to establish a formula such as:

\[
\left\langle \frac{dN}{d^3\vec{p}} \right\rangle = \text{LLog} \int [D\rho_1 D\rho_2] W_{Y_{\text{beam}} - y} [\rho_1] W_{y + Y_{\text{beam}}} [\rho_2] \left. \frac{dN}{d^3\vec{p}} \right|_{\text{LO}}
\]

with \( \frac{\partial}{\partial Y} W_Y = \mathcal{H} W \)

- All the leading logs of \( 1/x_{1,2} \) are absorbed in the \( W' \)s
- The \( W' \)s obey the JIMWLK evolution equation
Factorization in four easy steps

I : Express the single gluon spectrum at LO and NLO in terms of classical fields and small field fluctuations. Check that their boundary conditions are retarded.

II : Write the NLO terms as a perturbation of the initial value of the classical fields on the light-cone:

\[
\left. \frac{dN}{d^3\vec{p}} \right|_{\text{NLO}} = \left[ \frac{1}{2} \int_{\vec{u},\vec{v}\in\text{LC}} G(\vec{u}, \vec{v}) \mathbb{T}_u \mathbb{T}_v + \int_{\vec{u}\in\text{LC}} \beta(\vec{u}) \mathbb{T}_u \right] \left. \frac{dN}{d^3\vec{p}} \right|_{\text{LO}}
\]

III : For \( \vec{u}, \vec{v} \) on the same branch of the light-cone, one has:

\[
\frac{1}{2} \int_{\vec{u},\vec{v}\in\text{LC}} G(\vec{u}, \vec{v}) \mathbb{T}_u \mathbb{T}_v + \int_{\vec{u}\in\text{LC}} \beta(\vec{u}) \mathbb{T}_u = \log \left( \frac{\Lambda^+}{p^+} \right) \times \mathcal{H} + \text{finite terms}
\]

IV : There are no other logs. Factorization follows trivially.
Single gluon spectrum at LO

- LO results for the single gluon spectrum:
  - At LO, the single gluon spectrum can be expressed in terms of classical solutions of the field equation of motion.
  - These classical fields obey retarded boundary conditions.

\[
\left. \frac{dN}{d^3\vec{p}} \right|_{\text{LO}} = \lim_{t \to +\infty} \int d^3\vec{x} d^3\vec{y} e^{i\vec{p} \cdot (\vec{x} - \vec{y})} \cdots A^\mu(t, \vec{x}) A^\nu(t, \vec{y})
\]

\[
[D_\mu, F^{\mu\nu}] = J^\nu
\]

\[
\lim_{t \to -\infty} A^\mu(t, \vec{x}) = 0
\]
Retarded classical fields are sums of tree diagrams:
Retarded classical fields are sums of tree diagrams:

Note: the gluon spectrum is a functional of the value of the classical field just above the backward light-cone:

\[
\frac{dN}{d^3\vec{p}} = \mathcal{F}[A_{\text{initial}}]
\]
Single gluon spectrum at NLO

1-loop graphs contributing to the gluon spectrum at NLO:

\[
\frac{dN}{d^3\vec{p}}\bigg|_{\text{NLO}} = \lim_{t \to +\infty} \int d^3\bar{x} d^3\bar{y} \ e^{i\vec{p} \cdot (\bar{x}-\bar{y})} \ldots \left[ G^{\mu\nu}(x, y) + \beta^{\mu}(t, \bar{x}) A^{\nu}(t, \bar{y}) + A^{\mu}(t, \bar{x}) \beta^{\nu}(t, \bar{y}) \right]
\]

- \( G^{\mu\nu} \) is a 2-point function on top of the classical field
- \( \beta^{\mu} \) is a small field fluctuation driven by a 1-loop source
Single gluon spectrum at NLO

- The 2-point function $G^{\mu\nu}$ can be written as

$$G(x, y) = \int \frac{d^3 k}{(2\pi)^3 2E_k} a_{-k}(x) a_{+k}(y)$$

with

$$\begin{cases} 
\delta^2 S_{YM} \over \delta A^2 \cdot A_{\pm k} = 0 \\
\lim_{t \to -\infty} A_{\pm k}(t, \vec{x}) = \epsilon(k) e^{\pm ik \cdot x}
\end{cases}$$

- The equation of motion for $\beta^\mu$ reads

$$\delta^2 S_{YM} \over \delta A^2 \cdot \beta = \frac{\partial^3 S_{YM}(A)}{\partial A^3} \left( \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3 2E_k} a_{-k}(x) a_{+k}(x) \right)$$

3-gluon vertex in the background

$$\lim_{t \to -\infty} \beta(t, \vec{x}) = 0$$

value of the loop
The retarded nature of the field fluctuations allows a factorization between the initial condition (calculable analytically) and the evolution on top of $A^\mu$ (complicated):

$$a^\mu(x) = \left[ \int_{\bar{u} \in LC} a(u) \cdot \mathbb{T}_u \right] A^\mu(x)$$

- 'LC' is a surface just above the backward light-cone
- $\mathbb{T}_u$ is the generator of shifts of the initial value of the fields on this surface:

$$\mathcal{F}[A_{\text{initial}} + a] \equiv \exp \left[ \int_{\bar{u} \in LC} a(u) \cdot \mathbb{T}_u \right] \mathcal{F}[A_{\text{initial}}]$$
Single gluon spectrum at NLO

1-loop graphs contributing to the gluon spectrum at NLO:

\[ \frac{dN}{d^3\vec{p}} \bigg|_{\text{NLO}} = \left[ \frac{1}{2} \int_{\vec{u}, \vec{v} \in \text{LC}} G(\vec{u}, \vec{v}) \mathbb{T}_u \mathbb{T}_v + \int_{\vec{u} \in \text{LC}} \beta(\vec{u}) \mathbb{T}_u \right] \frac{dN}{d^3\vec{p}} \bigg|_{\text{LO}} \]

- The functions \( G(\vec{u}, \vec{v}) \) and \( \beta(\vec{u}) \) can be evaluated analytically.
If \( \vec{u}, \vec{v} \) belong to the same branch of the LC (e.g. \( u^- = v^- = \epsilon \)), the function \( G(\vec{u}, \vec{v}) \) contains

\[
G(\vec{u}, \vec{v}) \sim \int_0^{+\infty} \frac{dk^+}{k^+} \ldots e^{ik^- (u^+ - v^+)} \quad \text{with} \quad k^- \equiv \frac{k^2}{2k^+}
\]

\( k^- \) the integral converges at \( k^+ = 0 \) but not when \( k^+ \to +\infty \)

Note: the log is a \( \log(\Lambda^+ / p^+) \), where \( \Lambda^+ \) is the boundary between the hard color sources and the fields, and \( p^+ \) the longitudinal momentum of the produced gluon
JIMWLK Hamiltonian

- When $\vec{u}, \vec{v}$ are on the same branch of the LC, we have

$$\left[ \frac{1}{2} \int \mathcal{G}(\vec{u}, \vec{v}) \mathcal{T}_u \mathcal{T}_v + \int \beta(\vec{u}) \mathcal{T}_u \right]_{\vec{u}, \vec{v} \in \text{LC}} \approx \text{LLLog} \left( \frac{\Lambda^+}{p^+} \right) \times [\text{JIMWLK } \mathcal{H}]$$

- The configuration where $\vec{u}, \vec{v}$ are on the first branch of the LC can be rewritten as

$$\left. \frac{dN}{d^3\vec{p}} \right|_{\text{NLO}} = \text{LLLog} \left( \frac{\Lambda^+}{p^+} \right) \mathcal{H}_1 \left. \frac{dN}{d^3\vec{p}} \right|_{\text{LO}}$$

with $\mathcal{H}_1$ the JIMWLK Hamiltonian for the first nucleus

- Including also the configuration where both $\vec{u}, \vec{v}$ are on the second branch of the LC, we get

$$\left. \frac{dN}{d^3\vec{p}} \right|_{\text{NLO}} = \text{LLLog} \left[ \log \left( \frac{\Lambda^+}{p^+} \right) \mathcal{H}_1 + \log \left( \frac{\Lambda^-}{p^-} \right) \mathcal{H}_2 \right] \left. \frac{dN}{d^3\vec{p}} \right|_{\text{LO}}$$
The only remaining possibility is to have $\vec{u}$ and $\vec{v}$ on different branches of the LC. However, there is no log divergence in this case, since the $k^+$ integral is of the form:

$$\int \frac{dk^+}{k^+} \cdots e^{i k^+ (u^- - v^-)} e^{- i k^+ (u^+ - v^+)}$$

no mixing of the divergences of the two nuclei.

Therefore, one gets the expected factorization formula:

$$\left\langle \frac{dN}{d^3 \vec{p}} \right\rangle_{\text{LLog}} = \int \left[ D \rho_1 D \rho_2 \right] W_{Y_1}[\rho_1] W_{Y_2}[\rho_2] \frac{dN}{d^3 \vec{p}} \bigg|_{\text{LO}}$$

with $Y_1 = \log(\sqrt{s}/p^+)$, $Y_2 = \log(\sqrt{s}/p^-)$.
Extensions

■ One can prove similar factorization results for the inclusive two-gluon spectrum,

\[
\left\langle \frac{d^2 N}{d^3 \vec{p}_1 d^3 \vec{p}_2} \right\rangle_{\text{LLog}} = \int \left[ D\rho_1 D\rho_2 \right] W_{Y_1}[\rho_1] W_{Y_2}[\rho_2] \left. \frac{dN}{d^3 \vec{p}_1} \right|_{\text{LO}} \times \left. \frac{dN}{d^3 \vec{p}_2} \right|_{\text{LO}}
\]

(valid provided the two gluons are nearby in rapidity)

■ Obvious extensions of this result hold for the \( n \)-gluon spectrum

■ When there is a large rapidity separation between the measured gluons, additional large logs that are not resummed by this formula can exist
Summary
Summary

- Nucleus-nucleus collisions are not a good framework in order to probe saturation, but the physics of saturation is crucial in order to correctly assess what happens in the early stages of AA collisions
  - Leading order ➤ classical fields
    (retarded in the case of inclusive observables)
  - The resummation of Leading Logs of $1/x_{1,2}$ can be factorized in the evolved distribution of color sources

- Next lecture: among the higher order corrections, there are other terms that may become large due to an instability
  ➤ these terms must also be resummed
Extra bits

- Inclusive quark spectrum
Inclusive quark spectrum


- One can construct for quarks an operator $C_q$ that plays the same role as $C$ for the gluons

- By repeating the same arguments, we find two generic topologies contributing to the inclusive quark spectrum:

  (the blobs are sums of cut diagrams)

- The first topology cannot exist because the quark line is not closed on itself
  - the quark spectrum starts at one loop
Quark production at one loop

At lowest order (one loop), the quark spectrum reads:

\[
\frac{d\bar{N}_q}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot \vec{x}} \overline{u}(\vec{p}) (i \not\! \partial_x - m) S_{+-}(x,y) (i \not\! \partial_y + m) u(\vec{p}) e^{-i\vec{p} \cdot \vec{y}}
\]

where \( S_{+-} \) is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways.
Quark production at one loop

- At lowest order (one loop), the quark spectrum reads:

\[
\frac{d\bar{N}_q}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{ip \cdot x} \bar{u}(\vec{p}) \left( i \phi_x - m \right) S_{+-}(x, y) \left( i \phi_y + m \right) u(\vec{p}) e^{-ip \cdot y}
\]

where \( S_{+-} \) is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways.

- We need to calculate the sum of the following tree diagrams:
Quark production at one loop

- At lowest order (one loop), the quark spectrum reads:

\[
\frac{dN_q}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int_{x,y} e^{i\vec{p} \cdot \vec{x}} \bar{u}(\vec{p}) (i \hat{\phi}_x - m) S_{+-}(x, y) (i \hat{\phi}_y + m) u(\vec{p}) e^{-i\vec{p} \cdot \vec{y}}
\]

where \( S_{+-} \) is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways.

- We need to calculate the sum of the following tree diagrams:

- Perform a resummation of all the sub-diagrams that correspond to the retarded classical solution:

\[
\sum_{\text{trees}} \sum_{\text{cuts}} = \sum_{\text{trees}} = \ldots...
\]
Quark propagator

- The summation of all the classical field insertions can be done by solving a Lippmann-Schwinger equation:

\[ S_{\epsilon\epsilon'}(x, y) = S^0_{\epsilon\epsilon'}(x, y) - i g \sum_{\eta=\pm} (-1)^\eta \int d^4z \ S^0_{\epsilon\eta}(x, z) A_\mu(z) \gamma^\mu S_{\eta\epsilon'}(z, y) \]

- This equation is rather non-trivial to solve in this form, because of the mixing of the 4 components of the propagator. Perform a rotation on the \( \pm \) indices:

\[ S_{\epsilon\epsilon'} \rightarrow S_{\alpha\beta} = \sum_{\epsilon, \epsilon' = \pm} U_{\alpha\epsilon} U_{\beta\epsilon'} S_{\epsilon\epsilon'} \]

\[ (-1)^\epsilon \delta_{\epsilon\epsilon'} \rightarrow \Sigma_{\alpha\beta} = \sum_{\epsilon = \pm} U_{\alpha\epsilon} U_{\beta\epsilon} (-1)^\epsilon \]

- A useful choice for the rotation matrix \( U \) is:

\[ U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]
Quark propagator

- Under this rotation, the matrix propagator and field insertion become:

\[
S_{\alpha\beta} = \begin{pmatrix} 0 & S_A \\ S_R & S_D \end{pmatrix}, \quad \Sigma_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

where \( S_D^0(p) = 2\pi(p^+ + m)\delta(p^2 - m^2) \)

- The main simplification comes from the fact that \( S^0\Sigma \) is the sum of a diagonal matrix and a nilpotent matrix.

- One finds that \( S_R \) and \( S_A \) do not mix, i.e. they obey equations such as:

\[
S_R(x, y) = S_R^0(x, y) - ig \int d^4z \, S_R^0(x, z) A_\mu(z) \gamma^\mu S_R(z, y)
\]

- One can solve \( S_D \) in terms of \( S_R \) and \( S_A \):

\[
S_D = S_R \ast S_R^{-1} \ast S_D \ast S_A^{-1} \ast S_A
\]
Quark propagator

- In order to go back to $S_{+-}$, invert the rotation:

$$S_{+-} = \frac{1}{2} [S_A - S_R - S_D]$$

- At this point, we can rewrite the quark spectrum in terms of 
  retarded and advanced quark propagators in the classical 
  background

- Finally, one can rewrite it in terms of retarded solutions of the 
  Dirac equation on top of the background $A_\mu(x)$

$$\frac{d\bar{N}_q}{dY d^2\vec{p}_\perp} = \frac{1}{16\pi^3} \int \frac{d^3\vec{q}}{(2\pi)^3 2E_q} \left| \mathcal{M}(\vec{p}, \vec{q}) \right|^2$$

with

$$\mathcal{M}(\vec{p}, \vec{q}) = \lim_{x^0 \to +\infty} \int d^3\vec{x} \ e^{i\vec{p} \cdot \vec{x}} \ u^\dagger(\vec{p}) \psi_q(x)$$

$$\left( i\partial_x - gA(x) - m \right) \psi_q(x) = 0, \quad \psi_q(x^0, \vec{x}) \bigg|_{x^0 \to -\infty} = v(\vec{q}) e^{i\vec{q} \cdot \vec{x}}$$
- This calculation amounts to summing the following diagrams:
Background field

- Space-time structure of the classical color field:

- Region 0: $A^\mu = 0$
- Region 1: $A^\pm = 0$, $A^i = \frac{i}{g} U_1 \nabla^i_\perp U_1^\dagger$
- Region 2: $A^\pm = 0$, $A^i = \frac{i}{g} U_2 \nabla^i_\perp U_2^\dagger$
- Region 3: $A^\mu \neq 0$

- Notes:
  - In the region 3, $A^\mu$ is known only numerically
  - We must solve the Dirac equation numerically as well
Quark propagation

- Propagation through region 0:

\[ \psi_q(x) = v(a) e^{iq \cdot x} \]

▷ trivial because there is no background field
Quark propagation

- Propagation through region 1:

- Pure gauge background field

  \[ \psi_{q,1}(\tau_i) \] can be obtained analytically
Quark propagation

- Propagation through region 2:

  ▶ Pure gauge background field
  
  ▶ \( \psi_{q,2}(\tau_i) \) can be obtained analytically
Quark propagation

■ Propagation through region 3:

\[ \left[ i\partial_t - gA - m \right] \psi_q(\tau, \eta, \vec{x}_\perp) = 0 \]

▷ initial condition: \[ \psi_q(\tau_i) = \psi_{q,1}(\tau_i) + \psi_{q,2}(\tau_i) \]
\[ (\tau_i = 0^+ \text{ in practice}) \]
\[ g^2 \mu = 2 \text{ GeV} \quad (\ast) \quad g^2 \mu = 1 \text{ GeV} : \]
Spectra for various quark masses

\[ g^2 \mu = 2 \text{ GeV}, \tau = 0.25 \text{ fm} : \]

![Graph showing spectra for various quark masses]