QCD at finite Temperature

I – Quantum field theory at finite T

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General outline

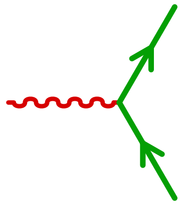
- Lecture I : Quantum field theory at finite T
- Lecture II : Collective phenomena in the QGP
- Lecture III : Out of equilibrium systems

- Introduction
- Perturbative expansion at finite T
- Matsubara formalism

Introduction

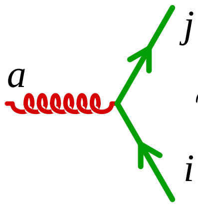
QCD : Quarks and gluons

- Electromagnetic interaction : Quantum electrodynamics
  - Matter : electron , interaction carrier : photon
  - Interaction :

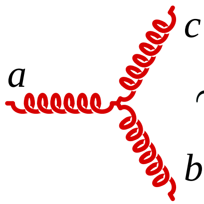


$\sim \; e$ (electric charge of the electron)

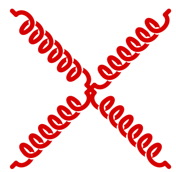
- Strong interaction : Quantum chromo-dynamics
  - Matter : quarks , interaction carriers : gluons
  - Interactions :



$\sim \; g\,(t^a)_{ij}$



$\sim \; g\,(T^a)_{bc}$

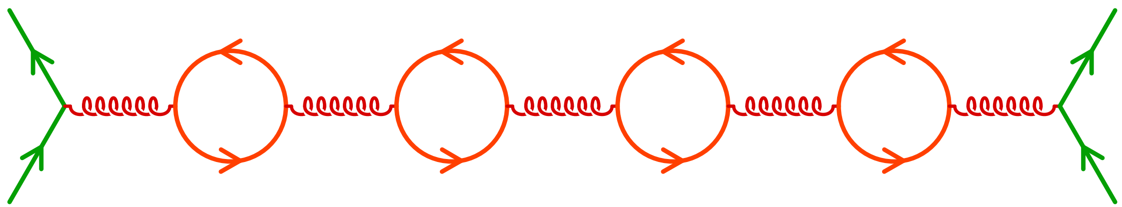


  - i, j : colors of the quarks (3 possible values)
  - a, b, c : colors of the gluons (8 possible values)
  - $(t^a)_{ij}$: 3×3 matrix , $(T^a)_{bc}$: 8×8 matrix

QCD : Asymptotic freedom

- Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(1/r\Lambda_{QCD})}$$



- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)

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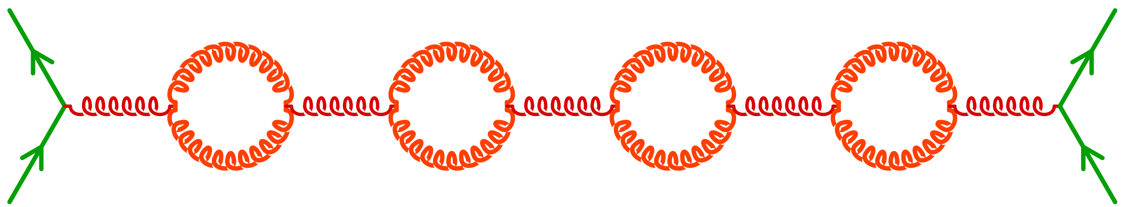
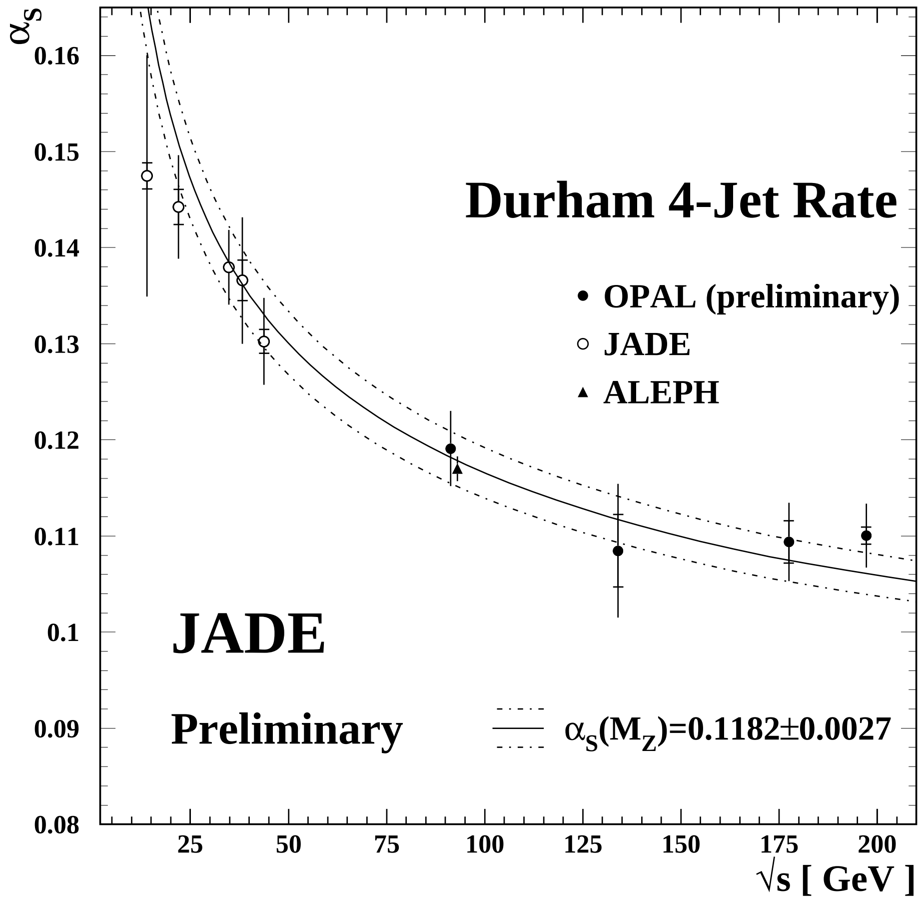


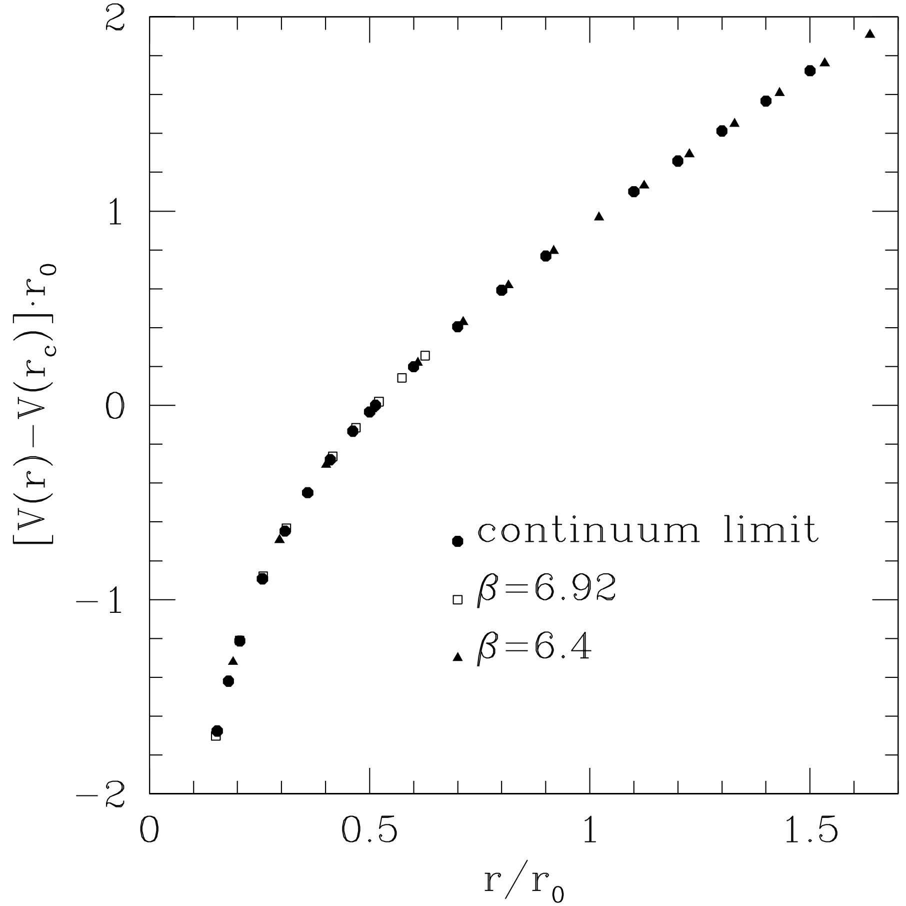
- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)
- But gluonic vacuum fluctuations produce an anti-screening (because of the non-abelian nature of their interactions)
- As long as $N_f < 11N_c/2 = 16.5$, the gluons win...

QCD : Asymptotic freedom



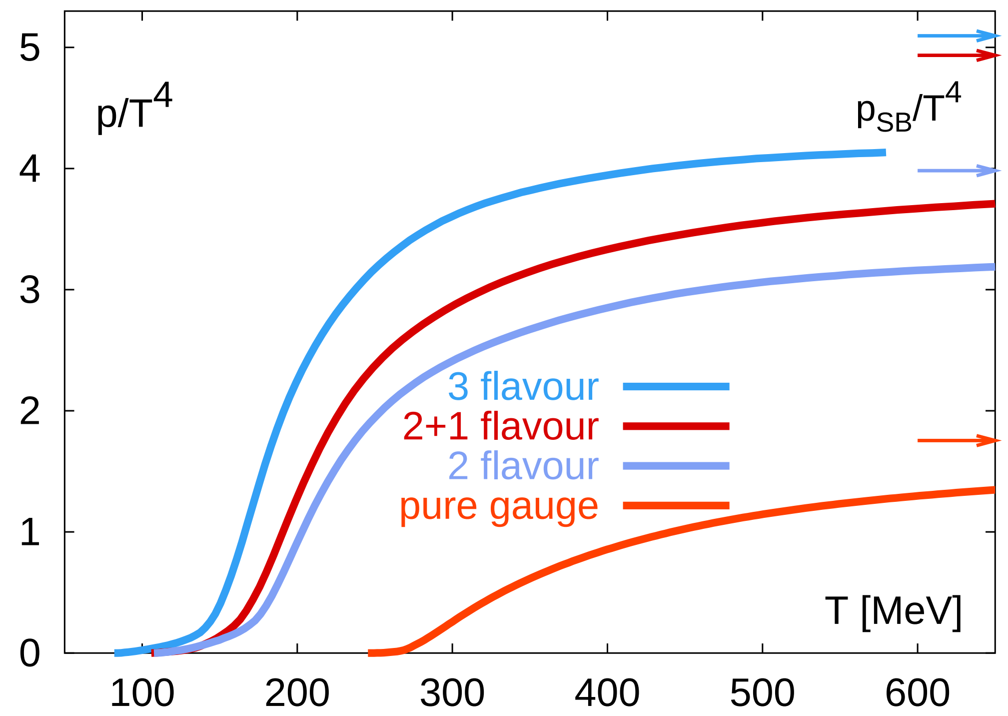
- The coupling constant is small at short distances
- At high density, a hadron gas may undergo deconfinement
  ▷ quark gluon plasma

QCD :Quark confinement



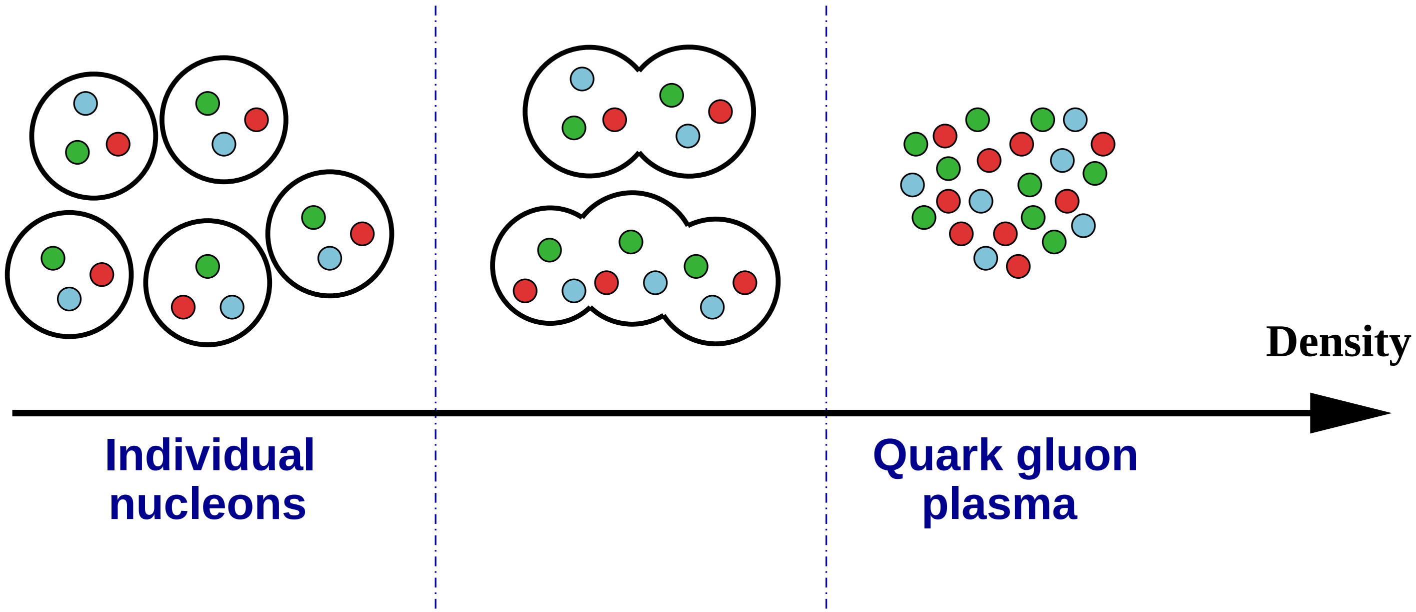
- The quark potential increases linearly with distance
- Quarks are confined into color singlet hadrons

Deconfinement transition



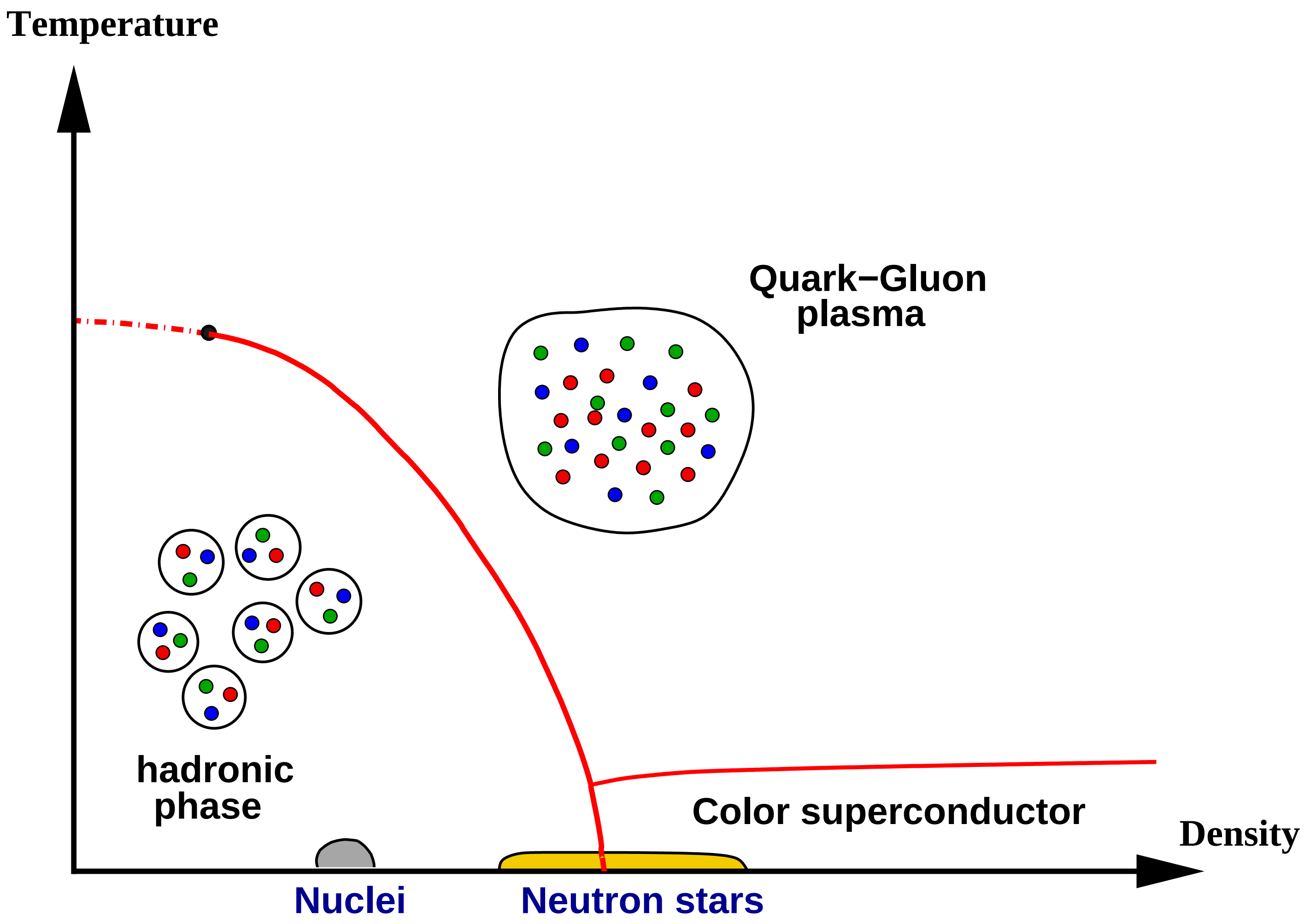
- Fast increase of the pressure :
  - at $T \sim 270$ MeV, if there are only gluons
  - at $T \sim 150\text{–}170$ MeV, depending on the number of light quarks

Deconfinement transition

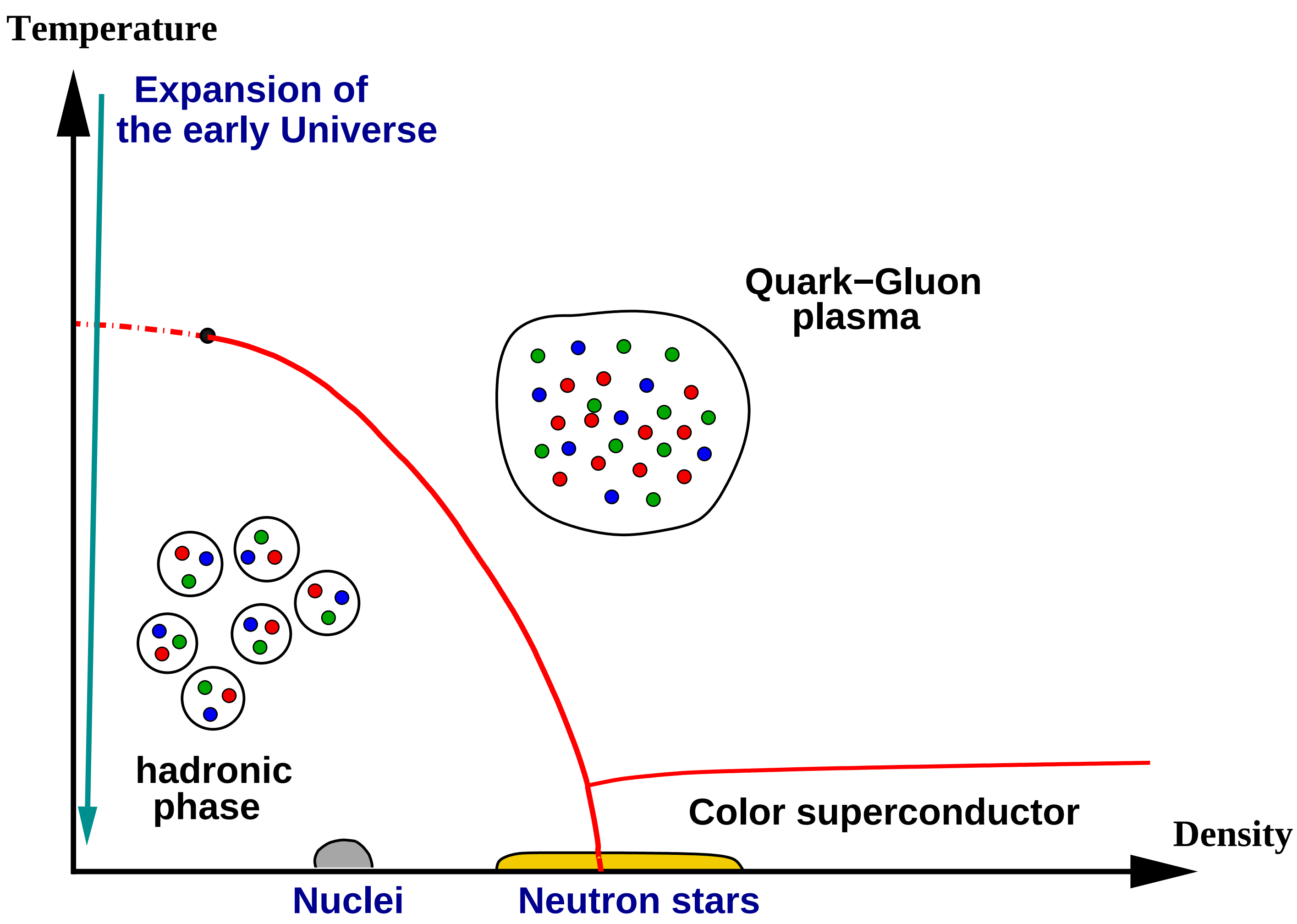


- When the nucleon density increases, they merge, enabling quarks and gluons to hop freely from a nucleon to its neighbors
- This phenomenon extends to the whole volume when the phase transition ends
- Note: if the transition is first order, it goes through a mixed phase containing a mixture of nucleons and plasma

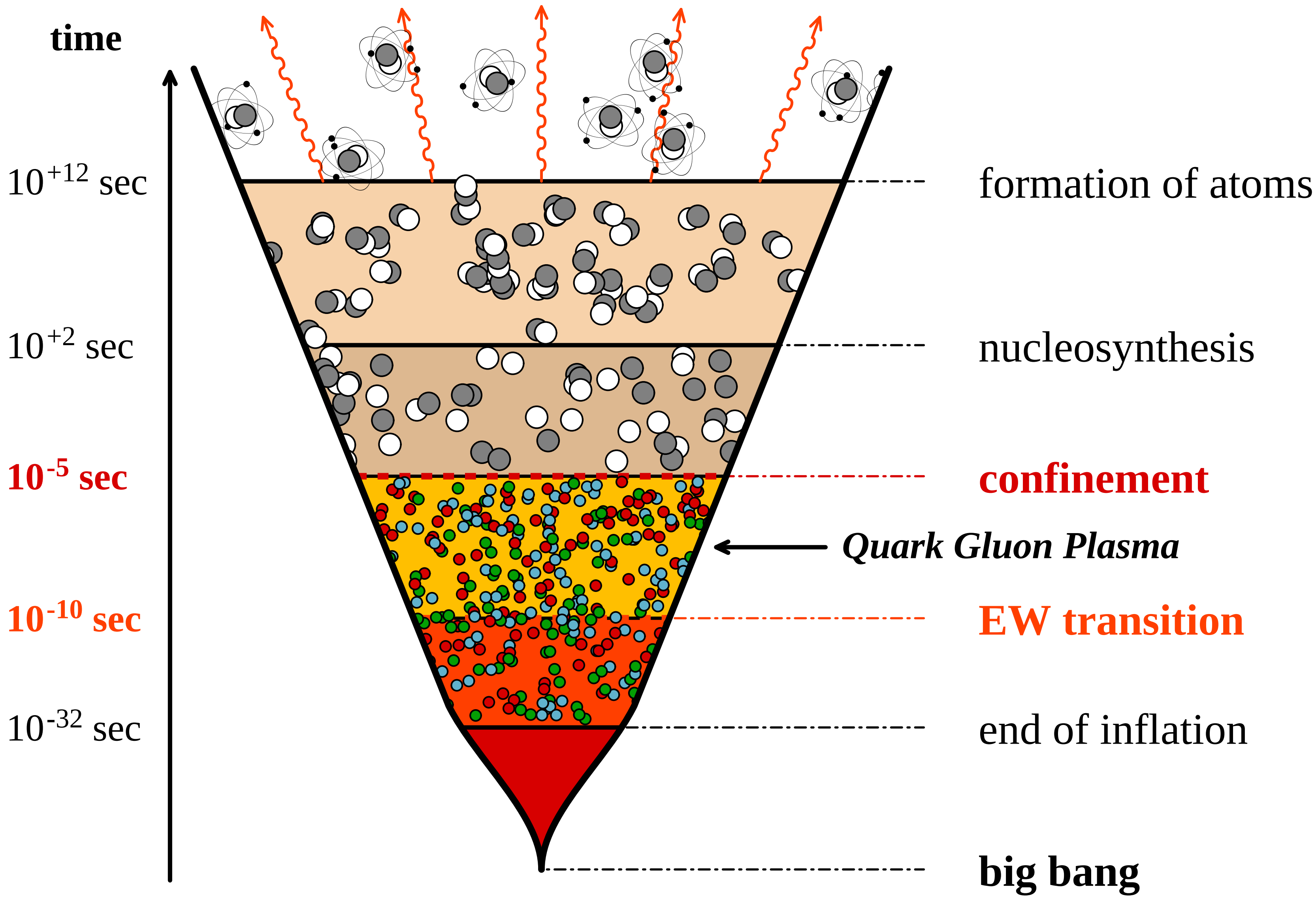
QCD phase diagram



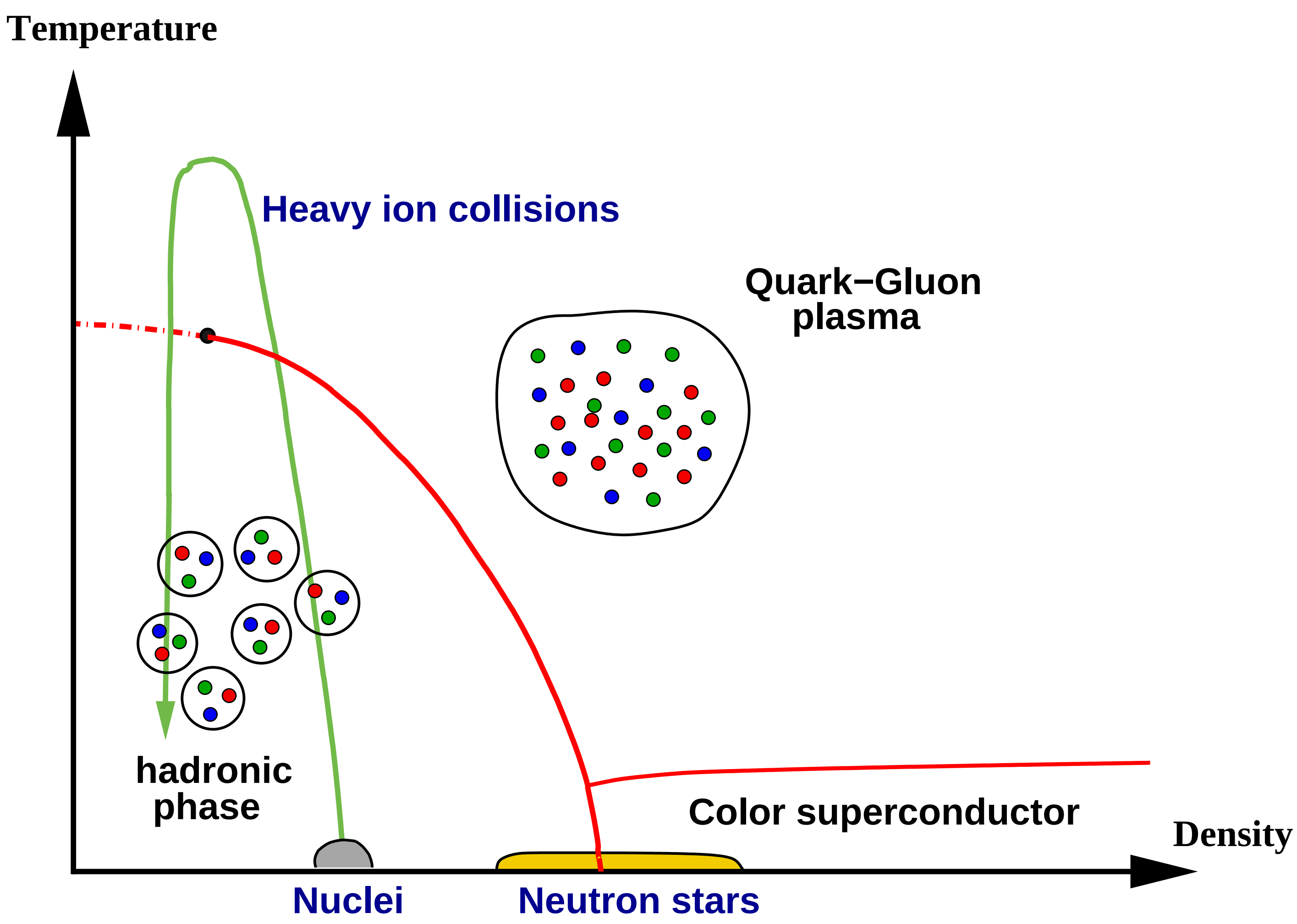
QGP in the early universe



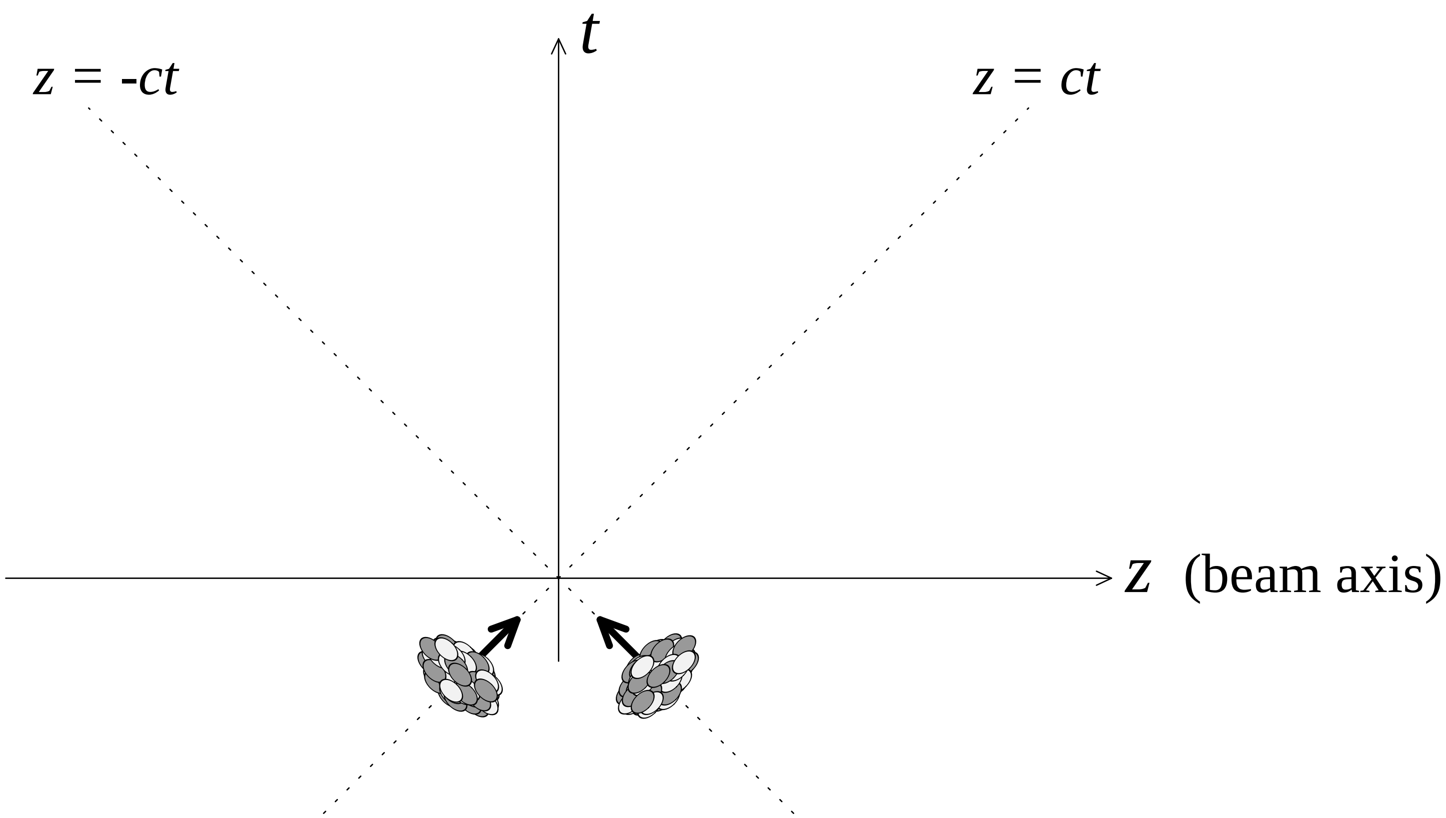
QGP in the early universe



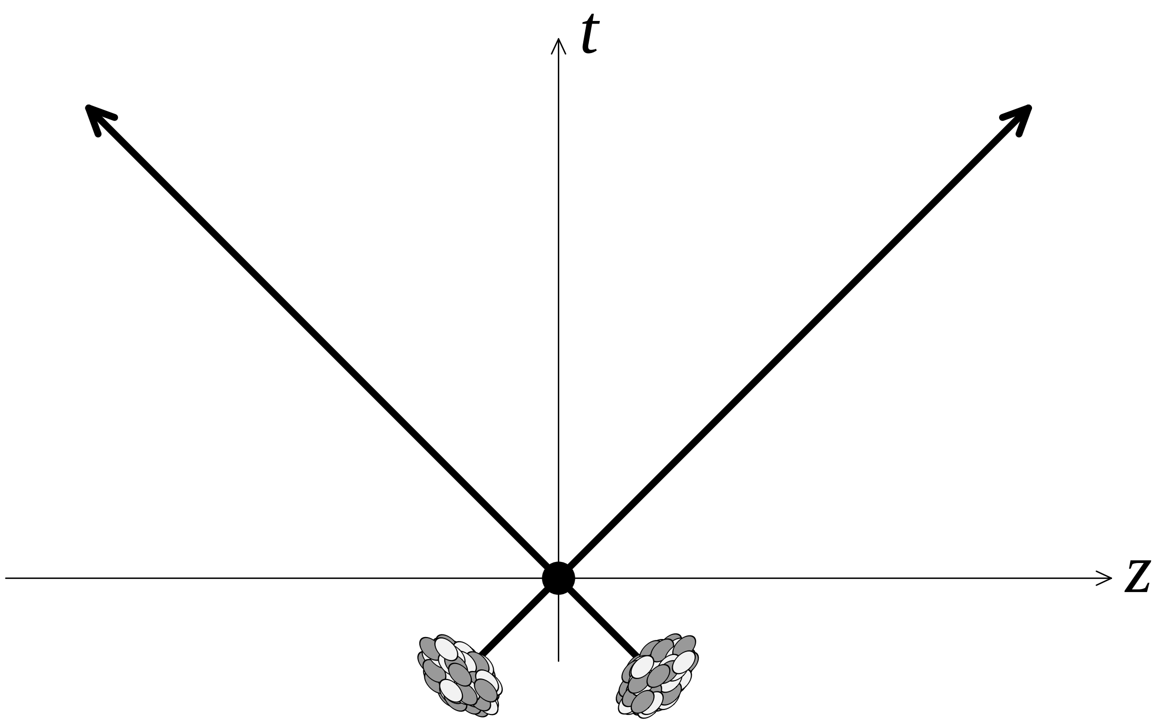
Heavy ion collisions



Stages of a nucleus-nucleus collision

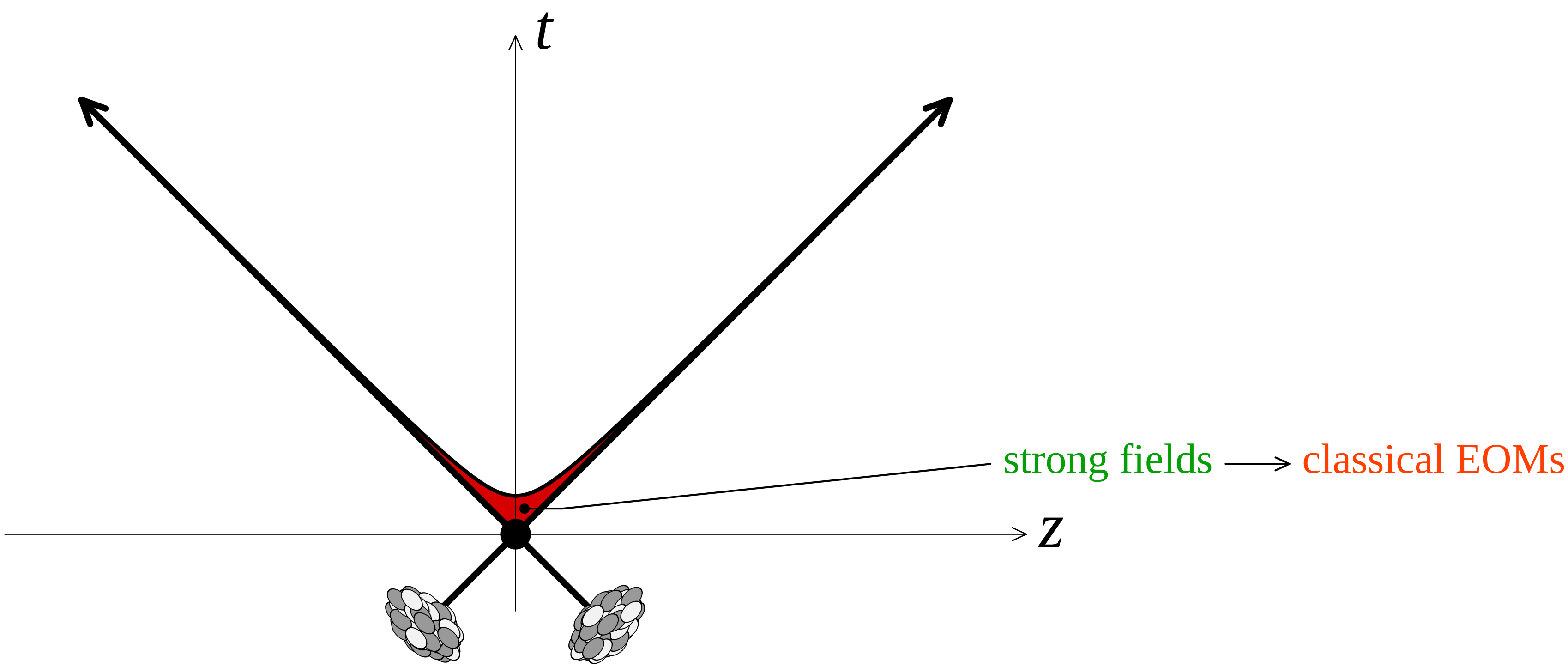


Stages of a nucleus-nucleus collision



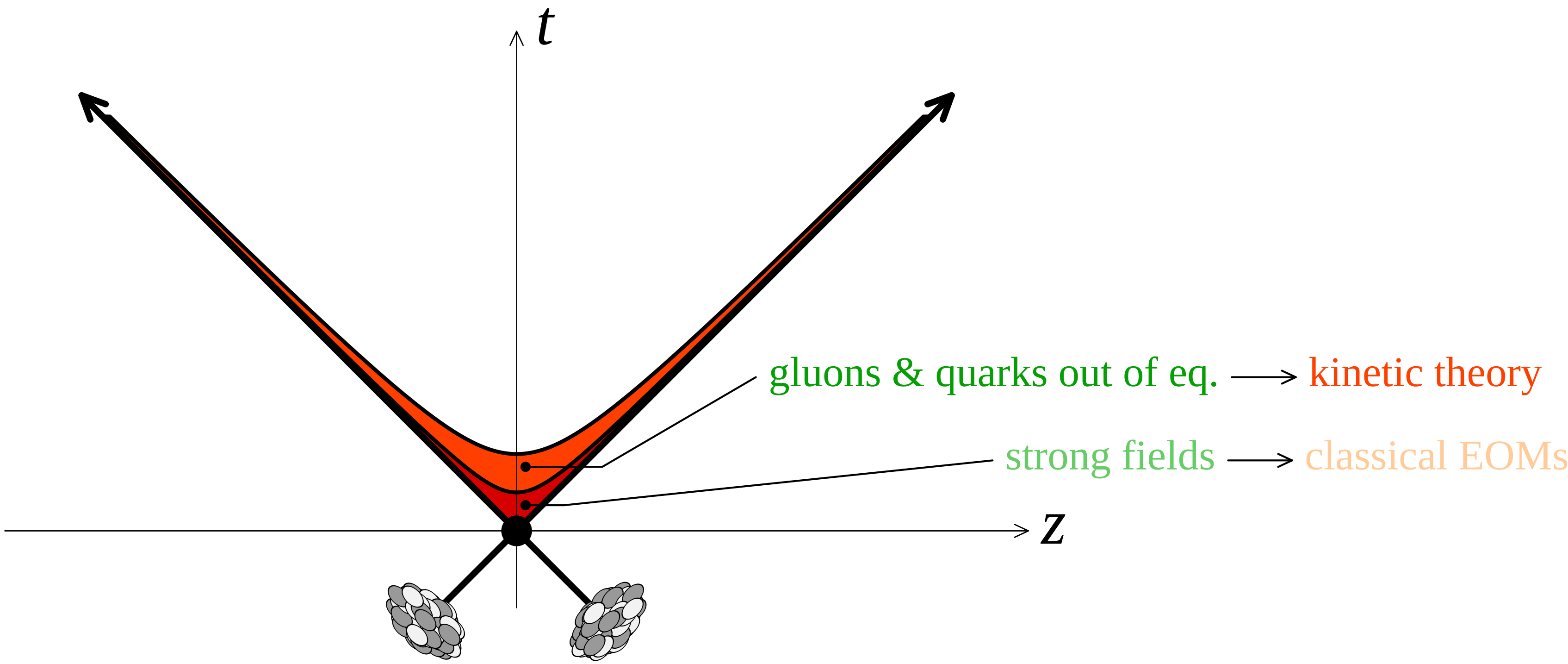
- $\tau \sim 0$ fm/c
- Production of hard particles :
  - jets, direct photons
  - heavy quarks
- calculable with perturbative QCD (leading twist)

Stages of a nucleus-nucleus collision



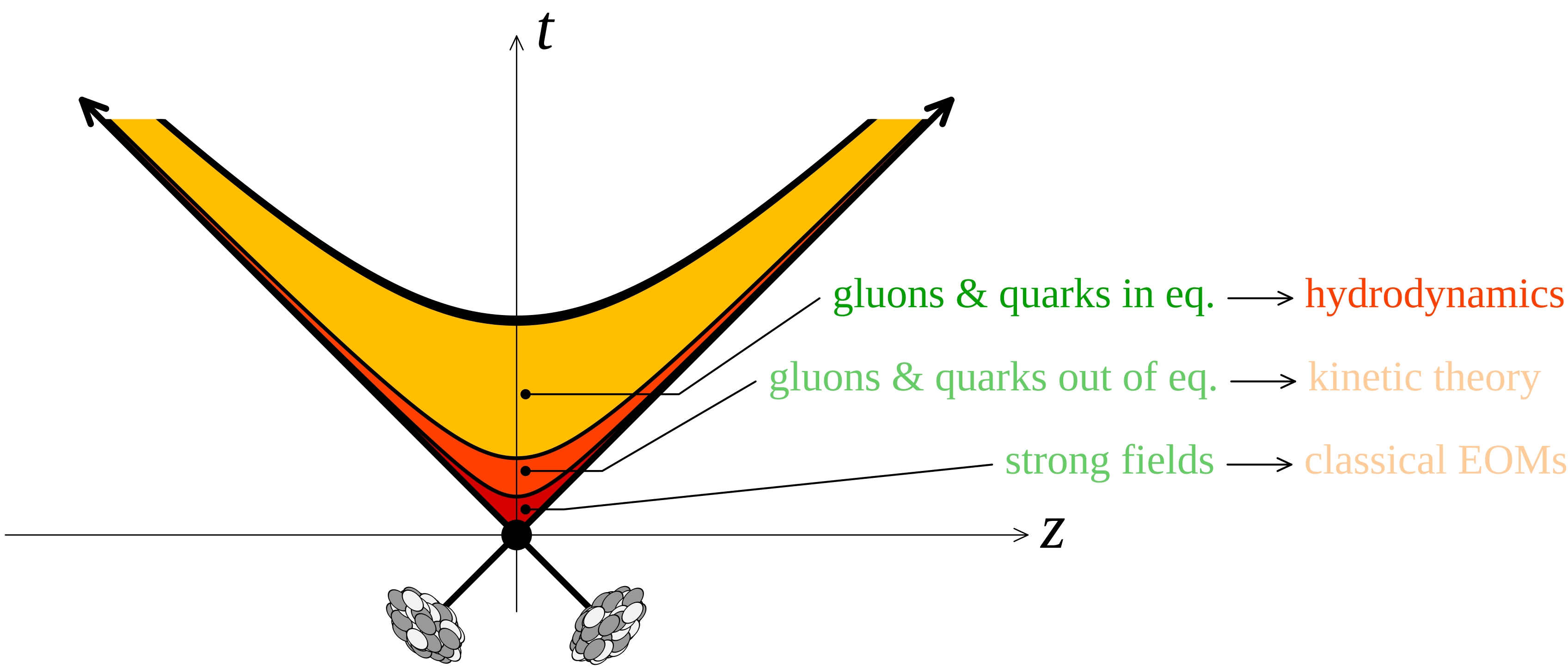
- $\tau \sim 0.2$ fm/c
- Production of semi-hard particles : gluons, light quarks
- relatively small momentum : $p_\perp \lesssim 2$–3 GeV
- make up for most of the multiplicity
- sensitive to the physics of saturation (higher twist)

Stages of a nucleus-nucleus collision



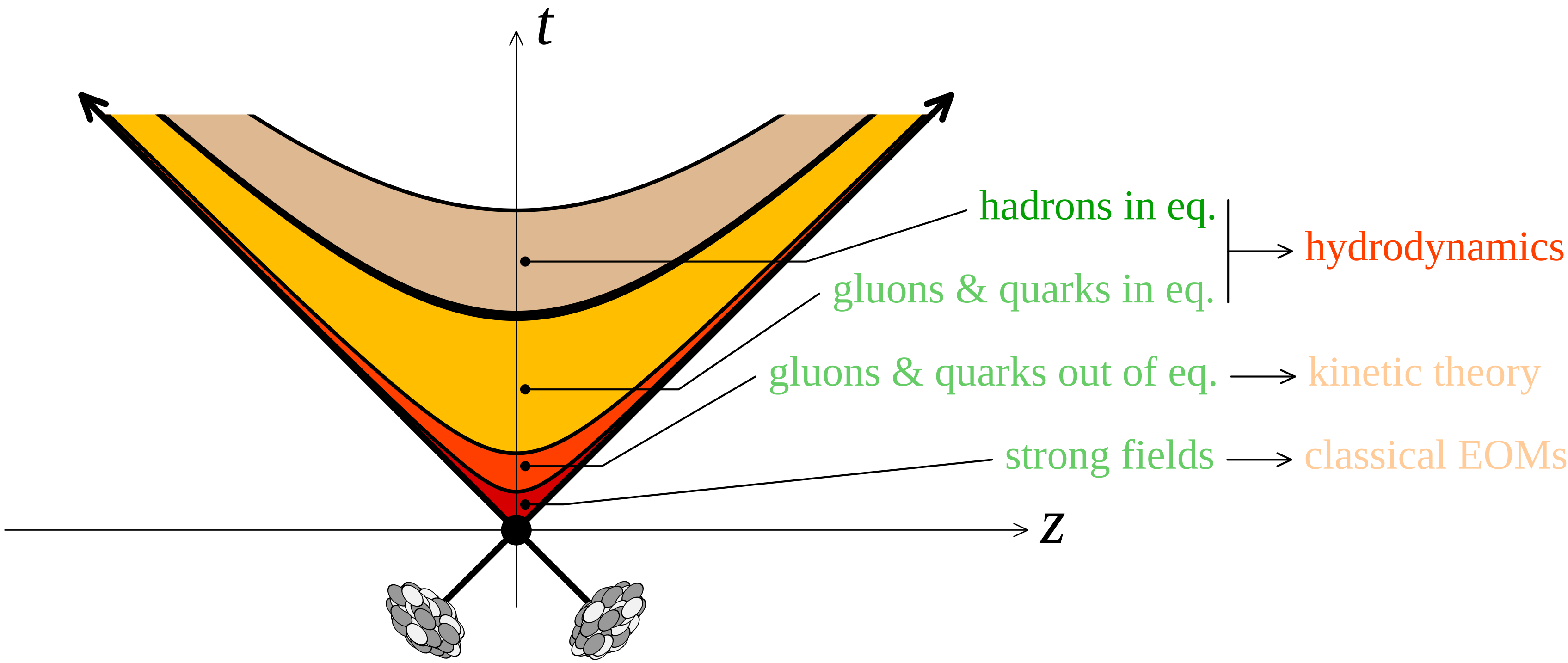
- $\tau \sim 1\text{–}2$ fm/c
- Thermalization
  - experiments suggest a fast thermalization
  - but this is still not well understood from QCD

Stages of a nucleus-nucleus collision



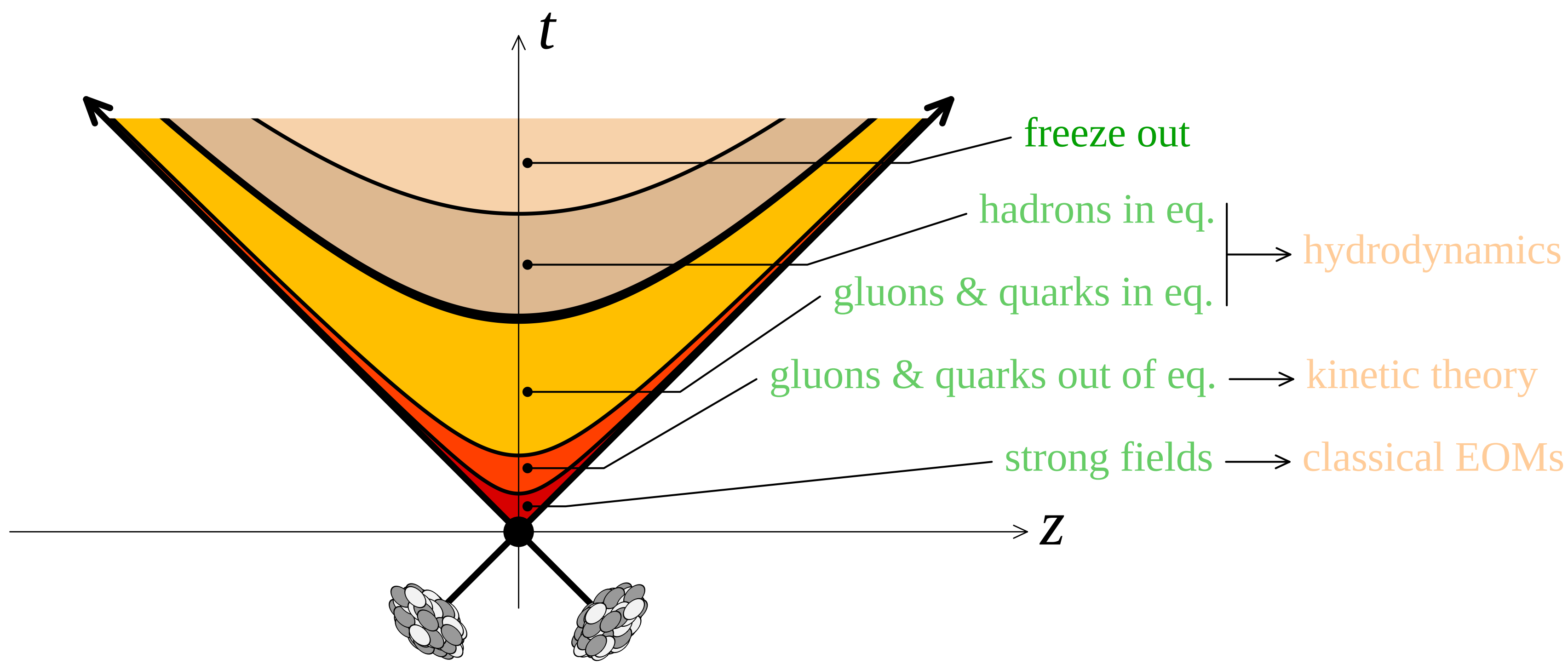
- $2 \leq \tau \lesssim 10$ fm/c
- Quark gluon plasma

Stages of a nucleus-nucleus collision



- $10 \lesssim \tau \lesssim 20$ fm/c
- Hot hadron gas

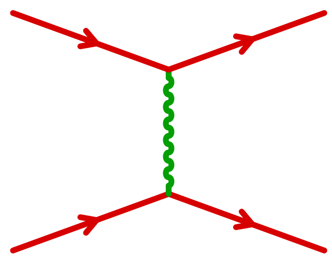
Stages of a nucleus-nucleus collision



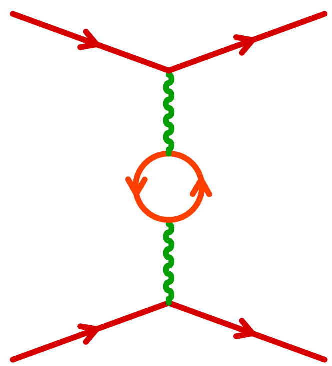
- $\tau \to +\infty$
- Chemical freeze-out :
  density too small to have inelastic interactions
- Kinetic freeze-out :
  no more elastic interactions

Quantum field theory at T=0

- In order to study collision processes involving a small number of particles, one uses Quantum Field Theory at zero temperature
- It can be used to calculate scattering amplitudes, such as $\left\langle \vec{\boldsymbol{p}}_1 \vec{\boldsymbol{p}}_{2\text{out}} \middle| \vec{\boldsymbol{k}}_1 \vec{\boldsymbol{k}}_{2\text{in}} \right\rangle$



- Besides the incoming particles, the only other fields that can be involved in the scattering process are quantum fluctuations of the vacuum



Quantum field theory at T=0

- A QFT is specified by its Lagrangian, that describes the interactions among its elementary constituents

- When the interactions are weak, one can compute observables in perturbation theory, i.e. as a series in the coupling constants

- LSZ reduction formulas : scattering amplitudes are obtained from the Fourier transform of the time-ordered correlators

$$\left\langle \vec{\boldsymbol{p}}_1 \vec{\boldsymbol{p}}_{2\,\mathrm{out}} \middle| \vec{\boldsymbol{k}}_1 \vec{\boldsymbol{k}}_{2\,\mathrm{in}} \right\rangle = \int_{x_1,x_2,y_1,y_2} e^{i(k_1\cdot x_1 + k_2\cdot x_2 - p_1\cdot y_1 - p_2\cdot y_2)}$$
$$\times \square_{x_1} \square_{x_2} \square_{y_1} \square_{y_2} \underbrace{\left\langle 0_{\mathrm{out}} \middle| \mathrm{T}\,\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2) \middle| 0_{\mathrm{in}} \right\rangle}_{\text{can be calculated perturbatively}}$$

Note : $\mathrm{T} =$ time ordering

Quantum field theory at T=0

- The perturbative expansion is a series in g^n. The g dependence can be extracted by writing the Heisenberg fields in terms of fields of the interaction representation :

$$\phi(x) \equiv U(-\infty, x^0)\phi_{\text{in}}(x)U(x^0, -\infty)$$

$$U(t_2, t_1) = \text{T} \exp i \int_{t_1}^{t_2} d^4x \; \underbrace{\mathcal{L}_I(\phi_{\text{in}}(x))}_{\text{interaction term, e.g. } g\phi_{\text{in}}^3(x)}$$

- One gets a series in g by expanding the exponential
- Feynman rules in coordinate space :
  - Vertices : $-ig \int d^4x$
  - Propagators : $G_F^0(x,y) = \langle 0 \big| \text{T}\, \phi_{\text{in}}(x)\phi_{\text{in}}(y) \big| 0 \rangle$

  Note : in momentum space,

$$G_F^0(p) \equiv \int d^4(x-y)\; e^{ip\cdot(x-y)}\; G_F^0(x,y) = \frac{i}{p^2 - m^2 + i\epsilon}$$

Quantum field theory at T=0 - Exercise

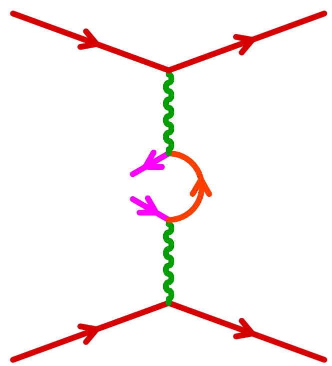
- Properties of $U(t_1, t_2)$:
  - $U(t,t) = \mathbf{1}$
  - $UU^\dagger = \mathbf{1}$
  - $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$
  - $U^{-1}(t_1,t_2) = U(t_2,t_1)$
- $\phi(x)$ and $\phi_{\rm in}(x)$ coincide when $x^0 \to -\infty$
- If $\phi(x)$ obeys the equation of motion with interactions, then $\phi_{\rm in}(x)$ is a free field :

$$(\square + m^2)\phi(x) - \frac{\partial \mathcal{L}_I(\phi(x))}{\partial \phi(x)} = U(-\infty, x^0)\Big[(\square + m^2)\phi_{\rm in}(x)\Big]U(x^0, -\infty)$$

Perturbation theory at finite T

Introduction

- Contrary to $T = 0$, particles from the thermal environment can participate in reactions :



- This phenomenon gives their temperature dependence to correlators
- The time-ordered correlators are now defined as

$$G(x_1, \cdots, x_n) \equiv \frac{\mathrm{Tr}\,(e^{-\beta H}\,\mathrm{T}\,\phi(x_1)\cdots\phi(x_n))}{\mathrm{Tr}\,(e^{-\beta H})}$$

(with $\beta \equiv 1/T$)

Introduction

- Contrary to $T = 0$, scattering amplitudes are not very interesting objects, because there are no asymptotically free states inside a sample of matter at non-zero temperature
- Interesting physical quantities :
  - Equation of state
  - Screening length
  - Quasi-particle spectral functions
  - Transport coefficients
- All these quantities can be obtained from the thermal correlators defined on the previous slide

T=0 limit

- The thermal correlators can be rewritten in terms of eigenstates of the Hamiltonian :

$$G(x_1,\cdots,x_n)=\frac{1}{\mathrm{Tr}\,(e^{-\beta H})}\sum_{\text{states }n} e^{-\beta E_n}\,\big\langle n\big|\mathrm{T}\,\phi(x_1)\cdots\phi(x_n)\big|n\big\rangle$$

- When $T\to 0$ (i.e. $\beta\to+\infty$), only the vacuum state $\big|0\big\rangle$ survives since it has the lowest energy. Thus

$$\lim_{T\to 0} G(x_1,\cdots,x_n)=\big\langle 0\big|\mathrm{T}\,\phi(x_1)\cdots\phi(x_n)\big|0\big\rangle$$

- Therefore, our definition of the thermal correlators is a natural extension of the definition used at zero temperature

Perturbative expansion

- In order to perform the perturbative expansion at finite T, we must identify all the sources of g dependence

- One of them is the interactions inside the field operator $\phi(x)$. This is identical to $T = 0$:

$$\phi(x) = U(-\infty, x^0)\phi_{\rm in}(x)U(x^0, -\infty)$$

$$U(t_2, t_1) \equiv {\rm T}\, \exp i \int_{t_1}^{t_2} d^4x\, \mathcal{L}_I(\phi_{\rm in}(x))$$

- At $T > 0$, another source of g-dependence is the density operator $\exp(-\beta H)$, since $H = H_0 + H_I$. One can prove

$$e^{-\beta H} = e^{-\beta H_0} \underbrace{{\rm T}\, \exp i \int_{-\infty}^{-\infty - i\beta} d^4x\, \mathcal{L}_I(\phi_{\rm in}(x))}_{U(-\infty - i\beta, -\infty)}$$

Perturbative expansion - Exercise

- Proof of

$$\underbrace{\exp(-\beta H)}_{A(\beta)} = \underbrace{\exp(-\beta H_0)\, U(-\infty - i\beta, -\infty)}_{B(\beta)}$$

- $B(\beta)$ can be rewritten as

$$B(\beta) = e^{-\beta H_0}\ \mathrm{T}\ \exp -i \int_{-\infty}^{-\infty - i\beta} dt\ H^I_{\rm in}(t)$$

with $\quad H^I_{\rm in}(t) = \exp(iH_0(t+\infty))H_{_I}\exp(-iH_0(t+\infty))$

- $A(\beta)$ and $B(\beta)$ are identical at $\beta = 0$ (trivial)
- Their first derivatives are identical at any β

$$A'(\beta) = -H\ A(\beta)$$

$$B'(\beta) = -H_0 B(\beta) - \underbrace{e^{-\beta H_0} H^I_{\rm in}(-\infty - i\beta)}_{H_{_I}\ e^{-\beta H_0}}\ \mathrm{T}\ \exp -i \int_{-\infty}^{-\infty - i\beta} dt\ H^I_{\rm in}(t)$$

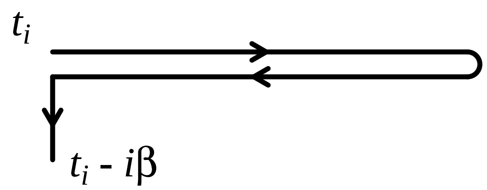
Perturbative expansion

- From the previous formulas, we can write :

$$e^{-\beta H}\,\mathrm{T}\,\phi(x_1)\cdots\phi(x_n) =$$

$$= e^{-\beta H_0}\,\mathrm{P}\,\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\;\exp i\int_{\mathcal{C}} d^4x\,\mathcal{L}_{_I}(\phi_{\rm in}(x))$$

$$\mathcal{C} = [t_i, +\infty]\cup[+\infty, t_i]\cup[t_i, t_i - i\beta] \quad :$$



(it is instructive to let the path start at an arbitrary $t_{\rm i}$ instead of $-\infty$)

- The symbol P denotes path ordering. The contour $\mathcal{C}$ is oriented, and the closest operator to the end of the path should be on the left of the product
- On the upper branch of the contour, the path ordering is equivalent to the usual time-ordering. The times $x_1^0, \cdots, x_n^0$ are on the upper branch of the path

Perturbative expansion

- From the previous formula, one sees that – in coordinate space – perturbation theory at finite T is very similar to perturbation theory at $T = 0$. The only difference is that the time integrations at the vertices run over the contour $\mathcal{C}$

- Feynman rules :
  - Vertices : $-ig \int_{\mathcal{C}} d^4x$
  - Propagator :

$$G^0(x, y) = \frac{\mathrm{Tr}\,(e^{-\beta H_0}\,\mathrm{P}\,\phi_{\mathrm{in}}(x)\phi_{\mathrm{in}}(y))}{\mathrm{Tr}\,(e^{-\beta H_0})}$$

- At the moment, it seems that the result may depend on the arbitrary initial time t_i we have just introduced. However, we will prove shortly that nothing depends on t_i

Perturbative expansion

- The free thermal propagator is obtained from the Fourier decomposition of the free field $\phi_{\rm in}(x)$:

$$\phi_{\rm in}(x) = \int \frac{d^3\vec{\boldsymbol p}}{(2\pi)^3 2E_{\boldsymbol p}} \left[a_{\rm in}(\vec{\boldsymbol p})\, e^{-ip\cdot x} + a^\dagger_{\rm in}(\vec{\boldsymbol p})\, e^{+ip\cdot x}\right]$$

- Exercise : prove the following relations

$$\begin{aligned}
\left[e^{-\beta H_0}, a_{\rm in}(\vec{\boldsymbol p})\right] &= e^{-\beta H_0}(1 - e^{-\beta E_{\boldsymbol p}})a_{\rm in}(\vec{\boldsymbol p}) \\
{\rm Tr}\,(e^{-\beta H_0}\, a_{\rm in}(\vec{\boldsymbol p})) &= 0 \\
{\rm Tr}\,(e^{-\beta H_0}\, a^\dagger_{\rm in}(\vec{\boldsymbol p}) a_{\rm in}(\vec{\boldsymbol p}')) &= (2\pi)^3\, 2E_{\boldsymbol p}\, n_B(E_{\boldsymbol p})\, \delta(\vec{\boldsymbol p} - \vec{\boldsymbol p}') \\
& \text{with } \; n_B(E) = \frac{1}{e^{\beta E} - 1}
\end{aligned}$$

- From there, it is easy to obtain :

$$\begin{aligned}
G^0(x,y) = \int \frac{d^3\vec{\boldsymbol p}}{(2\pi)^3 2E_{\boldsymbol p}} \Big[&(\theta_c(x^0 - y^0) + n_B(E_{\boldsymbol p}))\, e^{-ip\cdot(x-y)} \\
&+(\theta_c(y^0 - x^0) + n_B(E_{\boldsymbol p}))\, e^{+ip\cdot(x-y)}\Big]
\end{aligned}$$

KMS symmetry

- The density operator $\exp(-\beta H)$ can be seen as an evolution operator for an imaginary time shift :

$$e^{-\beta H}\phi(x^0-i\beta,\vec{\boldsymbol{x}})e^{\beta H}=\phi(x^0,\vec{\boldsymbol{x}})$$

- Consider the correlator $\mathcal{G}\equiv\mathrm{Tr}\,(e^{-\beta H}\,\mathrm{T}\,\phi(t_i,\vec{\boldsymbol{x}})\cdots)$
- t_i is the “smallest” time on $\mathcal{C}$: $\mathcal{G}=\mathrm{Tr}\,(e^{-\beta H}\,(\mathrm{T}\,\cdots)\phi(t_i,\vec{\boldsymbol{x}}))$
- Use the cyclicity of the trace, and the first relation : $\mathcal{G}=\mathrm{Tr}\,(e^{-\beta H}\,\phi(t_i-i\beta,\vec{\boldsymbol{x}})\,(\mathrm{T}\,\cdots))$
- $t_i-i\beta$ is the “largest” time : $\mathcal{G}=\mathrm{Tr}\,(e^{-\beta H}\,\mathrm{T}\,\phi(t_i-i\beta,\vec{\boldsymbol{x}})\cdots)$

  ▷ $\mathcal{G}$ has identical values at $x^0=t_i$ and $x^0=t_i-i\beta$ (Kubo-Martin-Schwinger symmetry)

- This property is true for equilibrium thermal correlators with any number of points, at any order of perturbation theory

Path deformations

- The free propagator does not depend explicitly on t_i
- It verifies the KMS symmetry
- Any graph contributing to a correlator $\mathcal{G}(x_1, \cdots, x_n)$ has a contribution of the form :

$$\mathcal{G} = \int_{\mathcal{C}} dy_1^0 \cdots dy_p^0 \; F(x_1, \cdots, x_n; y_1^0, \cdots, y_p^0)$$

  where the function F is (piece-wise) holomorphic and takes identical values at $y_i^0 = t_i$ and $y_i^0 = t_i - i\beta$
- The derivative of $\mathcal{G}$ with respect to t_i involves the difference of F at the endpoints of the contour, and is therefore zero

  Interpretation : t_i is the time at which the system is put in thermal equilibrium. By definition of thermal equilibrium, no measurement made afterwards can tell the value of t_i
- More general deformations of the contour $\mathcal{C}$ also leave the value of $\mathcal{G}$ unchanged

Conserved charges

- A field ϕ is charged (with charge q) under the operator Q if it obeys a relation of the form $[Q, \phi_{\rm in}(x)] = -q\phi_{\rm in}(x)$

  Note : Q is Hermitian and q is real. If the field ϕ is Hermitian, then q can only be zero. The simplest charged fields are complex scalars :

$$\phi_{\rm in}(x) = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[a_{\rm in}(\vec{\boldsymbol{p}})\, e^{-ip\cdot x} + b^\dagger_{\rm in}(\vec{\boldsymbol{p}})\, e^{+ip\cdot x}\right]$$

- When the charge Q is conserved, thermal averages should be calculated with the density operator $\exp(-\beta(H + \mu Q))$ where μ is the corresponding chemical potential
- For an external line that carry a conserved charge q, the KMS symmetry is $\mathcal{G}(\cdots t_i \cdots) = e^{\beta\mu q}\, \mathcal{G}(\cdots t_i - i\beta \cdots)$
- Since Q is conserved, it cannot depend on the coupling constants of the theory $\triangleright$ the contour $\mathcal{C}$ is not affected by chemical potentials

Conserved charges

- Exercise : derive the relations :

$$\mathrm{Tr}\,\big(e^{\beta(H_0+\mu Q)}\,a^\dagger_{\rm in}(\vec{\boldsymbol{p}})a_{\rm in}(\vec{\boldsymbol{p}}')\big) = (2\pi)^3\,2E_{\boldsymbol{p}}\,\frac{1}{e^{\beta(E_{\boldsymbol{p}}-\mu q)}-1}\delta(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')$$

$$\mathrm{Tr}\,\big(e^{\beta(H_0+\mu Q)}\,b^\dagger_{\rm in}(\vec{\boldsymbol{p}})b_{\rm in}(\vec{\boldsymbol{p}}')\big) = (2\pi)^3\,2E_{\boldsymbol{p}}\,\frac{1}{e^{\beta(E_{\boldsymbol{p}}+\mu q)}-1}\delta(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')$$

(all the other averages are zero)

- The free propagator now depends on μ :

$$G^0(x,y)=\int\frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\Big[\Big(\theta_c(x^0-y^0)+\frac{1}{e^{\beta(E_{\boldsymbol{p}}-\mu q)}-1}\Big)\,e^{-ip\cdot(x-y)} +\Big(\theta_c(y^0-x^0)+\frac{1}{e^{\beta(E_{\boldsymbol{p}}+\mu q)}-1}\Big)\,e^{+ip\cdot(x-y)}\Big]$$

Fermions

- Consider a spin 1/2 fermion :

$$\psi_{\rm in}(x) = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left[b^\lambda_{\rm in}(\vec{\boldsymbol{p}}) u^\lambda(\vec{\boldsymbol{p}})\, e^{-ip\cdot x} + d^{\lambda\dagger}_{\rm in}(\vec{\boldsymbol{p}}) v^\lambda(\vec{\boldsymbol{p}})\, e^{+ip\cdot x} \right]$$

  with $u^\lambda, v^\lambda (\lambda = 1,2)$ independent solutions of the free Dirac equations $(\not{p} - m) u^\lambda(\vec{\boldsymbol{p}}) = 0$, $(\not{p} + m) v^\lambda(\vec{\boldsymbol{p}}) = 0$

- For consistency, fermions must be quantized with anti-commutation relations ▷ Fermi-Dirac distributions

- Free propagator :

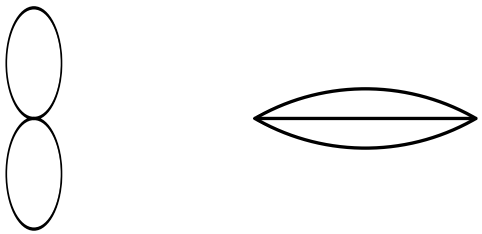
$$S^0(x,y) = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \Big[(E_{\boldsymbol{p}}\gamma^0 - \vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\gamma}} + m)\Big(\theta_c(x^0 - y^0) - \frac{1}{e^{\beta(E_{\boldsymbol{p}} - \mu q)} + 1}\Big)\, e^{-ip\cdot(x-y)}$$
$$+ (-E_{\boldsymbol{p}}\gamma^0 - \vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{\gamma}} + m)\Big(\theta_c(y^0 - x^0) - \frac{1}{e^{\beta(E_{\boldsymbol{p}} + \mu q)} + 1}\Big)\, e^{+ip\cdot(x-y)} \Big]$$

- KMS for fermions : $\mathcal{G}(\cdots t_i \cdots) = -e^{\beta\mu q}\, \mathcal{G}(\cdots t_i - i\beta \cdots)$

Matsubara formalism

Thermodynamical quantities

- Vacuum diagrams are diagrams without any external legs



- The sum of all the vacuum diagrams provides the partition function

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right)$$

- From Z, one can obtain other thermodynamical quantities :

$$E = -\frac{\partial Z}{\partial \beta}$$

$$S = \beta E + \ln(Z)$$

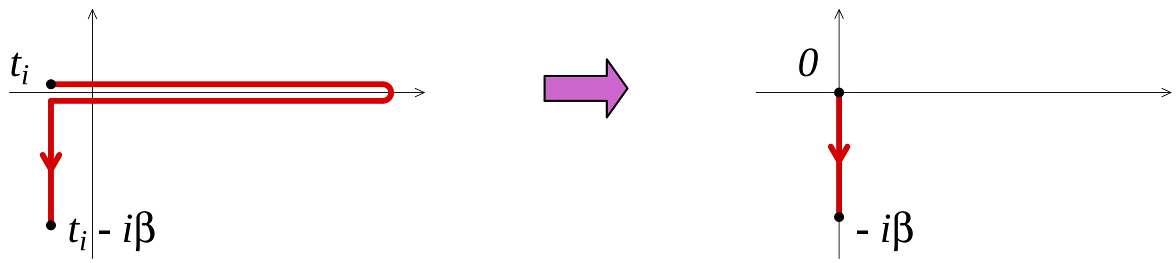
$$F = E - TS = -\frac{1}{\beta}\ln(Z)$$

Thermodynamical quantities

- Vacuum diagrams are pure numbers (they do not depend on any external coordinate)

  ▷ For this reason, we are not tied to using a contour $\mathcal{C}$ that contains the real axis

- We can deform the contour to make it simpler



- If we denote $x^0 = -i\tau$, the variable τ is real and spans the range $[0, \beta]$. The Feynman rules obtained with this choice of the contour $\mathcal{C}$ are known as “imaginary time formalism”
- Note : one could in principle use them to calculate non-vacuum diagrams, but beyond 2-point functions, the analytic continuation to real time is complicated

Matsubara frequencies

- The propagator – and more generally the integrand for any diagram – is β-periodic in the imaginary time τ
- Therefore, one can go to Fourier space by decomposing the time dependence in Fourier series and by doing an ordinary Fourier transform in space :

$$G^0(\tau_x, \vec{\boldsymbol{x}}, \tau_y, \vec{\boldsymbol{y}}) = T \sum_{n=-\infty}^{+\infty} \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, e^{i\omega_n(\tau_x - \tau_y)} e^{-i\vec{\boldsymbol{p}}\cdot(\vec{\boldsymbol{x}}-\vec{\boldsymbol{y}})}\, G^0(\omega_n, \vec{\boldsymbol{p}})$$

  with $\omega_n \equiv 2\pi n T$. Note : for fermions, $\omega_n = 2\pi(n+\frac{1}{2})T$
  If the line carries the conserved charge q, one must shift $\omega_n \to \omega_n - i\mu q$

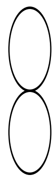
- Exercise : an explicit calculation gives :

$$G^0(\omega_n, \vec{\boldsymbol{p}}) = \frac{1}{\omega_n^2 + \vec{\boldsymbol{p}}^2 + m^2}$$

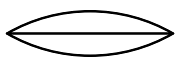
Matsubara formalism

- Feynman rules :
  - Propagators : $G^0(\omega_n, \vec{\boldsymbol{p}}) = 1/(\omega_n^2 + \vec{\boldsymbol{p}}^2 + m^2)$
  - Vertices : g + conservation of ω_n and $\vec{\boldsymbol{p}}$
  - Loops : $T \sum_n \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}$

- Examples (written here in the massless case) :



$$= \lambda T^2 \sum_{m,n} \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3} \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3} \frac{1}{(\omega_m^2 + \vec{\boldsymbol{p}}^2)(\omega_n^2 + \vec{\boldsymbol{q}}^2)}$$



$$= g^2 T^2 \sum_{m,n} \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3} \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3} \frac{1}{(\omega_m^2 + \vec{\boldsymbol{p}}^2)(\omega_n^2 + \vec{\boldsymbol{q}}^2)(\omega_{m+n}^2 + (\vec{\boldsymbol{p}} + \vec{\boldsymbol{q}})^2)}$$

Tips and tricks

- The calculation of the discrete sums can be a bit tedious...

- Method 1 : replace each propagator by

$$G^0(\omega_n,\vec{\boldsymbol{p}}) = \frac{1}{2E_{\boldsymbol{p}}}\int_0^\beta d\tau\; e^{-i\omega_n\tau}\Big[(1+n_B(E_{\boldsymbol{p}}))\,e^{-E_{\boldsymbol{p}}\tau}+n_B(E_{\boldsymbol{p}})\,e^{E_{\boldsymbol{p}}\tau}\Big]$$

- One should combine this trick with the formula

$$\sum_n e^{i\omega_n\tau} = \beta\sum_n \delta(\tau-n\beta)$$

which turns all the time dependence into combinations of delta functions. Then, all the time integrations are trivial

Tips and tricks

- Method 2 : use a function $P(\omega)$ that has simple poles of residue 1 at each $i\omega_n$. Then, write the discrete sums as

$$\sum_n f(i\omega_n) = \oint_\gamma \frac{dz}{2i\pi}\, f(z)P(z)$$

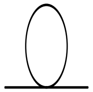
where γ is a path made of a small circle around each pole

Note : for instance $P(z) = \dfrac{\beta}{e^{\beta z} - 1}$

- If the function $f(z)$ has no pole on the imaginary axis, deform the contour γ in two lines along the imaginary axis
- Deform the contour to bring it along the real energy axis (beware of the poles lying away from the real axis!)

Tips and tricks

- Exercise. Tadpole in a $\lambda\phi^4$ theory :



$$
\begin{aligned}
&= \frac{\lambda T}{2}\sum_n \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{\omega_n^2+\vec{\boldsymbol{p}}^2} \\
&= \frac{\lambda T}{2}\sum_n \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\int_0^\beta d\tau\; e^{-i\omega_n\tau}\Big[(1+n_B(E_{\boldsymbol{p}}))e^{-E_{\boldsymbol{p}}\tau}+n_B(E_{\boldsymbol{p}})e^{E_{\boldsymbol{p}}\tau}\Big] \\
&= \frac{\lambda}{2}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\int_0^\beta d\tau \sum_n \delta(\tau-n\beta)\Big[(1+n_B(E_{\boldsymbol{p}}))e^{-E_{\boldsymbol{p}}\tau}+n_B(E_{\boldsymbol{p}})e^{E_{\boldsymbol{p}}\tau}\Big] \\
&= \frac{\lambda}{2}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\left[1+2n_B(E_{\boldsymbol{p}})\right]
\end{aligned}
$$

(the remaining integral is “elementary”)

- Note : in the last formula, the 1 gives the usual ultraviolet divergence, and the n_B gives a finite contribution that vanishes if $T\to 0$ $\triangleright$ this term is a medium effect

Tips and tricks

- With massless bosons, one frequently encounters integrals of the form :

$$I_{n,p} \equiv \int_0^\infty dx \, \frac{x^n}{(e^x - 1)^p}$$

- Exercise : prove the following formula :

$$I_{n,p} = \frac{n!}{(p-1)!} \sum_{i=0}^{p-1} \alpha_{p-1,i} \, \zeta(n+1-i)$$

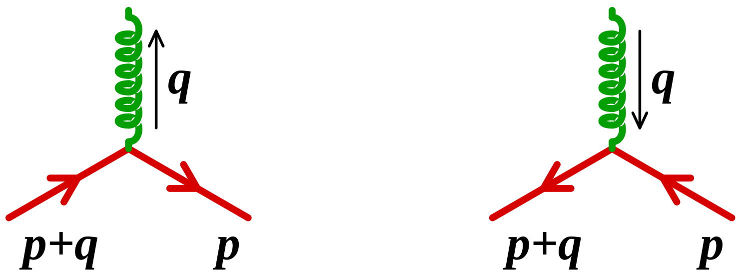
$$\begin{cases} (x-1)(x-2)\cdots(x-p+1) \equiv \sum_{i=0}^{p-1} \alpha_{p-1,i} \, x^i \\ \zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} \end{cases}$$

Tips and tricks

- The equilibrium thermal distributions obey some useful relations. The simplest one is :

$$(1+n_B(q^0))n_F(p^0+q^0)(1-n_F(p^0)) = n_B(q^0)n_F(p^0)(1-n_F(p^0+q^0))$$

- Notes :
  - These relations are closely related to KMS, and are valid only in equilibrium
  - They are the mathematical translation of "detailed balance" :



Lecture II : Collective effects in the QGP

- Length scales in the QGP
- Long distance effective theories
- Collective phenomena in the QGP
- Anisotropic plasmas

Lecture III : Out of equilibrium systems

- Schwinger-Keldysh formalism, Long time pathologies
- From fields to kinetic theory
- Collisionless kinetic equations
- Boltzmann equation
- Transport coefficients