Pre-equilibrium dynamics in heavy ion collisions

III – Instabilities and thermalization

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General outline

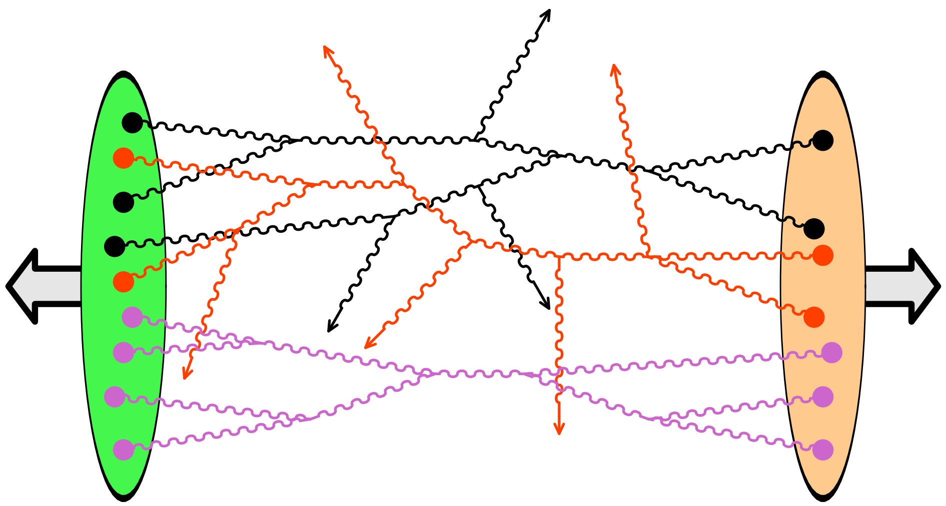
- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization
- Lecture IV : Kinetic theory, Near-Equilibrium dynamics

Lecture III : Instabilities, thermalization

- Reminder on initial gluon production
- Glasma instabilities
- Instabilities in anisotropic plasmas
- Thermalization ?

Reminder: initial gluon production

Relevant graphs in the saturated regime



- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial (+ pileup of many simultaneous scatterings)

Gluon spectrum at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)

- The gluon spectrum at LO is given by :

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} \propto \int_{x,y} e^{ip\cdot(x-y)} \;\cdots\; \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

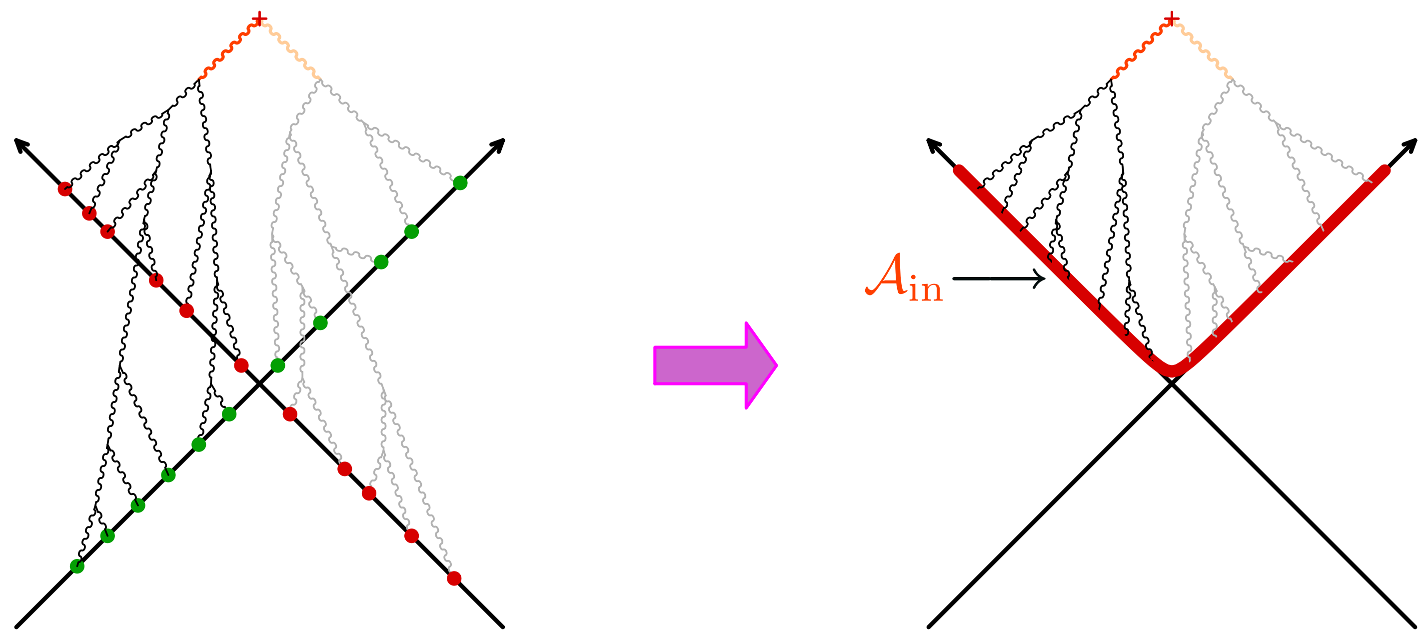
- $\mathcal{A}_\mu(x)$ is the retarded solution of Yang-Mills equations :

$$\begin{cases} \left[D_\mu, F^{\mu\nu}\right] = J^\nu \\ \mathcal{A}^\mu(x) \underset{t=-\infty}{=} 0 \end{cases}$$

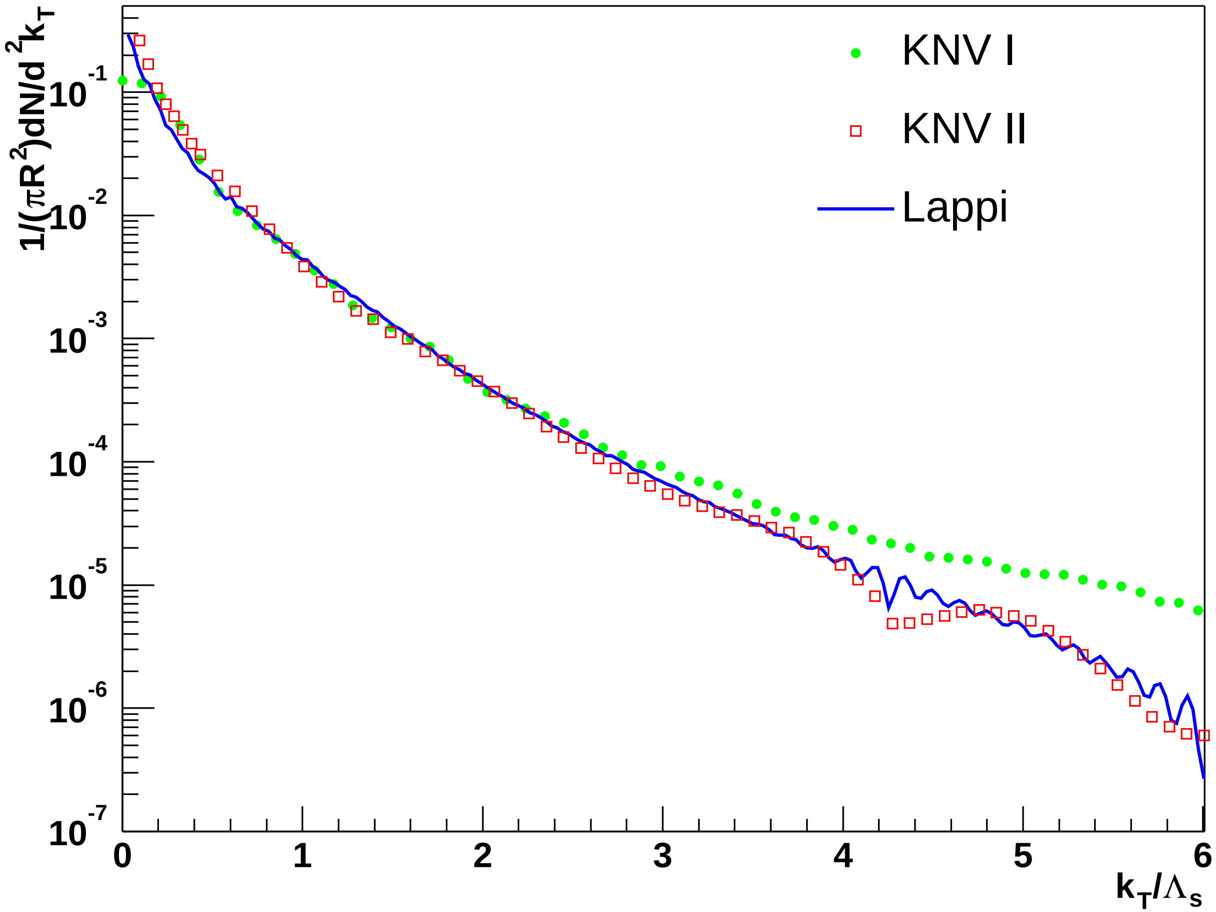
- $\overline{N}_{LO}$ is a functional of the gauge field $\mathcal{A}_{\rm in}$ on the light-cone
  Note : the functional $\overline{N}_{LO}[\mathcal{A}_{\rm in}]$ has no explicit dependence on the sources $\rho_{1,2}$, because there are no sources above the light-cone ($\rho_{1,2}$ are hidden in $\mathcal{A}_{\rm in}$)

Gluon spectrum at LO

- The calculation is done in the gauge : $x^+\mathcal{A}^- + x^-\mathcal{A}^+ = 0$
- In this gauge, one can find analytically the field $\mathcal{A}_{\rm in}$ just above the light-cone, at $\tau = 0^+$. Therefore, a numerical resolution is required only in the forward light-cone :

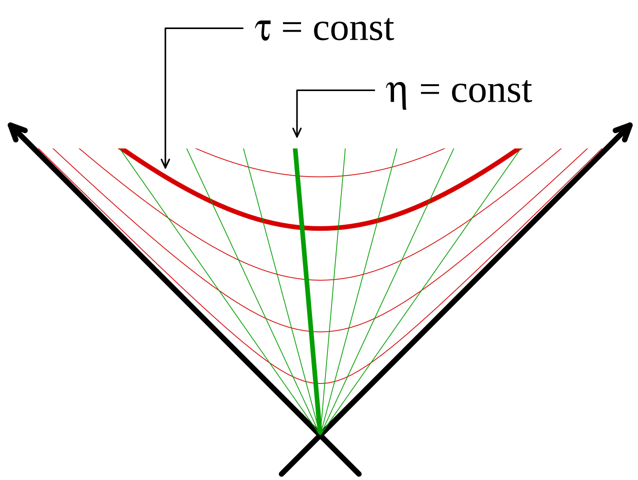


Gluon spectrum at LO



- Important softening at small $k_\perp$ compared to pQCD (saturation)

Boost invariance



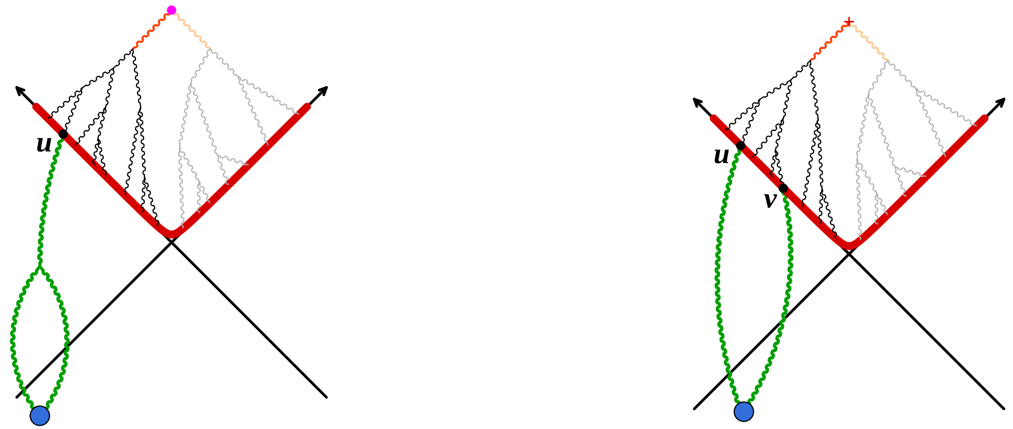
- Initial values at $\tau = 0^+$: the initial fields $\mathcal{A}_{\rm in}$ do not depend on the rapidity η

  ▷ they remain independent of η at all times (invariance under boosts in the z direction)

  ▷ numerical resolution performed in $1+2$ dimensions

1-loop corrections to $\overline{N}$

- One can divide the 1-loop corrections into a piece below the light-cone (calculable analytically) and a part above the light-cone (that must be solved numerically) :



  - Any divergence that happens in the part below the light-cone is related to the initial state, and we should try to absorb it in the “parton distributions” $W[\rho_{1,2}]$
  - Anything in the forward light-cone happens after the collision and has to do with the evolution of the final state

Divergences

- If taken at face value, this 1-loop correction is plagued by several divergences :
  - The pieces below the light-cone are infinite, because of an unbounded integration over a rapidity variable

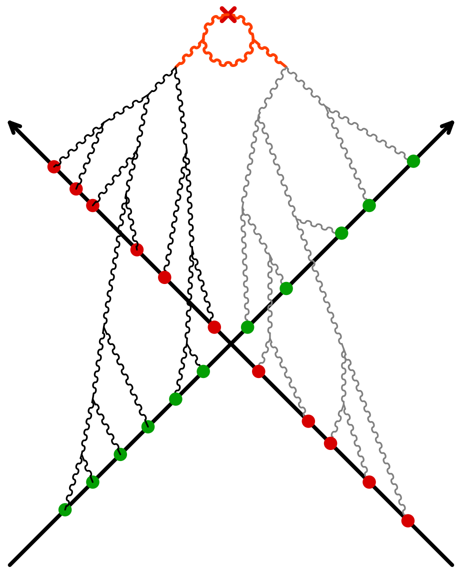
    It seems likely that this part can be absorbed in the evolution of the parton distributions $W[\rho_{1,2}]$

  - The loop corrections can be seen as a perturbation $a_{\rm in}$ of the initial field $\mathcal{A}_{\rm in}$ on the light-cone

    However, the boost invariant classical solution of Yang-Mills equations suffers from an instability under rapidity dependent perturbations (Romatschke, Venugopalan (2005))

Collinear and IR divergences

- A gluon spectrum is not a collinear and infrared safe quantity
  ▷ strictly speaking, $d\overline{N}/dY\,d^2\vec{\boldsymbol{p}}_\perp$ is not well defined
- This can also be seen at the 1-loop level, where one also gets IR and collinear divergences (in addition to the two kind of divergences already mentioned)
- They come from graphs where the loop is attached near the end point of the trees



Collinear and IR divergences

- Solution I : convolute the gluon spectrum with a fragmentation function. However, in heavy ion collisions, this would disregard all the dynamics that take place before hadronization

- Solution II : the gluon spectrum involves taking the limit $\tau \longrightarrow \infty$

$$\int d^4x \; e^{ip\cdot x} \; \square_x \, \mathcal{A}_\mu(x) = \lim_{x^0 \to +\infty} \int d^3\vec{\boldsymbol{x}} \; e^{ip\cdot x} \; \left[\partial_0 - iE_p\right] \mathcal{A}_\mu(x)$$

  Note that the IR and collinear divergences are due to this limit
  ▷ compute instead the gluon spectrum at a finite time τ

- Solution III : instead of a gluon spectrum, one can compute the density of energy-momentum tensor $T^{\mu\nu}(\tau, \vec{\boldsymbol{x}})$. Since $T^{\mu\nu}$ only keeps track of the flow of energy and momentum, it is unaffected by soft or collinear emission. Moreover, this is the relevant initial condition for hydrodynamics

Initial state factorization

- If the factorization of the logs $(1/x_{1,2})$ in the $W[\rho_{1,2}]$ work as expected, one can write

$$\frac{d\overline{N}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \int [D\rho_1]\,[D\rho_2]\;\; W_{Y_{\rm beam}-Y}[\rho_1]\;W_{Y_{\rm beam}+Y}[\rho_2] \times \frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]}{dY\,d^2\vec{\boldsymbol{p}}_\perp}$$

- Somewhat analogous to factorization in conventional pQCD :

$W_Y[\rho]$ $\longleftrightarrow$ parton distribution

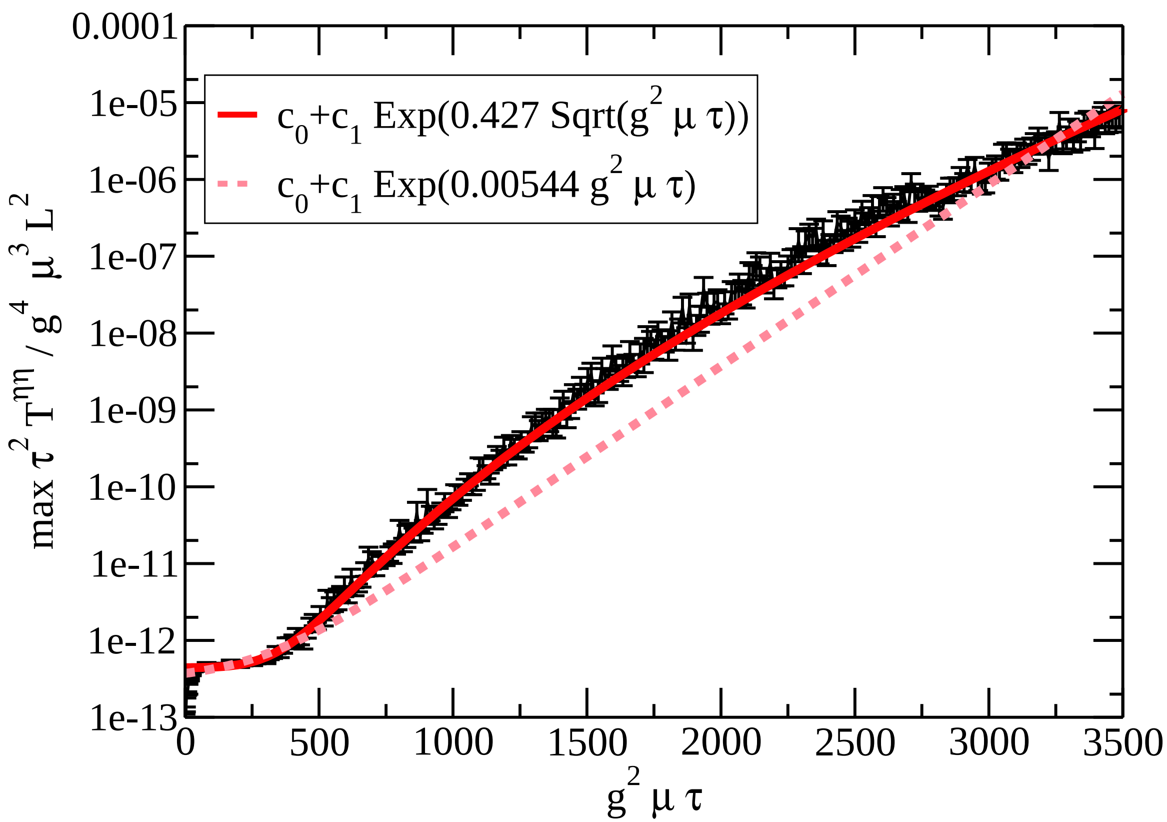
and it has the same conceptual importance, because it implies the universality of the distributions $W_Y[\rho]$ (e.g. that they are identical in eA and in AA collisions)

Glasma instabilities

Unstable modes

Romatschke, Venugopalan (2005)

- Rapidity dependent perturbations to the classical fields grow like $\exp(\sqrt{\mu\tau})$ until the non-linearities become important :



Unstable modes

- The instability of the boost-invariant classical field appears only at 1-loop. We therefore have to deal with terms that contain

$$\alpha_s \, e^{\sqrt{\mu\tau}} \qquad\qquad (\mu \sim Q_s)$$

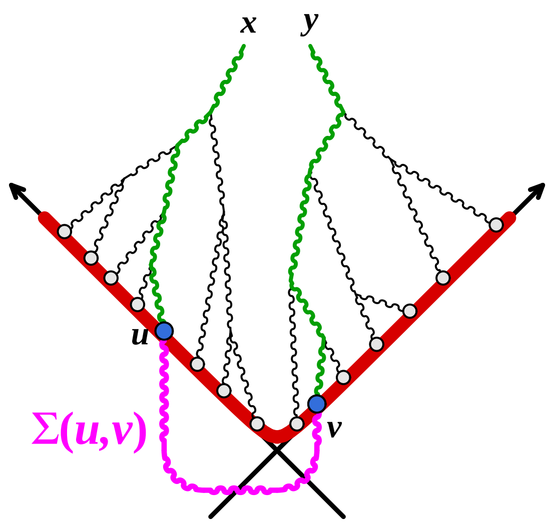
▷ breakdown of the plain perturbative expansion at the time :

$$\tau \sim Q_s^{-1} \, \ln^2\left(\frac{1}{\alpha_s}\right)$$

- Possible way out: identify among the n-loop diagrams the terms that have the fastest growth with time, $[\alpha_s \, e^{\sqrt{\mu\tau}}]^n$, and resum them
- If the resulting series in $\xi \equiv \alpha_s \, e^{\sqrt{\mu\tau}}$ can be analytically continued on the positive real axis, then the above time limit will have been pushed up

Power counting

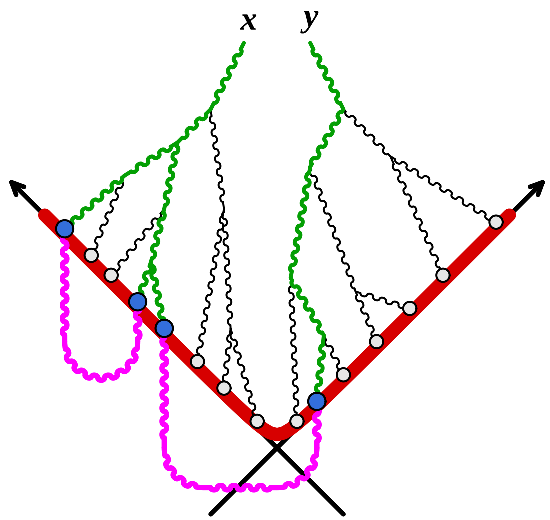
- Note : if a perturbation $a_{\rm in}$ on the light-cone is boost invariant, then it does not trigger the instability
- A 1-point function in a boost invariant background is itself boost invariant $\triangleright$ the 1-point function below the light-cone is boost invariant, and cannot trigger the instability
- The instabilities are triggered by the 2-point function :



Power counting

- At n-loop order, one must pick the terms that have the fastest growth in time

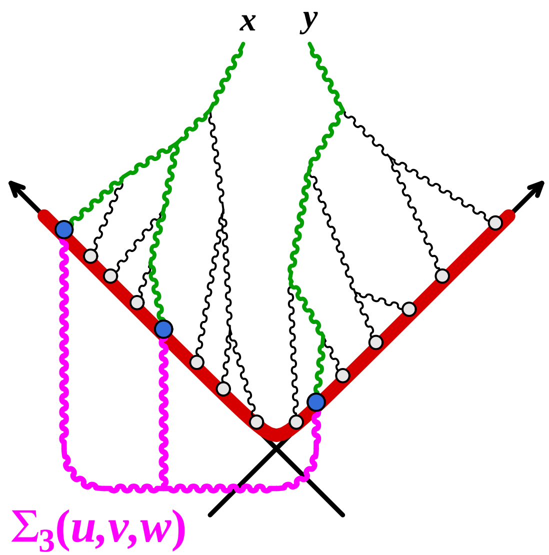
  ▷ one must maximize the number of locations where the initial field is perturbed on the light-cone, while minimizing the powers of α_s



- Power counting : $\Sigma \sim \mathcal{O}(1)$, $\bullet \sim \mathcal{O}(g\, e^{\sqrt{\mu\tau}})$

Power counting

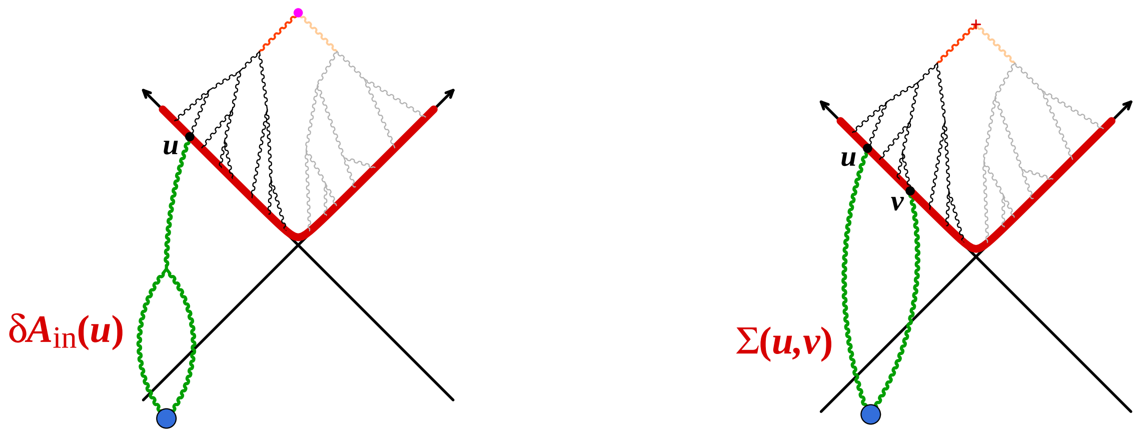
- Why higher (non-Gaussian) correlations may be suppressed :



- Power counting : $\Sigma_3 \sim \mathcal{O}(g)$, ● $\sim \mathcal{O}(g\ e^{\sqrt{\mu\tau}})$
  - ▷ more powers of g than powers of $e^{\sqrt{\mu\tau}}$

Instabilities from field fluctuations

- In order to go further in the study of these instabilities, we need a more explicit formula for the NLO correction to $\overline{N}$:



$$\delta\overline{N} = \left[\int\limits_{\vec{u}\,\in\,\text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{u})\; \boldsymbol{T}_{\vec{u}} + \int\limits_{\vec{u},\vec{v}\,\in\,\text{light cone}} \frac{1}{2}\; \boldsymbol{\Sigma}(\vec{u},\vec{v})\; \boldsymbol{T}_{\vec{u}}\, \boldsymbol{T}_{\vec{v}} \right] \overline{N}_{LO}$$

- $\boldsymbol{T}_{\vec{u}}$ is the generator of shifts of the initial condition at the point $\vec{u}$ on the light-cone, i.e. : $\boldsymbol{T}_{\vec{u}} \sim \delta/\delta\mathcal{A}_{\rm in}(\vec{u})$

Instabilities from field fluctuations

- Reminder : the 1-point function $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$ does not contribute to the instability

- If higher correlations are indeed suppressed, then the leading contributions are resummed simply by exponentiating the 1-loop one :

$$\left[\overline{N}\right]_{\substack{\rm unstable\\ \rm modes}} = \underbrace{\exp\left\{\frac{1}{2}\int_{\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}} \boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\, \boldsymbol{T}_{\vec{\boldsymbol{x}}}\boldsymbol{T}_{\vec{\boldsymbol{y}}}\right\}}_{Z[\boldsymbol{T}_{\vec{\boldsymbol{x}}}]}\ \overline{N}_{LO}\left[\mathcal{A}_{\rm in}(\rho_1,\rho_2)\right]$$

still not very illuminating...

Instabilities from field fluctuations

- This can be arranged in a more intuitive way by doing a Fourier transform of $Z[\boldsymbol{T}_{\vec{\boldsymbol{x}}}]$:

$$Z[\boldsymbol{T}_{\vec{\boldsymbol{x}}}] = \int [Da(\vec{\boldsymbol{x}})]\ \widetilde{Z}[a(\vec{\boldsymbol{x}})]\ e^{i\int_{\vec{\boldsymbol{x}}} a(\vec{\boldsymbol{x}})\ \boldsymbol{T}_{\vec{\boldsymbol{x}}}}$$

$$\begin{aligned}\left[\overline{N}\right]_{\substack{\text{unstable}\\ \text{modes}}} &= \int [Da(\vec{\boldsymbol{x}})]\ \widetilde{Z}[a(\vec{\boldsymbol{x}})]\ e^{i\int_{\vec{\boldsymbol{x}}} a(\vec{\boldsymbol{x}})\ \boldsymbol{T}_{\vec{\boldsymbol{x}}}}\ \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)] \\ &= \int [Da(\vec{\boldsymbol{x}})]\ \widetilde{Z}[a(\vec{\boldsymbol{x}})]\ \overline{N}_{LO}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]\end{aligned}$$

▷ summing these divergences simply requires to add fluctuations to the initial condition for the classical problem
▷ the fact that $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})$ does not contribute implies that the distribution $\widetilde{Z}[a(\vec{\boldsymbol{x}})]$ of fluctuations is real

- Note : if $Z[\boldsymbol{T}_{\vec{\boldsymbol{x}}}]$ is a Gaussian, then the distribution $\widetilde{Z}[a(\vec{\boldsymbol{x}})]$ is also a Gaussian

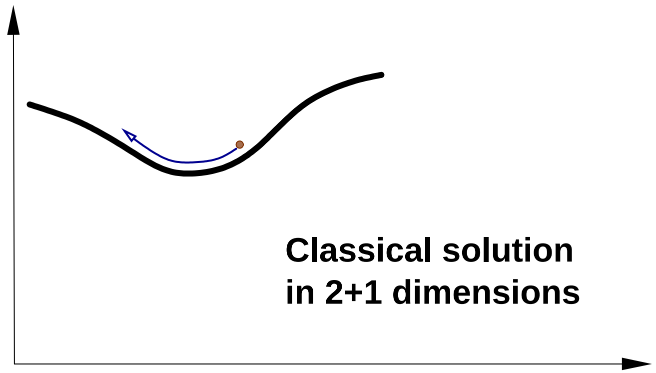
Physical interpretation

- Interpretation :

  Despite the fact that the fields are strong, the classical approximation alone is not good enough, because the classical solution has unstable modes that can be triggered by the quantum fluctuations

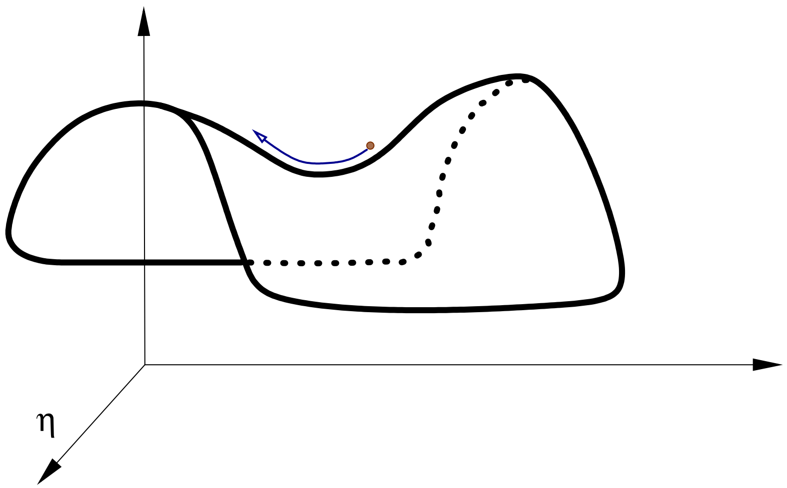
- After this resummation, the large time behavior does not have exponentially growing terms anymore :
  - These growing terms were due to the fact that, at fixed loop order, the fluctuations evolve according to the linearized equation for small perturbations
  - After the resummation, the fluctuations have been incorporated in the initial conditions for the fully non-linear field equations
  - Since the QCD Hamiltonian is bounded from below, the non-linear field equations have no runaway solutions

Physical interpretation

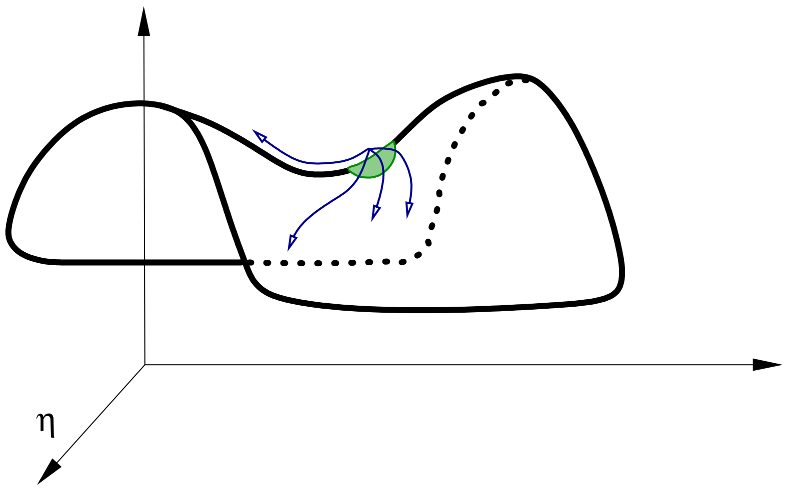


**Classical solution
in 2+1 dimensions**

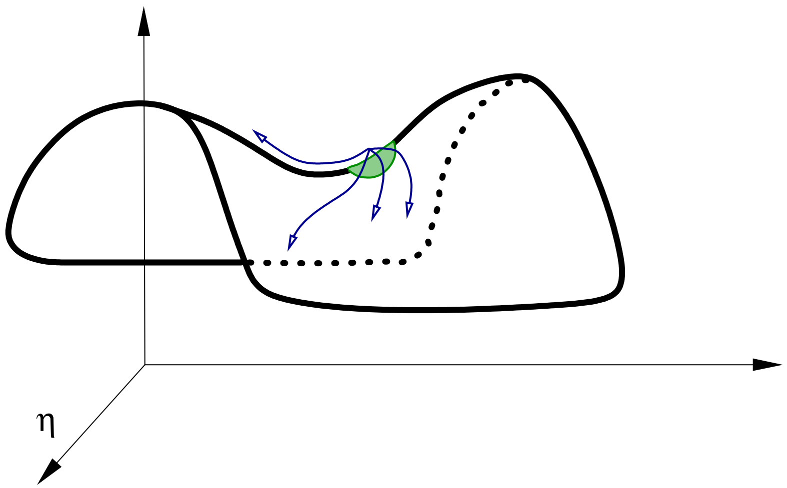
Physical interpretation



Physical interpretation



Physical interpretation



- Combining everything, one should write :

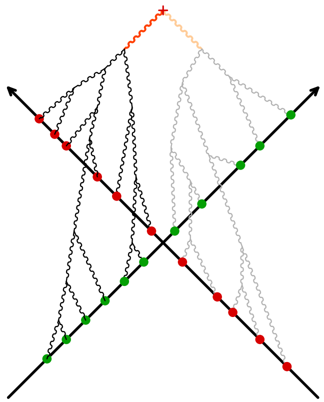
$$\frac{d\overline{N}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \int [D\rho_1]\,[D\rho_2]\; W_{Y_{\rm beam}-Y}[\rho_1]\, W_{Y_{\rm beam}+Y}[\rho_2] \times \int [Da]\;\; \widetilde{Z}[a]\;\; \frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)+a]}{dY\,d^2\vec{\boldsymbol{p}}_\perp}$$

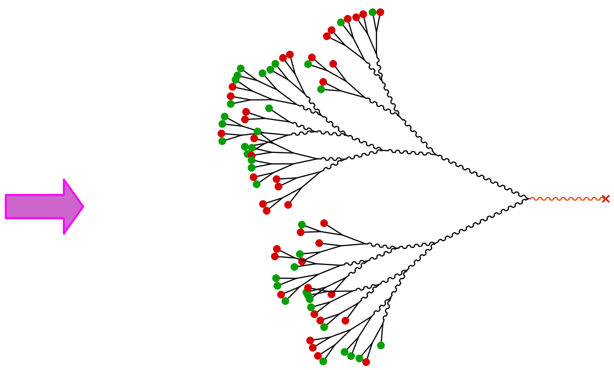
$W_Y[\rho]$ $\longleftrightarrow$ parton distribution

$\widetilde{Z}[a]$ $\longleftrightarrow$ fragmentation function

Instabilities and gluon splitting

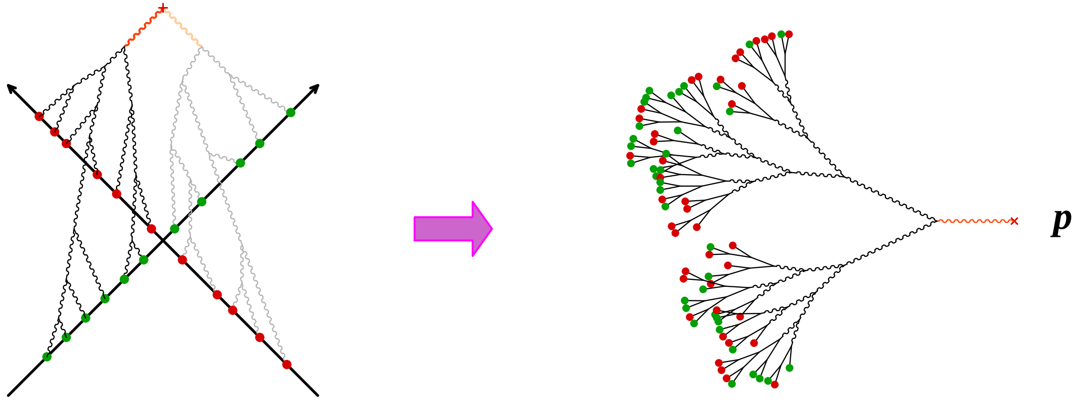
- Tree level :



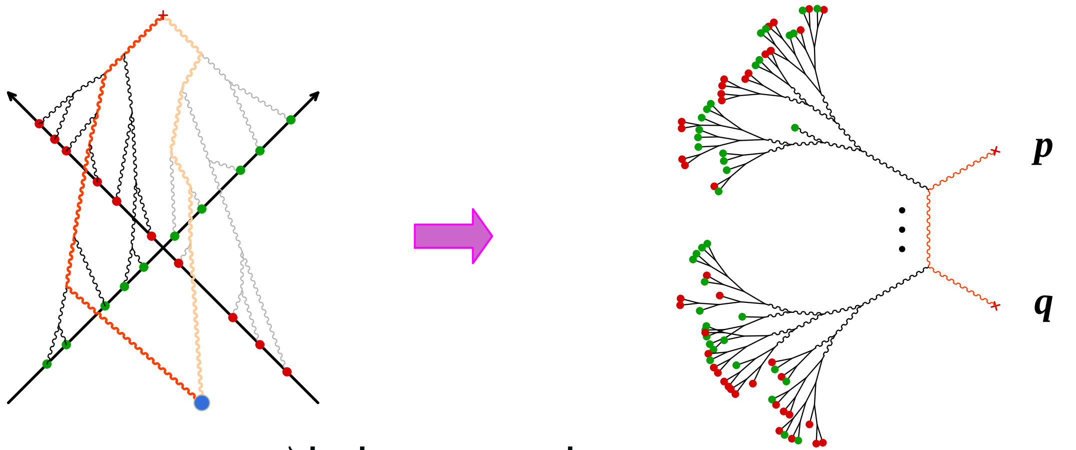


Instabilities and gluon splitting

- Tree level :



- One loop ▷ gluon pairs :



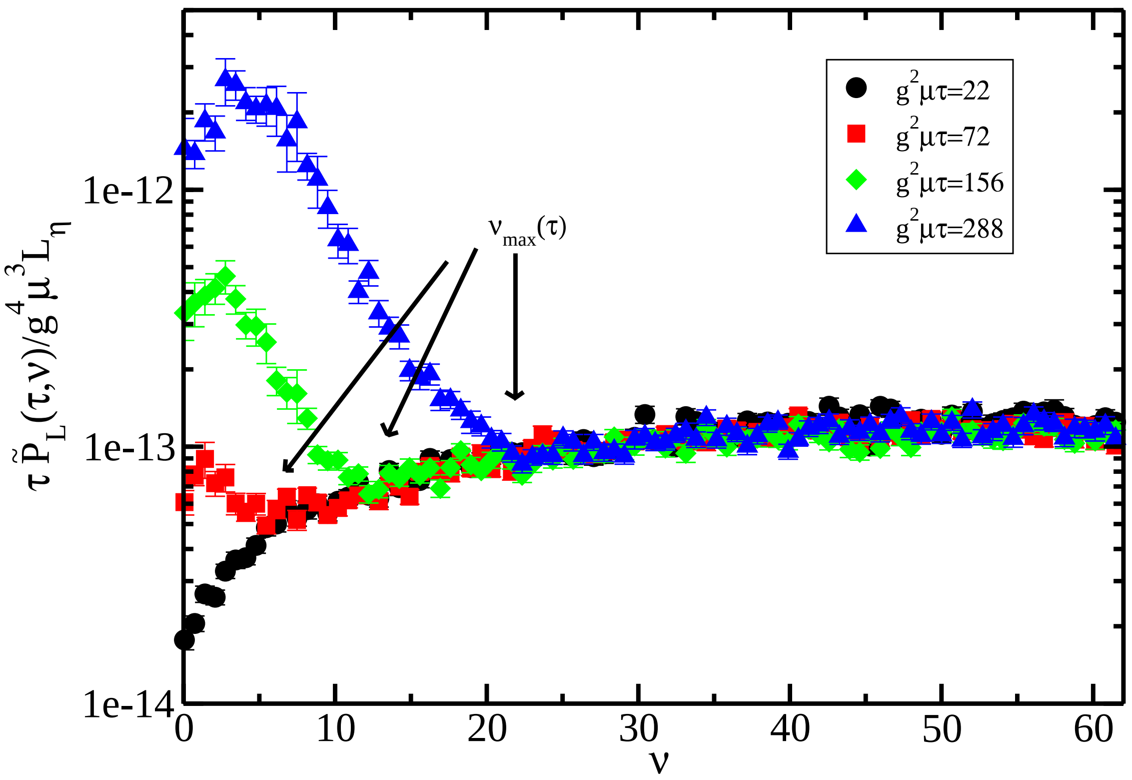
▷ The momentum $\vec{q}$ is integrated out

▷ If $\alpha_s^{-1} \lesssim |y_{\boldsymbol{p}} - y_{\boldsymbol{q}}|$, the correction is absorbed in $W[\rho_{1,2}]$

▷ If $|y_{\boldsymbol{p}} - y_{\boldsymbol{q}}| \lesssim \alpha_s^{-1}$: gluon splitting in the final state

Numerical results

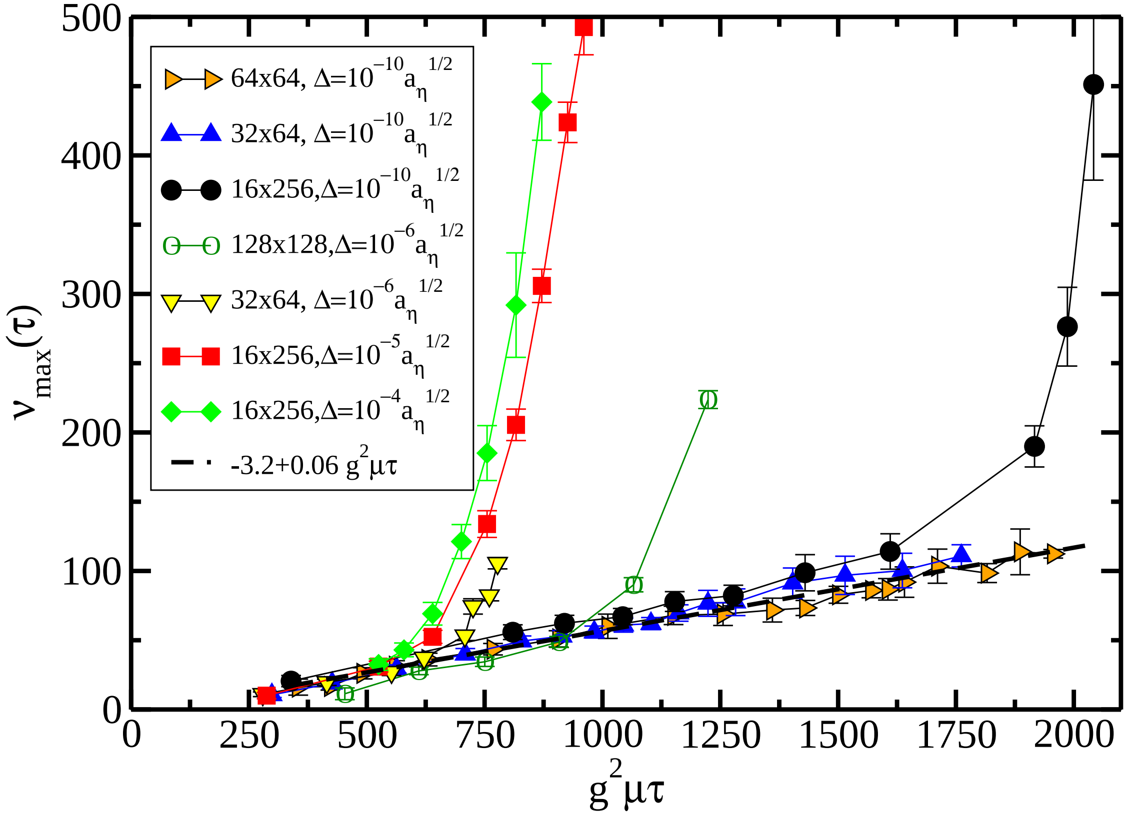
- Growing modes at different times :



▷ soft modes grow first

Numerical results

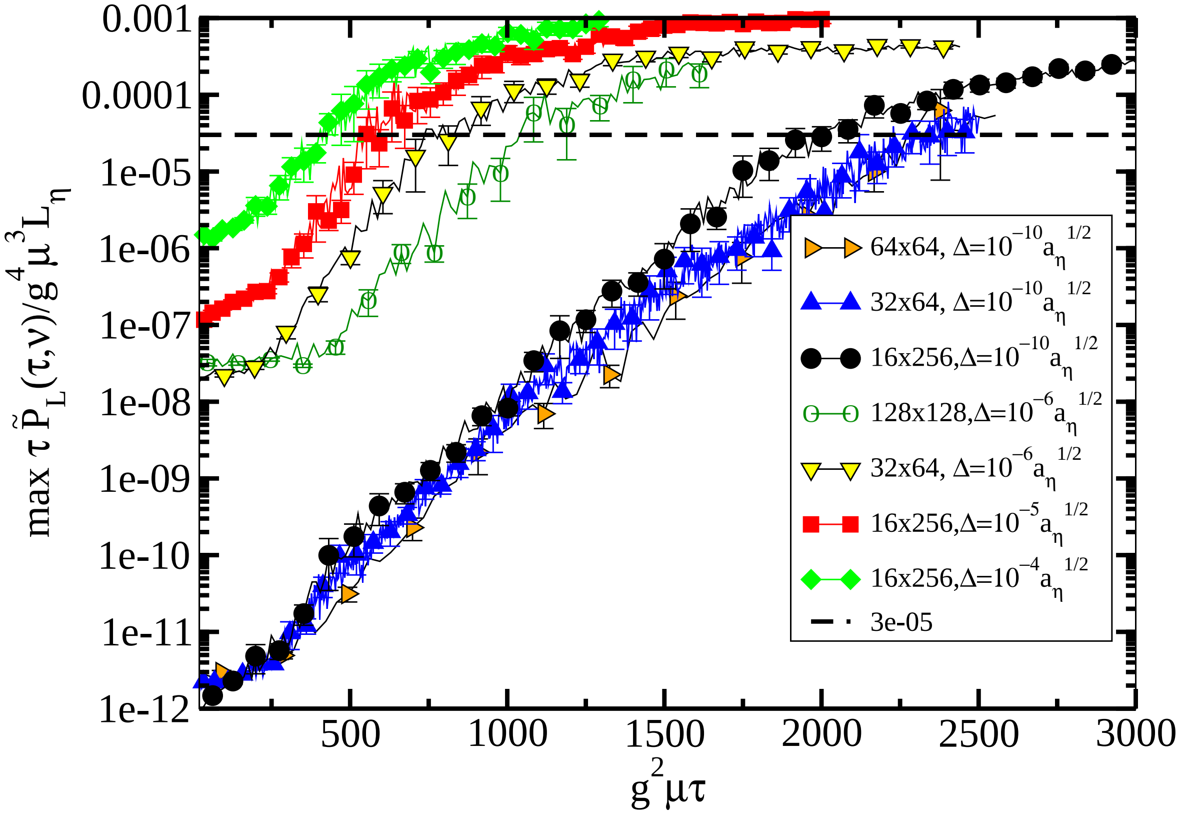
- Maximal growing mode as a function of time :



▷ eventually, explosion of the hard modes

Numerical results

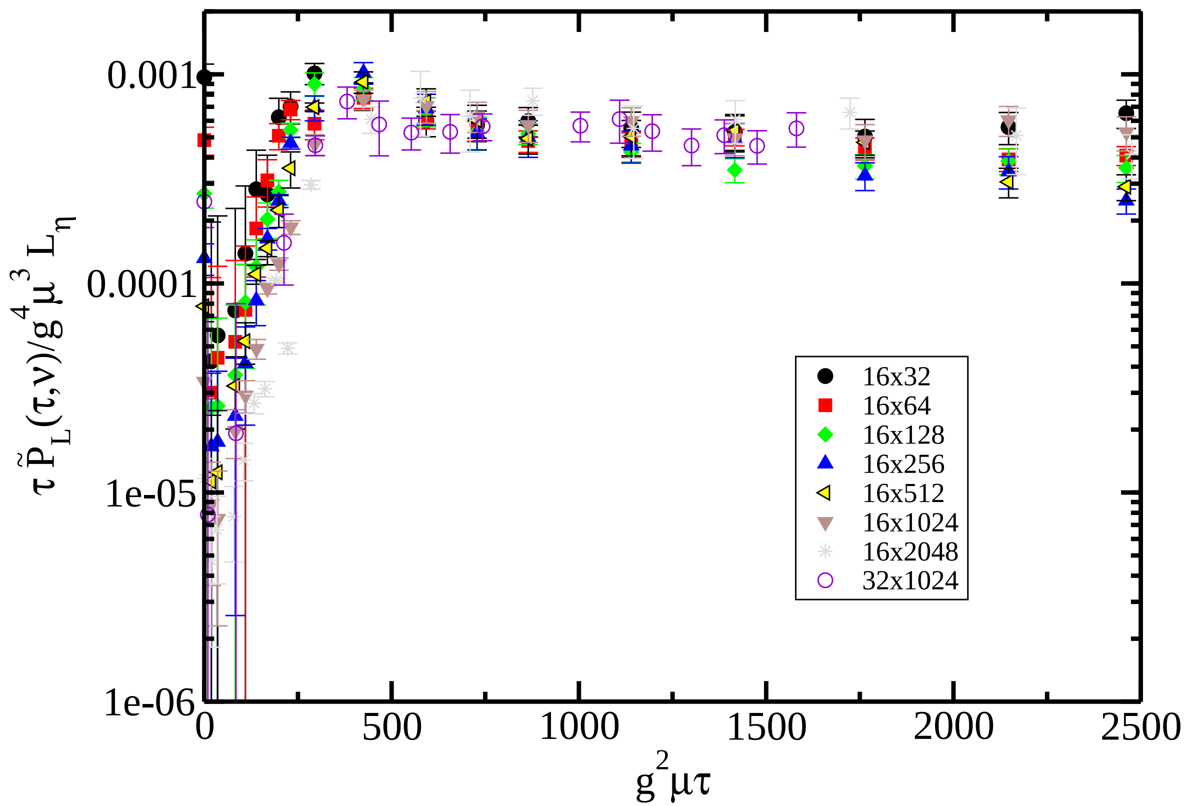
- Maximal amplitude as a function of time (weak anisotropy) :



▷ the UV explosion occurs when this amplitude reaches some fixed value

Numerical results

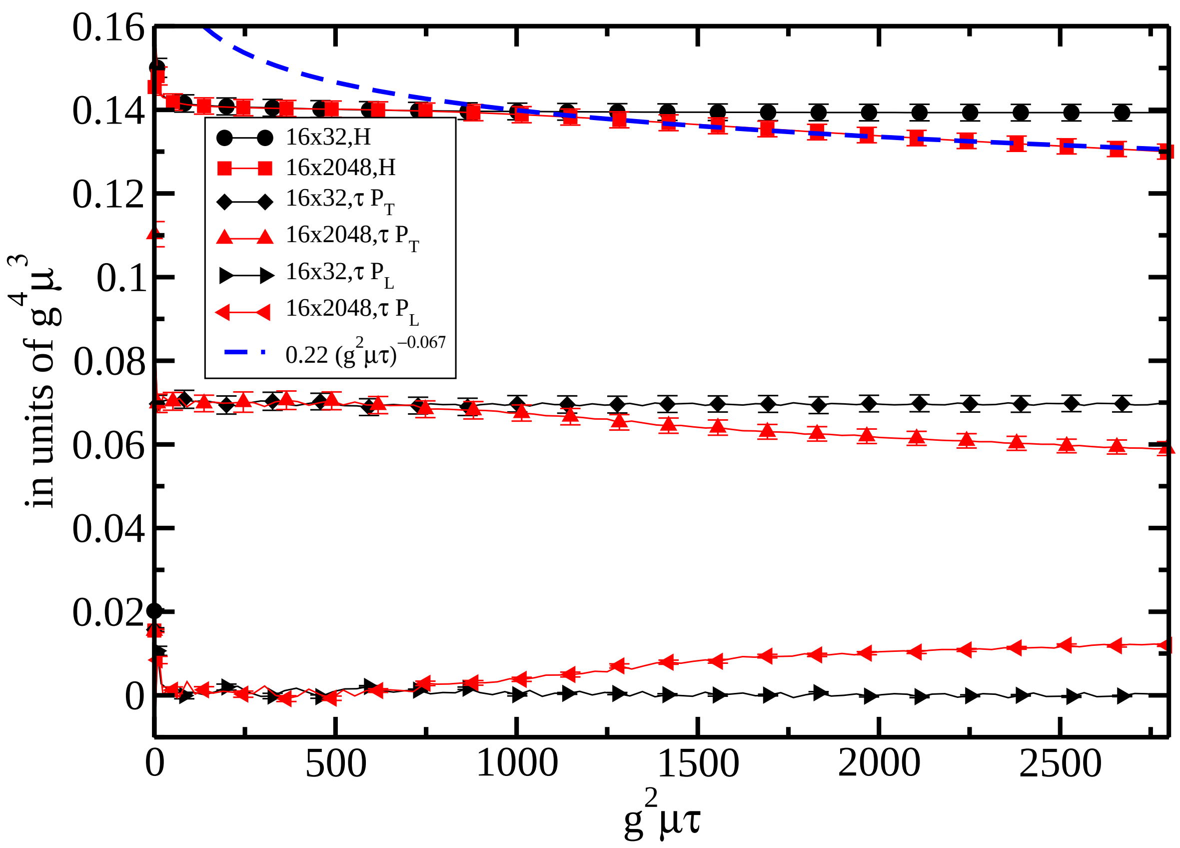
- Maximal amplitude as a function of time (larger anisotropy) :



▷ the longitudinal pressure grows faster for a larger initial anisotropy

Numerical results

- Time evolution of P_T, P_L and $\tau\epsilon$:



▷ the energy density drops slightly faster than τ^{-1}

Initial Gaussian fluctuations

- The previous numerical results have been obtained with a toy model for the distribution of initial fluctuations

- With some approximations, one can obtain a spectrum of Gaussian fluctuations characterized by :

$$\left\langle a_i(\eta,\vec{\boldsymbol{x}}_\perp)\,a_j(\eta',\vec{\boldsymbol{x}}'_\perp)\right\rangle = \\ = \frac{1}{\tau\sqrt{-(\partial_\eta/\tau)^2-\boldsymbol{\partial}_\perp^2}}\left[\delta_{ij}+\frac{\partial_i\partial_j}{(\partial_\eta/\tau)^2}\right]\delta(\eta-\eta')\,\delta(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{x}}'_\perp)$$

(Fukushima, FG, McLerran (2006))

- Numerical implementation under way by Lappi and Venugopalan, but more complicated due to UV divergences

Instabilities in anisotropic plasmas

Weibel instabilities

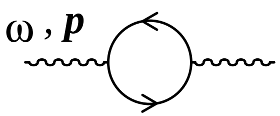
- Weibel (1959) : instability in anisotropic electron-ion plasmas
- Mrowczynski (1998-2003) : similar instabilities exist in QCD
- Romatschke-Strickland (2003-2004) : Weibel instability through the screening properties of the anisotropic QGP
- Arnold, Lenaghan, Moore, Yaffe (2005) : instabilities and thermalization
- Recent numerical investigations of these instabilities :
  Arnold, Moore, Yaffe
  Rebhan, Romatschke, Strickland
  Dumitru, Nara, Strickland
- Is there a relation between this instability, that occurs in anisotropic plasmas, and the Glasma instability discussed so far?

Dressed propagator (equilibrium)

- In order to assess how the medium affects the propagation of excitations, one must compute the photon (gluon) polarization tensor $\Pi^{\mu\nu}(x,y) \equiv \left\langle J^\mu(x) J^\nu(y) \right\rangle$

- Because one is after the long distance properties of the plasma, one also makes the approximation $|\vec{\boldsymbol{p}}| \ll |\vec{\boldsymbol{k}}|$ (Hard Thermal Loops : Braaten, Pisarski - 1990)

- For instance, the spatial part Π^{ij} of the polarization tensor reads :



$$= -e^2 T \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3}\, \widehat{\boldsymbol{v}}^i_{\boldsymbol{k}}\, \frac{\partial f(\vec{\boldsymbol{k}})}{\partial \boldsymbol{k}^l} \left[\delta^{jl} - \frac{\widehat{\boldsymbol{v}}^j_{\boldsymbol{k}} \widehat{\boldsymbol{v}}^l_{\boldsymbol{k}}}{\omega - \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \cdot \vec{\boldsymbol{p}} + i\epsilon}\right]$$

$(\widehat{\boldsymbol{v}}_{\boldsymbol{k}} \equiv \vec{\boldsymbol{k}}/|\vec{\boldsymbol{k}}|)$

  - Note : a similar formula exists for gluons
  - It depends on the distribution $f(\vec{\boldsymbol{k}})$ of particles in the plasma

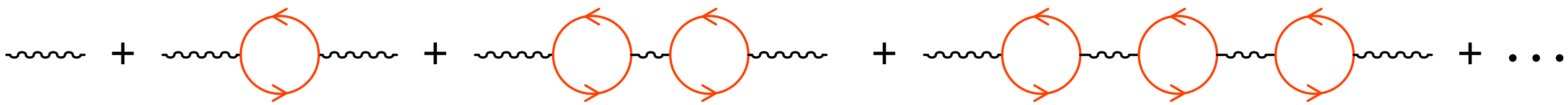
Dressed propagator (equilibrium)

- The transverse and longitudinal projections $\Pi_{T,L}(P)$ read :

$$\Pi_T(P) = \frac{e^2T^2}{6}\left[\frac{p_0^2}{p^2} + \frac{p_0}{2p}\left(1-\frac{p_0^2}{p^2}\right)\ln\left(\frac{p_0+p}{p_0-p}\right)\right]$$

$$\Pi_L(P) = \frac{e^2T^2}{3}\left[1-\frac{p_0^2}{p^2}\right]\left[1-\frac{p_0}{2p}\ln\left(\frac{p_0+p}{p_0-p}\right)\right]$$

- The photon (or gluon for QCD) self-energy can be resummed on the propagator. Diagrammatically, this amounts to summing :

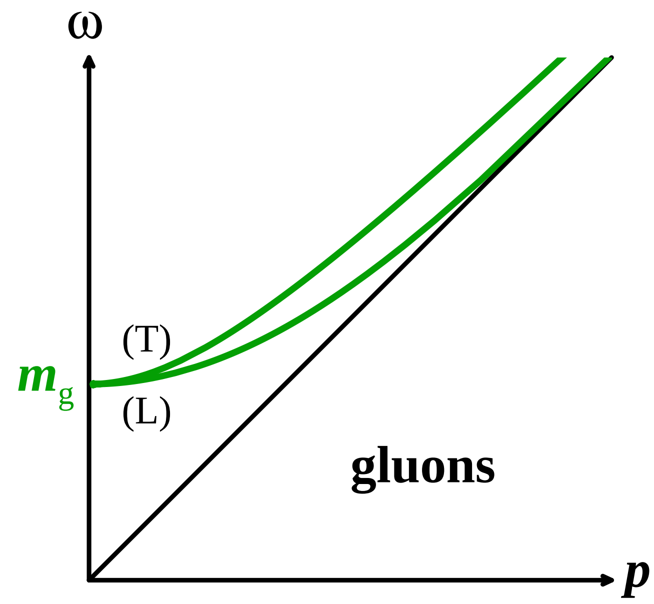


- This leads to the following in-medium photon propagator :

$$D^{\mu\nu}(P) = P_T^{\mu\nu}(P)\,\frac{1}{P^2-\Pi_T(P)} + P_L^{\mu\nu}(P)\,\frac{1}{P^2-\Pi_L(P)}$$

Quasi-particles (equilibrium)

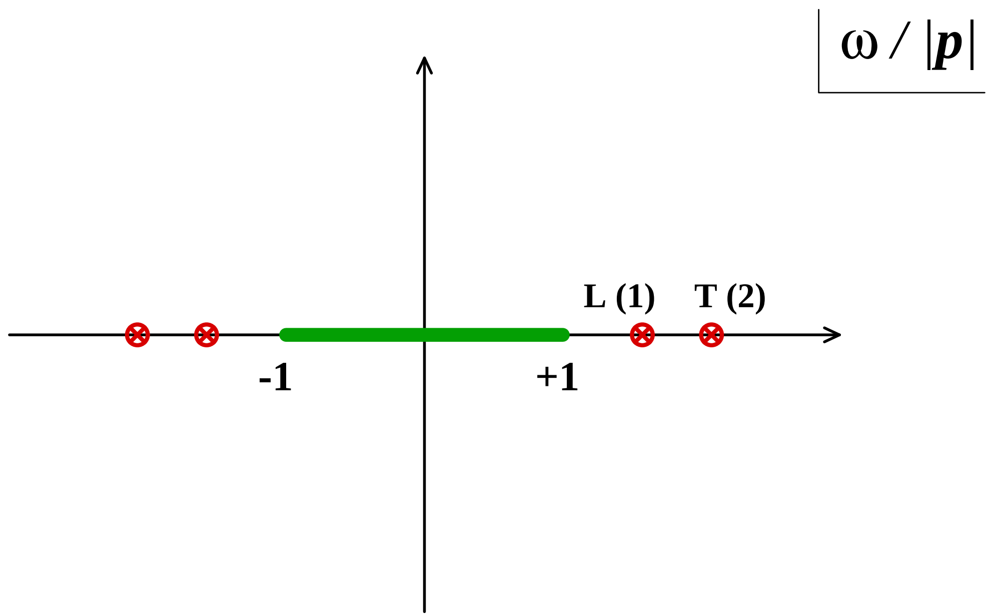
- Quasi-particles correspond to poles in the propagator. Their dispersion relation is the function $p_0 = \omega(\vec{\boldsymbol{p}})$ that defines the location of the pole
- Dispersion curves of gluons in the QGP :



- Thermal masses due to interactions with the other particles in the plasma : $m_{\rm g} \sim gT$

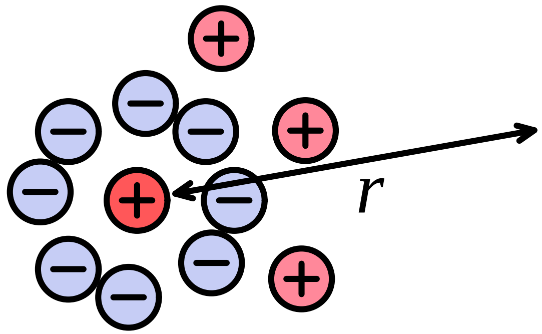
Singularities (equilibrium)

- In the complex plane of $\omega/|\vec{p}|$, the dressed propagator has poles (quasi-particles) and a cut (Landau damping) :



Debye screening (equilibrium)

- A test charge polarizes the particles of the plasma in its vicinity, in order to screen its charge :



$$V(r) = \frac{\exp(-m_{\mathrm{debye}}\, r)}{r}$$

- The Coulomb potential of the test charge decreases exponentially at large distance. The effective interaction range is :

$$\ell \sim 1/m_{\mathrm{debye}} \sim 1/gT$$

- Note : static magnetic fields are not screened by this mechanism (they are screened over length-scales $\ell_{\mathrm{mag}} \sim 1/g^2T$)

Debye screening (equilibrium)

- The Coulomb potential of a static charge reads :

$$V(\vec{\boldsymbol{r}}) = e \int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3} \, \frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{\vec{\boldsymbol{q}}^2 + \Pi_L(0,\vec{\boldsymbol{q}})}$$

- If we are in the vacuum, $\Pi_L = 0$, and the Fourier transform gives the usual Coulomb law :

$$V_{\rm vac}(\vec{\boldsymbol{r}}) = e \int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3} \, \frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{\vec{\boldsymbol{q}}^2} = \frac{e}{4\pi|\vec{\boldsymbol{r}}|}$$

- In a plasma, $\Pi_L(0,\vec{\boldsymbol{q}}) = \frac{e^2T^2}{3} \equiv m_D^2$. The Fourier transform can also be done exactly

$$V(\vec{\boldsymbol{r}}) = e \int \frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3} \, \frac{e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{r}}}}{\vec{\boldsymbol{q}}^2 + m_D^2} = \frac{e}{4\pi|\vec{\boldsymbol{r}}|} \, e^{-m_D|\vec{\boldsymbol{r}}|}$$

▷ the potential is unmodified at $r \ll 1/m_D$, but exponentially suppressed at large distance

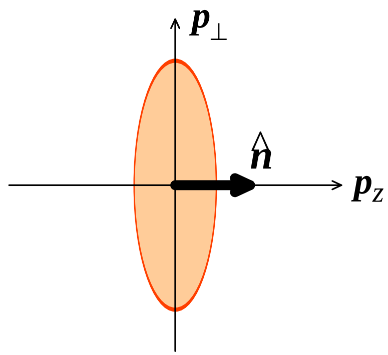
Medium effects (anisotropic)

- Most of the previous analysis can be carried through in the case of a plasma with an anisotropic distribution of particles. In particular, the formula for the polarization tensor in terms of $f(\vec{\boldsymbol{k}})$ remains valid :

$$\Pi^{ij}(\omega, \vec{\boldsymbol{p}}) = -e^2 T \int \frac{d^3\vec{\boldsymbol{k}}}{(2\pi)^3}\, \widehat{\boldsymbol{v}}^i_{\boldsymbol{k}}\, \frac{\partial f(\vec{\boldsymbol{k}})}{\partial \boldsymbol{k}^l} \left[\delta^{jl} - \frac{\widehat{\boldsymbol{v}}^j_{\boldsymbol{k}} \widehat{\boldsymbol{v}}^l_{\boldsymbol{k}}}{\omega - \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \cdot \vec{\boldsymbol{p}} + i\epsilon}\right]$$

- Model for an anisotropic distribution : start from a generic isotropic distribution $f(\boldsymbol{k}^2)$ and squeeze it :

$$f(\boldsymbol{p}^2) \quad \longrightarrow \quad f(\boldsymbol{p}^2 + \xi(\widehat{\boldsymbol{n}} \cdot \vec{\boldsymbol{p}})^2)$$



Medium effects (anisotropic)

- Within this model, it is easy to factorize the integration over the argument of f (i.e. $\boldsymbol{p}^2 + \xi(\widehat{\boldsymbol{n}} \cdot \vec{\boldsymbol{p}})^2$) :

$$\Pi^{ij}(\omega, \vec{\boldsymbol{p}}) = m_D^2 \int \frac{d^2\widehat{\boldsymbol{v}}_{\boldsymbol{k}}}{4\pi} \; \widehat{\boldsymbol{v}}_{\boldsymbol{k}}^i \; \frac{\widehat{\boldsymbol{v}}_{\boldsymbol{k}}^l + \xi(\widehat{\boldsymbol{v}}_{\boldsymbol{k}} \cdot \widehat{\boldsymbol{n}})\widehat{\boldsymbol{n}}^l}{(1 + \xi(\widehat{\boldsymbol{v}}_{\boldsymbol{k}} \cdot \widehat{\boldsymbol{n}})^2)^2} \left[\delta^{jl} - \frac{\widehat{\boldsymbol{v}}_{\boldsymbol{k}}^j \widehat{\boldsymbol{v}}_{\boldsymbol{k}}^l}{\omega - \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \cdot \vec{\boldsymbol{p}} + i\epsilon} \right]$$

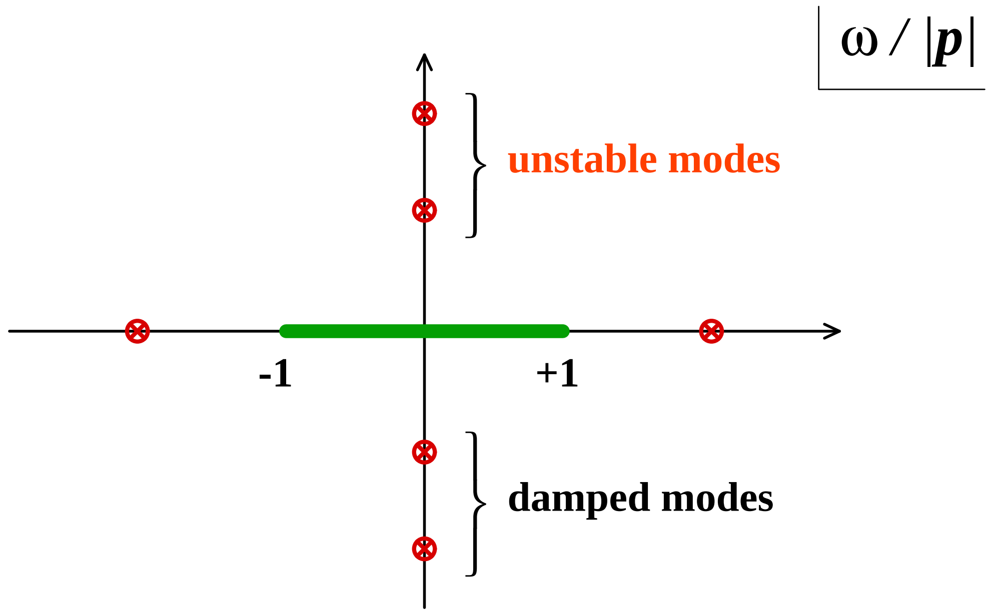
with

$$m_D^2 \equiv -\frac{e^2}{2\pi^2} \int_0^\infty dk \; k^2 \; \frac{df(k^2)}{dk^2}$$

- m_D sets the magnitude of all the medium effects on the gauge bosons
- Only the remaining integral over the unit vector $\widehat{\boldsymbol{v}}_{\boldsymbol{k}}$ is affected by the anisotropy $(\xi \neq 0)$
- The tensorial decomposition of $\Pi^{\mu\nu}$ is more complicated than in the isotropic case, because the vector $\widehat{\boldsymbol{n}}^\mu$ can be used in the construction of the basis

Singularities (anisotropic)

- In the anisotropic case, some poles of the dressed propagator have moved away from the real axis :



- Some poles have migrated to the upper half plane, and lead to instabilities
- These imaginary poles exist no matter how small the squeezing parameter ξ is (but their imaginary part goes to zero when $\xi \to 0$)

Instability

- The poles in the upper half plane lead to the indefinite growth of some small fluctuations. Indeed, the forward propagation of a small fluctuation $a(x)$ in the medium is given by :

$$a(x) = \int d^3\vec{\boldsymbol{y}}\; G_R(x,y)\left[\overleftarrow{\partial}_y^0 - \overrightarrow{\partial}_y^0\right] a_{\rm in}(y_0,\vec{\boldsymbol{y}})$$

  where $a_{\rm in}(y)$ is the initial condition for the fluctuation, and G_R the retarded propagator in the medium

- Consider a toy model for a propagator with such a pole :

$$G_R(k) \equiv \frac{1}{(k_0 - i\Gamma) - |\vec{\boldsymbol{k}}|^2} \qquad \text{with } \Gamma > 0$$

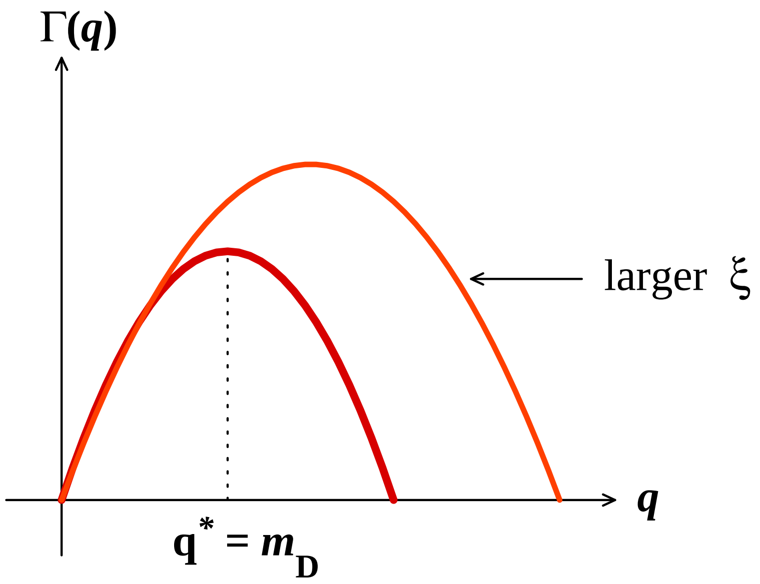
  and a plane wave as the initial condition $a_{\rm in}(y) \equiv \exp(-iq\cdot y)$

- One finds :

$$a(x) = e^{-i(q_0x_0 - \vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}})}\;\underbrace{e^{\Gamma(x_0-y_0)}}_{\text{unbounded growth of the fluctuation}}$$

Growth rate spectrum

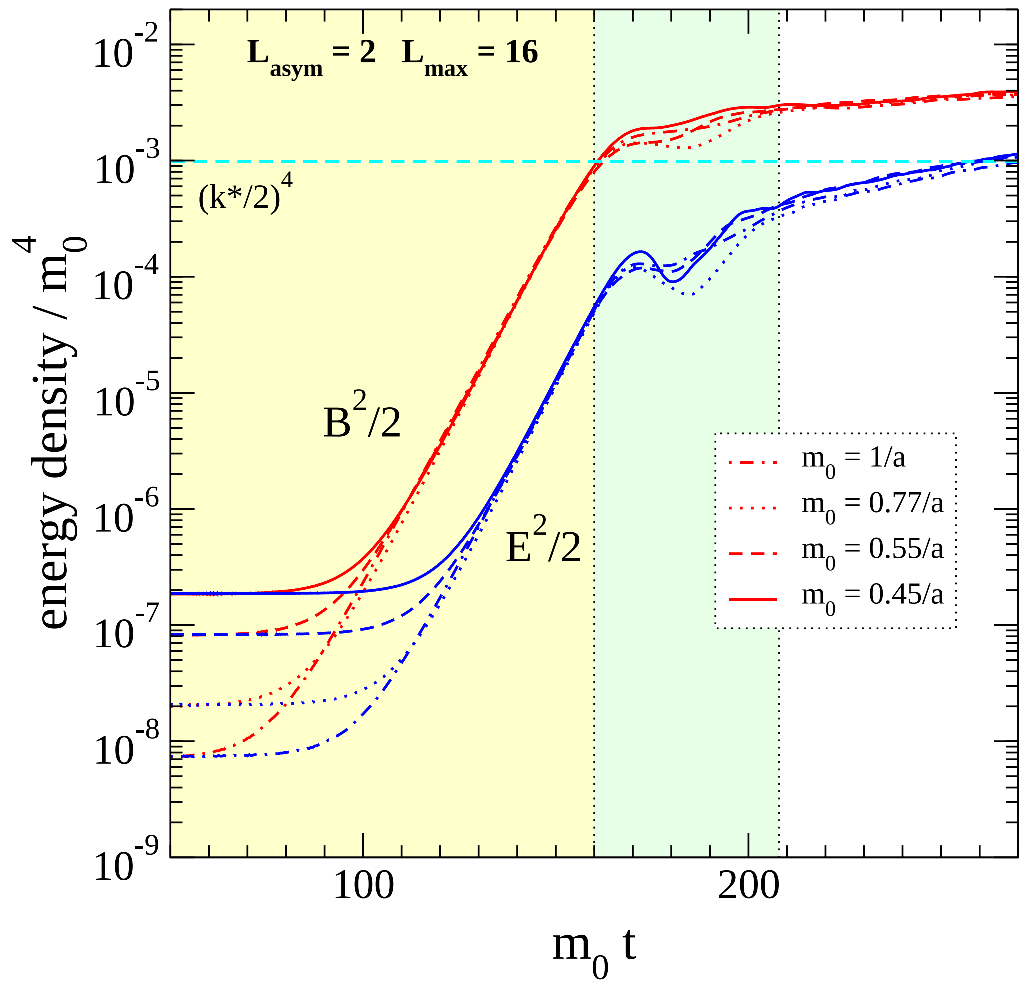
- In QED and QCD, the growth rate Γ depends on the momentum q of the plane wave fluctuation :



- At moderate anisotropies, the most unstable mode is of the same order as the Debye mass
- At very large anisotropies, all modes from soft to hard are unstable

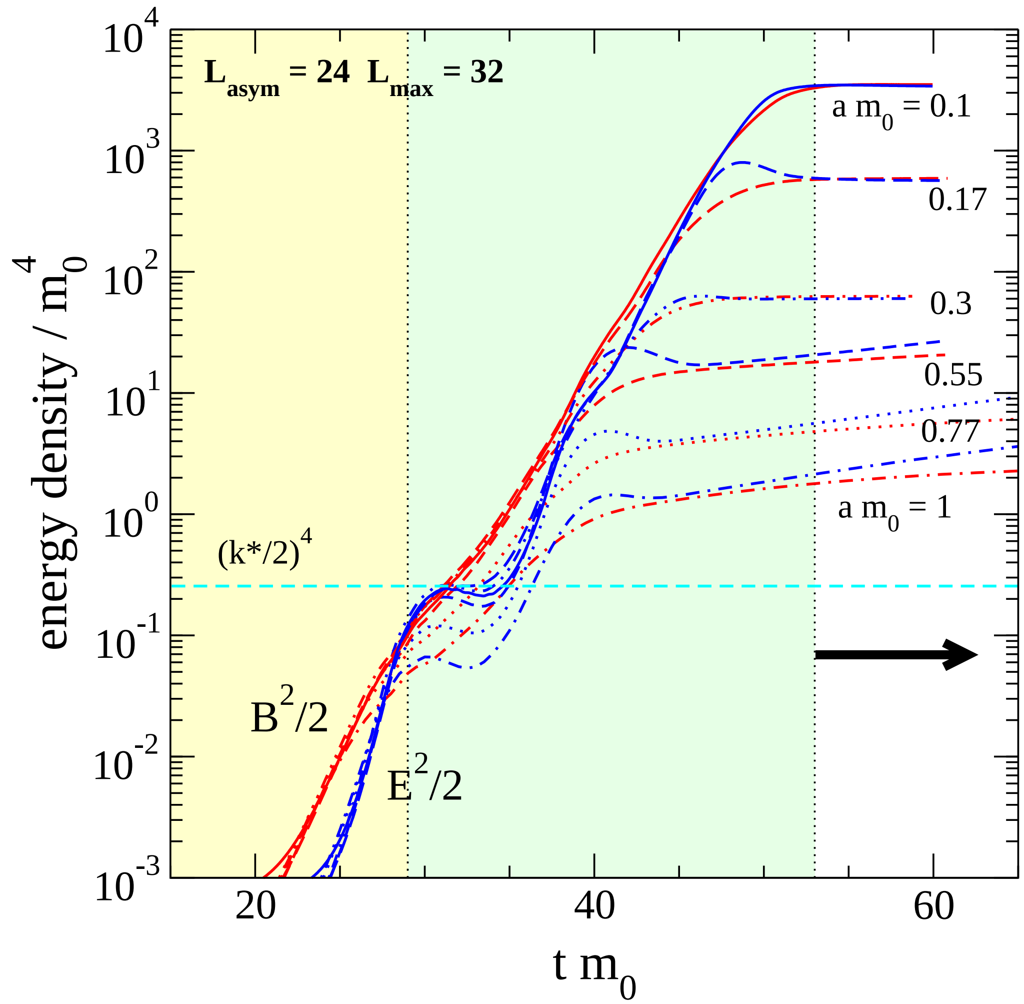
Numerical results

- Small anisotropy. From : Rummukainen (Trento, Jan. 2007)



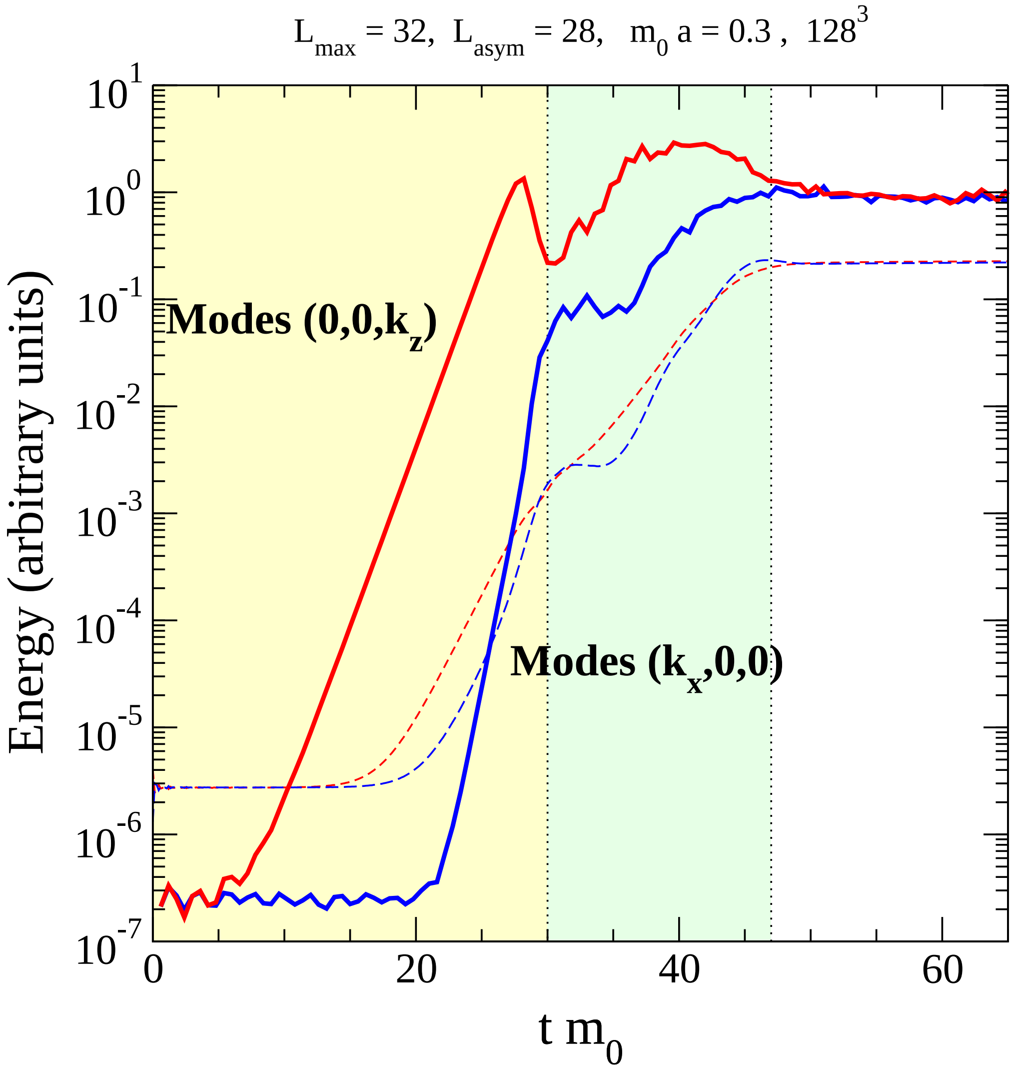
Numerical results

- Large anisotropy. From : Rummukainen (Trento, Jan. 2007)



Numerical results

- Isotropization. From : Rummukainen (Trento, Jan. 2007)

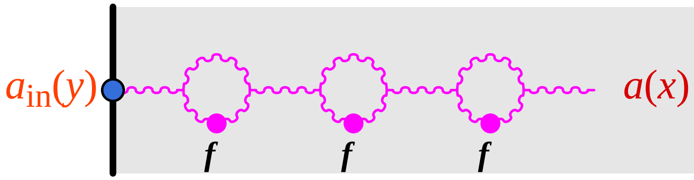


Relation to Glasma instabilities

- Reminder : in-medium propagation of a fluctuation :

$$a(x) = \int d^3\vec{\boldsymbol{y}}\; G_R(x,y)\,\left[\overleftarrow{\partial_y^0} - \overrightarrow{\partial_y^0}\right]\, a_{\rm in}(y_0,\vec{\boldsymbol{y}})$$

- This can be represented by diagrams such as :

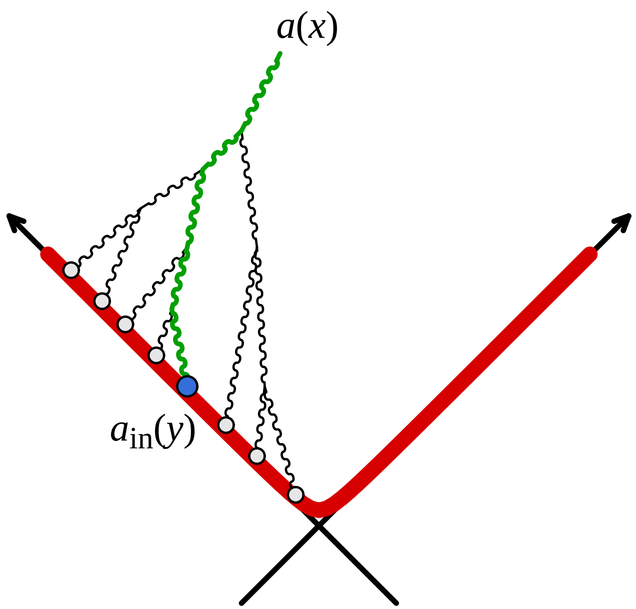


Note : the blob on one of the lines of the self-energies indicates the presence of one factor of the distribution $f(\vec{\boldsymbol{k}})$

Relation to Glasma instabilities

- Reminder : the Glasma instability also affects the propagation of small fluctuations in the forward light-cone, via diagrams such as :

$$a(x) \quad = $$

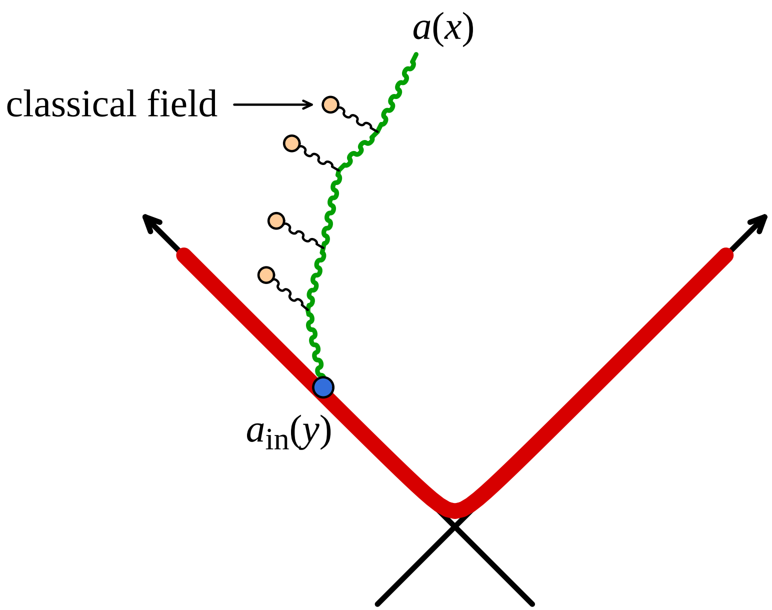


Note : each tree attached to the Green line (the propagator of the fluctuation in the background field) is an insertion of the classical field

Relation to Glasma instabilities

- Replace each such tree by a symbol denoting the background field at the point where the tree is attached :

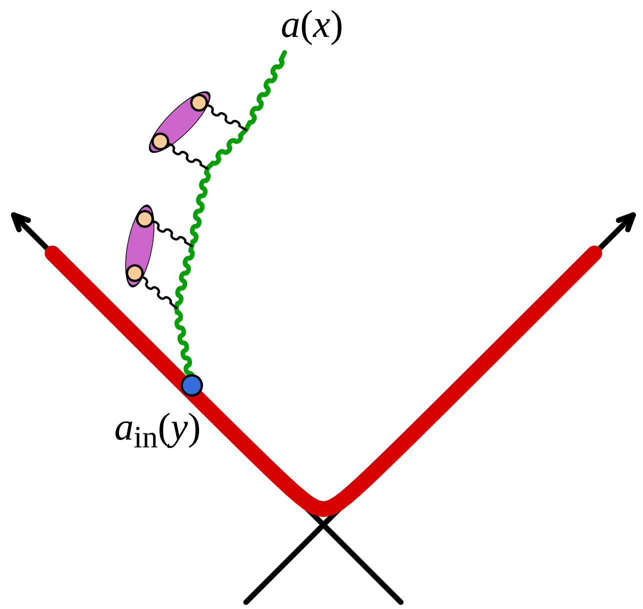
$$a(x) \quad =$$



Relation to Glasma instabilities

- The average over the sources ρ produces links between the classical fields. Some of the terms involve only connections among neighboring background field insertions :

$$a(x) \quad =$$



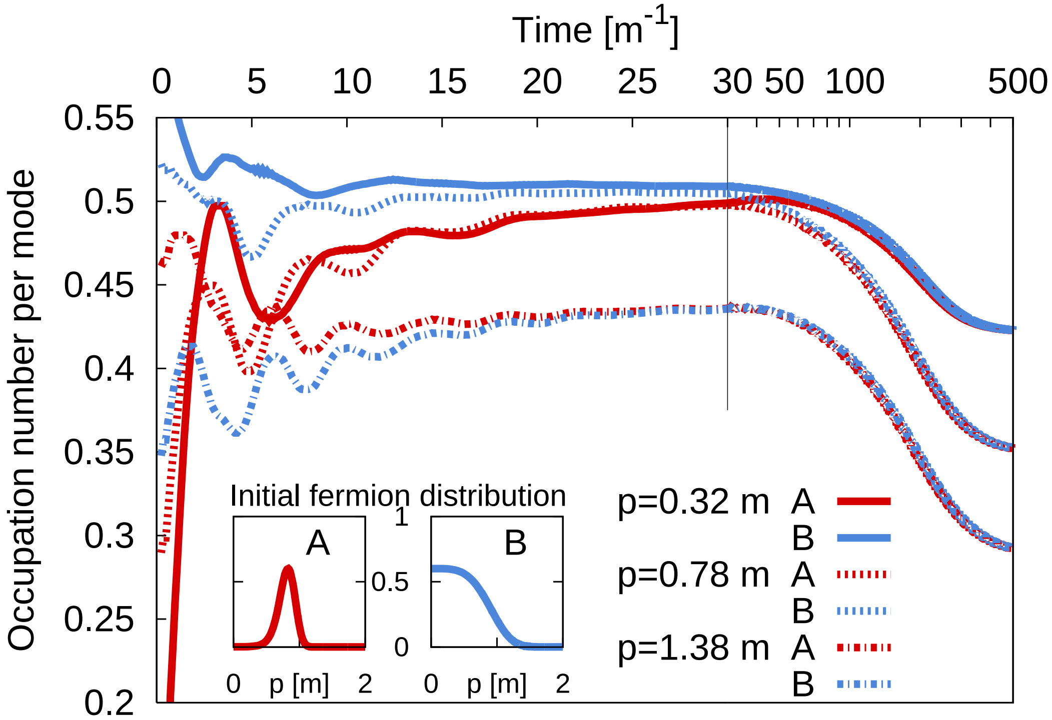
▷ We recover self-energy corrections very similar to the ones encountered in the study of the Weibel instability (the average of two classical fields produces an anisotropic “gluon distribution”)

Thermalization ?

Pre-thermalization

Berges, Borsányi, Wetterich (2005)

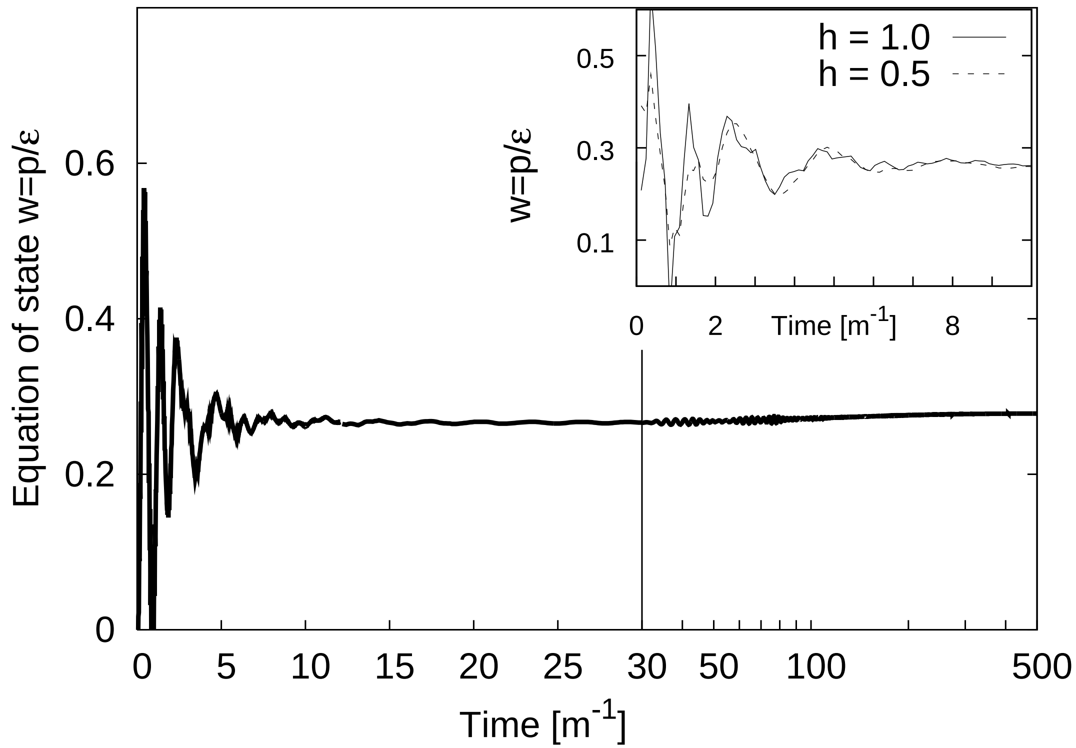
- Many systems exhibit a pre-thermal behavior, which is reached much faster than the true thermalization :



▷ the initial conditions are forgotten much before the true equilibrium is reached

Pre-thermalization

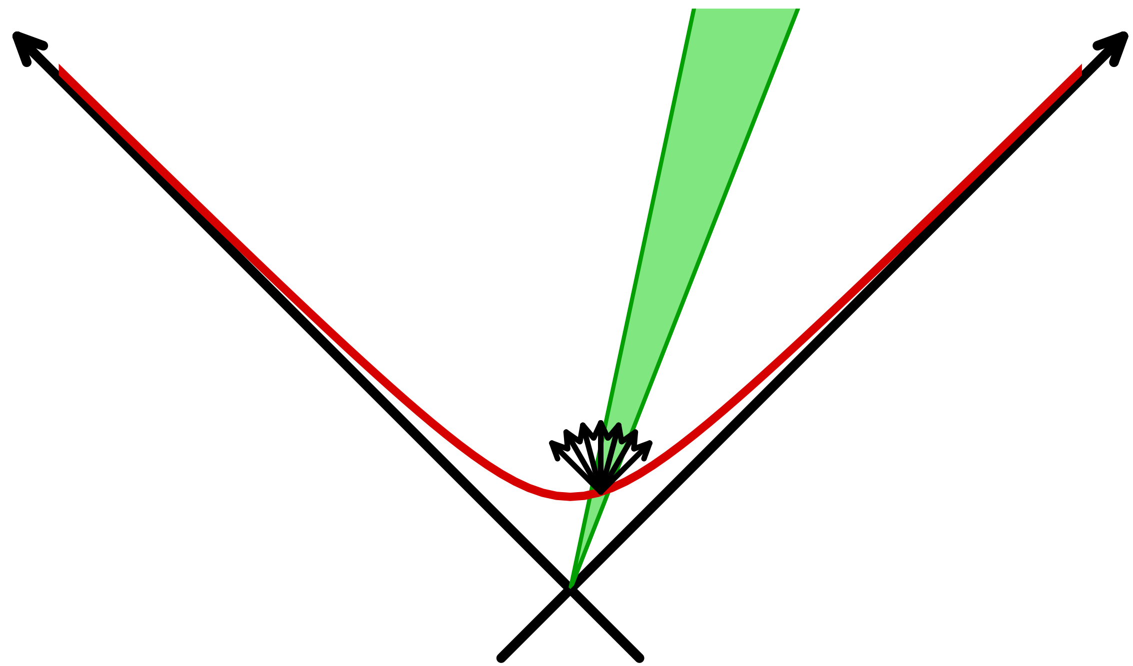
- The ratio of pressure to energy density becomes constant very quickly :



▷ an equation of state may be used long before the complete equilibrium is reached

Longitudinal expansion

- If nothing else happened, the distribution of produced particles would quickly become very anisotropic :



Longitudinal expansion

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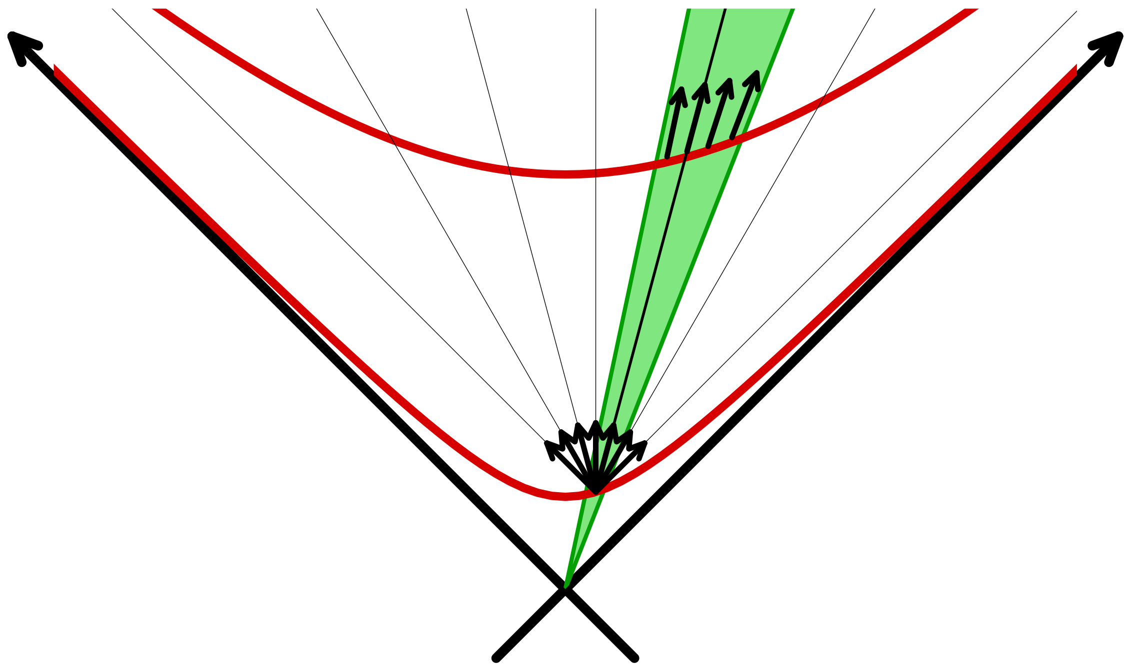
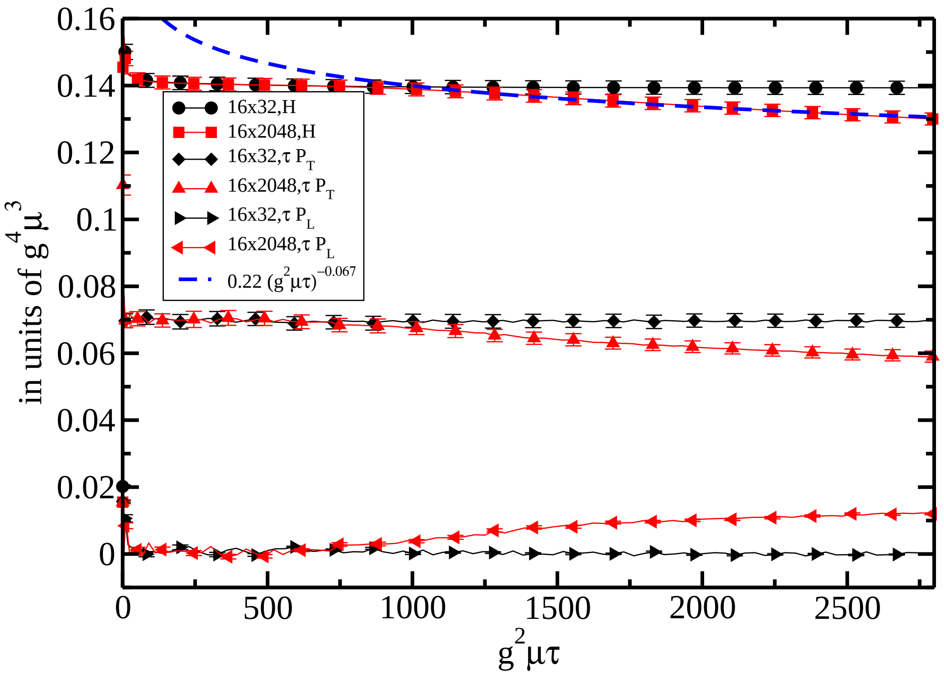


▷ if particles fly freely, only one longitudinal velocity can exist at a given η : $v_z = \tanh(\eta)$

▷ the longitudinal expansion of the system is the main obstacle to local isotropy

Glasma instability (expanding system)

- The Glasma instability seems to help fighting the expansion :



- The energy density drops slightly faster than τ^{-1}
  ($\tau^{-1.33}$ needed for local thermalization)
- This is for rather tiny initial fluctuations. In QCD, they are suppressed only by $\alpha_s \approx 0.3$

Lecture IV : Kinetic theory

- Collisionless kinetic equations
- Boltzmann equation
- Transport coefficients
- From kinetic theory to hydrodynamics

Complements

Action of Tu

- One can prove the following formula for the evolution of a small perturbation $a(x)$ on top of the classical field :

$$a(x) = \int\limits_{\vec{\boldsymbol{u}}\in\mathrm{LC}} \left[a_{\mathrm{in}}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\right] \mathcal{A}(x)$$

▷ from the classical field $\mathcal{A}(x)$, the operator $a_{\mathrm{in}}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}$ builds the fluctuation $a(x)$ whose initial condition on the light-cone is $a_{\mathrm{in}}(\vec{\boldsymbol{u}})$

- From this formula, one sees that $\boldsymbol{T}_{\vec{\boldsymbol{u}}}\mathcal{A}(x)$ determines how the fluctuation $a(x)$ is sensitive on its initial condition

- If there is an instability in the system, two fluctuations whose value on the light-cone differ by an amount $\sim \epsilon$ will differ by an amount $\sim \epsilon\exp(\#\sqrt{\tau})$ at the time τ

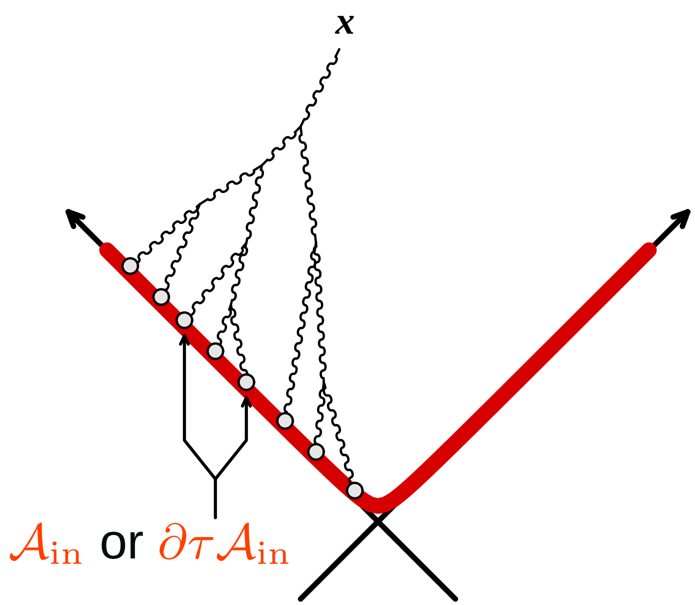
$$\triangleright \quad \boldsymbol{T}_{\vec{\boldsymbol{u}}}\mathcal{A}(\tau, \vec{\boldsymbol{y}}) \underset{\tau\to+\infty}{\sim} e^{\sqrt{\mu\tau}}$$

Action of Tu

- Green's formula for the retarded classical field $\mathcal{A}(x)$ above the light-cone :

$$\mathcal{A}(x) = \frac{g}{2}\int_{\mathrm{LC}+} d^4z\, G^0_R(x,z)\, \mathcal{A}^2(z) + \int_{\mathrm{LC}} d^3\vec{\boldsymbol{u}}\, G^0_R(x,u)\left[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\right]\mathcal{A}_{\mathrm{in}}(\vec{\boldsymbol{u}})$$

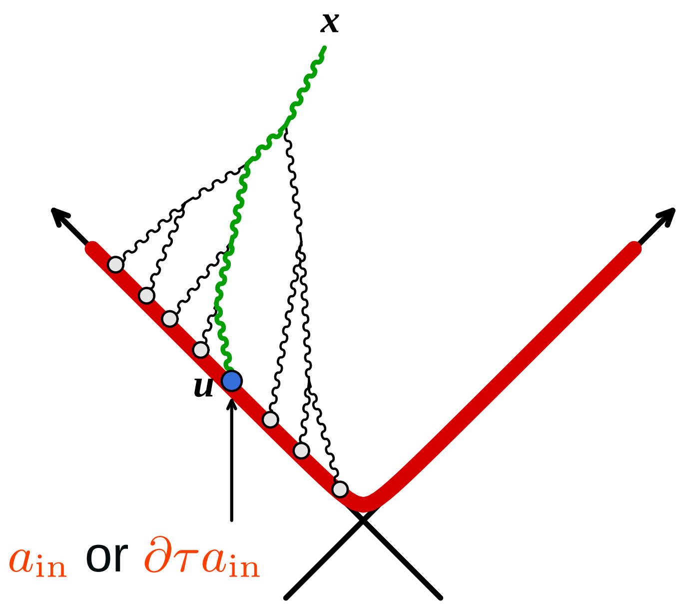
$\mathcal{A}(x) =$



Action of Tu

- Acting on $\mathcal{A}(x)$ with the operator $a_{\rm in} \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}$ replaces the initial $\mathcal{A}_{\rm in}$ or $\partial\tau\mathcal{A}_{\rm in}$ at point $\vec{\boldsymbol{u}}$ by $a_{\rm in}$ or $\partial\tau a_{\rm in}$ respectively :

$$\left[a_{\rm in} \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\right]\mathcal{A}(x) \quad =$$



- The line displayed in green can be seen as the retarded propagator of a fluctuation on top of the classical field between the point $\vec{\boldsymbol{u}}$ on the light-cone and the final point x

In-medium propagator

- Reminder : the photon polarization tensor $\Pi^{\mu\nu}$ is transverse. At $T = 0$, this implies :

$$\Pi^{\mu\nu}(P) \equiv \langle J^\mu(P) J^\nu(-P) \rangle = \left(g^{\mu\nu} - \frac{P^\mu P^\nu}{P^2} \right) \Pi(P^2)$$

  - this is due to current conservation and Lorentz invariance
  - this property ensures that the photon remains massless at all orders of perturbation theory

Dressed propagator (equilibrium)

- This formula is not valid at $T > 0$, because there is a preferred frame (in which the plasma velocity is zero)
  ▷ the tensorial decomposition of $\Pi^{\mu\nu}$ is more complicated

- At finite T, the tensorial decomposition of $\Pi^{\mu\nu}$ is :

$$\Pi^{\mu\nu}(P) = P_T^{\mu\nu}(P)\,\Pi_T(P) + P_L^{\mu\nu}(P)\,\Pi_L(P)$$

with the following projectors (in the plasma rest frame)

$$P_T^{ij}(P) = g^{ij} + \frac{p^i p^j}{\vec{\boldsymbol{p}}^2}\,, \quad P_T^{0i}(P) = 0\,, \quad P_T^{00}(P) = 0$$

$$P_L^{ij}(P) = -\frac{(p^0)^2 p^i p^j}{\vec{\boldsymbol{p}}^2 P^2}\,, \quad P_L^{0i}(P) = -\frac{p^0 p^i}{P^2}\,, \quad P_L^{00}(P) = -\frac{\vec{\boldsymbol{p}}^2}{P^2}$$

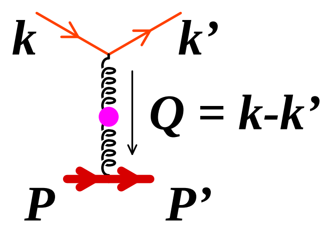
- Therefore, we have

$$\Pi^\mu{}_\mu(P) = 2\Pi_T(P) + \Pi_L(P) \quad , \quad \Pi^{00}(P) = -\frac{\vec{\boldsymbol{p}}^2}{P^2}\,\Pi_L(P)$$

Debye screening

- Place a quark of mass M at rest in the plasma, at $\vec{r} = 0$
- Scatter another quark off it. The scattering amplitude reads

$$\mathcal{M} = \left[e\,\overline{u}(\vec{k}')\gamma_\mu u(\vec{k})\right]\left[e\,\overline{u}(\vec{P}')\gamma_\nu u(\vec{P})\right] \sum_{\alpha=T,L} \frac{P^{\mu\nu}_\alpha(Q)}{Q^2 - \Pi_\alpha(Q)}$$



  - If $\vec{P} = 0$ (test charge at rest), only $\alpha = L$ contributes
  - From $(P+Q)^2 = M^2$, we get a $2\pi\delta(q_0)/2M$

- For the scattering off an external potential A^μ, the amplitude is $\mathcal{M} = \left[e\,\overline{u}(\vec{k}')\gamma_\mu u(\vec{k})\right] A^\mu(Q)$
- Thus, the potential created by the test charge at rest is :

$$A^\mu(Q) = \frac{2\pi e \delta^{\mu 0}\delta(q_0)}{\vec{q}^2 + \Pi_L(0, \vec{q})}$$