Quark-Gluon Plasma and Heavy Ion Collisions

I – Physics of the QGP, Field theory at finite $T$

François Gelis

CEA/Saclay and CERN
The PDF files of these lectures can be downloaded at:

http://www-spht.cea.fr/Images/Pisp/fgelis

(from there, go to "Talks")
General outline

- **Lecture I**: Physics of the QGP, Field theory at finite T

- **Lecture II**: Kinetic theory, Hydrodynamics, Phenomenology
Lecture I

- QCD reminder
- Confinement and deconfinement in QCD
- Heavy Ion Collisions
- Length scales in the QGP
- Field Theory at finite temperature
- Collective phenomena
QCD reminder
Quarks and gluons

- Electromagnetic interaction: **Quantum electrodynamics**
  - Matter: electron, interaction carrier: photon
  - Interaction:

    \[ \sim e \]  
    (electric charge of the electron)

- Strong interaction: **Quantum chromo-dynamics**
  - Matter: quarks, interaction carriers: gluons
  - Interactions:

    \[ \sim g (t^a)_{ij} \]
    \[ \sim g (T^a)_{bc} \]

  - \( i, j \): colors of the quarks (3 possible values)
  - \( a, b, c \): colors of the gluons (8 possible values)
  - \( (t^a)_{ij} \): 3 \times 3 matrix, \( (T^a)_{bc} \): 8 \times 8 matrix
QCD Lagrangian

- QCD Lagrangian:

\[ \mathcal{L} = -\frac{1}{2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) + \bar{\psi}(i\slashed{D} - m)\psi \]

- The gauge field \( A^\mu \) belongs to \( SU(3) \)
- \( D^\mu \equiv \partial^\mu - ig A^\mu \) is the covariant derivative
- \( F^{\mu\nu} \equiv i[D^\mu, D^\nu]/g = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu] \)

- The Lagrangian is invariant under gauge transformations:

\[ A^\mu (x) \rightarrow \Omega(x) A^\mu (x) \Omega^{-1}(x) + \frac{i}{g} \Omega(x) \partial^\mu \Omega^{-1}(x) \]

\[ \psi(x) \rightarrow \Omega(x) \psi(x) \]

where \( \Omega(x) \in SU(3) \)
- Note: the field strength is not invariant but transforms as:

\[ F^{\mu\nu}(x) \rightarrow \Omega(x) F^{\mu\nu}(x) \Omega^{-1}(x) \]
Quark confinement

- The quark potential increases linearly with distance
- Quarks are confined into color singlet hadrons
Asymptotic freedom

- Running coupling: \( \alpha_s = g^2 / 4\pi \)

\[
\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f) \log(1/r \Lambda_{QCD})}
\]

- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)
Asymptotic freedom

- Running coupling: \( \alpha_s = g^2 / 4\pi \)

\[
\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f) \log(1/r\Lambda_{QCD})}
\]

- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)
- But gluonic vacuum fluctuations produce an anti-screening (because of the non-abelian nature of their interactions)
- As long as \( N_f < 11N_c / 2 = 16.5 \), the gluons win...
Asymptotic freedom

- The coupling constant is small at short distances
- At high density, a hadron gas may undergo deconfinement
  - quark gluon plasma
Deconfinement transition
Deconfinement

- Fast increase of the pressure:
  - at $T \sim 270$ MeV, if there are only gluons
  - at $T \sim 150–170$ MeV, depending on the number of light quarks
Deconfinement transition

Individual nucleons  | Quark gluon plasma

When the nucleon density increases, they merge, enabling quarks and gluons to hop freely from a nucleon to its neighbors.

This phenomenon extends to the whole volume when the phase transition ends.

Note: if the transition is first order, it goes through a mixed phase containing a mixture of nucleons and plasma.
QCD phase diagram

- Temperature
- Density
- Hadronic phase
- Quark-Gluon plasma
- Color superconductor
- Nuclei
- Neutron stars
- Quark-Gluon hadronic phase
- Color superconductor
The QGP in the early universe

- Expansion of the early Universe
- Temperature
- Quark–Gluon plasma
- Color superconductor
- Hadronic phase
- Nuclei
- Neutron stars
- Density

Quark−Gluon hadronic phase Color superconductor

Nuclei Neutron stars

Expansion of the early Universe

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The QGP in the early universe

- **big bang**
- **end of inflation**
- **EW transition**
- **confinement**
- **nucleosynthesis**
- **formation of atoms**

Timeline:
- $10^{-32}$ sec
- $10^{-10}$ sec
- $10^{-5}$ sec
- $10^{+2}$ sec
- $10^{+12}$ sec

Events:
- Deconfinement transition
- Early Universe
- QCD phase diagram
- Heavy Ion Collisions
- Length scales in the QGP
- Field theory at finite T
- Collective phenomena
Heavy Ion Collisions
Heavy ion collisions

Quark–Gluon hadronic phase

Color superconductor

Quark–Gluon plasma

Temperature

Density

Heavy ion collisions

Nuclei

Neutron stars

hadronic phase

Nuclear

Neutron stars

Length scales in the QGP

Field theory at finite T

Collective phenomena

Heavy Ion Collisions

• RHIC
• LHC
• Successive stages

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Since 2000: RHIC

- QCD reminder
- Deconfinement transition
- Heavy Ion Collisions
  - RHIC
  - LHC
  - Successive stages
- Length scales in the QGP
- Field theory at finite $T$
- Collective phenomena
Since 2000: RHIC

- Collision of two gold ions at RHIC:

- Nucleus-nucleus collision event in the STAR detector:

(animations courtesy of Brookhaven National Laboratory)
Starting in 2008: LHC / ALICE
Stages of a nucleus-nucleus collision

\[ z = -ct \]

\[ z = ct \]

\[ \rightarrow z \ (\text{beam axis}) \]
Stages of a nucleus-nucleus collision

- $\tau \sim 0$ fm/c
- Production of hard particles:
  - jets, direct photons
  - heavy quarks
- calculable with perturbative QCD (leading twist)
Stages of a nucleus-nucleus collision

- $\tau \sim 0.2$ fm/c
- Production of semi-hard particles: gluons, light quarks
- relatively small momentum: $p_\perp \lesssim 2$–$3$ GeV
- make up for most of the multiplicity
- sensitive to the physics of saturation (higher twist)
Stages of a nucleus-nucleus collision

- $\tau \sim 1–2$ fm/c
- **Thermalization**
  - experiments suggest a fast thermalization
  - but this is still not understood from QCD
Stages of a nucleus-nucleus collision

- 2 \leq \tau \lesssim 10 \text{ fm/c}
- Quark gluon plasma
Stages of a nucleus-nucleus collision

- hadrons in eq. \rightarrow \text{hydrodynamics}
- gluons & quarks in eq. \rightarrow \text{kinetic theory}
- gluons & quarks out of eq.
- strong fields \rightarrow \text{classical EOMs}

- $10 \lesssim \tau \lesssim 20 \text{ fm/c}$
- Hot hadron gas
Stages of a nucleus-nucleus collision

- \( \tau \rightarrow +\infty \)
- **Chemical freeze-out**:
  density too small to have inelastic interactions
- **Kinetic freeze-out**:
  no more elastic interactions
Length scales in the QGP
Degrees of freedom

- **Quarks**: \(2\) (spin) \(\times\) \(3\) (color) = \(6\) (per flavor)

\[
\frac{dN_q}{d^3\vec{x}d^3\vec{k}} = \frac{1}{e^{\omega/T} + 1} \quad \text{(Fermi-Dirac)}
\]

- **Gluons**: \(3\) (spin) \(\times\) \(8\) (color) = \(24\)

\[
\frac{dN_g}{d^3\vec{x}d^3\vec{k}} = \frac{1}{e^{\omega/T} - 1} \quad \text{(Bose-Einstein)}
\]

- **Average energy per particle**: \(\langle \omega \rangle \sim T\)
- **Particle density**: \(\rho \sim T^3\)
- **Average distance between particles**: \(\ell \sim 1/T\)
Length scales

- $1/T$ : wavelength of particles in the plasma
- $1/gT$ : typical distance for collective phenomena
  - Thermal masses of quasi-particles
  - Screening phenomena
  - Damping of waves
- $1/g^2T$ : distance between two small angle scatterings
  - Color transport
  - Photon emission
- $1/g^4T$ : distance between two large angle scatterings
  - Momentum, electric charge transport
  - Characteristic scale of hydrodynamic modes

- In the \textit{weak coupling} limit ($g \ll 1$), there is a clear hierarchy between these scales
- Distinct \textit{effective theories} according to the characteristic scale of the problem under study
At distances scales $\ell \lesssim 1/T$, medium effects are irrelevant.

At such scales the dynamics is simply described by the usual QCD in the vacuum.
Distance scales $1/T \lesssim \ell \lesssim 1/gT$ control the bulk thermodynamic properties. The system can be studied by QCD at finite temperature.

The leading thermal effects can be treated by an effective theory that encompasses the main collective effects, and that has the form of a collision-less Vlasov equation.
Small angle scatterings

- When it is necessary to follow a plasma particle over distances \( \frac{1}{g^2 T} \lesssim \ell \), we must take into account soft (small angle) collisions with other particles of the plasma.

- This can be done simply by adding a collision term to the previous Vlasov equation.
Collision rate

- **Collisional width**:

\[
\Gamma_{\text{coll}} = \left| \begin{array}{c}
p_\perp \\
\end{array} \right|^2 \sim g^4 T^3 \int \frac{d^2 p_\perp}{p_\perp^4} \sim g^2 T
\]

- \( \lambda \equiv 1/\Gamma_{\text{coll}} \) is the mean free path between two small angle scatterings \( (\theta \sim g) \)

- Note: the mean free path between two large angle scatterings \( (\theta \sim 1) \) is \( \sim 1/g^4 T \)
Over distance scales $\ell \sim 1/g^4 T$, one must take into account the large angle collisions, that change significantly the direction of motion of the particle (this is necessary e.g. for calculating transport coefficients).

The most efficient way to describe the system over these scales is via a Boltzmann equation for color/spin averaged particle distributions.
The hydrodynamical regime is reached for length scales that are much larger than the mean free path: $1/g^4T \ll \ell$

In order to describe the system at such scales, one needs:
- Hydrodynamical equations (Euler, Navier-Stokes)
- Conservation equations for the various currents
- Equation of state, viscosity
Field theory at finite $T$
Field Theory at zero temperature

- Transition amplitudes can be written in terms of expectation values of \textit{time-ordered products of fields}, via \textit{reduction formulas} such as:

\[
\langle \vec{p}'_1 \vec{p}'_{2\text{out}} | \vec{p}_1 \vec{p}_{2\text{in}} \rangle = \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \ e^{i(p_1 \cdot x_1 + p_2 \cdot x_2)} \ e^{-i(p'_1 x_1 + p'_2 \cdot x_2)}
\times (\Box_1 + m^2) (\Box_2 + m^2) (\Box'_1 + m^2) (\Box'_2 + m^2) \ \langle 0_{\text{out}} | T\phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) | 0_{\text{in}} \rangle
\]

- Expectation values of products of fields are obtained from a generating functional \( Z[j] \):

\[
\langle 0_{\text{out}} | T\phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) | 0_{\text{in}} \rangle = \left. \frac{\delta}{\delta j(x_1)} \frac{\delta}{\delta j(x_2)} \frac{\delta}{\delta j(y_1)} \frac{\delta}{\delta j(y_2)} Z[j] \right|_{j=0}
\]

- The generating functional is given by:

\[
Z[j] \equiv \langle 0_{\text{out}} | Te^{i \int d^4 x j(x) \phi(x)} | 0_{\text{in}} \rangle = e^{i \int d^4 x L_{\text{int}} (\delta/\delta j(x))} e^{-\frac{1}{2} \int d^4 x d^4 y j(x) G(x-y) j(y)}
\]
Goals

- **Quantum field theory at** $T = 0$ **describes elementary processes taking place in the vacuum**

- **At finite temperature**, we would like to describe processes taking place in a bath of particles described by the canonical density operator $\rho \equiv e^{-H/T}$

  Note: we chose units in which the Boltzmann constant is $k_B = 1$ (i.e. temperature = energy)

- **Let us define the thermal Green's functions as**:

  $$G_T(x_1 \cdots x_n) = \left\langle T\phi(x_1) \cdots \phi(x_n) \right\rangle \equiv \frac{\text{tr} \left[ e^{-H/T} T\phi(x_1) \cdots \phi(x_n) \right]}{\text{tr} \left[ e^{-H/T} \right]}$$

  Note: we recover the usual definition when $T \to 0^+$:

  $$G_T(x_1 \cdots x_n) \propto \sum_{n=0}^{\infty} \sum_{\text{states with } n \text{ particles}} e^{-E_n/T} \left\langle n \left| T\phi(x_1) \cdots \phi(x_n) \right| n \right\rangle$$

  $$= \lim_{T \to 0^+} \left\langle 0 \left| T\phi(x_1) \cdots \phi(x_n) \right| 0 \right\rangle + \sum_{\text{states with 1 particle}} \sum_{\text{states with 1 particle}} e^{-E_1/T} \left\langle 1 \left| T\phi(x_1) \cdots \phi(x_n) \right| 1 \right\rangle + \cdots$$
Perturbation theory at finite T

- Relate the fields \( \phi \) in the Heisenberg picture to free field \( \phi_{\text{in}} \) in the interaction picture:

\[
\phi(x) = U(t_i, x^0)\phi_{\text{in}}(x)U(x^0, t_i)
\]

- \( U \) is an unitary operator defined by

\[
U(t_2, t_1) \equiv T \exp i \int_{t_1}^{t_2} d^4 x \mathcal{L}_{\text{int}}(\phi_{\text{in}}(x))
\]

- This operator obeys the following relations:

\[
U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3) , \quad U(t, t) = 1
\]

- One can then prove:

\[
(\Box_x + m^2)\phi(x) - \frac{\partial \mathcal{L}_{\text{int}}(\phi(x))}{\partial \phi(x)} = U(t_i, x^0) \left[(\Box_x + m^2)\phi_{\text{in}}(x)\right] U(x^0, t_i)
\]

- Note: \( t_i \) is the time at which \( \phi \) and \( \phi_{\text{in}} \) coincide (it can be arbitrary – no physical quantity should depend on this choice)
Perturbation theory at finite $T$

- The T-product of fields $\phi$ can be written in terms of $\phi_{\text{in}}$:

$$T \phi(x_1) \cdots \phi(x_n) = T \phi_{\text{in}}(x_1) \cdots \phi_{\text{in}}(x_n) \exp i \int_C d^4x \mathcal{L}_{\text{int}}(\phi_{\text{in}}(x))$$

Note: $C$ is a contour along the real time axis, going from $t_i$ to $+\infty$ and back to $t_i$.

- In order to perform the perturbative expansion, we must expand all the pieces that contain the coupling constant $g$:
  - the $\mathcal{L}_{\text{int}}$ in $U(t_1, t_2)$ (similar to $T = 0$)
  - the Hamiltonian $H$ in the density operator (specific to $T \neq 0$)

- The second source of $g$ dependence can be evaluated by noticing the similarity of $\rho$ with an evolution operator $e^{-H/T} = e^{i(i/T)H}$, which leads to

$$e^{-H/T} = e^{-H_0/T} T \exp i \int_{t_i}^{t_i-i/T} d^4x \mathcal{L}_{\text{int}}(\phi_{\text{in}}(x))$$

Note: $H_0$ is the free Hamiltonian.
Perturbation theory at finite T

- A generating functional can be defined for the thermal Green's functions:

\[ Z_T[j] \equiv \left< T e^{i \int d^4 x \ j(x) \phi(x)} \right> \]

\[ = \left< T \exp i \int_{C_T} d^4 x \ [j(x) \phi_{in}(x) + \mathcal{L}_{int}(\phi_{in}(x))] \right>_0 \]

Note: \( C_T = [t_i, +\infty] \cup [+\infty, t_i] \cup [t_i, t_i - i/T] \)

- We can deal with the interactions by writing:

\[ Z_T[j] = e^{i \int_{C_T} d^4 x \ \mathcal{L}_{int}(\delta/\delta j(x))} \left< T \exp i \int_{C_T} d^4 x \ j(x) \phi_{in}(x) \right>_0 \]

\( e^A e^B = e^{A+B} e^{\frac{1}{2} [A,B]} \) if \( [A, [A, B]] = [B, [A, B]] = 0 \)

- Campbell-Baker-Hausdorff formula:

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Field theory at finite T
- Reminder: zero T
- Goals
- Perturbation theory at finite T
- Main properties
- Momentum space
- Collective phenomena

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Lemma 1: if \([A, B] \) commutes with \(B\), then
\[
[A, f(B)] = [A, B] f'(B)
\]

Lemma 2: if \([A, B] \) commutes with \(A\) and \(B\), then
\[
\frac{d}{dz} e^{zA+B} = \left\{ A - \frac{1}{2} [A, B] \right\} e^{zA+B}
\]

Finally, define:
\[
F(z) \equiv e^{zA} e^B, \quad G(z) \equiv e^{zA+B} e^{\frac{z}{2} [A, B]}
\]

and check that
\[
F(0) = G(0) = e^B, \quad F'(z) = G'(z)
\]
Use the CBH formula to eliminate the $T$ product

$$Z_0[j] = \left\langle e^{i \int_{C_T} d^4x \ j(x) \phi_{\text{in}}(x)} \right\rangle_0 e^{-\frac{1}{2} \int_{C_T} d^4x d^4y \ \theta_c(x^0 - y^0) \ j(x) j(y) \ \left[ \phi_{\text{in}}(x), \phi_{\text{in}}(y) \right]}$$

- $\theta_c(x^0 - y^0)$ is a theta function defined on the contour $C_T$ – it is equal to 1 if $x^0$ is posterior to $y^0$ on the contour
- the second factor in the r.h.s. is a number, not an operator

Using: $\phi_{\text{in}}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left[ e^{-ip \cdot x} a^+(\vec{p}) + e^{ip \cdot x} a(\vec{p}) \right]$, write

$$\phi_{\text{in}}(x) \equiv \phi_{\text{in}}^{(+)}(x) + \phi_{\text{in}}^{(-)}(x)$$

creation op.  annihilation op.

and use CBH again to write the first factor in normal form:

$$Z_0[j] = \left\langle \left| e^{i \int_{C_T} d^4x \ j(x) \phi_{\text{in}}(x)} \right| \right\rangle_0$$

$$\times e^{\frac{1}{2} \int_{C_T} d^4x d^4y \ j(x) j(y) \ \left[ \phi_{\text{in}}^{(+)}(x), \phi_{\text{in}}^{(-)}(y) \right] - \theta_c(x^0 - y^0) \left[ \phi_{\text{in}}(x), \phi_{\text{in}}(y) \right]}$$
Perturbation theory at finite $T$

The bracket in the second exponential can be rearranged as follows:

$$\left[ \cdots \right] = \phi_{\text{in}}^{(+)}(y)\phi_{\text{in}}^{(+)}(x) + \phi_{\text{in}}^{(-)}(y)\phi_{\text{in}}^{(-)}(x) + \phi_{\text{in}}^{(+)}(y)\phi_{\text{in}}^{(-)}(x) + \phi_{\text{in}}^{(+)}(x)\phi_{\text{in}}^{(-)}(y)$$

$$- \theta(x^0 - y^0)\phi_{\text{in}}(x)\phi_{\text{in}}(y) - \theta(y^0 - x^0)\phi_{\text{in}}(y)\phi_{\text{in}}(x)$$
Perturbation theory at finite $T$

The bracket in in the second exponential can be rearranged as follows:

$$\cdots = \left\langle \phi_{\text{in}}^{(+)}(y)\phi_{\text{in}}^{(+)}(x) + \phi_{\text{in}}^{(-)}(y)\phi_{\text{in}}^{(-)}(x) + \phi_{\text{in}}^{(+)}(y)\phi_{\text{in}}^{(-)}(x) + \phi_{\text{in}}^{(+)}(x)\phi_{\text{in}}^{(-)}(y) - \theta(x^0 - y^0)\phi_{\text{in}}(x)\phi_{\text{in}}(y) - \theta(y^0 - x^0)\phi_{\text{in}}(y)\phi_{\text{in}}(x) \right\rangle_0$$

Note: this is a pure number $\implies$ it is equal to its average with $\left\langle \cdots \right\rangle_0$
Perturbation theory at finite T

The bracket in in the second exponential can be rearranged as follows:

\[
\begin{align*}
\langle \cdots \rangle &= \langle \phi_{\text{in}}(x) \phi_{\text{in}}(-x) \rangle + \langle \phi_{\text{in}}(-x) \phi_{\text{in}}(-x) \rangle + \langle \phi_{\text{in}}(-x) \phi_{\text{in}}(x) \rangle + \langle \phi_{\text{in}}(x) \phi_{\text{in}}(-x) \rangle \\
&\quad - \theta(x^0 - y^0)\phi_{\text{in}}(x)\phi_{\text{in}}(y) - \theta(y^0 - x^0)\phi_{\text{in}}(y)\phi_{\text{in}}(x) \bigg|_0
\end{align*}
\]

Note: this is a pure number \( \triangleright \) it is equal to its average with \( \langle \cdots \rangle_0 \)

Wick theorem. One can prove:

\[
\begin{align*}
&\langle a^\dagger(\vec{p})a(\vec{p}')\rangle_0 = (2\pi)^3 2E_p \delta(\vec{p} - \vec{p}') \frac{1}{e^{E_p/T} - 1} \\
&\langle a^\dagger(\vec{p})a(\vec{p}')\rangle_0 = \langle a(\vec{p})a(\vec{p}')\rangle_0 = 0 \\
&\langle \phi_{\text{in}}(x) \phi_{\text{in}}(x) \rangle_0 = \langle \phi_{\text{in}}(-x) \phi_{\text{in}}(-x) \rangle_0 = 0 \\
&\langle e^{i \int C_T d^4x j(x)\phi_{\text{in}}(x)} \rangle_0 = e^{-\int C_T d^4xd^4y j(x)j(y) \langle \phi_{\text{in}}(x) \phi_{\text{in}}(-x) \rangle_0}
\end{align*}
\]
Perturbation theory at finite $T$

- Finally, the **free generating functional** at finite $T$ reads:

  \[
  Z_0[j] = \exp \left(-\frac{1}{2} \int_{C_T} d^4x d^4y \, j(x) \, G_T(x, y) \, j(y) \right)
  \]

  with

  \[
  G_T(x, y) \equiv \langle T \phi_{\text{in}}(x) \phi_{\text{in}}(y) \rangle_0
  \]

  The result is formally identical to $T = 0$, except that the free propagator is given by a thermal average and that the time integration must be done on the contour $C_T$.

- **Feynman rules** in coordinate space:
  - **Vertex**: $-ig \int_{C_T} d^4x$
  - **Propagator**: $G_T(x, y)$

- Using: $\phi_{\text{in}}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left[ e^{-ip\cdot x} a^\dagger(\vec{p}) + e^{+ip\cdot x} a(\vec{p}) \right]$, one gets

  \[
  G_T(x, y) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \left[ e^{-ip\cdot(x-y)} \left[ \theta_c(x^0 - y^0) + n_B(E_p) \right] 
  + e^{+ip\cdot(x-y)} \left[ \theta_c(y^0 - x^0) + n_B(E_p) \right] \right]
  \]
Main properties

- Any field theory renormalizable at $T = 0$ is renormalizable at finite $T$, with the counterterms calculated at $T = 0$ physically, the short distance properties of the theory do not depend on whether there is an ensemble of particles around or just the vacuum.

- Thermal Green’s functions do not depend on the time $t_i$
  - one often chooses $t_i = -\infty$

- The contour $C_T$ can be deformed (with the endpoints $t_i$ and $t_i - i/T$ hold fixed)

- Kubo-Martin-Schwinger symmetry: thermal Green’s functions have equal (opposite for fermions) values at the two endpoints of the contour $C_T$:

  $$G_T(\cdots t_i \cdots) = G_T(\cdots t_i - i/T \cdots)$$

  - Proof: this property is immediate for the free thermal propagator, and is preserved by the integration over $C_T$ at the vertices.
Feynman rules in momentum space

- Feynman rules in coordinate space are complicated to use in practice
  - do the perturbation theory in momentum space

- At $T = 0$, one goes to momentum space by using the fundamental property that relates ordinary products in momentum space to convolution products in coordinate space:

  $f(x - y) \equiv \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x - y)} \tilde{f}(p)$

  then $\int d^4y \ f(x - y)g(y - z) = \int \frac{d^4p}{(2\pi)^4} e^{ip \cdot (x - z)} \tilde{f}(p)\tilde{g}(p)$

- Problem: this formula only works if the time integrations are on the real axis from $-\infty$ to $+\infty$. At finite $T$, the time integration runs on the contour $C_T$
Feynman rules in momentum space

- **Deform the contour** $C_T$ **into the vertical segment** $[t_i, t_i - i/T]$ (the value of $t_i$ does not matter)

- **Use the** $i/T$-**periodicity in order to decompose the Green’s functions in Fourier series**:

  $$G_T(x_1^0 \cdots x_p^0) = (iT)^p \sum_{n_1 \cdots n_p \in \mathbb{Z}} e^{\omega_{n_1} x_1^0} \cdots e^{\omega_{n_p} x_p^0} \tilde{G}_T(\omega_{n_1} \cdots \omega_{n_p})$$

  with frequencies $\omega_n \equiv 2\pi nT$ (Matsubara frequencies)

- **This formula can be inverted by**:

  $$\tilde{G}_T(\omega_{n_1} \cdots \omega_{n_p}) = \int_{C_T} dx_1^0 \cdots dx_p^0 e^{-\omega_{n_1} x_1^0} \cdots e^{-\omega_{n_p} x_p^0} G_T(x_1^0 \cdots x_p^0)$$

  Note: because of invariance under translations in time, the result is non-zero only if $n_1 + \cdots + n_p = 0$

- **For the spatial coordinates** (not written here), we do the usual Fourier transform $\triangleright$ the function $\tilde{G}_T$ depends on discrete frequencies and on the 3-momenta $\vec{p}_i$.
Feynman rules in momentum space

- This transformation applied to the free propagator leads to:

\[
\tilde{G}_T(\omega_n, \vec{p}) = -\frac{i}{\omega_n^2 + \vec{p}^2 + m^2}
\]

Note: the free Feynman propagator at \( T = 0 \) is \( i/(p_0^2 - \vec{p}^2 - m^2 + i\epsilon) \). Thus, one goes from the Matsubara propagator to the Feynman propagator by \( i\omega_n \rightarrow p_0 + ip_0\epsilon \).

- **Feynman rules** in this formalism:
  - Vertex: \(-ig(2\pi)^3\delta(\vec{p}_1 + \cdots + \vec{p}_p)\frac{1}{T}\delta n_1 + \cdots + n_p\)
  - Propagator: \(1/(\omega_n^2 + \vec{p}^2 + m^2)\)
  - Loops: \(T\sum_{n\in\mathbb{Z}} \int \frac{d^3\vec{p}}{(2\pi)^3}\)

- **Example in a scalar theory**:

\[
\omega_n, \vec{p} \quad = \quad g^2 T \sum_{n\in\mathbb{Z}} \int \frac{d^3\vec{k}}{(2\pi)^3} \quad \frac{1}{\omega_n^2 + \vec{k}^2 + m^2} \quad \frac{1}{\omega_{p-n}^2 + (\vec{p} - \vec{k})^2 + m^2}
\]
Collective phenomena
Collective phenomena

- Phenomena involving **many elementary constituents**
- **Long wavelength** compared to the typical distance between constituents
- **Small frequency or energy**

**Major collective phenomena:**
- Quasi-particles
- Debye screening
- Landau damping
- Collisional width
Dressed propagator

In order to study these collective effects, one must calculate the thermal self-energy of gauge bosons. In QED, it reads:

\[
\omega, p \rightarrow e^2 T \sum_{n \in \mathbb{Z}} \int \frac{d^3 \vec{k}}{(2\pi)^3} \, \text{Tr} \left[ \gamma^\mu \frac{K}{\omega_n^2 + \vec{k}^2} \gamma^\nu \frac{P - K}{\omega_{p-n}^2 + (\vec{p} - \vec{k})^2} \right]
\]

Reminder: the photon polarization tensor \( \Pi^{\mu\nu} \) is transverse. At \( T = 0 \), this implies:

\[
\Pi^{\mu\nu}(P) = \left( g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Pi(P^2)
\]

- this is due to gauge invariance and Lorentz invariance
- this property ensures that the photon remains massless at all orders of perturbation theory

This formula is not valid at \( T > 0 \), because there is a preferred frame (in which the plasma velocity is zero)

- the tensorial decomposition of \( \Pi^{\mu\nu} \) is more complicated, and the photon acquires a mass
Dressed propagator

- At finite $T$, the tensorial decomposition of $\Pi^{\mu\nu}$ is:

$$\Pi^{\mu\nu}(P) = P^{\mu\nu}_{\text{trans}}(P) \Pi_T(P) + P^{\mu\nu}_{\text{long}}(P) \Pi_L(P)$$

with the following projectors (in the plasma rest frame):

$$P^{ij}_{\text{trans}}(P) = g^{ij} + \frac{p^i p^j}{p^2}, \quad P^{0i}_{\text{trans}}(P) = 0, \quad P^{00}_{\text{trans}}(P) = 0$$

$$P^{ij}_{\text{long}}(P) = -\frac{(p^0)^2 p^i p^j}{p^2 P^2}, \quad P^{0i}_{\text{long}}(P) = -\frac{p^0 p^i}{P^2}, \quad P^{00}_{\text{long}}(P) = -\frac{p^2}{P^2}$$

- Therefore, we have

$$\Pi^{\mu\mu}(P) = 2\Pi_T(P) + \Pi_L(P), \quad \Pi^{00}(P) = -\frac{p^2}{P^2} \Pi_L(P)$$

- The calculation of $\Pi^{\mu\mu}$ and $\Pi^{00}$ is done for a discrete imaginary frequency $i\omega_p$, and one performs the analytic continuation $i\omega_p \rightarrow p_0$ afterwards.

- One also makes the approximation $|\vec{p}| \ll |\vec{k}|$
Dressed propagator

- The functions $\Pi_{T,L}(P)$ read:

\[
\Pi_T(P) = \frac{e^2 T^2}{6} \left[ \frac{p_0^2}{p^2} + \frac{p_0}{2p} \left( 1 - \frac{p_0^2}{p^2} \right) \ln \left( \frac{p_0 + p}{p_0 - p} \right) \right]
\]

\[
\Pi_L(P) = \frac{e^2 T^2}{3} \left[ 1 - \frac{p_0^2}{p^2} \right] \left[ 1 - \frac{p_0}{2p} \ln \left( \frac{p_0 + p}{p_0 - p} \right) \right]
\]

- The photon (or gluon for QCD) self-energy can be resummed on the propagator. Diagrammatically, this amounts to summing:

\[ \cdots + \cdots + \cdots + \cdots + \cdots + \cdots \]

- This leads to the following propagator:

\[
D^{\mu\nu}(P) = P^{\mu\nu}_{\text{trans}}(P) \frac{1}{P^2 - \Pi_T(P)} + P^{\mu\nu}_{\text{long}}(P) \frac{1}{P^2 - \Pi_L(P)}
\]
Quasi-particles

- **Quasi-particles** correspond to poles in the propagator. Their dispersion relation is the function $p_0 = \omega(\vec{p})$ that defines the location of the pole.

- The inverse of the imaginary part of $p_0$ is the lifetime of the quasi-particles (If $\text{Im}(p_0) = 0$, they are stable). In order to be able to talk about quasi-particles, one must have $\text{Im}(p_0) \ll \text{Re}(p_0)$.

- In order to find the poles of the propagator, one must solve the equations:

\[
\begin{align*}
p_0^2 - \vec{p}^2 &= \Pi_T(p_0, \vec{p}) \\
p_0^2 - \vec{p}^2 &= \Pi_L(p_0, \vec{p})
\end{align*}
\]

- this can be done numerically
- one finds that $\text{Im}(p_0) = 0$ at the poles ▶ stable quasi-particles
Quasi-particles

- Dispersion curves of particles in the plasma:

\[ \omega \]

\[ p \]

\[ m_q \]

\[ m_g \]

Quarks

Gluons

- Thermal masses due to interactions with the other particles in the plasma:

\[ m_q \sim m_g \sim gT \]

- One needs a non-zero energy to make a particle of the plasma move
Debye screening

- A test charge polarizes the particles of the plasma in its vicinity, in order to screen its charge:

\[ V(r) = \exp\left( -\frac{m_{\text{debye}}}{r} \right) \]

- The Coulomb potential of the test charge decreases exponentially at large distance. The effective interaction range is:

\[ \ell \sim \frac{1}{m_{\text{debye}}} \sim \frac{1}{gT} \]

- Note: static magnetic fields are not screened by this mechanism (they are screened over length-scales

\[ \ell_{\text{mag}} \sim \frac{1}{g^2T} \]
Debye screening

- The Debye screening can be studied more quantitatively from the dressed gluon propagator we have constructed.
- Place a quark of mass $M$ at rest in the plasma, at $\vec{r} = 0$.
- Scatter another quark off it. The scattering amplitude reads

$$
\mathcal{M} = \left[ g\bar{u}(\vec{k}') \gamma_\mu u(\vec{k}) \right] \left[ g\bar{u}(\vec{P}') \gamma_\nu u(\vec{P}) \right] \sum_{\alpha = T, L} \frac{P_\alpha^{\mu\nu}(Q)}{Q^2 - \Pi_\alpha(Q)}
$$

- If $\vec{P} = 0$ (test charge at rest), only $\alpha = L$ contributes.
- From $(P + Q)^2 = M^2$, we get a $\frac{2\pi \delta(q_0)}{2M}$.

- For the scattering off an external potential $A^\mu$, the amplitude is

$$
\mathcal{M} = \left[ g\bar{u}(\vec{k}') \gamma_\mu u(\vec{k}) \right] A^\mu(Q)
$$

- Thus, the potential created by the test charge at rest is:

$$
A^\mu(Q) = \frac{g}{2M} \frac{\bar{u}(\vec{P}') \gamma_\nu u(\vec{P})}{\vec{q}^2 + \Pi_L(0, \vec{q})} \frac{2\pi \delta(q_0) P^{\mu\nu}_\text{long}(0, \vec{q})}{\vec{q}^2 + \Pi_L(0, \vec{q})} = \frac{2\pi g \delta^{\mu0} \delta(q_0)}{\vec{q}^2 + \Pi_L(0, \vec{q})}
$$
Debye screening

- By a Fourier transform, we obtain the Coulomb potential:

\[ A^0(t, \vec{r}) = g \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{e^{i\vec{q} \cdot \vec{r}}}{\vec{q}^2 + \Pi_L(0, \vec{q})} \]

- If we are in the vacuum, \( \Pi_L = 0 \), and the Fourier transform gives the usual Coulomb law:

\[ A^0_{\text{vac}}(t, \vec{r}) = g \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{e^{i\vec{q} \cdot \vec{r}}}{\vec{q}^2} = \frac{g}{4\pi |\vec{r}|} \]

- In a plasma, \( \Pi_L(0, \vec{q}) = \frac{g^2 T^2}{3} \equiv m_D^2 \). The Fourier transform can also be done exactly

\[ A^0(t, \vec{r}) = g \int \frac{d^3 \vec{q}}{(2\pi)^3} \frac{e^{i\vec{q} \cdot \vec{r}}}{\vec{q}^2 + m_D^2} = \frac{g}{4\pi |\vec{r}|} e^{-m_D |\vec{r}|} \]

▷ the potential is unmodified at \( r \ll 1/m_D \), but exponentially suppressed at large distance
Landau damping

- The self-energies $\Pi_{L,T}(p_0, \vec{p})$ have an imaginary part when $|p_0| \leq |\vec{p}|$. This implies that the propagation of space-like modes is attenuated.

- A wave propagating through the plasma is damped because its quanta may be absorbed by particles of the plasma:

  $\omega_c \sim gT$

  ![Wave damping diagram]