QCD at finite Temperature

III – Out of equilibrium systems

François Gelis

CERN and CEA/Saclay

General outline

- Lecture I : Quantum field theory at finite T
- Lecture II : Collective phenomena in the QGP
- Lecture III : Out of equilibrium systems

Lecture III : Out of equilibrium systems

- Schwinger-Keldysh formalism, Long time pathologies
- From fields to kinetic theory
- Collisionless kinetic equations
- Boltzmann equation
- Transport coefficients

Schwinger-Keldysh formalism

Long time pathologies

Reminder: equilibrium

- In equilibrium, the free density operator is $\boldsymbol{\rho} \equiv \exp(-\beta H_0)$:

$$\boldsymbol{\rho} = \exp \; - \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3} \; \beta \; E_{\boldsymbol{k}} \; a^\dagger_{\rm in}(\vec{\boldsymbol{k}}) a_{\rm in}(\vec{\boldsymbol{k}})$$

Note : the interactions contained in the full H lead to the vertical branch of the time contour

- The fact that $\boldsymbol{\rho}$ is the equilibrium density operator is reflected in the KMS symmetry of thermal correlators :

$$G(\cdots, t_i, \cdots) = G(\cdots, t_i - i\beta, \cdots)$$

- The free scalar propagator reads :

$$G^0(x,y) = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \Big[(\theta_c(x^0 - y^0) + n_B(E_{\boldsymbol{p}}))\, e^{-ip\cdot(x-y)} + (\theta_c(y^0 - x^0) + n_B(E_{\boldsymbol{p}}))\, e^{+ip\cdot(x-y)} \Big]$$

More remarks on equilibrium

- $\boldsymbol{\rho}$ (and n_B) represent the statistical properties of the system at the initial time t_i. This precision is pointless in equilibrium - because the particle distribution is time independent - but is crucial out-of-equilibrium

- The equilibrium density operator is extremely peculiar. All the information about the distribution of particles in the system is contained in the single particle phase-space density n_B. All the higher correlations are trivial in equilibrium

  ▷ this is the reason why the Feynman rules at finite T are very similar to those at $T = 0$ (modification of the time integration contour, and of the free propagator)

- For a completely generic $\boldsymbol{\rho}$, one may have non-Gaussian initial correlations. In the Feynman rules, they would appear in the form of additional vertices (usually non-local). For instance, non-trivial 2-particle correlations would be encoded in a 4-point vertex

Non-equilibrium Gaussian systems

- There are no systematic studies of non-equilibrium systems with non-Gaussian initial correlations, mostly because of the complexity of the Feynman rules

  It is generally believed that these non-Gaussian correlations affect the system only during a very short transient regime

- In this lecture, I consider only Gaussian correlations. The most general Gaussian density operator can be written as

$$\boldsymbol{\rho} = \exp -\int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3}\; \beta_{\boldsymbol{k}}\; E_{\boldsymbol{k}}\; a^\dagger_{\rm in}(\vec{\boldsymbol{k}})a_{\rm in}(\vec{\boldsymbol{k}})$$

  - Because this $\boldsymbol{\rho}$ does not contain any interaction term, the time contour is simply $[t_i, +\infty] \cup [+\infty, t_i]$
  - The propagator is the same as in equilibrium, with the substitution

$$n_B(E_{\boldsymbol{k}}) \to f_{\boldsymbol{k}} \equiv \frac{1}{e^{\beta_{\boldsymbol{k}} E_{\boldsymbol{k}}} - 1}$$

  Note : any function $f_{\boldsymbol{k}}$ can be parameterized in this form

Schwinger-Keldysh formalism

- The diagrammatic expansion is the same as in equilibrium
- At each vertex : $-ig\int_{\mathcal{C}} d^4x$
  - The contour is now limited to the two horizontal branches
  - The KMS symmetry, and the freedom to deform the contour at will - that one had in equilibrium - are lost
- Free propagator :

$$G^0(x,y)=\int\frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\Big[(\theta_c(x^0-y^0)+f_{\boldsymbol{p}})\,e^{-ip\cdot(x-y)} +(\theta_c(y^0-x^0)+f_{\boldsymbol{p}})\,e^{+ip\cdot(x-y)}\Big]$$

where $f_{\boldsymbol{p}}$ is the initial particle distribution

Momentum space formulation

- Let us assume that the initial time t_i is $t_i = -\infty$
- Then, one can compute the diagrams in momentum space. Because the time contour has two branches, there are four possible combinations for the propagator, depending on which branch hosts the endpoints. In momentum space, they read

$$G^0_{++}(p) = \frac{i}{p^2 - m^2 + i\epsilon} + 2\pi f_{\boldsymbol{p}} \delta(p^2 - m^2)$$

$$G^0_{--}(p) = \frac{-i}{p^2 - m^2 - i\epsilon} + 2\pi f_{\boldsymbol{p}} \delta(p^2 - m^2)$$

$$G^0_{+-}(p) = 2\pi(\theta(-p^0) + f_{\boldsymbol{p}})\delta(p^2 - m^2)$$

$$G^0_{-+}(p) = 2\pi(\theta(+p^0) + f_{\boldsymbol{p}})\delta(p^2 - m^2)$$

- The vertices are $-ig$ or $+ig$ depending on whether the time is on the upper or lower branch (the opposite sign is due to the fact that the lower branch is oriented in the opposite direction)

Momentum space formulation - Exercise

- Check the following formula :

$$\begin{pmatrix} G^0_{++} & G^0_{+-} \\ G^0_{-+} & G^0_{--} \end{pmatrix} = U \begin{pmatrix} G^0_{_F} & 0 \\ 0 & G^{0*}_{_F} \end{pmatrix} U$$

with

$$U(p) \equiv \begin{pmatrix} \sqrt{1+f_{\boldsymbol{p}}} & \frac{\theta(-p^0)+f_{\boldsymbol{p}}}{\sqrt{1+f_{\boldsymbol{p}}}} \\ \frac{\theta(+p^0)+f_{\boldsymbol{p}}}{\sqrt{1+f_{\boldsymbol{p}}}} & \sqrt{1+f_{\boldsymbol{p}}} \end{pmatrix}$$

and

$$G^0_{_F}(p) \equiv \frac{i}{p^2 - m^2 + i\epsilon}$$

Momentum space formulation

- For each graph, assign $\pm$ signs to the vertices in all the possible ways (2^n possibilities for n vertices)
- Connect the vertices by the corresponding $G^0_{\epsilon\epsilon'}$ propagators
- Integrate over the momenta of all the independent loops
- Notes :
  - The same formalism can be used in equilibrium
  - However, the contribution of the vertical part of the time contour brings a small modification to the propagator :

$$n_B(E_{\boldsymbol{p}}) \to n_B(|p^0|)$$

KMS symmetry

- The n-point correlators in the Schwinger-Keldysh formalism obey the following relation :

$$\sum_{\epsilon_1\cdots\epsilon_n=\pm}\Big[\prod_{\{i|\epsilon_i=-\}}(-1)\Big]\,G_{\epsilon_1\cdots\epsilon_n}(k_1,\cdots,k_n)=0$$

Note : this relation is true even out of equilibrium

- A second relation - related to KMS - is satisfied in equilibrium :

$$\sum_{\epsilon_1\cdots\epsilon_n=\pm}\Big[\prod_{\{i|\epsilon_i=-\}}(-e^{-\beta k_i^0})\Big]\,G_{\epsilon_1\cdots\epsilon_n}(k_1,\cdots,k_n)=0$$

- Note : amputated correlators obey the same relations, without the minus signs

Pathologies - Exercise

- The propagators of the Schwinger-Keldysh formalism in momentum space are linear combinations of the distributions

$$\mathrm{P}\,\frac{1}{p^2 - m^2} \quad , \qquad \delta(p^2 - m^2)$$

- Show that the square of these distributions is ill-defined
- However, some bilinear combinations are well defined :

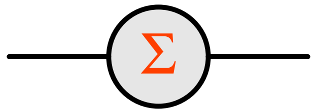
$$2\left[\mathrm{P}\,\frac{1}{x}\right]\delta(x) = -\frac{d}{dx}\delta(x)$$

$$\pi^2\delta^2(x) - \left[\mathrm{P}\,\frac{1}{x}\right]^2 = \frac{d}{dx}\left[\mathrm{P}\,\frac{1}{x}\right]$$

- For consistency, all the ill-defined products of distributions must cancel when calculating graphs in the Schwinger-Keldysh formalism

Pathologies

- Example : insertion of a self-energy. Consider :



$$= \sum_{\epsilon,\epsilon'=\pm} G^0_{+\epsilon}(p)\,\Sigma_{\epsilon\epsilon'}(p)\,G^0_{\epsilon'+}(p)$$

- This expression contains $\delta^2(p^2-m^2)$ terms (that cannot be combined with others to make finite objects) whose sum is proportional to (for $p^0>0$)

$$2f_{\boldsymbol{p}}(1+f_{\boldsymbol{p}})\Big[\Sigma_{++}+\Sigma_{--}\Big]+(1+2f_{\boldsymbol{p}})\Big[(1+f_{\boldsymbol{p}})\Sigma_{+-}+f_{\boldsymbol{p}}\Sigma_{-+}\Big]$$

- Using the first relation among the $\Sigma_{\epsilon\epsilon'}$'s (which is always true), this coefficient becomes

$$(1+f_{\boldsymbol{p}})\Sigma_{+-}-f_{\boldsymbol{p}}\Sigma_{-+}$$

▷ This is zero only if the KMS identity holds, i.e. if the system is in equilibrium!

Pathologies

- One can learn a bit more by resumming the self energy on the propagator. Define :

$$\mathbb{G}^0 \equiv \begin{pmatrix} G^0_{++} & G^0_{+-} \\ G^0_{-+} & G^0_{--} \end{pmatrix} \quad , \quad \mathbb{D} \equiv \begin{pmatrix} G^0_F & 0 \\ 0 & G^{0*}_F \end{pmatrix} \quad , \quad \$ \equiv \begin{pmatrix} \Sigma_{++} & \Sigma_{+-} \\ \Sigma_{-+} & \Sigma_{--} \end{pmatrix}$$

- We want to calculate :

$$\mathbb{G} \equiv \sum_{n=0}^{\infty} \Big[\mathbb{G}^0(-i\$)\Big]^n \, \mathbb{G}^0 = U \sum_{n=0}^{\infty} \Big[-i\mathbb{D}U\$U\Big]^n \, \mathbb{D}U$$

- For $p^0 > 0$, we have $\mathbb{D}U\$U = \begin{pmatrix} G^0_F \Sigma_F & G^0_F \widetilde{\boldsymbol{\Sigma}} \\ 0 & G^{0*}_F \Sigma^*_F \end{pmatrix}$

$$\text{with} \quad \begin{cases} \Sigma_F \equiv \Sigma_{++} + \Sigma_{+-} \\ \widetilde{\boldsymbol{\Sigma}} \equiv \dfrac{1}{1+f_{\boldsymbol{p}}}\Big[(1+f_{\boldsymbol{p}})\Sigma_{+-} - f_{\boldsymbol{p}}\Sigma_{-+}\Big] \end{cases}$$

Pathologies

- $\mathbb{D}U\$U$ is the sum of a diagonal and a nilpotent matrix
  - ▷ the calculation of its n-th power is easy
- The resummed propagator matrix is :

$$\mathbb{G} = U \begin{pmatrix} G_F & G_F \widetilde{\Sigma} G_F^* \\ 0 & G_F^* \end{pmatrix} U \qquad \text{with} \quad G_F(p) \equiv \frac{i}{p^2 - m^2 - \Sigma_F + i\epsilon}$$

- In equilibrium $\widetilde{\Sigma} = 0$ thanks to KMS, and the resummed propagator matrix is diagonalized with the same matrix U. This was expected since, in equilibrium, interactions do not change the particle distribution
- Out of equilibrium, the propagator matrix is no longer diagonalizable with U. Moreover, G_F and G_F^* have mirror poles with respect to the real energy axis
  - ▷ pinch singularities if $\mathrm{Im}\,\Sigma_F = 0$

Interpretation

- Compare the bare and resummed propagators :

$$\mathbb{G}^0 = \begin{pmatrix} G_F^0 & \theta(-p^0)(G_F^0 + G_F^{0*}) \\ \theta(+p^0)(G_F^0 + G_F^{0*}) & G_F^{0*} \end{pmatrix} + (G_F^0 + G_F^{0*})\, f_{\boldsymbol{p}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbb{G} = \begin{pmatrix} G_F & \theta(-p^0)(G_F + G_F^{*}) \\ \theta(+p^0)(G_F + G_F^{*}) & G_F^{*} \end{pmatrix} + (G_F + G_F^{*})\, f_{\boldsymbol{p}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$+ \Big[(1 + f_{\boldsymbol{p}})\Sigma_{+-} - f_{\boldsymbol{p}}\Sigma_{-+}\Big] G_F G_F^* \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- The pinch term gives an equal contribution to the four components of the propagator matrix, exactly like the distribution $f_{\boldsymbol{p}}$ $\triangleright$ this suggests that this term can be absorbed in a redefinition of $f_{\boldsymbol{p}}$

Interpretation

- Strictly speaking, the Schwinger-Keldysh formalism with $t_i = -\infty$ makes sense only in equilibrium

  It is possible to check this in the space-time representation : the previous calculation gives a finite result even out of equilibrium as long as t_i is finite, but the limit $t_i \to -\infty$ is finite only in equilibrium

- In fact, the pinch singularities tell us that we are trying to do something a bit stupid: we are trying to calculate a certain process taking place at a time x^0 in an out of equilibrium medium, in terms of the particle distribution $f_{\boldsymbol{p}}$ at the time t_i. This is in principle feasible, but extremely unnatural

  The pinch singularities suggest that it would be much simpler to compute this process in terms of the particle distribution at the time x^0 instead

- By working in coordinate space, we will see that the self-energy resummation amounts - in a certain approximation - to let $f_{\boldsymbol{p}}$ have a time dependence governed by a Boltzmann equation

From fields to kinetic theory

Dyson-Schwinger equations

- In coordinate space, the resummation of the self-energy can be done via the Dyson-Schwinger equations :

$$G(x,y) = G^0(x,y) + \int_{\mathcal{C}} d^4u d^4v \; G^0(x,u)\Big(-i\Sigma(u,v)\Big)G(v,y)$$

$$G(x,y) = G^0(x,y) + \int_{\mathcal{C}} d^4u d^4v \; G(x,u)\Big(-i\Sigma(u,v)\Big)G^0(v,y)$$

- Apply $\square_x + m^2$ to the first equation, using the fact that $(\square_x + m^2)G^0(x,y) = -i\delta_c(x-y)$:

$$(\square_x + m^2)G(x,y) = -i\delta_c(x-y) - \int_{\mathcal{C}} d^4v \; \Sigma(x,v) \; G(v,y)$$

- Similarly,

$$(\square_y + m^2)G(x,y) = -i\delta_c(x-y) - \int_{\mathcal{C}} d^4v \; G(x,v) \; \Sigma(v,y)$$

Wigner transform

- Out of equilibrium, 2-point functions depend separately on their two arguments (in equilibrium they depend only on the difference $x-y$)

- However, it is useful to perform a Fourier transform with respect to the difference $s \equiv x-y$. The Wigner transform of $F(x,y)$ is defined as

$$F(X,p) \equiv \int d^4s\; e^{ip\cdot s}\; F(X+\frac{s}{2}, X-\frac{s}{2})$$

- Derivatives with respect to x and y can be written in terms of derivatives with respect to X and s :

$$\partial_x = \frac{1}{2}\partial_X + \partial_s \quad , \quad \partial_y = \frac{1}{2}\partial_X - \partial_s$$

$$\square_x = \frac{1}{4}\square_X + \partial_X\cdot\partial_s + \square_s \quad , \quad \square_y = \frac{1}{4}\square_X - \partial_X\cdot\partial_s + \square_s$$

Wigner transform - Exercise

- Wigner transform of a convolution. Consider :

$$H(x,y) \equiv \int d^4z\ F(x,z)\,G(z,y)$$

- Prove that :

$$H(X,p) = e^{\frac{i}{2}\left[\partial_{X_1}\cdot\partial_{p_2}-\partial_{X_2}\cdot\partial_{p_1}\right]}\ F(X_1,p_1)\,G(X_2,p_2)\Big|_{\substack{X_1=X_2=X\\ p_1=p_2=p}}$$

- By expanding the exponential, one gets the gradient expansion of the Wigner transform of the convolution product

Gradient expansion

- The derivatives with respect to X ($\partial_X, \square_X$) characterize the space and time scales over which the particle distribution changes significantly
- We assume that these scales are much larger than the De Broglie wavelength of the particles, i.e. that $\partial_X \ll p, \square_X \ll p^2$
- Note : typically, ∂_X is at most of the order of the inverse transport mean free path, i.e. g^4T
- As we shall see, the relevant self-energy in transport phenomena is of order g^4T^2, while the typical particle momentum is of order T

  ▷ it is sufficient to expand the convolution product in the r.h.s. to zeroth order in gradients

Gradient expansion

- By taking the difference of the Dyson-Schwinger equations w.r.t. x and y, and by breaking it down into its $\pm$ components, one finds

$$-2ip\cdot\partial_X\left(G_{+-}(X,p)-G_{-+}(X,p)\right)=0$$
$$-2ip\cdot\partial_X\left(G_{+-}(X,p)+G_{-+}(X,p)\right)=2\Big[G_{-+}\Sigma_{+-}-G_{+-}\Sigma_{-+}\Big]$$

- Quasi-particle ansatz : by analogy with the free theory, one assumes that (for $p^0>0$)

$$G_{-+}(X,p)=(1+f(X,p))\rho(X,p)$$
$$G_{+-}(X,p)=f(X,p)\rho(X,p)$$

where $\rho(X,p)\equiv G_{-+}(X,p)-G_{+-}(X,p)$

- This assumption is valid when the quasi-particles are long-lived. This usually requires that the coupling be small

Boltzmann equation

- Thus, we get a Boltzmann equation :

$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{x}}}\right] f(X,p) = \frac{i}{2E_{\boldsymbol{p}}}\Big[(1+f(X,p))\Sigma_{+-} - f(X,p)\Sigma_{-+}\Big]$$

  where $\vec{\boldsymbol{v}}_{\boldsymbol{p}} \equiv \vec{\boldsymbol{p}}/E_{\boldsymbol{p}}$

- In the r.h.s (collision term), we see the same combination as in the KMS condition ▷ it is zero in equilibrium

- The collision term is a (spatially local) functional of the particle distribution $f(X,p)$ ▷ the Boltzmann equation is an approximation of the Dyson-Schwinger equations in which the degrees of freedom are on-shell particles

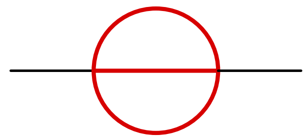
- The combination $\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{x}}}$ is the transport derivative

  It is zero on any function whose t and $\vec{\boldsymbol{x}}$ dependence arise only in the combination $\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_{\boldsymbol{p}} t$

Boltzmann equation - Exercise

- Consider a scalar theory with a $\lambda\phi^4$ interaction

- Show that the first non-zero contribution to the collision term arises at 2-loops, in the diagram



- Calculate the corresponding collision term, and show that it is given by

$$\frac{\lambda^2}{4E_{\boldsymbol{p}}}\int\frac{d^3\vec{\boldsymbol{p}}_1}{(2\pi)^3 2E_1}\int\frac{d^3\vec{\boldsymbol{p}}_2}{(2\pi)^3 2E_2}\int\frac{d^3\vec{\boldsymbol{p}}_3}{(2\pi)^3 2E_3}(2\pi)^4\delta(p-p_1-p_2-p_3)$$
$$\times\Big[f(p_1)f(p_2)(1+f(p_3))(1+f(p)) - f(p_3)f(p)(1+f(p_1))(1+f(p_2))\Big]$$

(General structure : Gain term – Loss term)

Boltzmann-Vlasov equation

- Our derivation must be slightly modified when the self-energy $\Sigma(u,v)$ contains a local part :

$$\Sigma(u,v) = \Phi(u)\delta_c(u-v) + \Pi(u,v)$$

- In the derivation of the Boltzmann equation, one needs the Wigner transform of

$$\Phi(y)G(x,y) - \Phi(x)G(x,y)$$

Exercise : show that to lowest order in the gradient expansion, this Wigner transform is

$$i\partial_X \Phi(X) \cdot \partial_p G(X,p)$$

- The modified Boltzmann equation reads :

$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\vec{\boldsymbol{x}}}\right] f + \underline{\frac{1}{2E_{\boldsymbol{p}}}\partial_X \Phi \cdot \partial_p f} = \frac{i}{2E_{\boldsymbol{p}}}\Big[(1+f)\Sigma_{+-} - f\Sigma_{-+}\Big]$$

Collisionless kinetic equations

Free transport

- Free transport is a regime in which the particles do not interact. Given an initial $f(t_0, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$, the particles propagate on straight lines, at constant velocity
- The kinetic equation that describes this regime reads :

$$p \cdot \partial_X f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = 0$$

or, equivalently :

$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = 0 \qquad \text{with } \vec{\boldsymbol{v}}_{\boldsymbol{p}} \equiv \frac{\vec{\boldsymbol{p}}}{E_{\boldsymbol{p}}}$$

- This equation can be solved trivially from its initial condition :

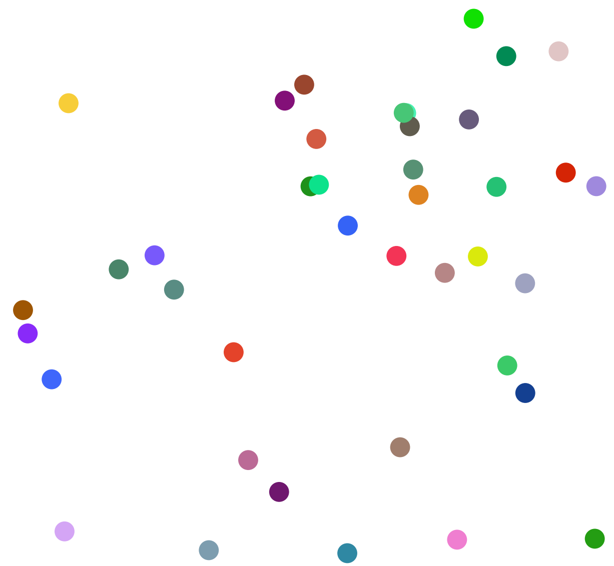
$$f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = f(t_0, \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_{\boldsymbol{p}}(t - t_0), \vec{\boldsymbol{p}})$$

Interpretation :

- The momentum $\vec{\boldsymbol{p}}$ of the particles does not change
- If a particle of momentum $\vec{\boldsymbol{p}}$ is at the position $\vec{\boldsymbol{x}}$ at time t, it comes from the position $\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_{\boldsymbol{p}}(t - t_0)$ at the time t_0

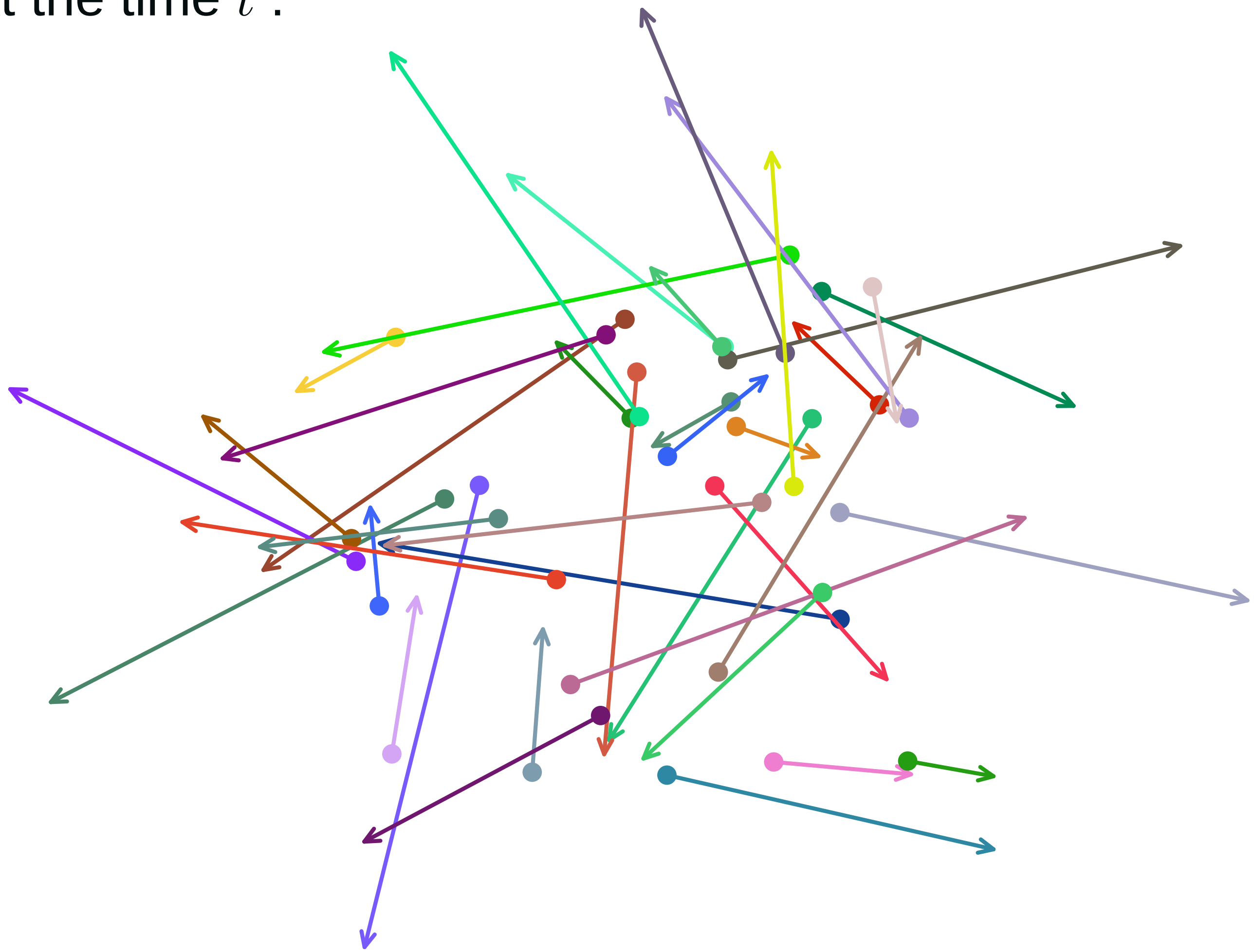
Free transport

- At the time t_0 :



Free transport

- At the time t :



Vlasov equation

- The Vlasov equation describes the time evolution of a distribution of particles under the influence of a force $\vec{\boldsymbol{F}}$

- The Vlasov equation reads :

$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) + \underline{\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{p}} f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})} = 0$$

- When the force is externally applied, it can be solved formally by :

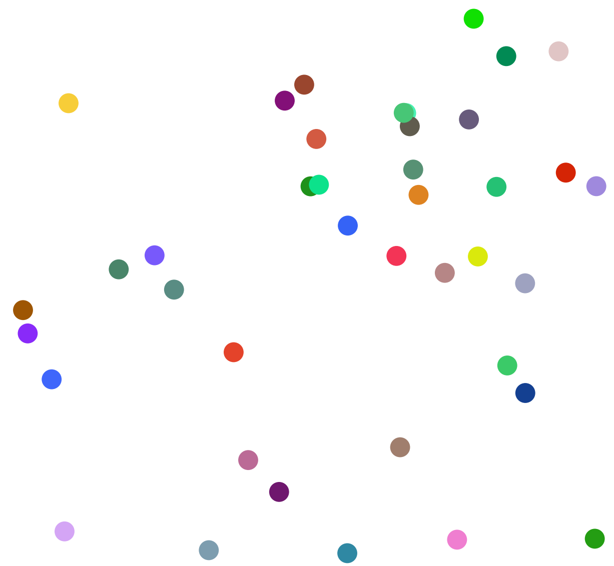
$$f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = f(t_0, \vec{\boldsymbol{x}}_0, \vec{\boldsymbol{p}}_0)$$

  where $(\vec{\boldsymbol{x}}_0, \vec{\boldsymbol{p}}_0)$ is the position in phase space at time t_0 that leads to $(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$ at time t under the effect of the force $\vec{\boldsymbol{F}}$. If $(\vec{\boldsymbol{x}}(\tau), \vec{\boldsymbol{p}}(\tau))$ denotes the trajectory between t_0 and t, one has

$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_0 + \int_{t_0}^{t} d\tau \, \frac{\vec{\boldsymbol{p}}(\tau)}{E_{\boldsymbol{p}}(\tau)} \quad , \quad \vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_0 + \int_{t_0}^{t} d\tau \, \vec{\boldsymbol{F}}(\tau, \vec{\boldsymbol{x}}(\tau))$$

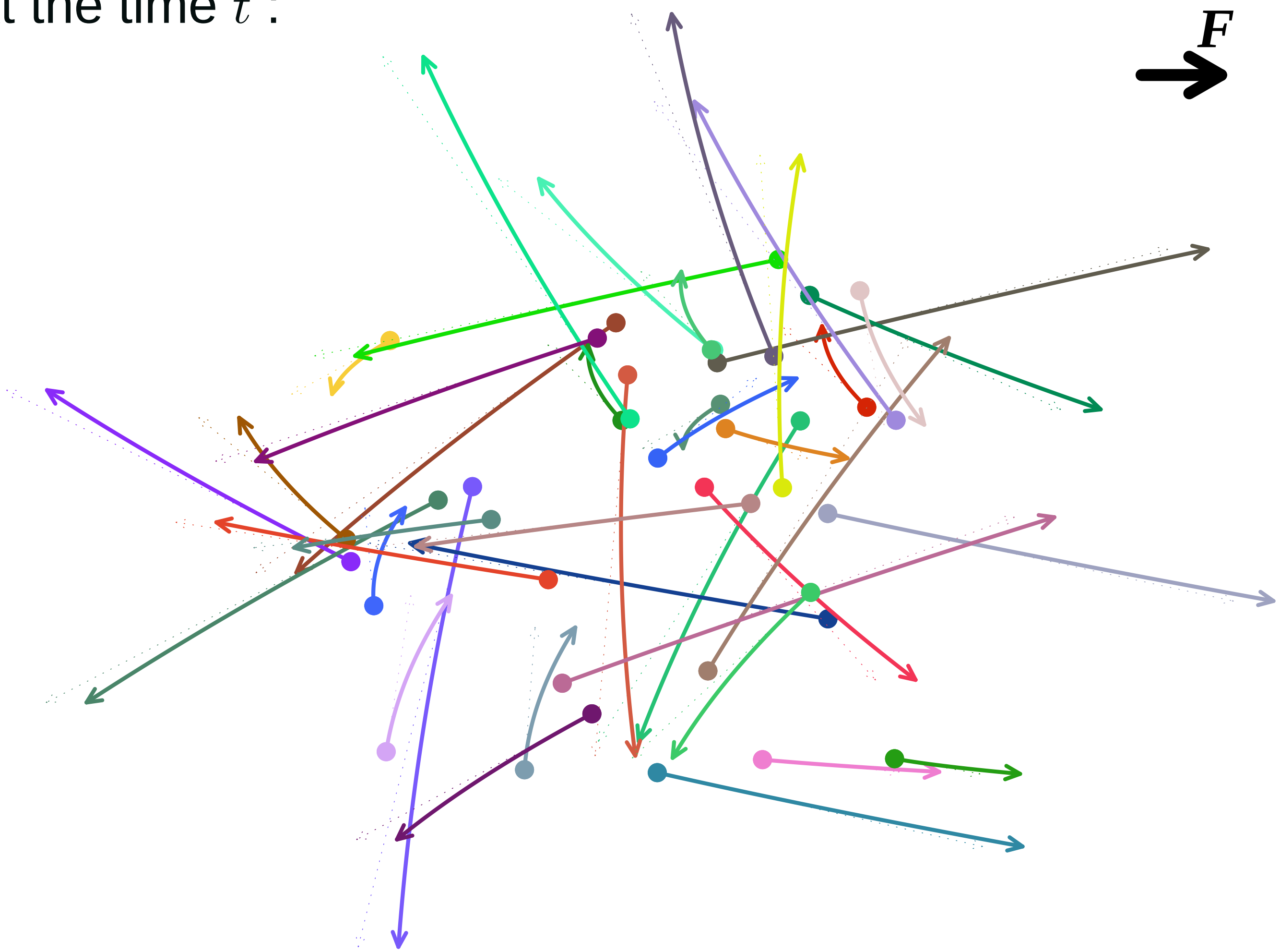
Vlasov equation

- At the time t_0 :



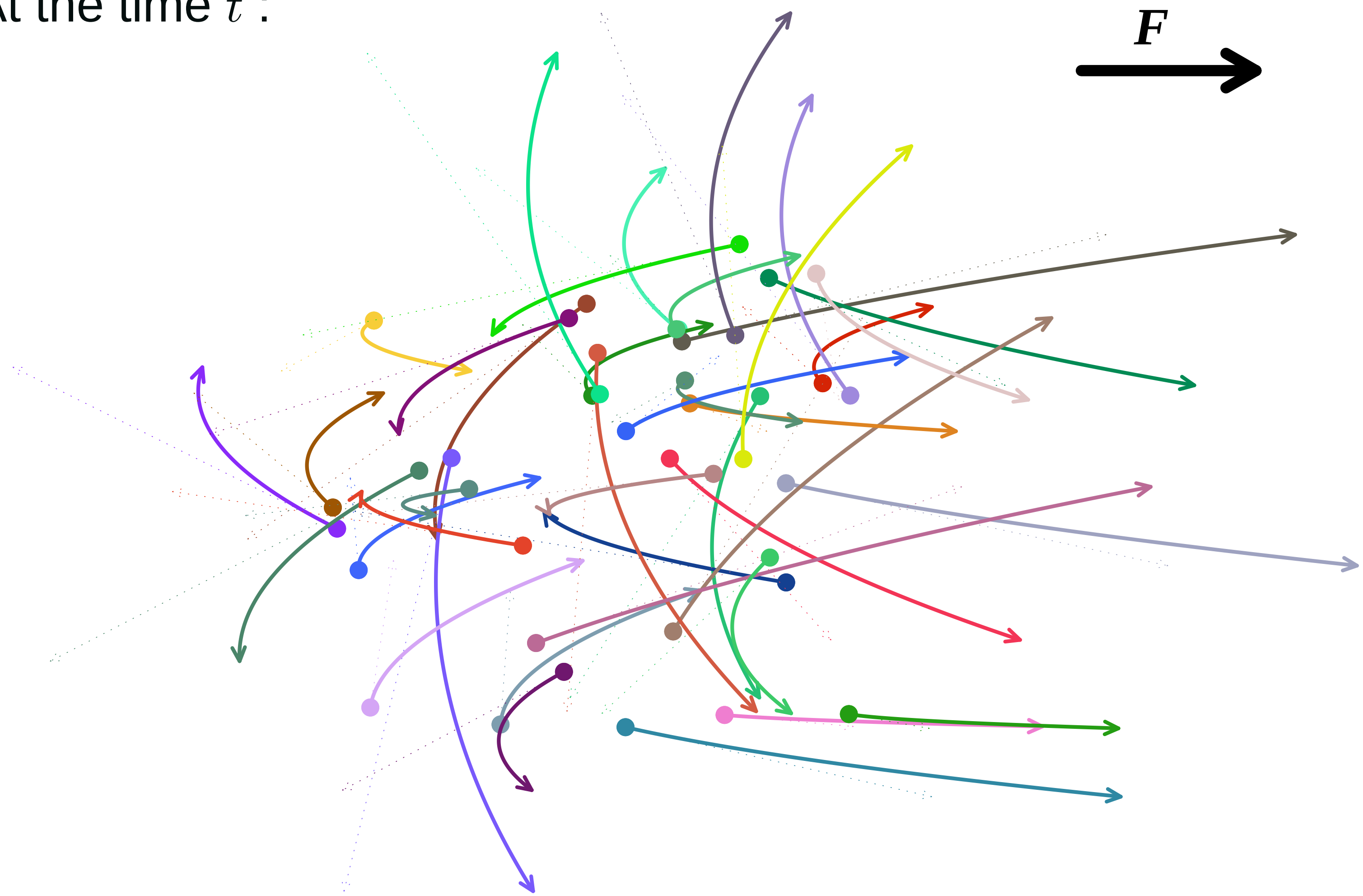
Vlasov equation

- At the time t :



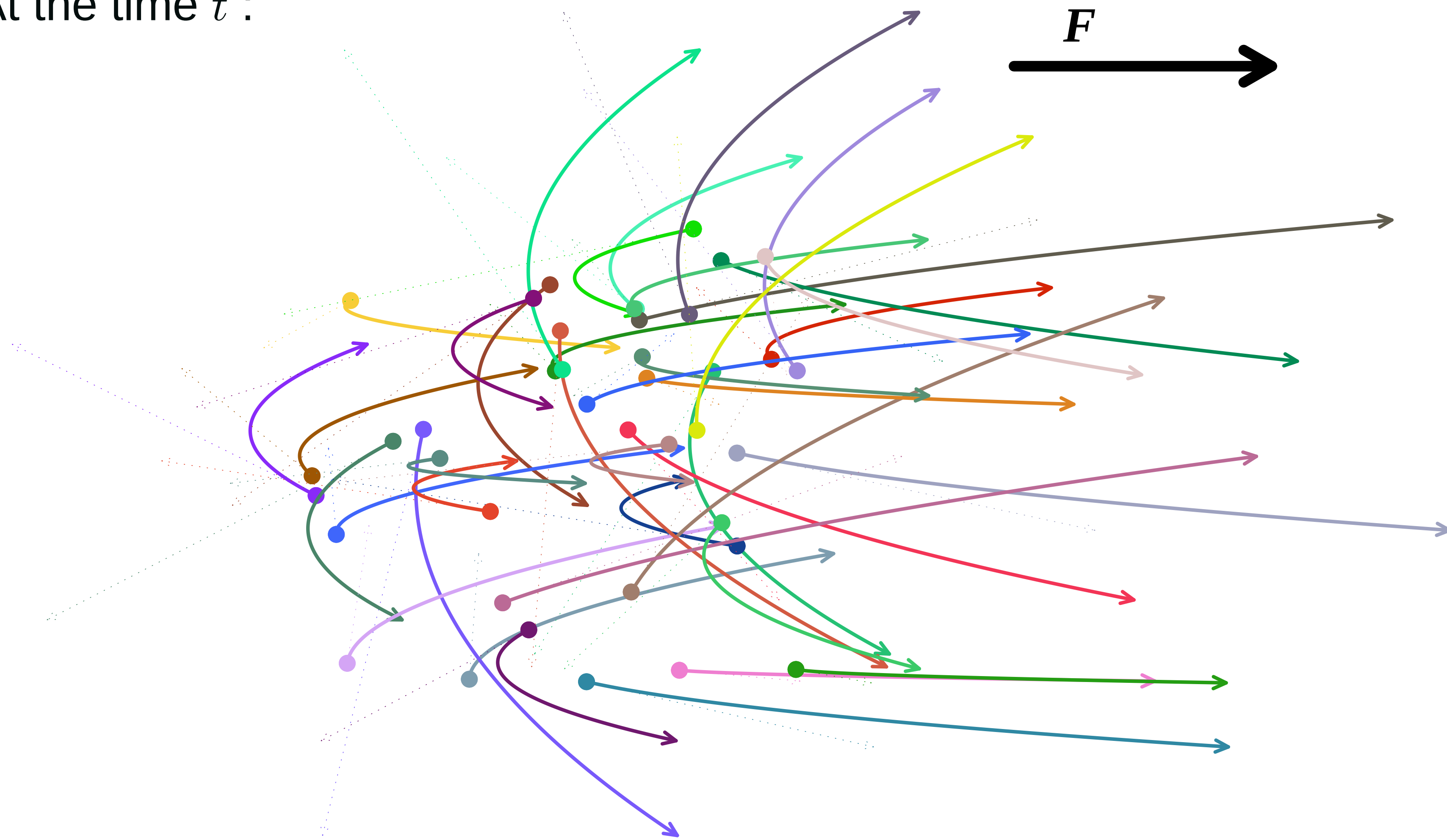
Vlasov equation

- At the time t :



Vlasov equation

- At the time t :



Vlasov equation + mean field

- In many applications, the force $\vec{\boldsymbol{F}}$ is not externally applied, but results from the action of all the other particles

- Example : for electro-magnetic interactions among the particles in the system, the force term in the Vlasov equation reads

$$\underbrace{e\, v_{\boldsymbol{p}}^{\mu}\, F_{\mu\nu}}\ \partial_p^{\nu} f(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{p}})$$

Lorentz force in covariant form

with

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

$$\partial_{\mu} F^{\mu\nu}(x) = \underbrace{e \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, v_{\boldsymbol{p}}^{\nu}\, f(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{p}})} \qquad \text{(Maxwell's equation)}$$

EM current created by the particles

Boltzmann equation

Collision term

- The Boltzmann equation takes into account the collisions among particles. It is valid when these collisions are sufficiently local (i.e. no long range interactions among pairs of particles). Thanks to the Debye screening, this is a valid assumption for a neutral plasma

- The Boltzmann equation reads :

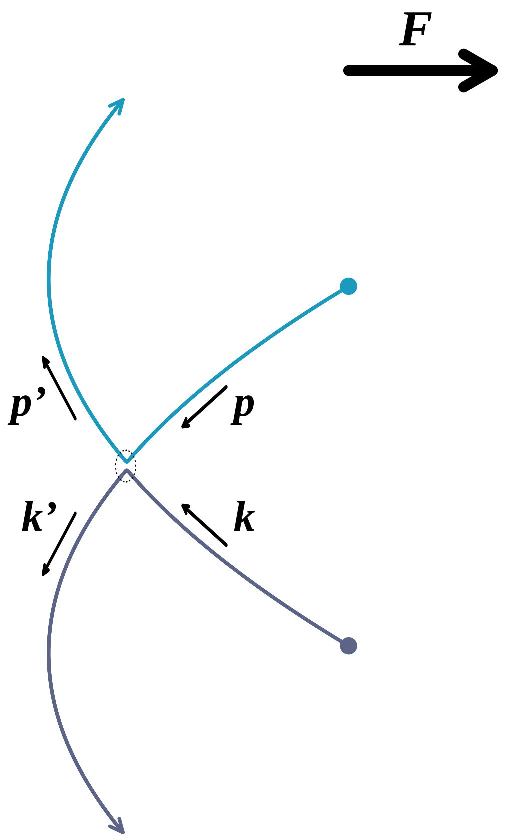
$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) + \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{p}} f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = \mathcal{C}_{\boldsymbol{p}}[f]$$

▷ the functional $\mathcal{C}_{\boldsymbol{p}}[f]$ is the collision term. For $2 \to 2$ collisions, it can be written as :

$$\mathcal{C}_{\boldsymbol{p}}[f] = \frac{1}{2E_{\boldsymbol{p}}} \int \frac{d^3\vec{\boldsymbol{p}}'}{(2\pi)^3 2E_{\boldsymbol{p}'}} \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}} \int \frac{d^3\vec{\boldsymbol{k}}'}{(2\pi)^3 2E_{\boldsymbol{k}'}} (2\pi)^4 \delta(p+k-p'-k')$$
$$\times \Big[f(X,\vec{\boldsymbol{p}}') f(X,\vec{\boldsymbol{k}}')(1+f(X,\vec{\boldsymbol{p}}))(1+f(X,\vec{\boldsymbol{k}}))$$
$$- f(X,\vec{\boldsymbol{p}}) f(X,\vec{\boldsymbol{k}})(1+f(X,\vec{\boldsymbol{k}}'))(1+f(X,\vec{\boldsymbol{p}}'))\Big] \left|\mathcal{M}\right|^2$$

Collision term

- Elementary 2-body collision :



Note : microscopic collisions are reversible

Diluteness assumption

- In order to be able to neglect 3-body collisions and higher, the system under study must be sufficiently dilute

- For a system of N hard spheres of radius r, the Boltzmann equation is valid in the limit :

$$\left\{ \begin{array}{rcr} Nr^2 & = & \text{const} \\ Nr^3 & \to & 0 \end{array} \right.$$

  (Boltzmann-Grad limit)

- The first condition means that the mean free path is fixed ($\lambda = 1/n\sigma,\ n = N/V,\ \sigma = 2\pi r^2$)

- The second condition means that the volume occupied by the particles tend to zero

Molecular chaos assumption

- Strictly speaking, the collision term should contain the probability to find a pair of particles of momenta $\vec{\boldsymbol{p}}, \vec{\boldsymbol{k}}$ at the point $(t, \vec{\boldsymbol{x}})$ before the collision

  ▷ one should have used the 2-particle phase-space distribution :

$$f_2(X, \vec{\boldsymbol{p}}; X, \vec{\boldsymbol{k}})$$

  that contains information about the 2-particle correlations

- By writing :

$$f_2(X, \vec{\boldsymbol{p}}; X, \vec{\boldsymbol{k}}) = f(X, \vec{\boldsymbol{p}}) f(X, \vec{\boldsymbol{k}})$$

  one assumes that the two colliding particles have uncorrelated momenta before the collision

- Although the microscopic processes are reversible, the Boltzmann equation is not, because the two momenta become correlated after the collision

Collisions or mean field ?

- Given a two-body interaction between particles, should we treat it as part of the mean field force term, or as part of the collision term?

- Bobylev, Illner : for inverse power forces in r^{-s}
  - the collision term prevails if $s > 3$
  - the mean-field term prevails if $s < 3$

- This indicates that short-range interactions should be treated as collisions, while long range interactions go in the mean-field term

- Examples :

Debye screened forces	$\longrightarrow$	collisions
Hard sphere interactions	$\longrightarrow$	collisions
Gravitational forces	$\longrightarrow$	mean-field

Collisional invariants

- Consider a quantity $I(\vec{\boldsymbol{p}})$, and the integral

$$\mathcal{I}[f] \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\; \mathcal{C}_{\boldsymbol{p}}[f]\; I(\vec{\boldsymbol{p}})$$

- By symmetry under the exchange $(\vec{\boldsymbol{p}},\vec{\boldsymbol{p}}') \leftrightarrow (\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}')$ and antisymmetry under $(\vec{\boldsymbol{p}},\vec{\boldsymbol{k}}) \leftrightarrow (\vec{\boldsymbol{p}}',\vec{\boldsymbol{k}}')$, we can write

$$\mathcal{I}[f] = \frac{1}{4}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\; \mathcal{C}_{\boldsymbol{p}}[f]\; \Big[I(\vec{\boldsymbol{p}}) + I(\vec{\boldsymbol{k}}) - I(\vec{\boldsymbol{p}}') - I(\vec{\boldsymbol{k}}')\Big]$$

- A quantity $I(\vec{\boldsymbol{p}})$ for which the bracket $[\cdots]$ vanishes is called a collisional invariant
- Collisional invariants :
  - $I(\vec{\boldsymbol{p}}) = 1$ (elastic collisions conserve the number of particles)
  - $I(\vec{\boldsymbol{p}}) = p^\mu$ (energy-momentum conservation)

Local conservation laws

- Define the density and current at point X for the quantity I :

$$I(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\; I(\vec{\boldsymbol{p}})\; f(X,\vec{\boldsymbol{p}})$$

$$\vec{\boldsymbol{J}}_{I}(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\; I(\vec{\boldsymbol{p}})\; \vec{\boldsymbol{v}}_{\boldsymbol{p}}\; f(X,\vec{\boldsymbol{p}})$$

- Multiply the Boltzmann equation by $I(\vec{\boldsymbol{p}})$ and integrate it over all the momenta $\boldsymbol{p}$:
  - The collision term gives zero for a collisional invariant
  - If there is no force term, then one obtains

$$\partial_t I(X) + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} \cdot \vec{\boldsymbol{J}}_{I}(X) = 0$$

  ▷ continuity equation for the local conservation of the quantity I

- Note : if there is a force term, the number of particles is locally conserved, but not their momentum

H theorem

- Define the quantities

$$h(X,\vec{\boldsymbol{p}}) \equiv (1+f(X,\vec{\boldsymbol{p}}))\ln(1+f(X,\vec{\boldsymbol{p}})) - f(X,\vec{\boldsymbol{p}})\ln(f(X,\vec{\boldsymbol{p}}))$$

$$H(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, h(X,\vec{\boldsymbol{p}}) \quad , \quad \vec{\boldsymbol{J}}_H(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, h(X,\vec{\boldsymbol{p}})\,\vec{\boldsymbol{v}}_{\boldsymbol{p}}$$

- From the Boltzmann equation, we get

$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] h + \vec{\boldsymbol{F}}(X)\cdot\vec{\boldsymbol{\nabla}}_{\boldsymbol{p}} h = \mathcal{C}_{\boldsymbol{p}}[f]\ln\left(\frac{1+f(X,\vec{\boldsymbol{p}})}{f(X,\vec{\boldsymbol{p}})}\right)$$

$$\partial_t H + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\cdot\vec{\boldsymbol{J}}_H = \sigma_H$$

with

$$\sigma_H \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\,\mathcal{C}_{\boldsymbol{p}}[f]\,\ln\left(\frac{1+f(X,\vec{\boldsymbol{p}})}{f(X,\vec{\boldsymbol{p}})}\right)$$

H theorem

- Using the symmetry properties of the collision term, we can rewrite σ_H as

$$\sigma_H = \frac{1}{4}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\,\mathcal{C}_{\boldsymbol{p}}[f]\,\Bigg[\ln\left(\frac{1+f(X,\vec{\boldsymbol{p}})}{f(X,\vec{\boldsymbol{p}})}\right)+\ln\left(\frac{1+f(X,\vec{\boldsymbol{k}})}{f(X,\vec{\boldsymbol{k}})}\right) - \ln\left(\frac{1+f(X,\vec{\boldsymbol{p}}')}{f(X,\vec{\boldsymbol{p}}')}\right)-\ln\left(\frac{1+f(X,\vec{\boldsymbol{k}}')}{f(X,\vec{\boldsymbol{k}}')}\right)\Bigg]$$

- In the right hand side, one can rewrite the factors that depend on f as follows :

$$f(X,\vec{\boldsymbol{p}})f(X,\vec{\boldsymbol{k}})(1+f(X,\vec{\boldsymbol{p}}'))(1+f(X,\vec{\boldsymbol{k}}'))\left[\frac{\alpha_{\boldsymbol{p}}\alpha_{\boldsymbol{k}}}{\alpha_{\boldsymbol{p}'}\alpha_{\boldsymbol{k}'}}-1\right]\ln\left(\frac{\alpha_{\boldsymbol{p}}\alpha_{\boldsymbol{k}}}{\alpha_{\boldsymbol{p}'}\alpha_{\boldsymbol{k}'}}\right)$$

with $\alpha_{\boldsymbol{p}} \equiv (1+f(X,\vec{\boldsymbol{p}}))/f(X,\vec{\boldsymbol{p}})$

- Since $(X-1)\ln(X)\geq 0$, we have $\sigma_H \geq 0$
  ▷ the quantity H has a positive source term

H theorem

- Interpretation :
  - $H(X)$ is the entropy density, and $\vec{J}_H(X)$ its current
  - Because the continuity equation for H has a right hand side σ_H, it is not a conserved quantity
  - Because $\sigma_H \geq 0$, the total amount of H in the system can only increase
- Remarks :
  - This seems to contradict Poincaré's recurrence theorem : *"Any system with a finite volume phase-space will return arbitrarily close to its initial conditions in a finite time"*

    ▷ where does the irreversibility come from in the Boltzmann eq.?
  - Molecular chaos assumption : the Boltzmann equation is an approximation of the full dynamical evolution of the system, in which one neglects correlations among particles prior to collisions. By dropping these correlations, one loses the information necessary to reverse the time evolution of the system

Equilibrium state

- When the equilibrium is reached, $\sigma_H = 0$
  - ▷ $\ln((1+f_{\rm eq})/f_{\rm eq})$ is a collisional invariant
  - ▷ it is a linear combination of 1 and p^μ :

$$\ln\left(\frac{1+f_{\rm eq}(X,\vec{\boldsymbol{p}})}{f_{\rm eq}(X,\vec{\boldsymbol{p}})}\right) = \alpha+\beta_\mu p^\mu \quad \Rightarrow \quad f_{\rm eq}(X,\vec{\boldsymbol{p}}) = \frac{1}{e^{\alpha+\beta_\mu p^\mu}-1}$$

(Bose-Einstein distribution)

- $\beta_\mu p^\mu$ is the Lorentz covariant form of $p^0/T \quad (\beta = 1/T)$
- α is a chemical potential associated to the conservation of the number of particles

Transport coefficients

Transport coefficients

- The Boltzmann equation is a powerful tool for calculating transport coefficients such as conductivity, viscosity, diffusion constants
- These transport coefficients can also be calculated in quantum field at finite temperature. Example for the electric conductivity :
  - $\sigma_{\rm el}$ is the coefficient of proportionality between the induced electric current and the applied electric field :

$$\vec{\boldsymbol{j}}_{\rm el} = \sigma_{\rm el}\ \vec{\boldsymbol{E}}$$

  - It is given by a current-current correlator (Kubo's formula) :

$$\sigma_{\rm el} = \frac{1}{6}\lim_{\omega\to 0}\int d^4x\ e^{i\omega t}\ \Big\langle j^i_{\rm el}(t,\vec{\boldsymbol{x}})\ j^i_{\rm el}(0,\vec{\boldsymbol{0}})\Big\rangle_T$$

  - This correlation function can be evaluated from Feynman diagrams at finite temperature, but one needs to sum an infinite series of graphs ▷ quite difficult

Transport coefficients

- In the evaluation of $\sigma_{\rm el}$ from the Boltzmann equation, one perturbs a system at equilibrium by a small electric field
  ▷ it enters in the Boltzmann equation via the force $\vec{\boldsymbol{F}} \equiv e\vec{\boldsymbol{E}}$
- This force induces a departure of f away from $f_{\rm eq}$. It is convenient to parameterize it by

$$f(X,\vec{\boldsymbol{p}}) \equiv f_{\rm eq}(X,\vec{\boldsymbol{p}}) + f_{\rm eq}(X,\vec{\boldsymbol{p}})(1+f_{\rm eq}(X,\vec{\boldsymbol{p}}))\; f_1(X,\vec{\boldsymbol{p}})$$

- Since the applied field is small, the deviation f_1 is also small
  ▷ linearize the collision term in f_1 :

$$\begin{aligned}
\mathcal{C}_{\boldsymbol{p}}[f] &= \boldsymbol{L}_{\boldsymbol{p}}\cdot f_1 + \mathcal{O}(f_1^2)\\
&\equiv \frac{1}{2E_{\boldsymbol{p}}}\int\frac{d^3\vec{\boldsymbol{p}}'}{(2\pi)^3 2E_{\boldsymbol{p}'}}\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\int\frac{d^3\vec{\boldsymbol{k}}'}{(2\pi)^3 2E_{\boldsymbol{k}'}}(2\pi)^4\delta(p+k-p'-k')\\
&\qquad\times f_{\rm eq}(X,\vec{\boldsymbol{p}})f_{\rm eq}(X,\vec{\boldsymbol{k}})(1+f_{\rm eq}(X,\vec{\boldsymbol{k}}'))(1+f_{\rm eq}(X,\vec{\boldsymbol{p}}'))\\
&\qquad\times\Big[f_1(X,\vec{\boldsymbol{p}})+f_1(X,\vec{\boldsymbol{k}})-f_1(X,\vec{\boldsymbol{p}}')-f_1(X,\vec{\boldsymbol{k}}')\Big]\,\Big|\mathcal{M}\Big|^2
\end{aligned}$$

Transport coefficients

- We apply an uniform electric field, hence $\vec{\nabla}_{\boldsymbol{x}} f(X, \vec{\boldsymbol{p}}) = 0$
- In order to have $\vec{\boldsymbol{j}}_{\rm el} = \sigma_{\rm el}\, \vec{\boldsymbol{E}}$, we must reach the stationary regime. Therefore $\partial_t f(X, \vec{\boldsymbol{p}}) = 0$
- Since the applied field is small, it is legitimate to replace f by $f_{\rm eq}$ in the force term. Thus, the linearized Boltzmann equation reads :

$$\boldsymbol{L}_{\boldsymbol{p}} \cdot f_1 = e\vec{\boldsymbol{E}} \cdot \vec{\nabla}_{\boldsymbol{p}}\, f_{\rm eq}(X, \vec{\boldsymbol{p}})$$

- Solve this equation (not easy, but doable numerically). Since it is a linear equation, the solution f_1 is linear in $\vec{\boldsymbol{E}}$
- Then, one calculates the current induced by this perturbation of the particle distribution,

$$\vec{\boldsymbol{j}}_{\rm el} = e \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, \vec{\boldsymbol{v}}_{\boldsymbol{p}}\, f_{\rm eq}(X, \vec{\boldsymbol{p}})(1 + f_{\rm eq}(X, \vec{\boldsymbol{p}}))\, f_1(X, \vec{\boldsymbol{p}})$$

▷ read $\sigma_{\rm el}$ as the coefficient of proportionality between $\vec{\boldsymbol{j}}_{\rm el}$ and $\vec{\boldsymbol{E}}$