Pre-equilibrium dynamics in heavy ion collisions

IV – Kinetic theory, Near-Equilibrium dynamics

François Gelis

CERN and CEA/Saclay

General outline

- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization
- Lecture IV : Kinetic theory, Near-Equilibrium dynamics

Lecture IV : Kinetic theory

- Collisionless kinetic equations
- Boltzmann equation
- Transport coefficients
- From kinetic theory to hydrodynamics

Introduction

Factorization formula

- By resumming all the large $\log s(1/x_{1,2})$ and the terms affected by the instability, one gets :

$$\frac{d\overline{N}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \int [D\rho_1]\,[D\rho_2]\;\; W_{Y_{\rm beam}-Y}[\rho_1]\; W_{Y_{\rm beam}+Y}[\rho_2] \times \int [Da]\;\; \widetilde{Z}[a]\;\; \frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]}{dY\,d^2\vec{\boldsymbol{p}}_\perp}$$

- The gluon spectrum for given (ρ_1, ρ_2) and a is given by :

$$\frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]}{dY\,d^2\vec{\boldsymbol{p}}_\perp} \propto \int_{x,y} e^{ip\cdot(x-y)}\;\cdots\;\mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

- $\mathcal{A}_\mu(x)$ is the retarded solution of Yang-Mills equations :

$\left[D_\mu, F^{\mu\nu}\right] = 0$ with initial condition : $\mathcal{A} \underset{\tau=0^+}{=} \mathcal{A}_{\rm in}(\rho_1,\rho_2) + a$

Kinetic theory

- So far, our description of the early time dynamics of the system is based on fields (classical fields + quantum fluctuations)

  The fields are then Fourier transformed (with an on-shell momentum) and a distribution of particles can be extracted

- However, this semi-classical description cannot apply until very late times, because some important dynamics is missing : collisions

- The fact that collisions are missing would show up at two loops, under the form of secular terms :
  - Secular terms increase like powers of the time, and must be resummed in order to keep the description of the system under control at late time
  - This resummation leads to a Boltzmann equation

Kinetic theory

- In kinetic theory, the system is described by distributions of particles

$$f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) \equiv \frac{dN}{d^3\vec{\boldsymbol{x}}\, d^3\vec{\boldsymbol{p}}}$$

- Implicit hypothesis :
  - The particle distributions vary slowly with t and $\vec{\boldsymbol{x}}$ (gradients in $t, \vec{\boldsymbol{x}}$ are much smaller than the typical momentum $\vec{\boldsymbol{p}}$)
  - On-shell particles propagate freely between two collisions

- Kinetic equations describe the time evolution of these distributions under the influence of external forces, or of the mutual interactions of the particles

- Kinetic equations :
  - Free transport equation
  - Vlasov equation
  - Boltzmann equation

Collisionless kinetic equations

Free transport

- Free transport is a regime in which the particles do not interact. Given an initial $f(t_0, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$, the particles propagate on straight lines, at constant velocity
- The kinetic equation that describes this regime reads :

$$p \cdot \partial_X f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = 0$$

or, equivalently :

$$\Big[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\Big] f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = 0 \qquad \text{with } \vec{\boldsymbol{v}}_{\boldsymbol{p}} \equiv \frac{\vec{\boldsymbol{p}}}{E_{\boldsymbol{p}}}$$

- This equation can be solved trivially from its initial condition :

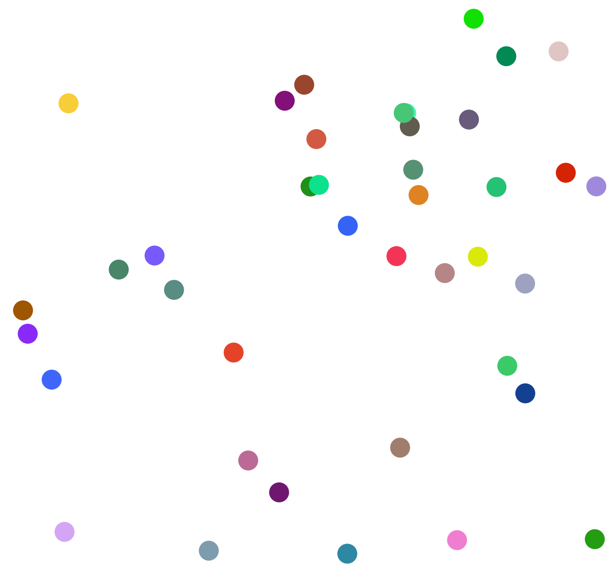
$$f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = f(t_0, \vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_{\boldsymbol{p}}(t - t_0), \vec{\boldsymbol{p}})$$

Interpretation :

- The momentum $\vec{\boldsymbol{p}}$ of the particles does not change
- If a particle of momentum $\vec{\boldsymbol{p}}$ is at the position $\vec{\boldsymbol{x}}$ at time t, it comes from the position $\vec{\boldsymbol{x}} - \vec{\boldsymbol{v}}_{\boldsymbol{p}}(t - t_0)$ at the time t_0

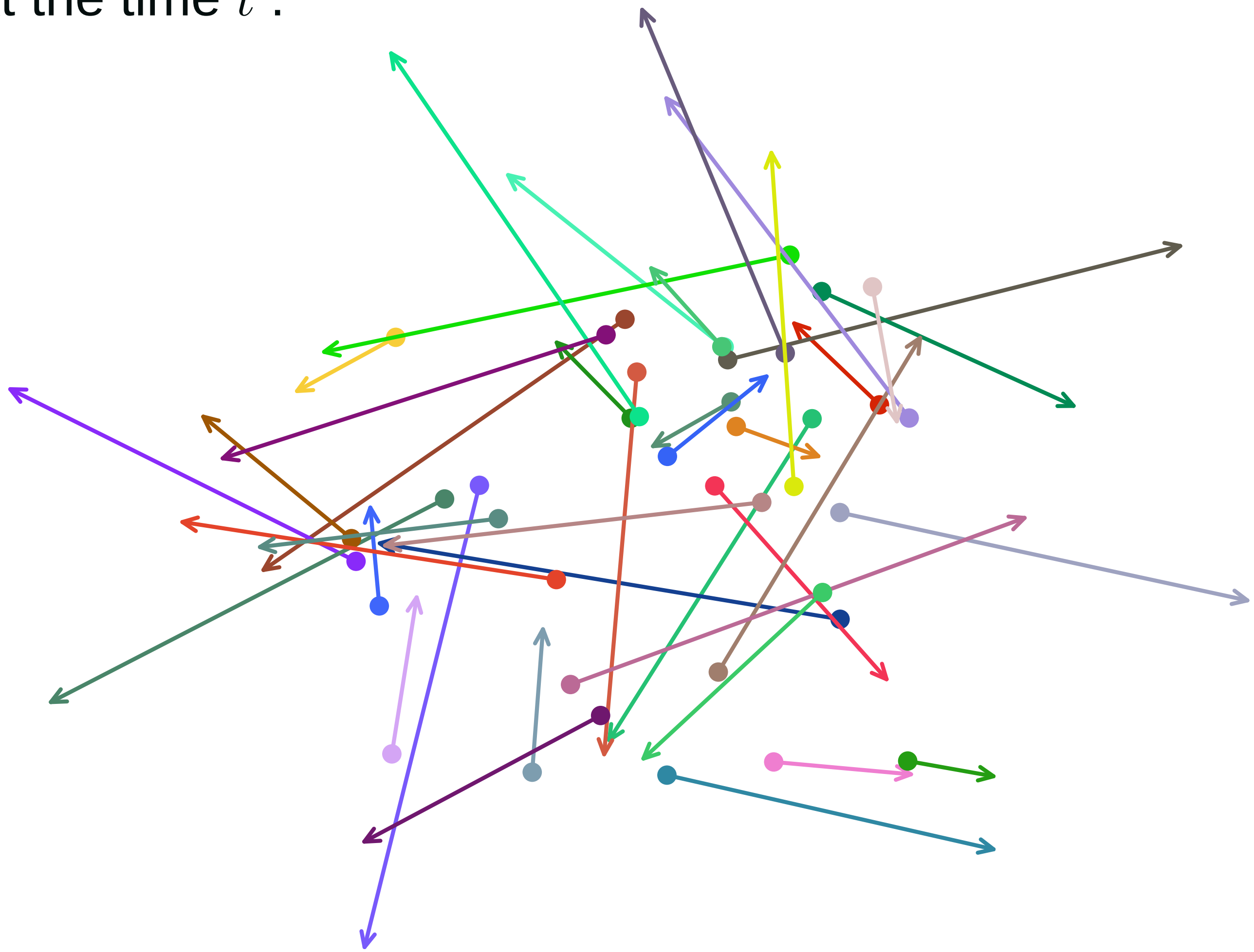
Free transport

- At the time t_0 :



Free transport

- At the time t :



Vlasov equation

- The Vlasov equation describes the time evolution of a distribution of particles under the influence of a force $\vec{\boldsymbol{F}}$

- The Vlasov equation reads :

$$\Big[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\Big] f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) + \underline{\vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{p}} f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})} = 0$$

- When the force is externally applied, it can be solved formally by :

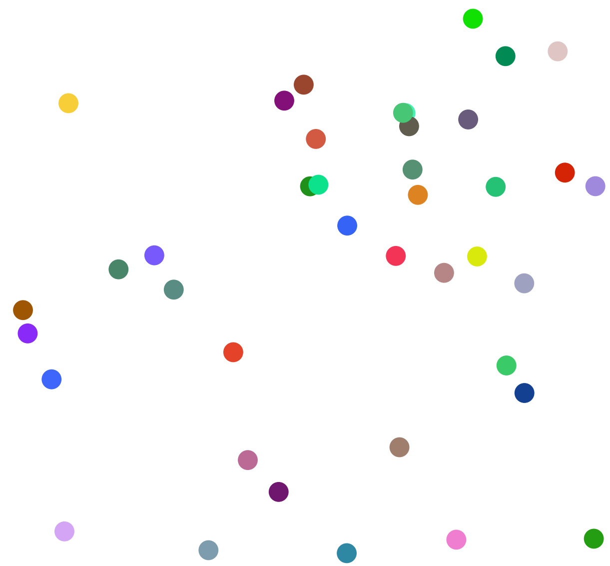
$$f(t, \vec{\boldsymbol{x}}, \vec{\boldsymbol{p}}) = f(t_0, \vec{\boldsymbol{x}}_0, \vec{\boldsymbol{p}}_0)$$

where $(\vec{\boldsymbol{x}}_0, \vec{\boldsymbol{p}}_0)$ is the position in phase space at time t_0 that leads to $(\vec{\boldsymbol{x}}, \vec{\boldsymbol{p}})$ at time t under the effect of the force $\vec{\boldsymbol{F}}$. If $(\vec{\boldsymbol{x}}(\tau), \vec{\boldsymbol{p}}(\tau))$ denotes the trajectory between t_0 and t, one has

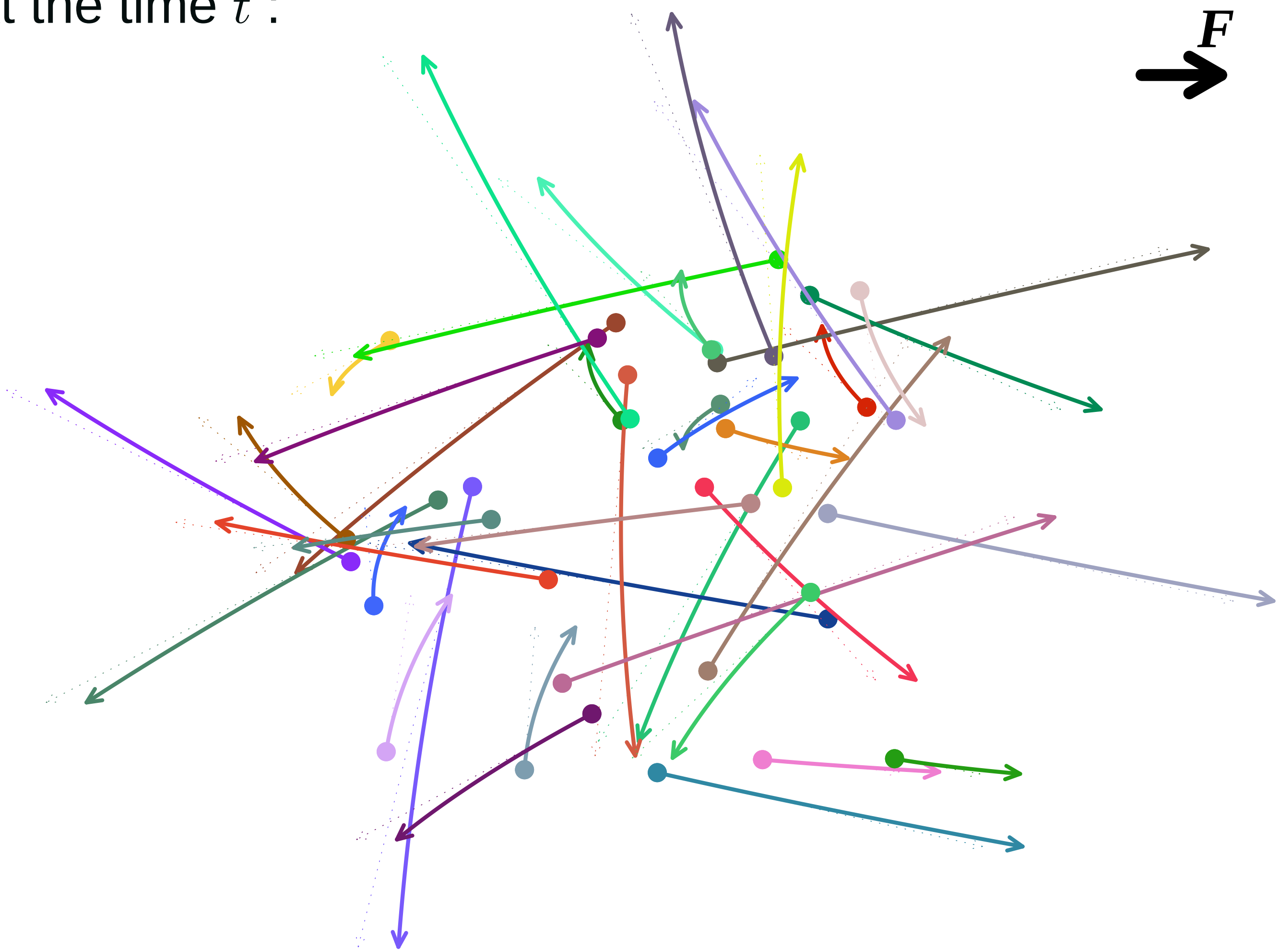
$$\vec{\boldsymbol{x}} = \vec{\boldsymbol{x}}_0 + \int_{t_0}^{t} d\tau\, \frac{\vec{\boldsymbol{p}}(\tau)}{E_{\boldsymbol{p}}(\tau)} \quad , \quad \vec{\boldsymbol{p}} = \vec{\boldsymbol{p}}_0 + \int_{t_0}^{t} d\tau\, \vec{\boldsymbol{F}}(\tau, \vec{\boldsymbol{x}}(\tau))$$

Vlasov equation

- At the time t_0 :

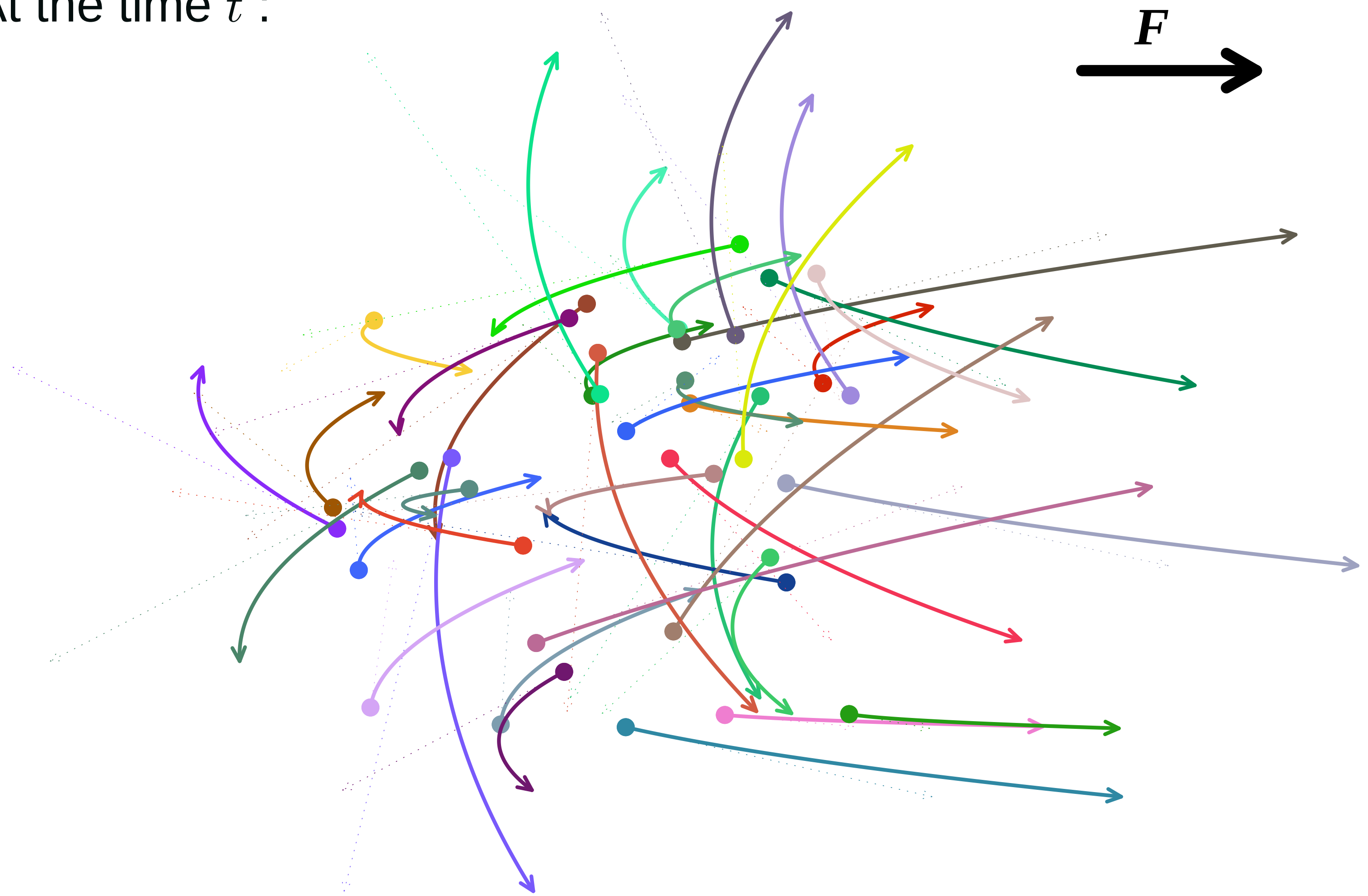


- At the time t :



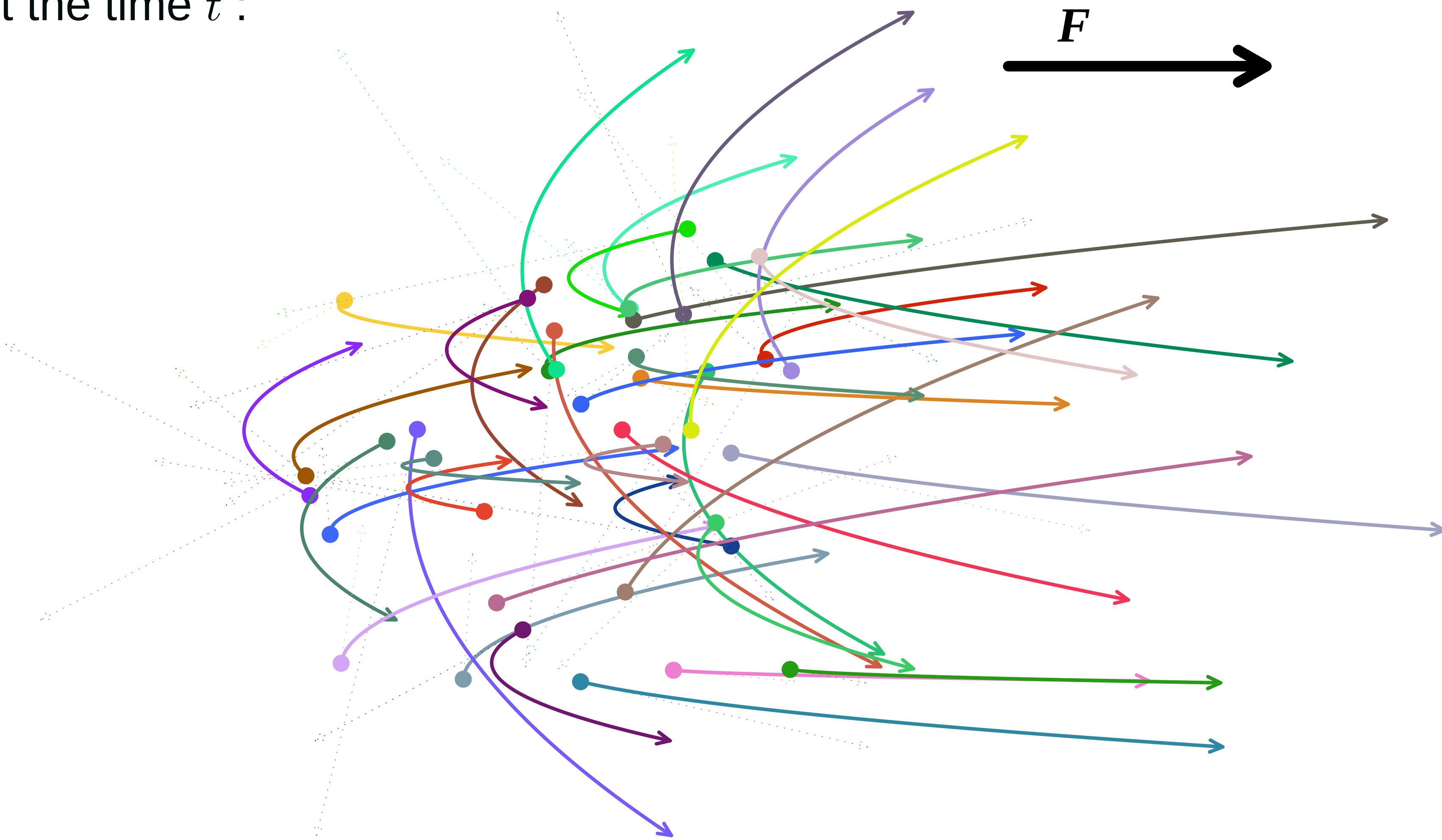
Vlasov equation

- At the time t :



Vlasov equation

- At the time t :



Vlasov equation + mean field

- In many applications, the force $\vec{\boldsymbol{F}}$ is not externally applied, but results from the action of all the other particles

- Example : for electro-magnetic interactions among the particles in the system, the force term in the Vlasov equation reads

$$\underbrace{e\, v_{\boldsymbol{p}}^{\mu}\, F_{\mu\nu}}_{\text{Lorentz force in covariant form}}\ \partial_p^{\nu} f(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{p}})$$

Lorentz force in covariant form

with

$$F^{\mu\nu} \equiv \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$$

$$\partial_{\mu} F^{\mu\nu}(x) = \underbrace{e \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\ v_{\boldsymbol{p}}^{\nu}\ f(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{p}})}_{\text{EM current created by the particles}} \qquad \text{(Maxwell's equation)}$$

EM current created by the particles

Boltzmann equation

Collision term

- The Boltzmann equation takes into account the collisions among particles. It is valid when these collisions are sufficiently local (i.e. no long range interactions among pairs of particles). Thanks to the Debye screening, this is a valid assumption for a neutral plasma
- The Boltzmann equation reads :

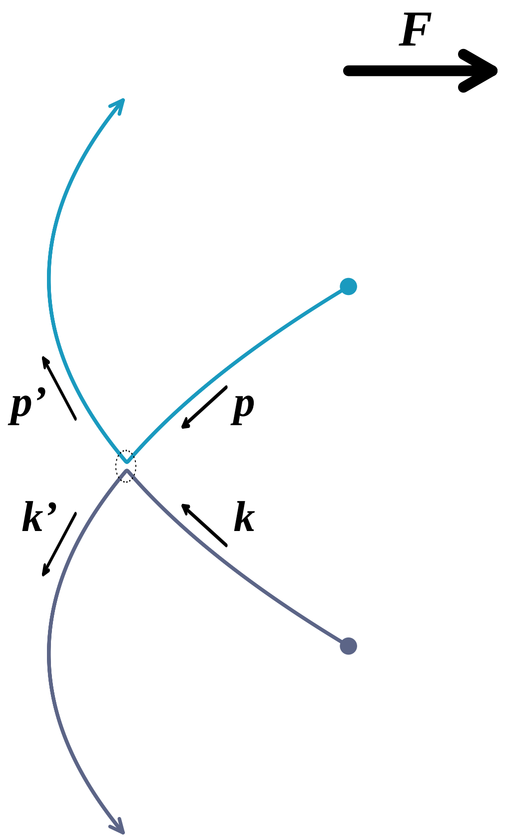
$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] f(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{p}}) + \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{p}} f(t,\vec{\boldsymbol{x}},\vec{\boldsymbol{p}}) = \mathcal{C}_{\boldsymbol{p}}[f]$$

▷ the functional $\mathcal{C}_{\boldsymbol{p}}[f]$ is the collision term. For $2 \to 2$ collisions, it can be written as :

$$\mathcal{C}_{\boldsymbol{p}}[f] = \frac{1}{2E_{\boldsymbol{p}}} \int \frac{d^3\vec{\boldsymbol{p}}'}{(2\pi)^3 2E_{\boldsymbol{p}'}} \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}} \int \frac{d^3\vec{\boldsymbol{k}}'}{(2\pi)^3 2E_{\boldsymbol{k}'}} (2\pi)^4 \delta(p+k-p'-k')$$
$$\times \Big[f(X,\vec{\boldsymbol{p}}') f(X,\vec{\boldsymbol{k}}')(1+f(X,\vec{\boldsymbol{p}}))(1+f(X,\vec{\boldsymbol{k}}))$$
$$- f(X,\vec{\boldsymbol{p}}) f(X,\vec{\boldsymbol{k}})(1+f(X,\vec{\boldsymbol{k}}'))(1+f(X,\vec{\boldsymbol{p}}')) \Big] \left|\mathcal{M}\right|^2$$

Collision term

- Elementary 2-body collision :



Note : microscopic collisions are reversible

Historical note

- A weaker form of the Boltzmann equation first appeared in the work of Maxwell (1866), who derived equations for momentum averaged quantities such as :

$$\varphi(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\; \phi(\vec{\boldsymbol{p}})\; f(X,\vec{\boldsymbol{p}})$$

- Later, Boltzmann obtained an equation for $f(X,\vec{\boldsymbol{p}})$ itself, and provided the corresponding physical interpretation

Diluteness assumption

- In order to be able to neglect 3-body collisions and higher, the system under study must be sufficiently dilute

- For a system of N hard spheres of radius r, the Boltzmann equation is valid in the limit :

$$\left\{ \begin{array}{rcr} Nr^2 & = & \text{const} \\ Nr^3 & \to & 0 \end{array} \right.$$

(Boltzmann-Grad limit)

- The first condition means that the mean free path is fixed ($\lambda = 1/n\sigma,\ n = N/V,\ \sigma = 2\pi r^2$)

- The second condition means that the volume occupied by the particles tend to zero

Molecular chaos assumption

- Strictly speaking, the collision term should contain the probability to find a pair of particles of momenta $\vec{\boldsymbol{p}}, \vec{\boldsymbol{k}}$ at the point $(t, \vec{\boldsymbol{x}})$ before the collision

  ▷ one should have used the 2-particle phase-space distribution :

$$f_2(X, \vec{\boldsymbol{p}}; X, \vec{\boldsymbol{k}})$$

  that contains information about the 2-particle correlations

- By writing :

$$f_2(X, \vec{\boldsymbol{p}}; X, \vec{\boldsymbol{k}}) = f(X, \vec{\boldsymbol{p}}) f(X, \vec{\boldsymbol{k}})$$

  one assumes that the two colliding particles have uncorrelated momenta before the collision

- Although the microscopic processes are reversible, the Boltzmann equation is not, because the two momenta become correlated after the collision

Collisions or mean field ?

- Given a two-body interaction between particles, should we treat it as part of the mean field force term, or as part of the collision term?
- Bobylev, Illner : for inverse power forces in r^{-s}
  - the collision term prevails if $s > 3$
  - the mean-field term prevails if $s < 3$
- This indicates that short-range interactions should be treated as collisions, while long range interactions go in the mean-field term
- Examples :

Debye screened forces	$\longrightarrow$	collisions
Hard sphere interactions	$\longrightarrow$	collisions
Gravitational forces	$\longrightarrow$	mean-field

Collisional invariants

- Consider a quantity $I(\vec{\boldsymbol{p}})$, and the integral

$$\mathcal{I}[f] \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, \mathcal{C}_{\boldsymbol{p}}[f]\; I(\vec{\boldsymbol{p}})$$

- By symmetry under the exchange $(\vec{\boldsymbol{p}},\vec{\boldsymbol{p}}') \leftrightarrow (\vec{\boldsymbol{k}},\vec{\boldsymbol{k}}')$ and antisymmetry under $(\vec{\boldsymbol{p}},\vec{\boldsymbol{k}}) \leftrightarrow (\vec{\boldsymbol{p}}',\vec{\boldsymbol{k}}')$, we can write

$$\mathcal{I}[f] = \frac{1}{4}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, \mathcal{C}_{\boldsymbol{p}}[f]\, \Big[I(\vec{\boldsymbol{p}}) + I(\vec{\boldsymbol{k}}) - I(\vec{\boldsymbol{p}}') - I(\vec{\boldsymbol{k}}')\Big]$$

- A quantity $I(\vec{\boldsymbol{p}})$ for which the bracket $[\cdots]$ vanishes is called a collisional invariant
- Collisional invariants :
  - $I(\vec{\boldsymbol{p}}) = 1$ (elastic collisions conserve the number of particles)
  - $I(\vec{\boldsymbol{p}}) = p^\mu$ (energy-momentum conservation)

Local conservation laws

- Define the density and current at point X for the quantity I :

$$I(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, I(\vec{\boldsymbol{p}})\; f(X,\vec{\boldsymbol{p}})$$

$$\vec{\boldsymbol{J}}_{I}(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, I(\vec{\boldsymbol{p}})\; \vec{\boldsymbol{v}}_{\boldsymbol{p}}\; f(X,\vec{\boldsymbol{p}})$$

- Multiply the Boltzmann equation by $I(\vec{\boldsymbol{p}})$ and integrate it over all the momenta $\boldsymbol{p}$:
  - The collision term gives zero for a collisional invariant
  - If there is no force term, then one obtains

$$\partial_t I(X) + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} \cdot \vec{\boldsymbol{J}}_{I}(X) = 0$$

▷ continuity equation for the local conservation of the quantity I

- Note : if there is a force term, the number of particles is locally conserved, but not their momentum

H theorem

- Define the quantities

$$h(X,\vec{\boldsymbol{p}}) \equiv (1+f(X,\vec{\boldsymbol{p}}))\ln(1+f(X,\vec{\boldsymbol{p}})) - f(X,\vec{\boldsymbol{p}})\ln(f(X,\vec{\boldsymbol{p}}))$$

$$H(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, h(X,\vec{\boldsymbol{p}}) \quad , \quad \vec{\boldsymbol{J}}_H(X) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, h(X,\vec{\boldsymbol{p}})\, \vec{\boldsymbol{v}}_{\boldsymbol{p}}$$

- From the Boltzmann equation, we get

$$\left[\partial_t + \vec{\boldsymbol{v}}_{\boldsymbol{p}}\cdot\vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] h + \vec{\boldsymbol{F}}(X)\cdot\vec{\boldsymbol{\nabla}}_{\boldsymbol{p}} h = \mathcal{C}_{\boldsymbol{p}}[f]\ln\left(\frac{1+f(X,\vec{\boldsymbol{p}})}{f(X,\vec{\boldsymbol{p}})}\right)$$

$$\partial_t H + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\cdot\vec{\boldsymbol{J}}_H = \sigma_H$$

with $$\sigma_H \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, \mathcal{C}_{\boldsymbol{p}}[f]\, \ln\left(\frac{1+f(X,\vec{\boldsymbol{p}})}{f(X,\vec{\boldsymbol{p}})}\right)$$

H theorem

- Using the symmetry properties of the collision term, we can rewrite σ_H as

$$\sigma_H = \frac{1}{4}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, \mathcal{C}_{\boldsymbol{p}}[f]\, \Bigg[\ln\left(\frac{1+f(X,\vec{\boldsymbol{p}})}{f(X,\vec{\boldsymbol{p}})}\right) + \ln\left(\frac{1+f(X,\vec{\boldsymbol{k}})}{f(X,\vec{\boldsymbol{k}})}\right) - \ln\left(\frac{1+f(X,\vec{\boldsymbol{p}}')}{f(X,\vec{\boldsymbol{p}}')}\right) - \ln\left(\frac{1+f(X,\vec{\boldsymbol{k}}')}{f(X,\vec{\boldsymbol{k}}')}\right)\Bigg]$$

- In the right hand side, one can rewrite the factors that depend on f as follows :

$$f(X,\vec{\boldsymbol{p}})f(X,\vec{\boldsymbol{k}})(1+f(X,\vec{\boldsymbol{p}}'))(1+f(X,\vec{\boldsymbol{k}}'))\left[\frac{\alpha_{\boldsymbol{p}}\alpha_{\boldsymbol{k}}}{\alpha_{\boldsymbol{p}'}\alpha_{\boldsymbol{k}'}}-1\right]\ln\left(\frac{\alpha_{\boldsymbol{p}}\alpha_{\boldsymbol{k}}}{\alpha_{\boldsymbol{p}'}\alpha_{\boldsymbol{k}'}}\right)$$

with $\alpha_{\boldsymbol{p}} \equiv (1+f(X,\vec{\boldsymbol{p}}))/f(X,\vec{\boldsymbol{p}})$

- Since $(X-1)\ln(X) \geq 0$, we have $\sigma_H \geq 0$
  $\triangleright$ the quantity H has a positive source term

H theorem

- Interpretation :
  - $H(X)$ is the entropy density, and $\vec{J}_H(X)$ its current
  - Because the continuity equation for H has a right hand side σ_H, it is not a conserved quantity
  - Because $\sigma_H \geq 0$, the total amount of H in the system can only increase
- Remarks :
  - This seems to contradict Poincaré's recurrence theorem :
    "Any system with a finite volume phase-space will return arbitrarily close to its initial conditions in a finite time"
    $\triangleright$ where does the irreversibility come from in the Boltzmann eq.?
  - Molecular chaos assumption : the Boltzmann equation is an approximation of the full dynamical evolution of the system, in which one neglects correlations among particles prior to collisions. By dropping these correlations, one loses the information necessary to reverse the time evolution of the system

H theorem

- Loschmidt's paradox :
  - Evolve a gas of particles from $t = 0$ to $t = t_0$
  - At $t = t_0$, reverse all the velocities
  - Evolve the system from $t = t_0$ to $t = 2t_0$
- In a mechanistic description, the entropy at $2t_0$ should be the same as the entropy at 0, because all the particles went back to their original location
- With the Boltzmann equation, the H theorem implies $H(2t_0) > H(0)$ (having reversed the velocities at $t = t_0$ does not alter this conclusion)

H theorem

- Loschmidt's paradox :
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- Resolution : the assumption that pairs of particles are non-correlated before the collisions breaks down when one tries to reverse the time direction in the Boltzmann equation

Equilibrium state

- When the equilibrium is reached, $\sigma_H = 0$
  - ▷ $\ln((1+f_{\rm eq})/f_{\rm eq})$ is a collisional invariant
  - ▷ it is a linear combination of 1 and p^μ :

$$\ln\left(\frac{1+f_{\rm eq}(X,\vec{\boldsymbol{p}})}{f_{\rm eq}(X,\vec{\boldsymbol{p}})}\right) = \alpha+\beta_\mu p^\mu \quad \Rightarrow \quad f_{\rm eq}(X,\vec{\boldsymbol{p}}) = \frac{1}{e^{\alpha+\beta_\mu p^\mu}-1}$$

(Bose-Einstein distribution)

- $\beta_\mu p^\mu$ is the Lorentz covariant form of p^0/T $\;(\beta = 1/T)$
- α is a chemical potential associated to the conservation of the number of particles

Transport coefficients

Transport coefficients

- The Boltzmann equation is a powerful tool for calculating transport coefficients such as conductivity, viscosity, diffusion constants
- These transport coefficients can also be calculated in quantum field at finite temperature. Example for the electric conductivity :
  - $\sigma_{\rm el}$ is the coefficient of proportionality between the induced electric current and the applied electric field :

$$\vec{\boldsymbol{j}}_{\rm el} = \sigma_{\rm el}\, \vec{\boldsymbol{E}}$$

  - It is given by a current-current correlator (Kubo's formula) :

$$\sigma_{\rm el} = \frac{1}{6} \lim_{\omega\to 0} \int d^4x\; e^{i\omega t} \left\langle j^i_{\rm el}(t,\vec{\boldsymbol{x}})\; j^i_{\rm el}(0,\vec{\boldsymbol{0}})\right\rangle_T$$

  - This correlation function can be evaluated from Feynman diagrams at finite temperature, but one needs to sum an infinite series of graphs ▷ quite difficult

Transport coefficients

- In the evaluation of $\sigma_{\rm el}$ from the Boltzmann equation, one perturbs a system at equilibrium by a small electric field
  ▷ it enters in the Boltzmann equation via the force $\vec{\boldsymbol{F}} \equiv e\vec{\boldsymbol{E}}$
- This force induces a departure of f away from $f_{\rm eq}$. It is convenient to parameterize it by

$$f(X,\vec{\boldsymbol{p}}) \equiv f_{\rm eq}(X,\vec{\boldsymbol{p}}) + f_{\rm eq}(X,\vec{\boldsymbol{p}})(1+f_{\rm eq}(X,\vec{\boldsymbol{p}}))\; f_1(X,\vec{\boldsymbol{p}})$$

- Since the applied field is small, the deviation f_1 is also small
  ▷ linearize the collision term in f_1 :

$$\begin{aligned}\mathcal{C}_{\boldsymbol{p}}[f] &= \boldsymbol{L}_{\boldsymbol{p}}\cdot f_1 + \mathcal{O}(f_1^2)\\ &\equiv \frac{1}{2E_{\boldsymbol{p}}}\int\frac{d^3\vec{\boldsymbol{p}}'}{(2\pi)^3 2E_{\boldsymbol{p}'}}\int\frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3 2E_{\boldsymbol{k}}}\int\frac{d^3\vec{\boldsymbol{k}}'}{(2\pi)^3 2E_{\boldsymbol{k}'}}(2\pi)^4\delta(p+k-p'-k')\\ &\quad\times f_{\rm eq}(X,\vec{\boldsymbol{p}})f_{\rm eq}(X,\vec{\boldsymbol{k}})(1+f_{\rm eq}(X,\vec{\boldsymbol{k}}'))(1+f_{\rm eq}(X,\vec{\boldsymbol{p}}'))\\ &\quad\times\Big[f_1(X,\vec{\boldsymbol{p}})+f_1(X,\vec{\boldsymbol{k}})-f_1(X,\vec{\boldsymbol{p}}')-f_1(X,\vec{\boldsymbol{k}}')\Big]\,\big|\mathcal{M}\big|^2\end{aligned}$$

Transport coefficients

- We apply an uniform electric field, hence $\vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} f(X, \vec{\boldsymbol{p}}) = 0$
- In order to have $\vec{\boldsymbol{j}}_{\rm el} = \sigma_{\rm el}\, \vec{\boldsymbol{E}}$, we must reach the stationary regime. Therefore $\partial_t f(X, \vec{\boldsymbol{p}}) = 0$
- Since the applied field is small, it is legitimate to replace f by $f_{\rm eq}$ in the force term. Thus, the linearized Boltzmann equation reads :

$$\boldsymbol{L}_{\boldsymbol{p}} \cdot f_1 = e\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{p}}\, f_{\rm eq}(X, \vec{\boldsymbol{p}})$$

- Solve this equation (not easy, but doable numerically). Since it is a linear equation, the solution f_1 is linear in $\vec{\boldsymbol{E}}$
- Then, one calculates the current induced by this perturbation of the particle distribution,

$$\vec{\boldsymbol{j}}_{\rm el} = e \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\, \vec{\boldsymbol{v}}_{\boldsymbol{p}}\, f_{\rm eq}(X, \vec{\boldsymbol{p}})(1 + f_{\rm eq}(X, \vec{\boldsymbol{p}}))\, f_1(X, \vec{\boldsymbol{p}})$$

$\triangleright$ read $\sigma_{\rm el}$ as the coefficient of proportionality between $\vec{\boldsymbol{j}}_{\rm el}$ and $\vec{\boldsymbol{E}}$

From Boltzmann to Hydrodynamics

Energy-momentum tensor

- In a theory invariant under translations in time and position, the energy and the momentum are conserved quantities
- For each direction ν, there is a conserved current, denoted $T^{\mu\nu}$, called the energy-momentum tensor, that obeys

$$\partial_\mu T^{\mu\nu} = 0$$

- The integral over space of the zero component gives the 4-momentum of the system

$$P^\nu = \int d^3\vec{\boldsymbol{x}}\; T^{0\mu}(t, \vec{\boldsymbol{x}})$$

- The vector $T^{i\mu}$ (i=1,2,3) represents the flow of the component μ of momentum. For $\mu = 0$, this is an energy flow. For $\mu = 1, 2, 3$, this is a 3-momentum flow and it is thus related to pressure

Ideal hydrodynamics

- Hydrodynamics is a macroscopic description of a system based on the local conservation laws
- Energy and momentum. This conservation law can be expressed as :

$$\partial_\mu T^{\mu\nu} = 0$$

- There are additional continuity equations for every conserved quantum number. Example : baryon number
- Number of equations : 4 + 1 for each extra conserved charge
- In order to turn these conservation laws into hydrodynamic equations, one needs to know the form of the energy-momentum tensor in terms of local quantities :
  - energy density $\epsilon(X)$
  - pressure $p(X)$
  - flow velocity $\vec{v}(X)$

Ideal hydrodynamics

- Consider a fluid cell at rest, of volume δV. It has an energy $\delta P^0 = \epsilon\,\delta V$ and a 3-momentum $\delta\vec{\boldsymbol{P}} = 0$. This can be achieved if the energy momentum tensor has the following components :

$$T^{00} = \epsilon \quad , \quad T^{0i} = 0$$

- The flow of momentum P^i across an element of surface $d\vec{\boldsymbol{S}}$ is $dP^i = dS^j T^{ji}$. From the definition of the pressure p, this must be equal to $p dS^i$. Hence $T^{ij} = p\delta^{ij}$.

- Therefore, in the local rest frame of the fluid :

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

Ideal hydrodynamics

- In an arbitrary frame where the fluid 4-velocity is v^μ, the energy-momentum tensor can only be built from the symmetric tensors $g^{\mu\nu}$ and $v^\mu v^\nu$. In the local rest frame ($v^\mu = (1, 0, 0, 0)$), we must recover the previous expression. Therefore :

$$T^{\mu\nu} = (p + \epsilon)\, v^\mu\, v^\nu - p\, g^{\mu\nu}$$

- Note : this expression is valid only for an ideal fluid, with no dissipative phenomena. In a viscous fluid, there can be a transport of momentum due to the friction of fluid layers that move at different velocities. This is taken into account by additional terms in $T^{\mu\nu}$ that are proportional to the derivatives $\partial_i v^j$, multiplied by the viscosity η.

Ideal hydrodynamics

- In the non-relativistic limit, the energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ leads to :
  - $v^\mu \approx (1, \vec{\boldsymbol{v}})$
  - ϵ becomes the mass density ρ
  - the pressure p is much smaller than the energy density ϵ

It is easy to check that the above equation is equivalent to the continuity equation for mass and to Euler's equation :

$$\nu = 0 \;:\qquad \partial_t \rho + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}} \cdot (\rho \vec{\boldsymbol{v}}) = 0$$

$$\nu = i \;:\qquad \partial_t(\rho v^i) + \partial_j(\rho v^i v^j) + \partial_i p = 0$$

Note : the second equation can be cast into the more familiar form

$$\rho \left[\partial_t + \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\right] \vec{\boldsymbol{v}} + \vec{\boldsymbol{\nabla}}_{\boldsymbol{x}}\, p = 0$$

Ideal hydrodynamics

- In the previous equations, the unknown functions are :
  - $p(t,\vec{\boldsymbol{x}}), \epsilon(t,\vec{\boldsymbol{x}})$
  - $v^\mu(t,\vec{\boldsymbol{x}})$ (3 unknowns only, since $v_\mu v^\mu = 1$)
- $\partial_\mu T^{\mu\nu} = 0$ gives only 4 equations
- An additional constraint comes from the equation of state of the matter under consideration, as a relation between the local pressure p and energy density ϵ
- An initial condition $p_0(\vec{\boldsymbol{x}}), \epsilon_0(\vec{\boldsymbol{x}}), \vec{\boldsymbol{v}}_0(\vec{\boldsymbol{x}})$ must be specified at a certain time t_0. Since the relativistic Euler equation contains only first derivatives in time, this is sufficient to obtain the solution at any time $t > t_0$.

Kinetic point of view

- In kinetic theory, the expression of the energy-momentum tensor is :

$$T^{\mu\nu}(X) = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3} \frac{p^\mu p^\nu}{E_{\boldsymbol{p}}} f(X,\vec{\boldsymbol{p}})$$

  Note : this quantity transforms as a rank 2 tensor thanks to the fact that $d^3\vec{\boldsymbol{p}}/E_{\boldsymbol{p}}$ and $f(X,\vec{\boldsymbol{p}})$are Lorentz invariants

- This tensor combines the density $I(X)$ and the flow vector $\boldsymbol{J}_I(X)$ associated to the four collisional invariants p^ν

- The discussion on collisional invariants automatically implies that $\partial_\mu T^{\mu\nu} = 0$ at all times in kinetic theory

- But Boltzmann does not automatically imply that ideal hydrodynamics applies. One needs in addition that $T^{\mu\nu}$ takes the form $T^{\mu\nu} = \mathrm{Diag}\,(\epsilon, p, p, p)$

Kinetic point of view

- Assume that the Boltzmann equation has led to a situation of local thermal equilibrium characterized by :

$$f_{\rm eq}(X,\vec{\boldsymbol{p}}) = \frac{1}{e^{\beta_\mu(X)p^\mu} - 1}$$

- Find the velocity $-v^\mu(X)$ of the boost that transforms $\beta_\mu(X)p^\mu$ into $\beta(X)p^0$. $v^\mu(X)$ is the local flow velocity

- In this frame, the kinetic energy-momentum tensor is diagonal, and it is easy to compute ϵ and p as a function of $T \equiv 1/\beta$. For massless particles, one finds :

$$p = \frac{\epsilon}{3}$$

(equation of state of an ideal gas of massless particles)

Kinetic point of view

- Note : the conditions of application of ideal hydrodynamics are also realized under less restrictive conditions, when the particle distribution is isotropic in the local rest frame :

$$f(X,\vec{\boldsymbol{p}}) = f(X,|\vec{\boldsymbol{p}}|)$$

- With such a particle distribution, one gets $T^{0i} = 0$ and $T^{ij} = p\,\delta^{ij}$, with

$$p = \frac{1}{3}\int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\;|\vec{\boldsymbol{p}}|\; f(X,|\vec{\boldsymbol{p}}|)$$

- Similarly, one gets the energy density :

$$\epsilon = T^{00} = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3}\;|\vec{\boldsymbol{p}}|\; f(X,|\vec{\boldsymbol{p}}|)$$

  ▷ one obtains again $p = \epsilon/3$

Summary

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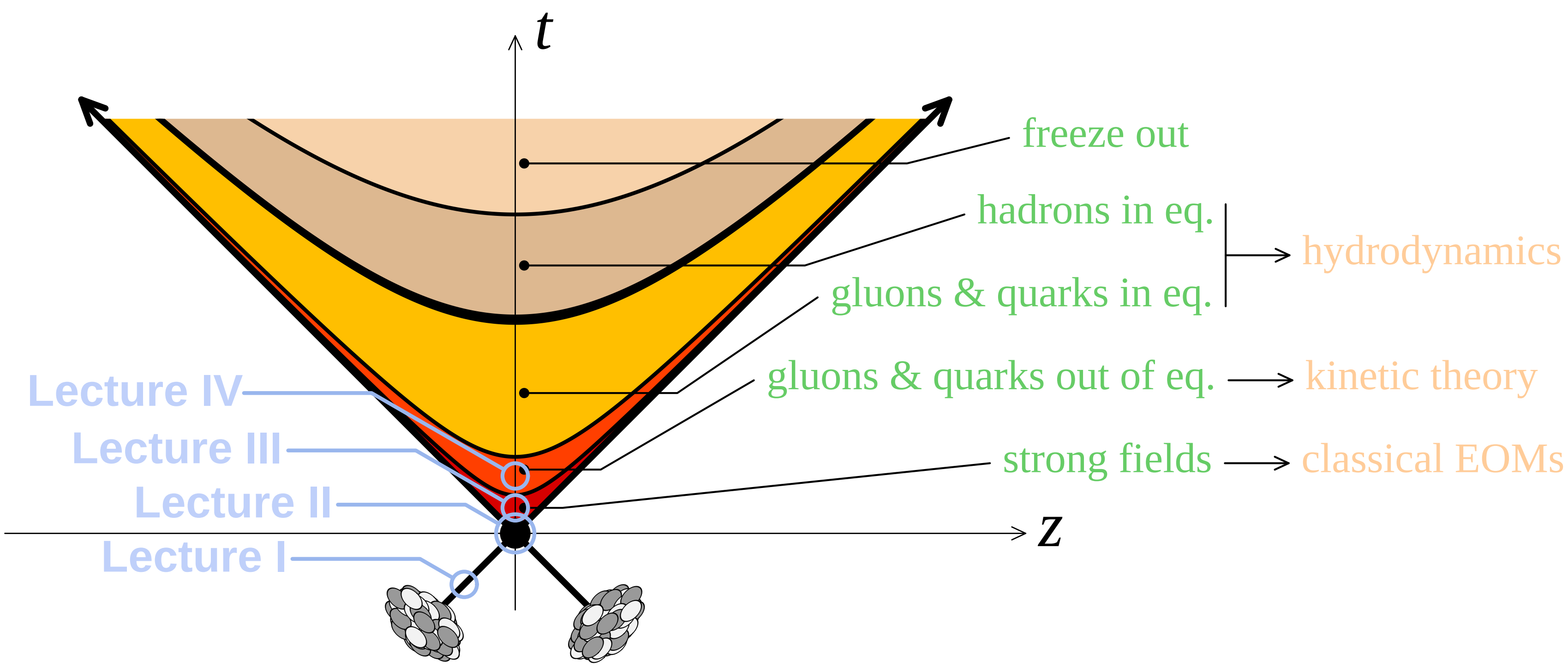

- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization
- Lecture IV : Kinetic theory, Near-Equilibrium dynamics