Pre-equilibrium dynamics in heavy ion collisions

II – Initial particle production

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General outline

- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization
- Lecture IV : Kinetic theory, Near-Equilibrium dynamics

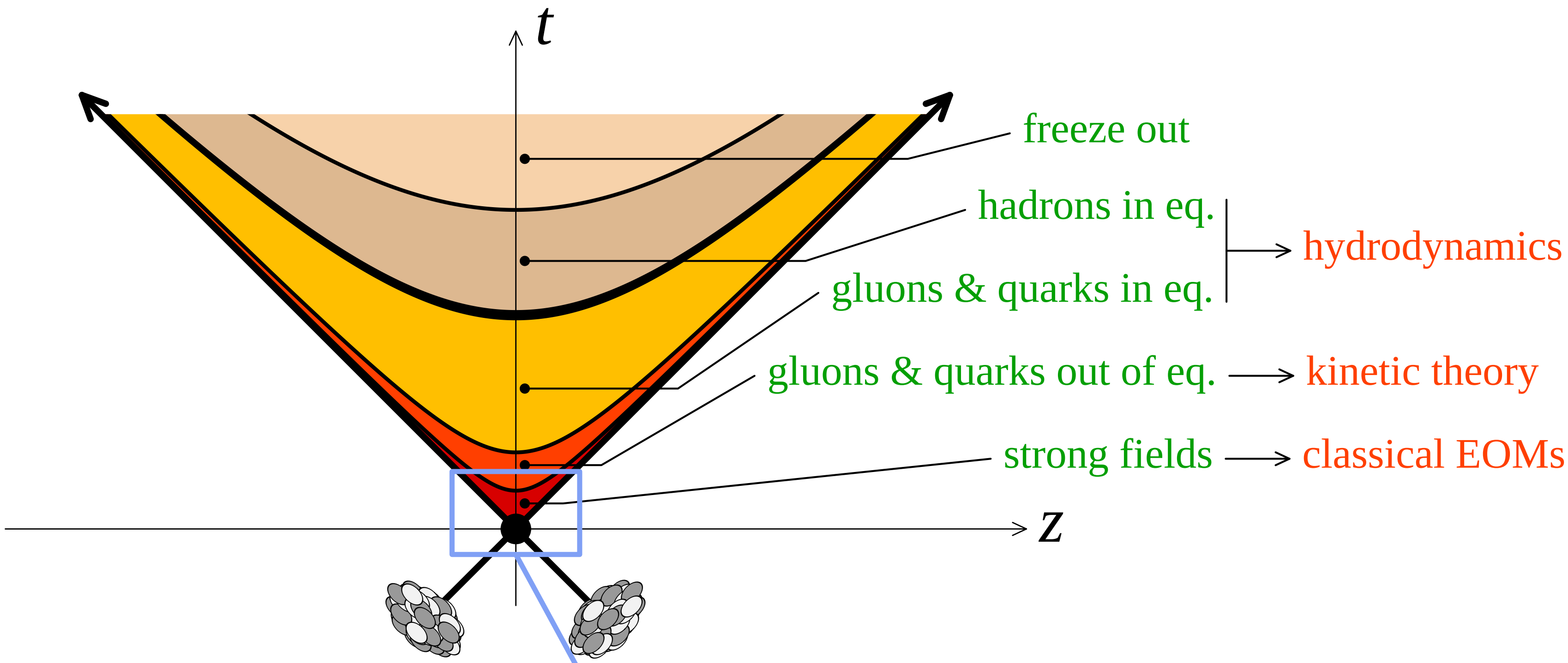
Lecture II : Initial particle production

- Introduction to nucleus-nucleus collisions
- Power counting and bookkeeping
- Inclusive gluon spectrum
- Loop corrections and factorization

Introduction

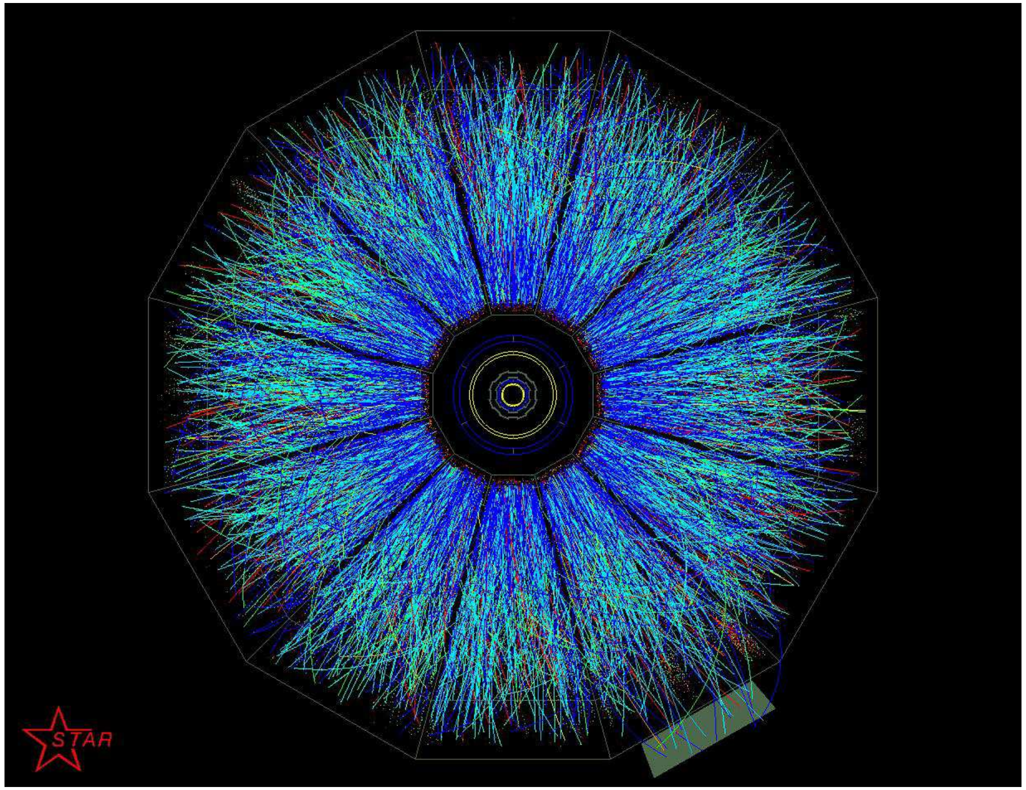
Initial particle production

- Introduction
- Bookkeeping
- Inclusive gluon spectrum
- Loop corrections
- Motivation for lecture III



Can we calculate the initial particle spectrum in QCD ?

Small x QCD in AA collisions



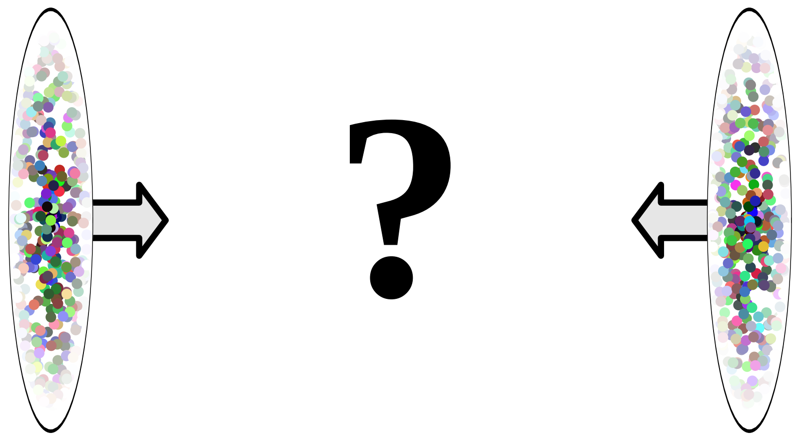
- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- The bulk of of particle production comes from (very) low x
  - ▷ high gluon density (even more so in nuclei : $G_A/G_\mathrm{p} \approx A$)

Small x QCD in AA collisions

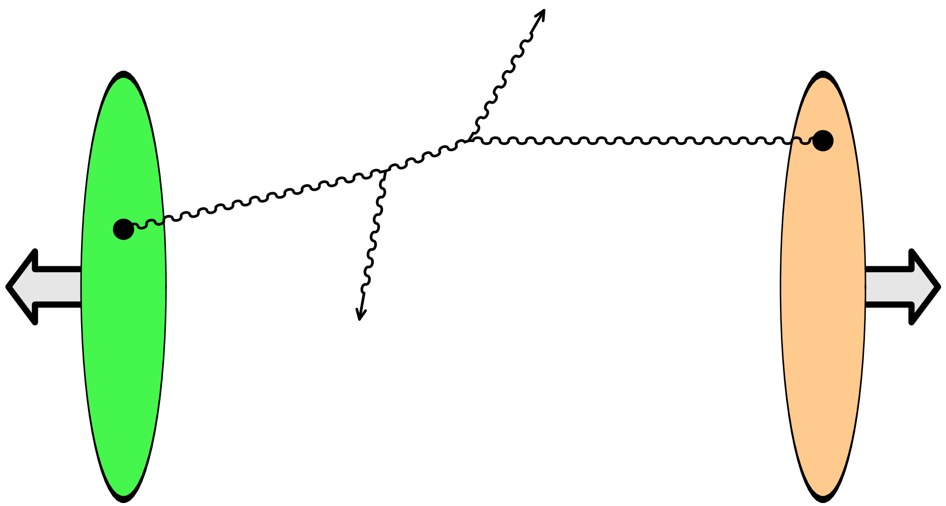
- Saturation affects the early stages of heavy ion collisions, up to a time $\tau \sim Q_s^{-1}$
- The dynamics that takes place afterwards blurs the physics of saturation (for instance, if the system reaches thermalization, it does not remember the details of the dynamics at early times)
  $\triangleright$ Saturation controls only inclusive observables, like the overall multiplicity and its energy dependence
  $\triangleright$ Nucleus-nucleus collisions are a limited framework in order to probe saturation
- The Color Glass Condensate provides a (consistent?) framework in order to compute the spectrum of the particles that are produced initially, which is then used as an initial condition for the rest of the evolution

Initial particle production

- Main difficulty : studying the collision of two densely occupied projectiles is much more complicated than the asymmetric cases involving an elementary probe

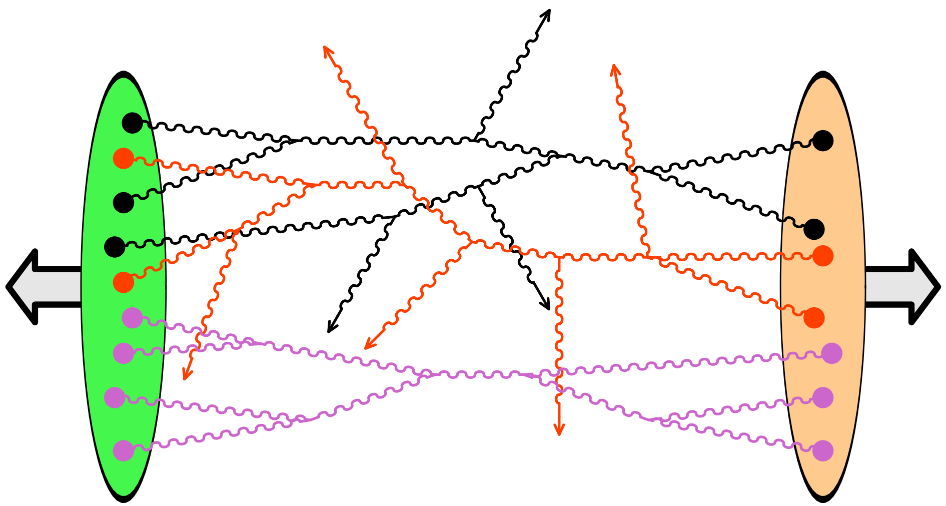


Initial particle production



- Dilute regime : one parton in each projectile interact

Initial particle production

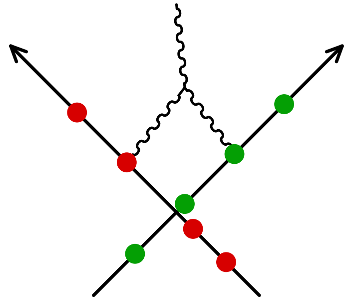


- Dilute regime : one parton in each projectile interact
- Dense regime : multiparton processes become crucial
(+ pileup of many simultaneous scatterings)

Description of AA collisions

- For symmetric collisions (e.g. nucleus-nucleus collisions), the two projectiles should be treated on the same footing
- For nucleus-nucleus collisions, there are two strong sources that contribute to the color current :

$$J^\mu \equiv \delta^{\mu +}\delta(x^-)\,\rho_1(\vec{\boldsymbol{x}}_\perp) + \delta^{\mu -}\delta(x^+)\,\rho_2(\vec{\boldsymbol{x}}_\perp)$$



- Average over the sources ρ_1, ρ_2

$$\langle \mathcal{O}_Y \rangle = \int [D\rho_1]\,[D\rho_2]\; W_{Y_{\mathrm{beam}}-Y}[\rho_1]\; W_{Y+Y_{\mathrm{beam}}}[\rho_2]\; \mathcal{O}[\rho_1,\rho_2]$$

- Can this factorization formula be justified ?
- How to compute $\mathcal{O}[\rho_1,\rho_2]$?

Kinematics

- If the produced object is characterized by $(M_\perp, Y)$, it comes from partons with momentum fractions

$$x_{1,2} = \frac{M_\perp}{\sqrt{s}}\, e^{\pm Y}$$

- The rapidity of a nucleon in the beam is :

$$Y_{\rm beam} \equiv \frac{1}{2}\ln\left(\frac{P^+}{P^-}\right) = \frac{1}{2}\ln\left(\frac{s}{m_N^2}\right)$$

- The rapidities of the incoming partons are given by :

$$Y_1 \equiv \frac{1}{2}\ln\left(\frac{k_1^+}{k_1^-}\right) = Y_{\rm beam} + \ln(x_1) + \frac{1}{2}\ln\left(\frac{m_N^2}{\boldsymbol{k}_{1\perp}^2}\right)$$

$$Y_2 \equiv \frac{1}{2}\ln\left(\frac{k_2^+}{k_2^-}\right) = -Y_{\rm beam} - \ln(x_2) - \frac{1}{2}\ln\left(\frac{m_N^2}{\boldsymbol{k}_{2\perp}^2}\right)$$

Kinematics

- The mid value and difference are :

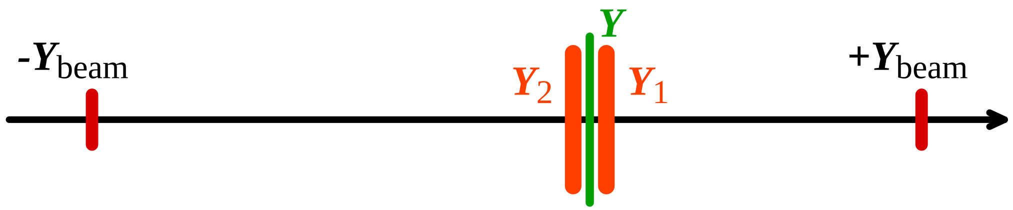
$$\frac{Y_1+Y_2}{2} = Y + \frac{1}{4}\ln\left(\frac{\boldsymbol{k}_{2\perp}^2}{\boldsymbol{k}_{1\perp}^2}\right)$$

$$Y_1 - Y_2 = \ln\left(\frac{M_\perp^2}{\boldsymbol{k}_{1\perp}\boldsymbol{k}_{2\perp}}\right)$$

- When one looks at the bulk of particle production, $M_\perp, k_{1\perp}, k_{2\perp}$ are all of the order of the saturation scale, and the remaining logs in these formulae are small. Hence,

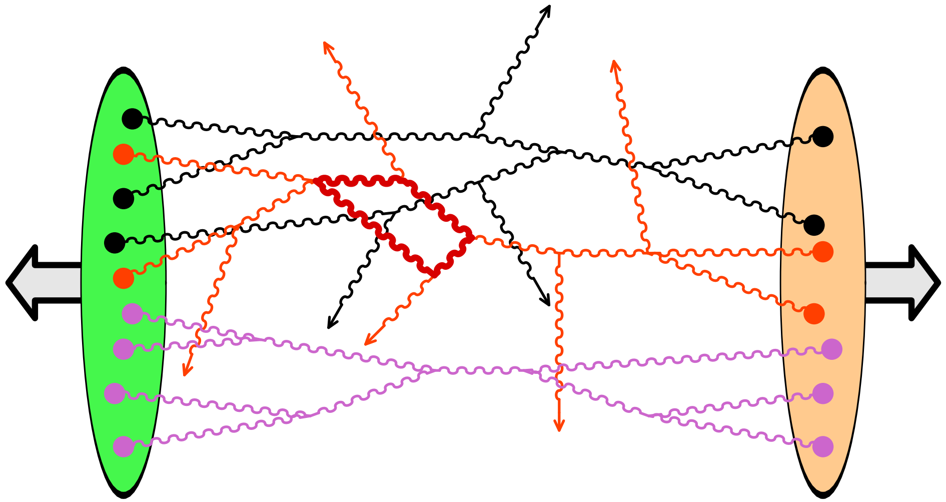
$$Y_1 \approx Y_2 \approx Y$$

$\triangleright$ the incoming partons (or color sources in the CGC context) sit at rapidities very close to the rapidity of the produced object

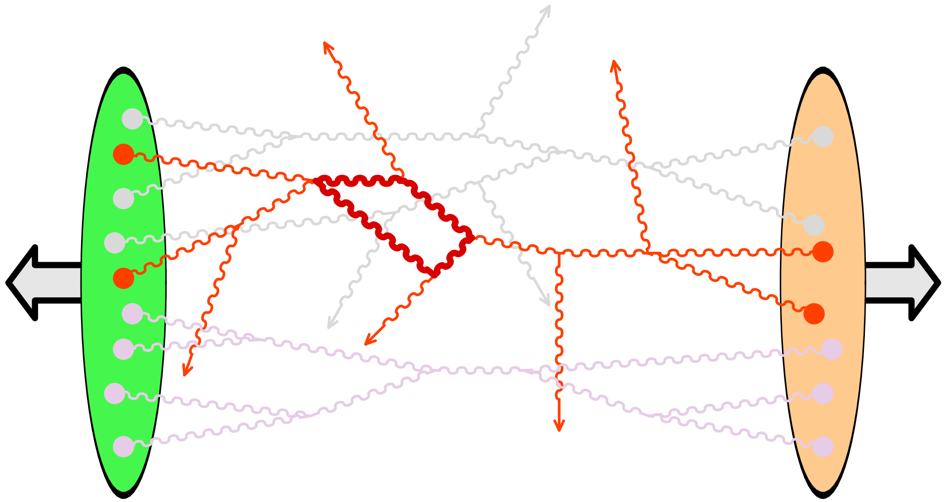


Power counting and Bookkeeping

Power counting



Power counting

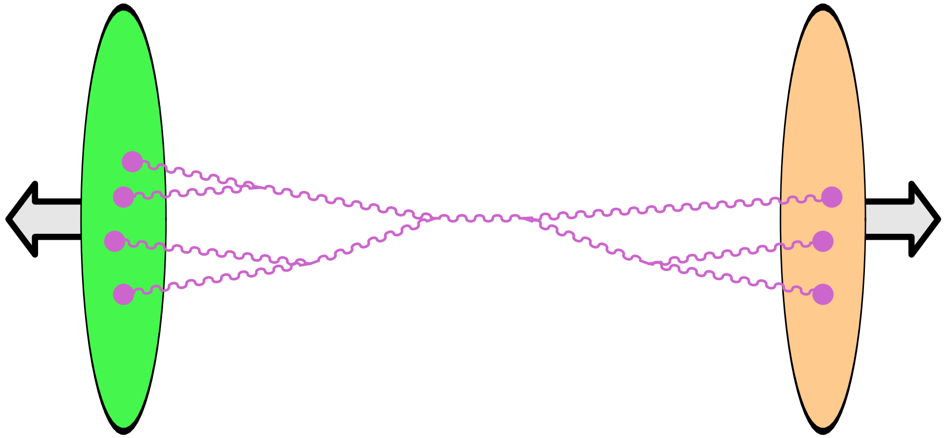


- In the saturated regime, the sources are of order $1/g$ (because $\langle\rho\rho\rangle \sim$ occupation number $\sim 1/\alpha_s$)
- The order of each connected diagram is given by :

$$\frac{1}{g^2}\;g^{\#\ \text{produced gluons}}\;g^{2(\#\ \text{loops})}$$

- The total order of a graph is the product of the orders of its disconnected subdiagrams $\triangleright$ somewhat messy...

Vacuum diagrams



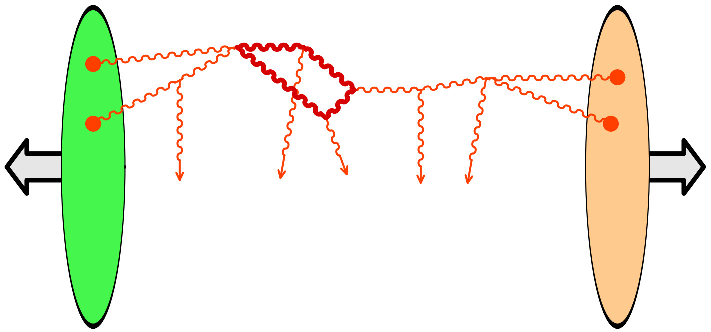
- Vacuum diagrams do not produce any gluon. They are contributions to the vacuum to vacuum amplitude $\langle 0_{\rm out} | 0_{\rm in} \rangle$
- The order of a connected vacuum diagram is given by :

$$g^{-2} \quad g^{2(\#\ \text{loops})}$$

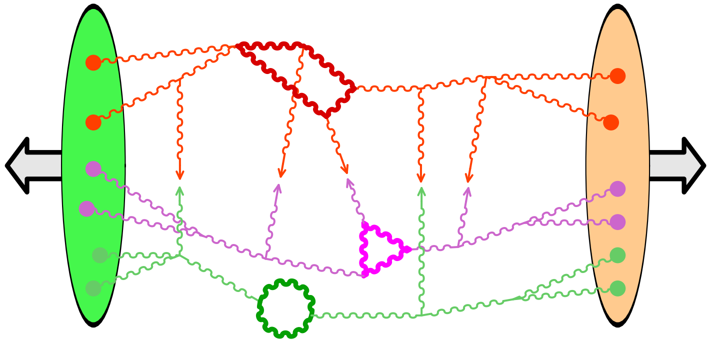
- Relation between connected and non connected vacuum diagrams :

$$\sum \begin{pmatrix} \text{all the vacuum} \\ \text{diagrams} \end{pmatrix} = \exp\left\{ \sum \begin{pmatrix} \text{connected} \\ \text{vacuum diagrams} \end{pmatrix} \right\} = e^{iV[j]}$$

Bookkeeping

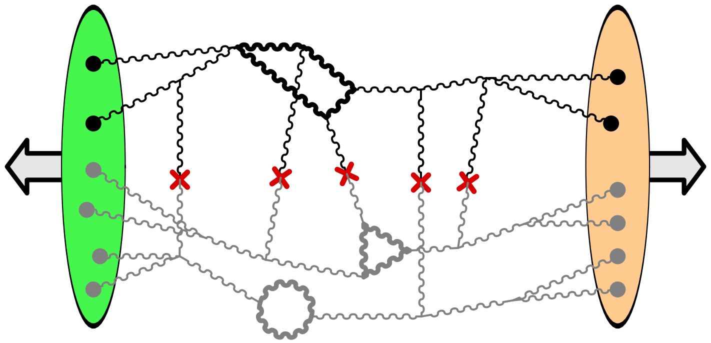


Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

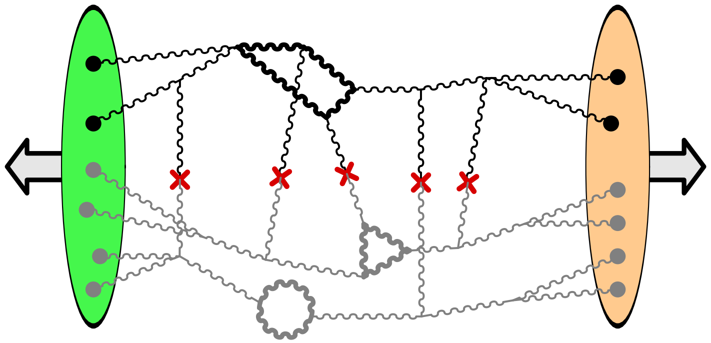
Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves

- See them as cuts vacuum diagrams. Cutting lines amounts to putting them on-shell (cut propagator : $2\pi\theta(-p^0)\delta(p^2)$)

Bookkeeping



- Consider squared amplitudes (including interference terms) rather than the amplitudes themselves
- See them as cuts vacuum diagrams. Cutting lines amounts to putting them on-shell (cut propagator : $2\pi\theta(-p^0)\delta(p^2)$)
- The sum of the vacuum diagrams, $\exp(iV[j])$, is the generating functional for time-ordered products of fields :

$$\left\langle 0_{\rm out}\middle|TA(x_1)\cdots A(x_n)\middle|0_{\rm in}\right\rangle = \frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_n)}\; e^{iV[j]}$$

Bookkeeping

- The probability P_n of producing exactly n particles can be obtained by acting n times on vacuum diagrams with a "cut operator" $\mathcal{C}$

$$P_n = \frac{1}{n!}\,\mathcal{C}^n\,\underbrace{e^{iV[j_+]}}_{\text{amplitude}}\;\underbrace{e^{-iV^*[j_-]}}_{\text{c.c. amplitude}}\Big|_{j_+=j_-=j}$$

- The sum of all the cut vacuum diagrams, with sources j_+ on one side of the cut and j_- on the other side, can be written as :

$$e^{\mathcal{C}}\;e^{iV[j_+]}\;e^{-iV^*[j_-]} \;=\; \sum\begin{pmatrix}\text{all the cut}\\ \text{vacuum diagrams}\end{pmatrix}$$

▷ Note : if we set $j_+ = j_- = j$, then this is $\sum_n P_n = 1$

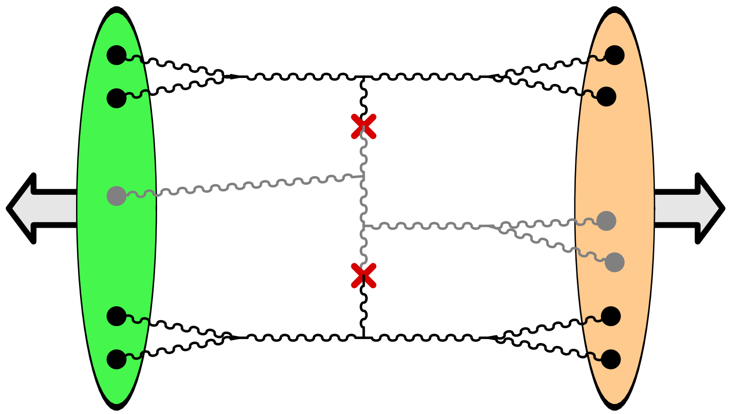
Bookkeeping

- With more details :

$$\mathcal{C} \equiv \int_{x,y} \underbrace{G^0_{+-}(x,y)} \; \Box_x \Box_y \frac{\delta}{\delta j_+(x)} \frac{\delta}{\delta j_-(y)}$$

$$G^0_{+-}(x,y) \equiv \int \frac{d^4p}{(2\pi)^4} \; e^{-ip\cdot(x-y)} \; \underbrace{2\pi\theta(-p^0)\delta(p^2)}_{\text{cut propagator}}$$

- Consider a generic cut vacuum diagram :



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Bookkeeping
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Bookkeeping

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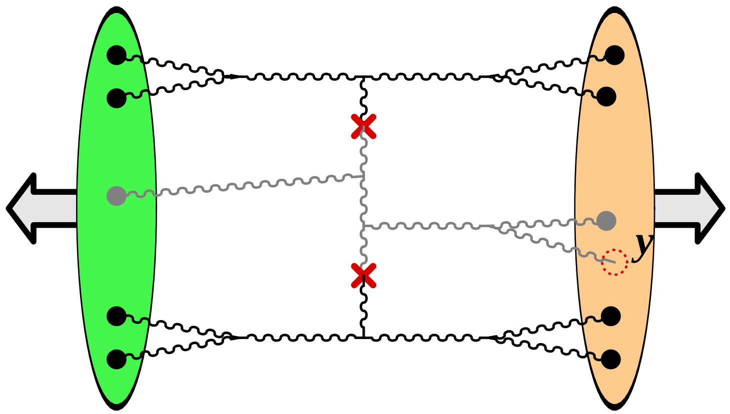

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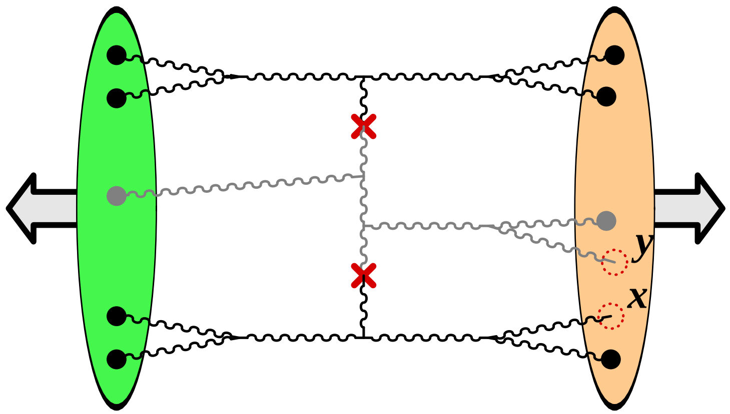

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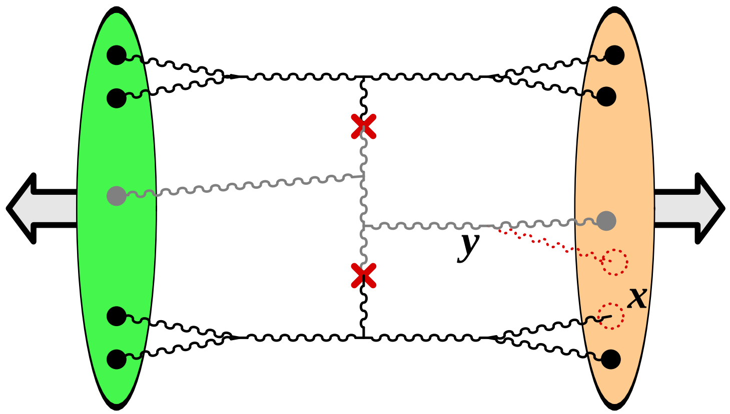

Bookkeeping

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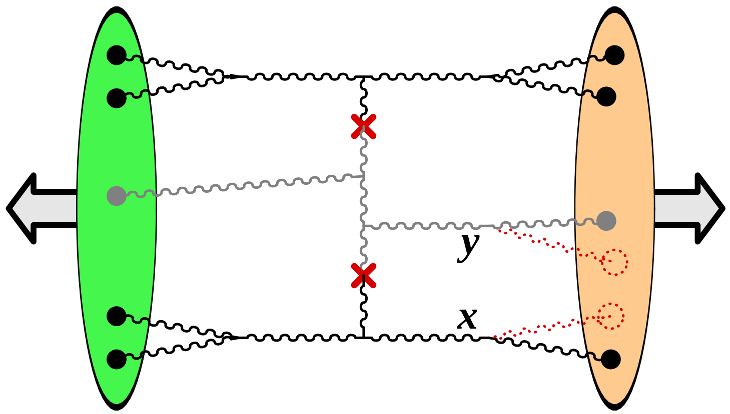

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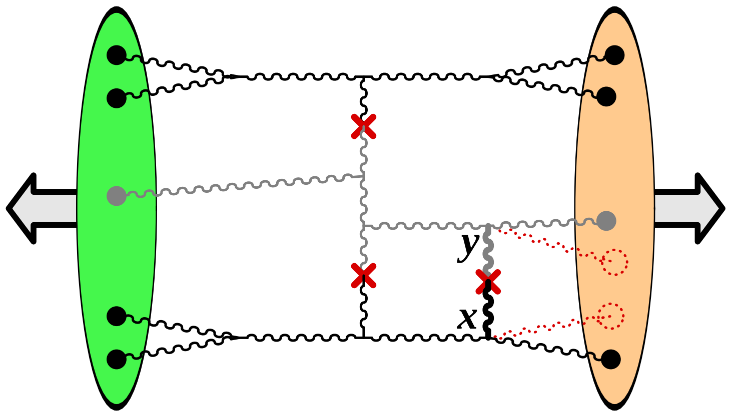
$G^0_{+-}(x,y) \quad \longrightarrow$ 

$\triangleright$ $\mathcal{C}$ removes two sources (one in the amplitude and one in the complex conjugated amplitude), and creates a new cut propagator

Bookkeeping

- The operator $\mathcal{C}$ can be used to derive many useful formulas :

$$F(z) = \sum_{n=0}^{+\infty} z^n \, P_n = e^{z\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

▷ sum of all cut vacuum graphs, where each cut is weighted by z

$$\overline{N} = F'(1) = \mathcal{C} \; e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}\Big|_{j_+=j_-=j}$$

- Benefits :
  - The tracking of infinite sets of Feynman diagrams has been replaced by simple algebraic manipulations
  - The use of the identity $e^{\mathcal{C}} \; e^{iV[j_+]} \; e^{-iV^*[j_-]}\Big|_{j_+=j_-} = 1$ renders automatic an important cancellation that would be hard to see from the diagrams (Abramovsky-Gribov-Kancheli)

Inclusive gluon spectrum

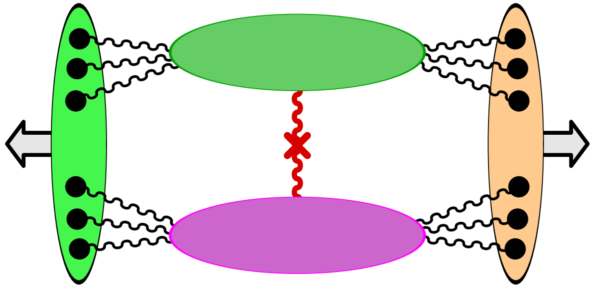
First moment of the distribution

- Reminder :

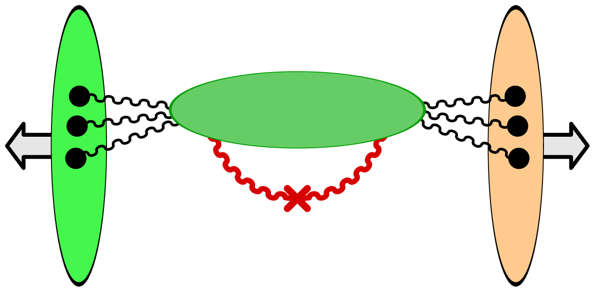
$$\overline{N} = \sum_n n\, P_n = \mathcal{C}\left\{ \underbrace{e^{\mathcal{C}}\; e^{iV[j_+]}\; e^{-iV^*[j_-]}} \right\}_{j_+=j_-=j}$$

sum of all the cut vacuum graphs

- There are two kind of terms :
  - $\mathcal{C}$ picks two sources in two distinct connected cut diagrams



  - $\mathcal{C}$ picks two sources in the same connected cut diagram

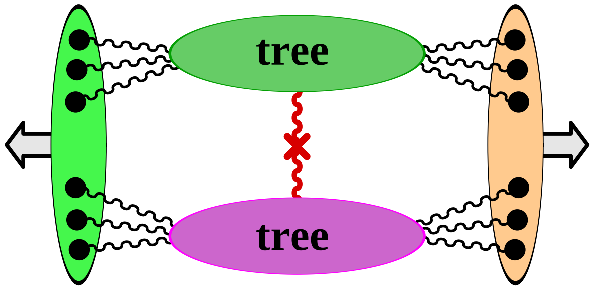


First moment at LO

- At LO, only tree diagrams contribute (each loop costs an α_s)
  ▷ the second type of topologies can be neglected (they have at least one loop)

- In each blob, we must sum over all the tree diagrams, and over all the possible cuts :

$$\overline{N}_{LO} = \sum_{\text{trees}} \sum_{\text{cuts}}$$



- Reminder : at the end, the sources on both sides of the cut must be set equal :

$$j_+ = j_-$$

First moment at LO

- One can prove easily that, for each graph, the sum over the cuts amounts to replacing all the propagators by retarded propagators. This comes from the fact that :


$=$ **retarded** propagator

- The sum of all the tree graphs ending at a point x is a solution $\mathcal{A}^\mu(x)$ of the classical equations of motion of the theory. In the case of QCD, they are the Yang-Mills equations (analogue of the Maxwell equations of electrodynamics)

- Note : the possibility of neglecting quantum effects comes from the fact that we are dealing with large occupation numbers, i.e. large fields :

$$\left[\mathcal{A}^\mu(x), \mathcal{A}^\nu(y)\right] \sim \mathcal{O}(1) \ll \mathcal{A}^\mu(x)\mathcal{A}^\nu(y)$$

- Because the propagators are retarded, the solution we need obeys the following boundary condition : $\mathcal{A}^\mu(x) \underset{t\to-\infty}{=} 0$

Gluon spectrum at LO

Krasnitz, Nara, Venugopalan (1999 – 2001), Lappi (2003)

- The gluon spectrum at LO is given by :

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y\,\sum_\lambda \epsilon_\lambda^\mu\epsilon_\lambda^\nu\;\mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

where $\mathcal{A}_\mu(x)$ is the retarded solution of Yang-Mills equations that vanishes in the remote past

- Note : the spectrum depends only on the fields at late time

$$\int d^4x\; e^{ip\cdot x}\;\square_x\,\mathcal{A}_\mu(x) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\;e^{ip\cdot x}\;\left[\partial_0 - iE_p\right]\mathcal{A}_\mu(x)$$

Gluon spectrum at LO

- The retarded nature of the classical fields involved in this approach makes the numerical resolution straightforward :
  - Discretize the spatial coordinates and put the fields on a lattice :

    $$\mathcal{A}_\mu(x_0, x, y, z) \to \mathcal{A}_\mu(x_0, i, j, k)$$

  - Write the Yang-Mills equations as

    $$\frac{\partial}{\partial x^0}\mathcal{A} = F(\mathcal{A}, \vec{\boldsymbol{\nabla}}\mathcal{A})$$

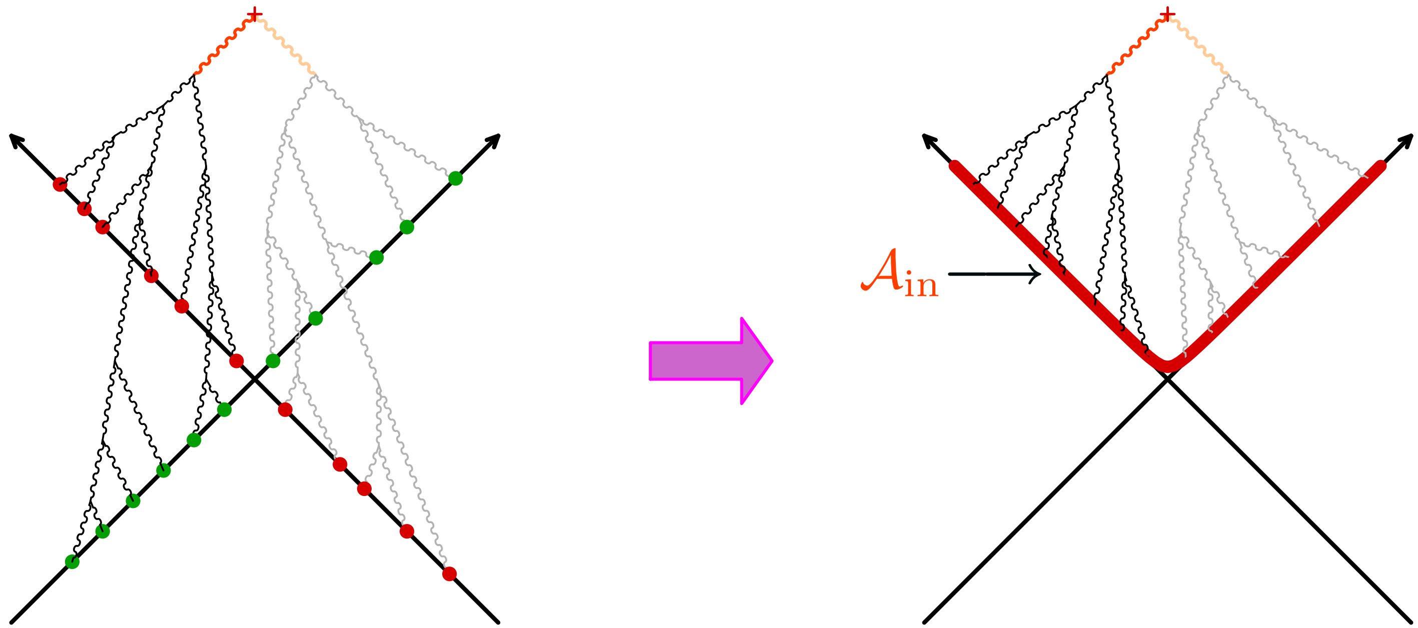
  - Start at some large negative time $x^0_{\rm ini}$ with $\mathcal{A} = 0$
  - To update from x^0 to $x^0 + \Delta x^0$, do :

    $$\mathcal{A}(x^0 + \Delta x^0) = \mathcal{A}(x^0) + \Delta x^0\; F(\mathcal{A}(x^0), \vec{\boldsymbol{\nabla}}\mathcal{A}(x^0))$$

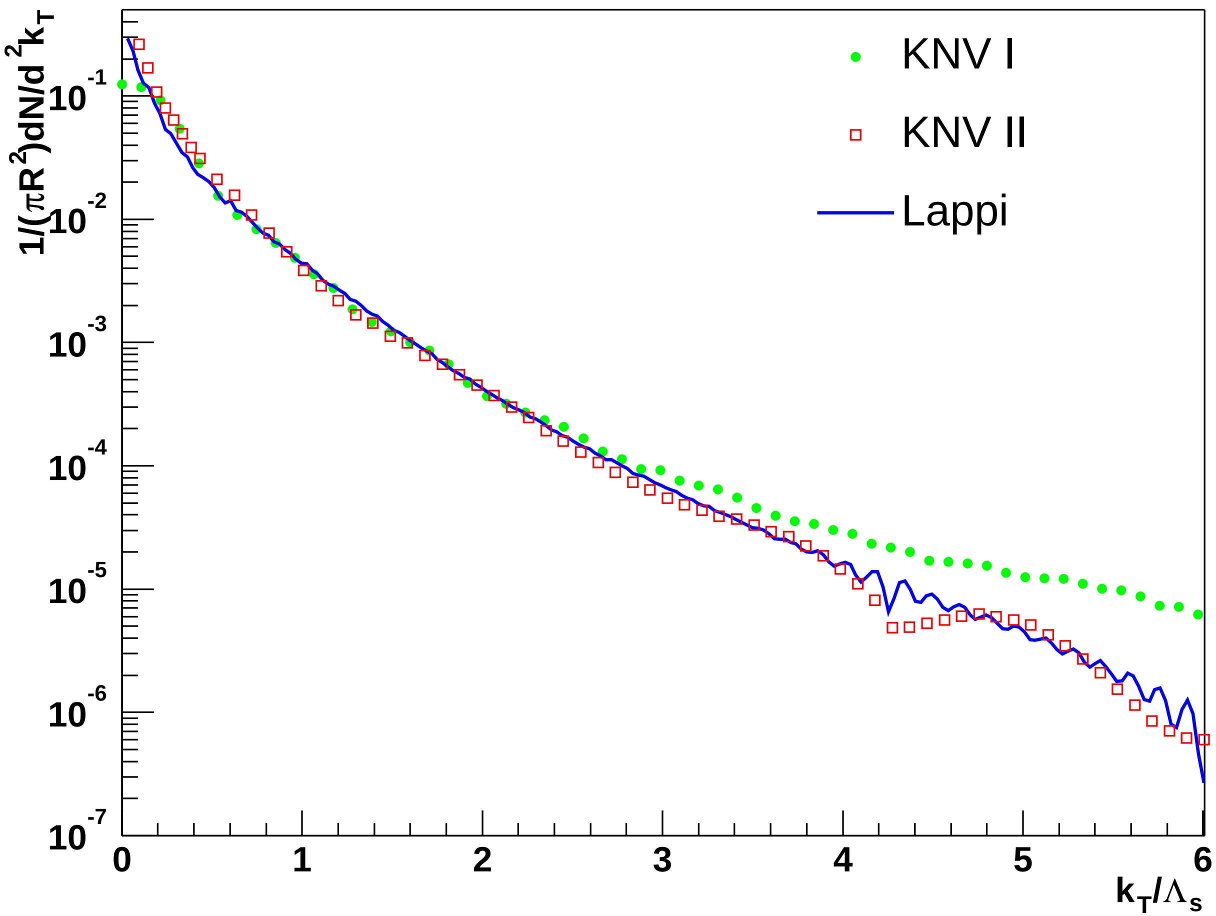
  - At a large positive time, perform a Fourier decomposition of the field $\mathcal{A}$, and compute the gluon spectrum

Gluon spectrum at LO

- The calculation is done in the gauge : $x^+\mathcal{A}^- + x^-\mathcal{A}^+ = 0$
  - ▷ $\mathcal{A}^- = 0$ at $z = t$ and $\mathcal{A}^+ = 0$ at $z = -t$
  - ▷ the produced gauge field does not modify the currents J^+, J^-
- In this gauge, one can find analytically the field just above the light-cone, at $\tau = 0^+$. Therefore, the previous algorithm needs to be implemented only in the forward light-cone :

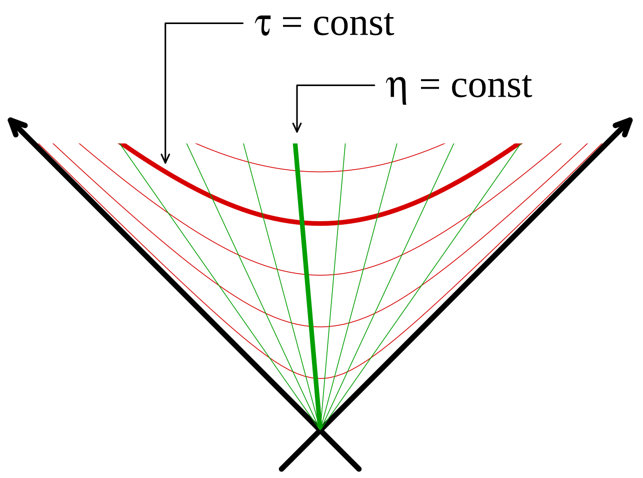


Gluon spectrum at LO



- Important softening at small $k_\perp$ compared to pQCD (saturation)
- Lattice artefacts at large momentum
  (they do not affect much the overall number of gluons)

Boost invariance



- Initial values at $\tau = 0^+$: the initial fields $\mathcal{A}_{\rm in}$ do not depend on the rapidity η

  ▷ they remain independent of η at all times (invariance under boosts in the z direction)

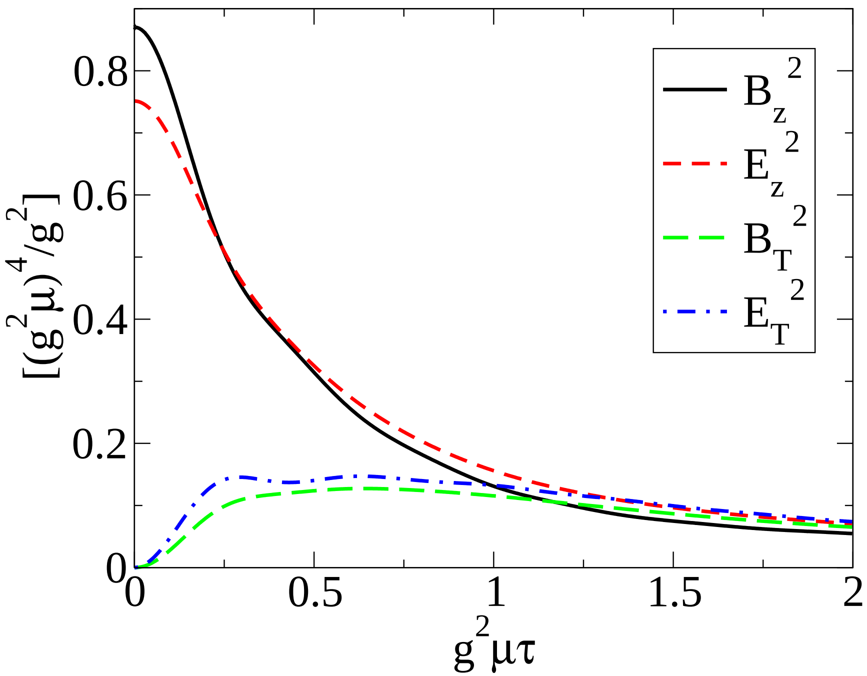
  ▷ numerical resolution performed in $1+2$ dimensions

Initial Glasma fields

Lappi, McLerran (2006) (Semantics : Glasma $\equiv$ Glas[s - plas]ma)

- Before the collision, the chromo-$\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields are localized in two sheets transverse to the beam axis
- Immediately after the collision, the chromo-$\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$ fields have become longitudinal :

$$\boldsymbol{E}^z = ig\left[\mathcal{A}_1^i, \mathcal{A}_2^i\right] \quad , \qquad \boldsymbol{B}^z = ig\epsilon^{ij}\left[\mathcal{A}_1^i, \mathcal{A}_2^i\right]$$



Loop corrections

WARNING : work in progress !!

(FG, Lappi, Venugopalan)

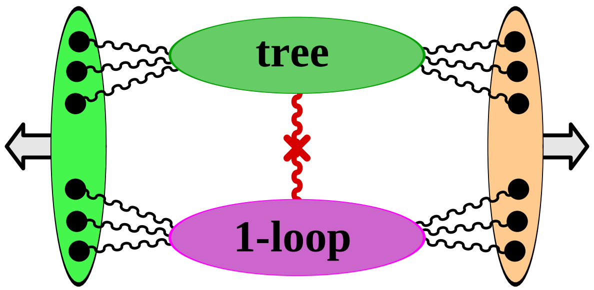
Reasons for studying loop corrections

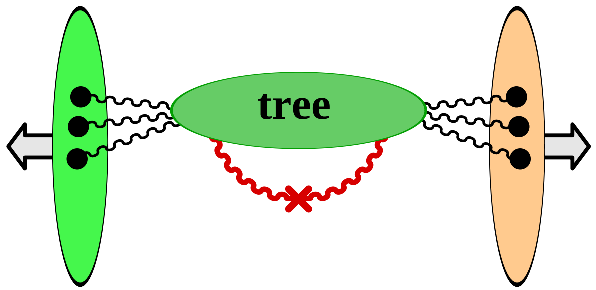
- Factorization. Proving "factorization at leading order" in fact requires to look at loop corrections :
  - Loop corrections in QCD have large logarithms of $1/x_{1,2}$
  - These large logs can compensate the smallness of the coupling constant α_s, i.e. $\alpha_s \log(1/x_{1,2}) \sim 1$ even though $\alpha_s \ll 1$
  - Factorizability in this context means that all the powers $[\alpha_s \log(1/x_{1,2})]^n [Q_s/p_\perp]^n$ can be resummed by letting the generalized "parton distributions" $W[\rho_{1,2}]$ evolve with rapidity

- The boost invariant solution found at tree level is a very peculiar configuration. Moreover, it is known from numerical studies that it is unstable if perturbed by rapidity dependent fluctuations. Loop corrections generate such perturbations!

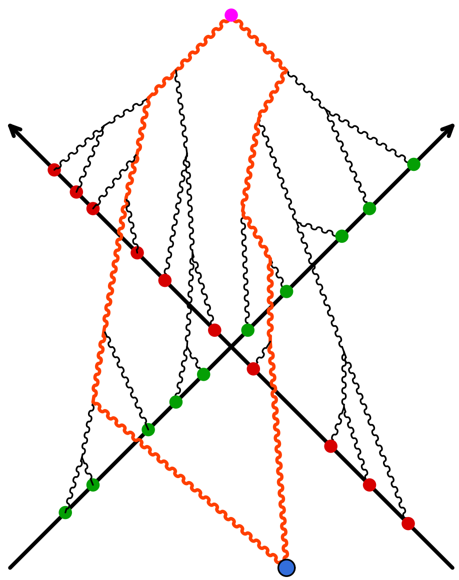
1-loop corrections to $\overline{N}$

- 1-loop diagrams for $\overline{N}$



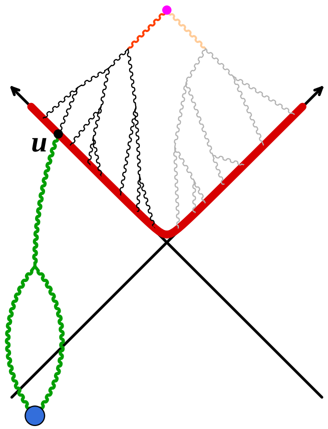


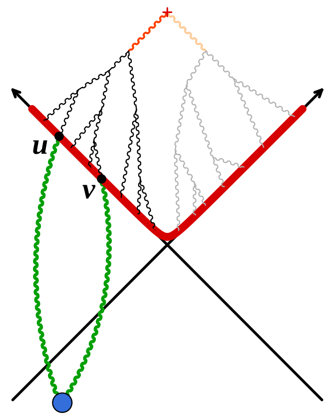
- This involves diagrams such as :



1-loop corrections to $\overline{N}$

- It is useful to divide the 1-loop corrections into a piece below the light-cone (calculable analytically) and a part above the light-cone (that must be solved numerically) :





  - Any divergence that happens in the part below the light-cone is related to the initial state, and we should try to absorb it in the "parton distributions" $W[\rho_{1,2}]$
  - Anything in the forward light-cone happens after the collision and has to do with the evolution of the final state

Divergences

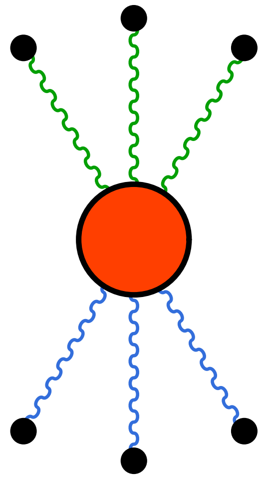
- If taken at face value, the 1-loop corrections are plagued by several divergences :
  - The pieces below the light-cone are infinite, because of an unbounded integration over a rapidity variable
  - The loop corrections can be seen as a perturbation of the initial field on the light-cone

    However, the boost invariant classical solution of Yang-Mills equations suffers from an instability under rapidity dependent perturbations (Romatschke, Venugopalan (2005))

    ▷ see lecture III : Glasma instabilities

Initial state factorization

- Anatomy of the full calculation :



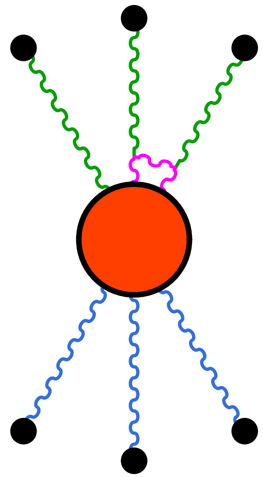
$$\mathbf{W}_{\mathbf{Y}_{\mathrm{beam}}\,\mathbf{-Y}}[\rho_1]$$

$$\mathbf{N}[\ \mathbf{A}_{\mathrm{in}}(\rho_1\ ,\ \rho_2)\ ]$$

$$\mathbf{W}_{\mathbf{Y}_{\mathrm{beam}}\,\mathbf{+Y}}[\rho_2]$$

Initial state factorization

- Anatomy of the full calculation :



$\mathbf{W}_{\mathbf{Y}_{\mathrm{beam}}\,\mathbf{-Y}}[\rho_1]$

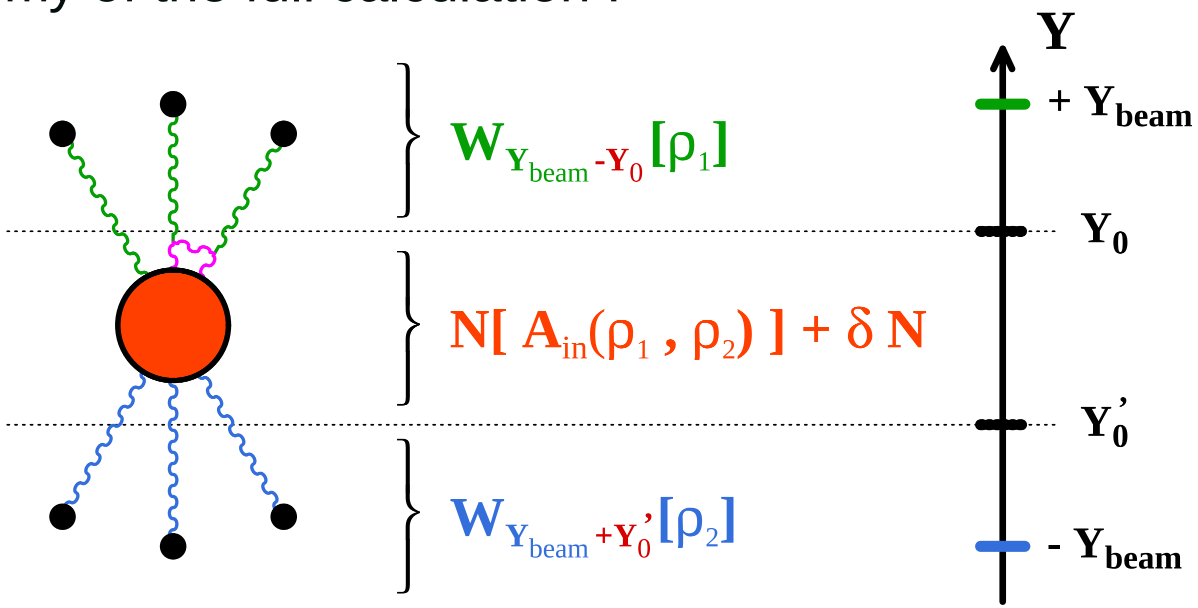
$\mathbf{N}[\,\mathbf{A}_{\mathrm{in}}(\rho_1\,,\,\rho_2)\,] + \delta\,\mathbf{N}$

$\mathbf{W}_{\mathbf{Y}_{\mathrm{beam}}\,\mathbf{+Y}}[\rho_2]$

- When the observable $\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s \int dY$ in $\delta\overline{N}$

Initial state factorization

- Anatomy of the full calculation :



- When the observable $\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]$ is corrected by an extra gluon, one gets divergences of the form $\alpha_s \int dY$ in $\delta\overline{N}$

- Put some arbitrary cutoffs Y_0 and Y_0' between the "observable" and the "source distributions" : the dependence on Y_0, Y_0' should cancel between the various factors

Initial state factorization

- In order to prove the factorization of these divergences in the initial state distributions of sources, one must be able to relate them to the Hamiltonian that governs the rapidity dependence of the distributions $W_{Y}[\rho_{1,2}]$ in the following way :

$$\left[\delta\overline{N}\right]_{\substack{\text{divergent}\\ \text{coefficients}}} = \left[(Y_0 - Y)\,\mathcal{H}^\dagger[\rho_1] + (Y - Y_0')\,\mathcal{H}^\dagger[\rho_2]\right]\ \overline{N}_{LO}$$

where $\mathcal{H}[\rho]$ is the Hamiltonian that governs the rapidity dependence of the source distribution $W_{Y}[\rho]$:

$$\frac{\partial W_{Y}[\rho]}{\partial Y} = \mathcal{H}[\rho]\, W_{Y}[\rho]$$

Initial state factorization

- If everything works as expected, one can write

$$\frac{d\overline{N}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \int [D\rho_1]\,[D\rho_2]\;W_{Y_{\rm beam}-Y}[\rho_1]\;W_{Y_{\rm beam}+Y}[\rho_2] \times \frac{d\overline{N}[\mathcal{A}_{\rm in}(\rho_1,\rho_2)]}{dY\,d^2\vec{\boldsymbol{p}}_\perp}$$

- Somewhat analogous to factorization in conventional pQCD :

$$W_Y[\rho] \qquad \longleftrightarrow \qquad \text{parton distribution}$$

  and it has the same conceptual importance, because it implies the universality of the distributions $W_Y[\rho]$ (e.g. that they are identical in eA and in AA collisions)

- In lecture III, we will see that this formula must be slightly modified because of the Glasma instabilities

Motivation for lecture III

Motivation for lecture III

- The instabilities alluded to in this lecture have the potential to ruin the whole approach:
  - Formally, they arise at one-loop and are suppressed by α_s
  - They also come with an exponentially growing factor of time $\exp\sqrt{\mu\tau}$ with $\mu \sim Q_s$
- Does the CGC framework become completely useless after a time $\tau \sim Q_s^{-1}\log^2(1/\alpha_s)$?
- Or can we make it work beyond this limit by a resummation ?
- What is the connection between this instability and the usual Weibel instability in plasma physics?
- Does it help to reach local thermalization?

Lecture III : Instabilities, thermalization

- Reminder on initial gluon production
- Glasma instabilities
- Instabilities in anisotropic plasmas
- Thermalization ?

Lecture IV : Kinetic theory

- Collisionless kinetic equations
- Boltzmann equation
- Transport coefficients
- From kinetic theory to hydrodynamics

Complements

Power counting

- Consider a connected graph with :
  - n_E external lines, n_I internal lines
  - n_3 3-gluon vertices (g), n_4 4-gluon vertices (g^2)
  - n_ρ color sources (g^{-1})
  - n_L independent loops

- These numbers are related by :

$$3n_3 + 4n_4 + n_\rho = n_E + 2n_I$$

$$n_L = n_I - (n_3 + n_4) - n_\rho + 1$$

- Therefore, the order of the diagram in g :

$$g^{n_3}\; g^{2n_4}\; g^{-n_\rho} = g^{-2}\; g^{n_E}\, g^{2(n_L - 1)}$$

Sum over the cuts

- In the previous diagrams, one must sum over all the possible ways of cutting inside the blobs
- This can be achieved via Cutkosky's cutting rules :
  - A vertex is $-ig$ on one side of the cut, and $+ig$ on the other side
  - There are four propagators, depending on the location w.r.t. the cut of the vertices they connect :

$$G^0_{++}(p) = i/(p^2 - m^2 + i\epsilon) \qquad \text{(standard Feynman propagator)}$$
$$G^0_{--}(p) = -i/(p^2 - m^2 - i\epsilon) \qquad \text{(complex conjugate of } G^0_{++}(p))$$
$$G^0_{+-}(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2)$$
$$G^0_{-+}(p) = 2\pi\theta(p^0)\delta(p^2 - m^2)$$

  - At each vertex of a given diagram, sum over the types $+$ and $-$ (2^n terms for a diagram with n vertices)
- Note : this is also the zero-temperature version of the Schwinger-Keldysh formalism

Sum over the cuts

- In the sum over the cuts, we get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \frac{\boldsymbol{i}}{\boldsymbol{p^2 - m^2 + i\epsilon}} - 2\pi\theta(-p^0)\delta(p^2 - m^2)$$

Sum over the cuts

- In the sum over the cuts, we get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \mathrm{PP}\left[\frac{i}{p^2-m^2}\right] + \pi\delta(p^2-m^2) - 2\pi\theta(-p^0)\delta(p^2-m^2)$$

$$\underbrace{}_{\boldsymbol{1 = \theta(p^0) + \theta(-p^0)}}$$

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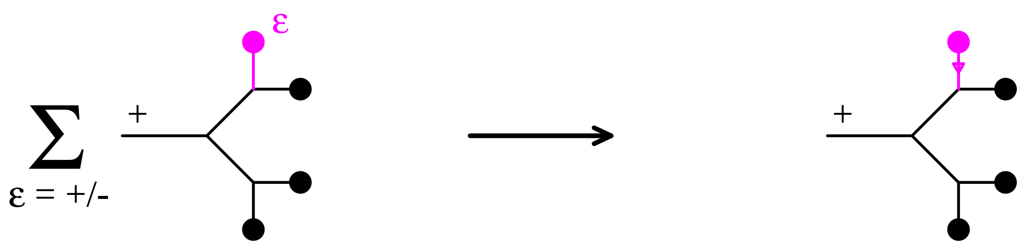
Sum over the cuts

- In the sum over the cuts, we get combinations of propagators such as :

$$G^0_{++}(p) - G^0_{+-}(p) = \boldsymbol{G^0_R(p)}$$

- Similarly : $G^0_{-+}(p) - G^0_{--}(p) = G^0_R(p)$

- Starting from the "leaves" of the trees, use these formulas recursively in order to replace all the $G^0_{\pm\pm}$ propagators by retarded propagators. Example :



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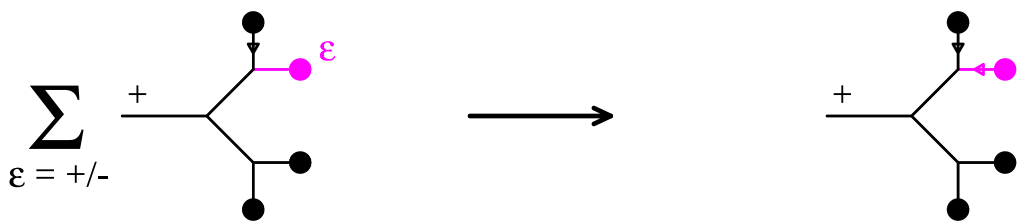

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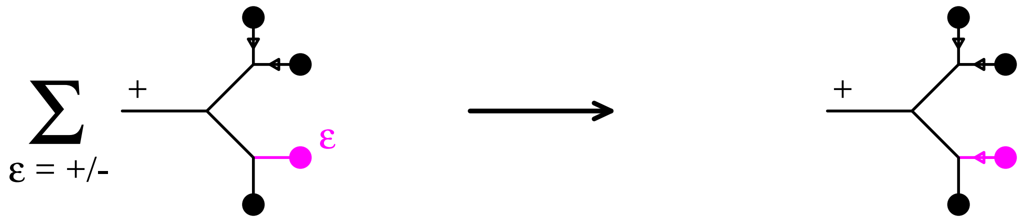

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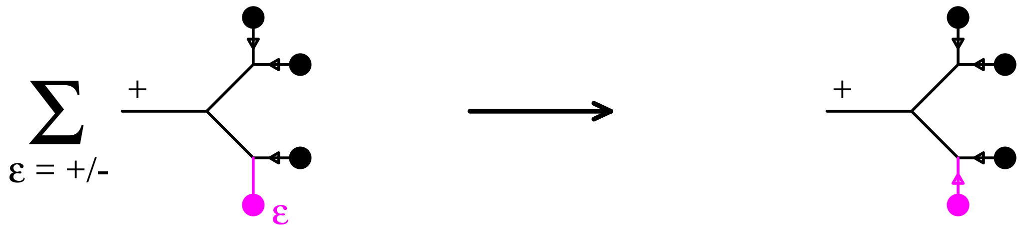

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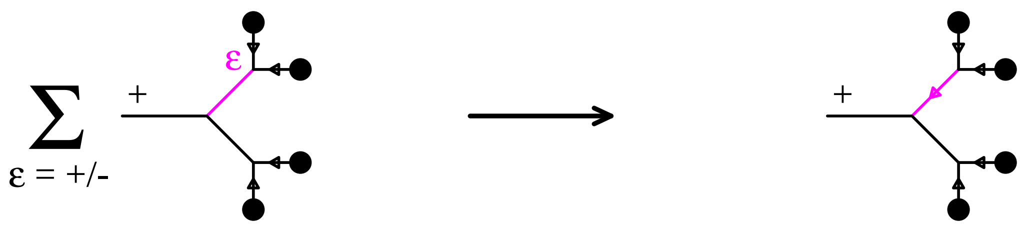

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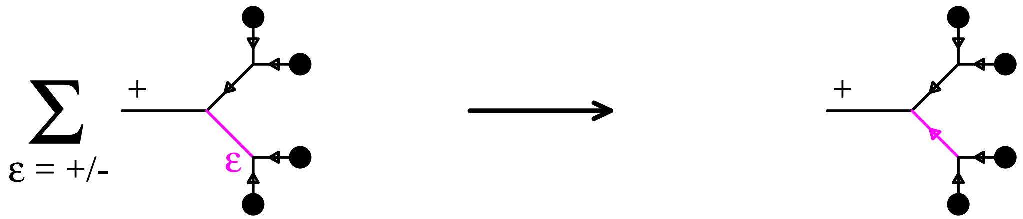

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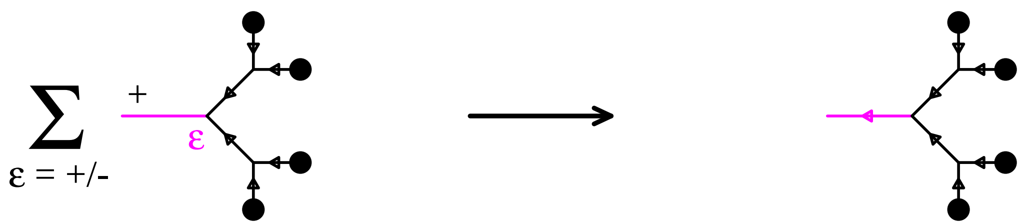


▷ we have a sum of tree diagrams with retarded propagators

Retarded classical solution

- The sum of all the tree diagrams constructed with retarded propagators is the solution of Yang-Mills equations,

  $[D_\mu, F^{\mu\nu}] = J^\nu$, with retarded boundary condition $A^\mu(x_0 = -\infty) = 0$

- Proof (for a scalar theory). The classical EOM reads

$$(\square + m^2)\,\varphi(x) + \frac{g}{2}\varphi^2(x) = j(x)$$

- Write the Green's formula for the retarded solution that obeys $\varphi(x) = 0$ at $x^0 = -\infty$:

$$\varphi(x) = \int d^4y\; G^0_R(x-y)\left[-i\,\frac{g}{2}\varphi^2(y) + i\,j(y)\right]$$

  where $G^0_R(x-y)$ is the free retarded propagator

Retarded classical solution

- Order g^0 :

$$\varphi_{(0)}(x) = \int d^4y\, G_R^0(x-y)\, i\, j(y)$$

- Order g^1 :

$$\varphi_{(0)}(x) + \varphi_{(1)}(x) = \int d^4y\, G_R^0(x-y) \left[-i\, \frac{g}{2} \varphi_{(0)}^2(y) + i\, j(y)\right]$$

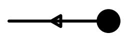
i.e.

$$\varphi_{(1)}(x) = -i\, \frac{g}{2} \int d^4y\, G_R^0(x-y) \left[\int d^4z\, G_R^0(y-z)\, i\, j(z)\right]^2$$

- One can construct the solution iteratively, by using in the r.h.s. the solution found in the previous orders

Retarded classical solution

- The diagrammatic expansion of this classical solution is :



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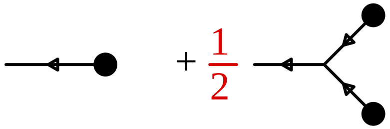

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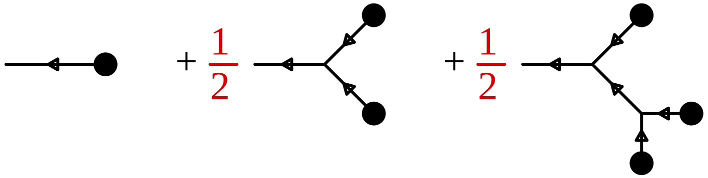

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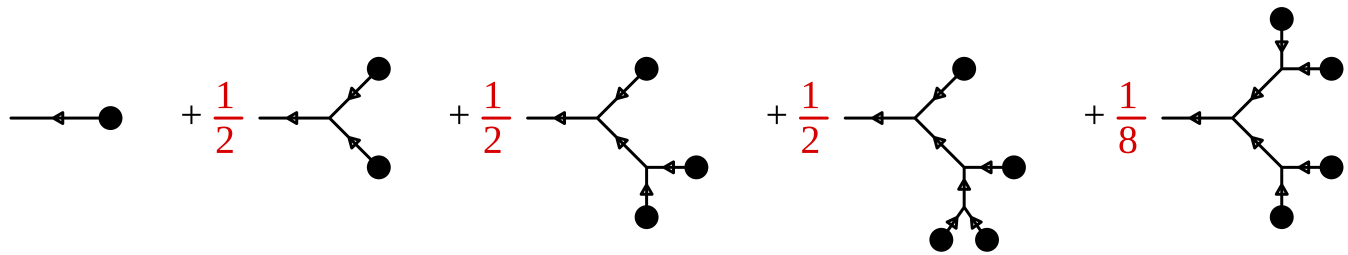

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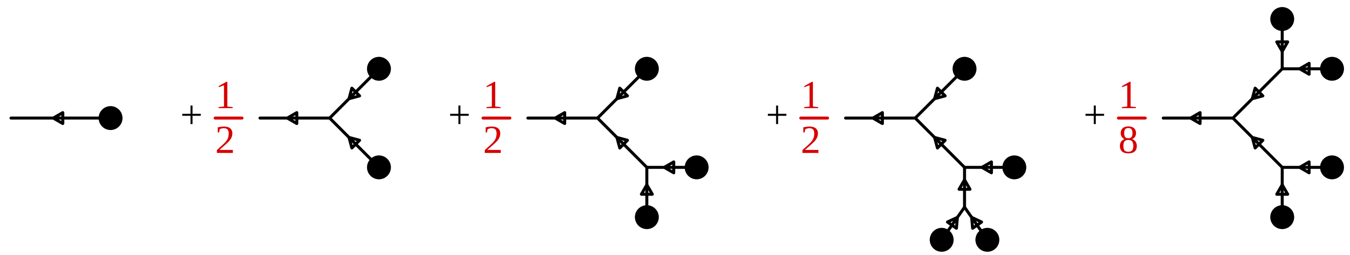
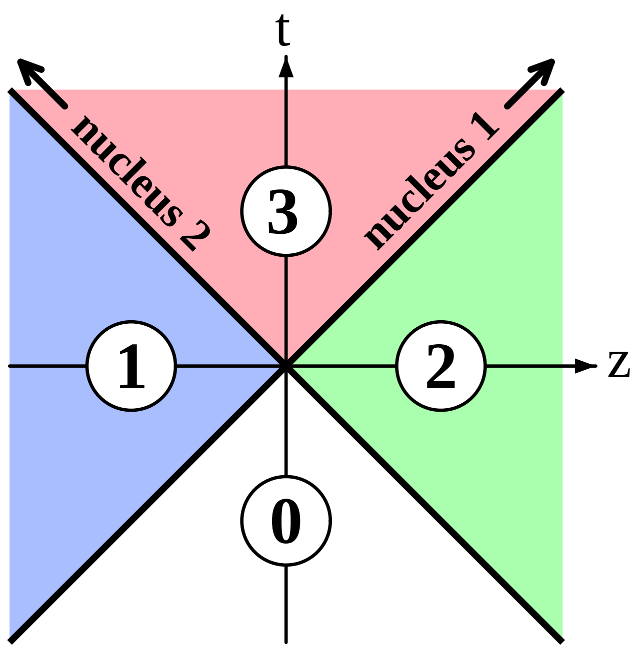


- The classical solution is given by the sum of all the tree diagrams with retarded propagators

Gluon spectrum at LO

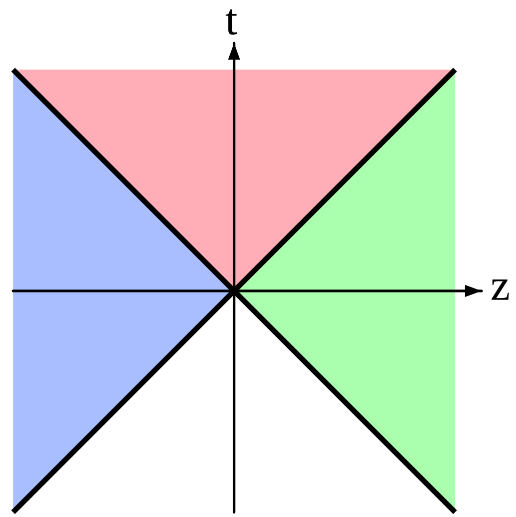
- Space-time structure of the classical color field:



  - Region 0 : no causal relation to either nuclei
  - Region 1 : causal relation to the 1st nucleus only
  - Region 2 : causal relation to the 2nd nucleus only
  - Region 3 : causal relation to both nuclei

Gluon spectrum at LO

- Propagation through region 0:

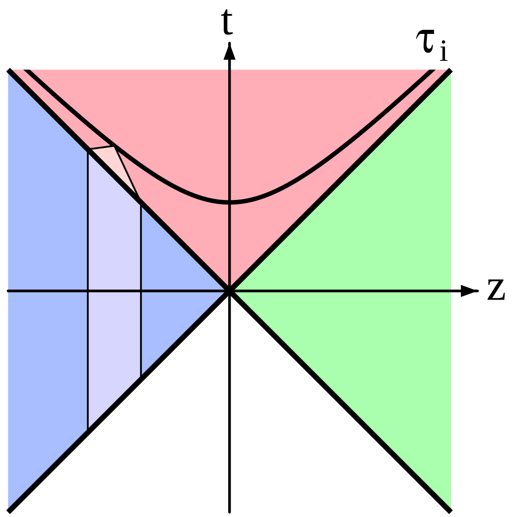


▷ trivial : the classical field is entirely determined by the initial condition, i.e.

$$\mathcal{A}^{\mu} = 0$$

Gluon spectrum at LO

- Propagation through region 1:



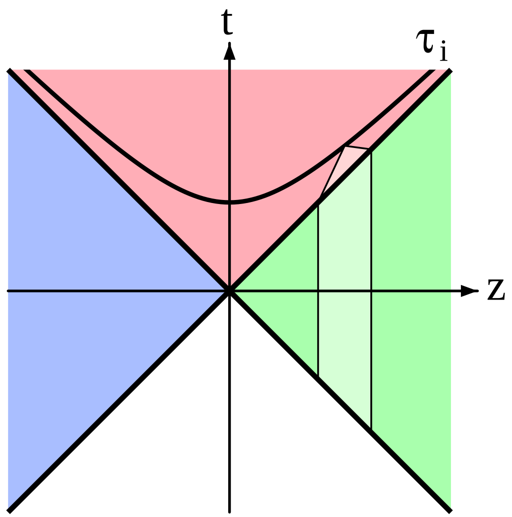
▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

$$\mathcal{A}_1^+ = \mathcal{A}_1^- = 0 \quad , \quad \mathcal{A}_1^i = \frac{i}{g} U_1(\vec{\boldsymbol{x}}_\perp)\partial^i U_1^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $\quad U_1(\vec{\boldsymbol{x}}_\perp) = T_+ \exp\; ig \int dx^+ T^a \frac{1}{\boldsymbol{\nabla}_\perp^2} \rho_1^a(x^+, \vec{\boldsymbol{x}}_\perp)$

Gluon spectrum at LO

- Propagation through region 2:



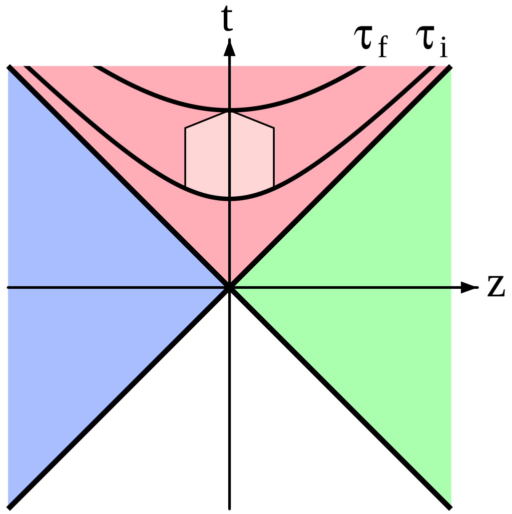
▷ the Yang-Mills equation can be solved analytically when there is only one nucleus :

$$\mathcal{A}_2^+ = \mathcal{A}_2^- = 0 \quad , \quad \mathcal{A}_2^i = \frac{i}{g} U_2(\vec{\boldsymbol{x}}_\perp)\partial^i U_2^\dagger(\vec{\boldsymbol{x}}_\perp)$$

with $U_2(\vec{\boldsymbol{x}}_\perp) = T_-\ \exp\ ig \int dx^- T^a \frac{1}{\boldsymbol{\nabla}_\perp^2} \rho_2^a(x^-, \vec{\boldsymbol{x}}_\perp)$

Gluon spectrum at LO

- Propagation through region 3 (forward light cone):



▷ one must solve numerically the Yang-Mills equations with the following initial condition at $\tau_i = 0^+$:

$$\mathcal{A}^i(\tau=0,\vec{\boldsymbol{x}}_\perp) = \mathcal{A}_1^i(\vec{\boldsymbol{x}}_\perp) + \mathcal{A}_2^i(\vec{\boldsymbol{x}}_\perp)$$

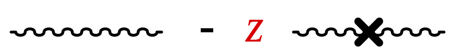
$$\mathcal{A}^\eta(\tau=0,\vec{\boldsymbol{x}}_\perp) = \frac{ig}{2}\left[\mathcal{A}_1^i(\vec{\boldsymbol{x}}_\perp)\,,\,\mathcal{A}_2^i(\vec{\boldsymbol{x}}_\perp)\right]$$

$\mathcal{A}^\tau = 0$ (gauge condition)

Less inclusive quantities

- One can also compute less inclusive quantities at Leading Order in terms of classical solutions of Yang-Mills equations, but with complicated boundary conditions in general

- Example: Generating function
  - Definition : $F(z) \equiv \sum_{n=0}^{\infty} P_n \; z^n$
  - $F'(z)/F(z)$ has the same diagrammatic expansion as $\overline{N}$, but with each cut propagator multiplied by z
  - Performing the sum over the cuts does not give retarded propagators anymore :



$\neq$ **retarded** propagator

  - $F'(z)/F(z)$ can be written in terms of classical solutions of Yang-Mills, but they must obey boundary conditions both at $t = -\infty$ and $t = +\infty$

Definition

- One can encode the information about all the probabilities P_n in a generating function defined as :

$$F(z) \equiv \sum_{n=0}^{\infty} P_n \; z^n$$

- From the expression of P_n in terms of the operator $\mathcal{C}$, we can write :

$$F(z) \quad = \quad e^{z\mathcal{C}} \; e^{iV} \; e^{-iV^*}$$

- Reminder :
  - $e^{\mathcal{C}} \; e^{iV} \; e^{-iV^*}$ is the sum of all the cut vacuum diagrams
  - The cuts are produced by the action of $\mathcal{C}$

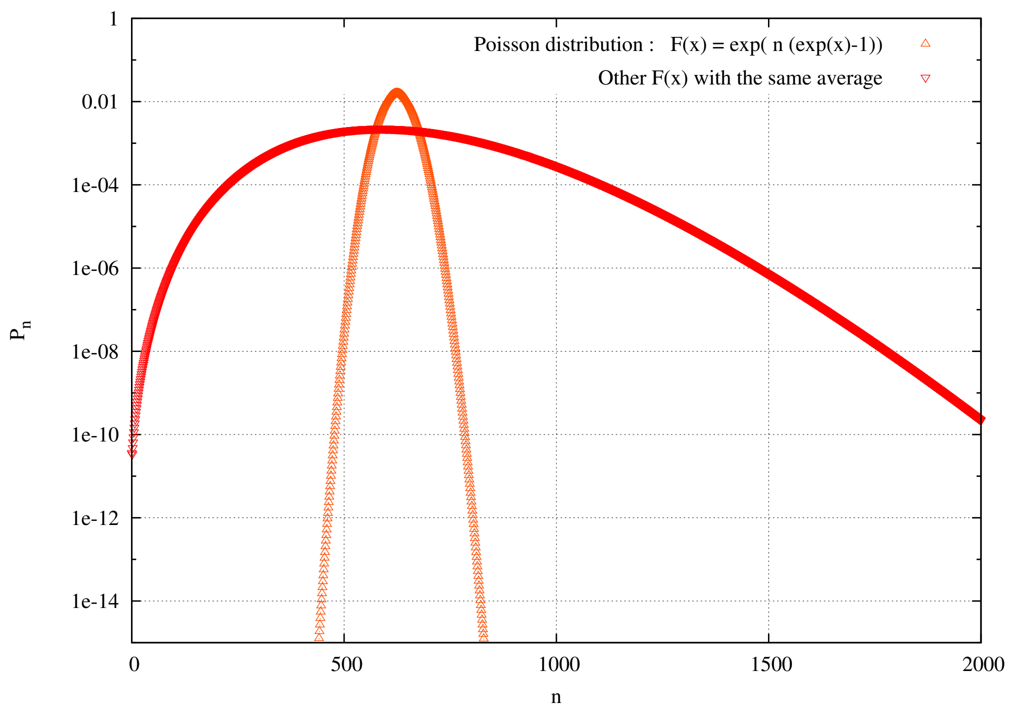
- Therefore, $F(z)$ is the sum of all the cut vacuum diagrams in which each cut line is weighted by a factor z

What would it be good for ?

- Let us pretend that we know the generating function $F(z)$. We could get the probability distribution as follows :

$$P_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta\; e^{-in\theta}\;\; F(e^{i\theta})$$

Note : this is trivial to evaluate numerically by a FFT :

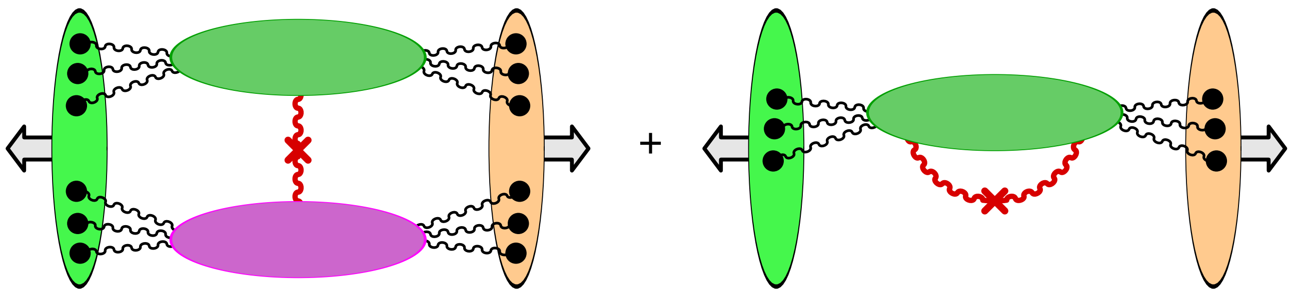


F(z) at Leading Order

- We have : $F'(z) = \mathcal{C}\ \left\{e^{z\mathcal{C}}\ e^{iV}\ e^{-iV^*}\right\}$

- By the same arguments as in the case of $\overline{N}$, we get :

$$\frac{F'(z)}{F(z)} =$$



- The major difference is that the cut graphs that must be evaluated have a factor z attached to each cut line

- At tree level (LO), we can write $F'(z)/F(z)$ in terms of solutions of the classical Yang-Mills equations, but these solutions are not retarded anymore, because :

retarded propagator

F(z) at Leading Order

- The derivative F'/F has an expression which is formally identical to that of $\overline{N}$,

$$\frac{F'(z)}{F(z)} = \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int_{x,y} e^{ip\cdot(x-y)}\,\square_x\square_y \sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^\nu \; A_\mu^{(+)}(x) A_\nu^{(-)}(y) \,,$$

with $A_\mu^{(\pm)}(x)$ two solutions of the Yang-Mills equations

- If one decomposes these fields into plane-waves,

$$A_\mu^{(\varepsilon)}(x) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \left\{ f_+^{(\varepsilon)}(x^0,\vec{\boldsymbol{p}})e^{-ip\cdot x} + f_-^{(\varepsilon)}(x^0,\vec{\boldsymbol{p}})e^{ip\cdot x} \right\}$$

the boundary conditions are :

$$f_+^{(+)}(-\infty,\vec{\boldsymbol{p}}) = f_-^{(-)}(-\infty,\vec{\boldsymbol{p}}) = 0$$

$$f_+^{(-)}(+\infty,\vec{\boldsymbol{p}}) = z\, f_+^{(+)}(+\infty,\vec{\boldsymbol{p}}) \quad , \quad f_-^{(+)}(+\infty,\vec{\boldsymbol{p}}) = z\, f_-^{(-)}(+\infty,\vec{\boldsymbol{p}})$$

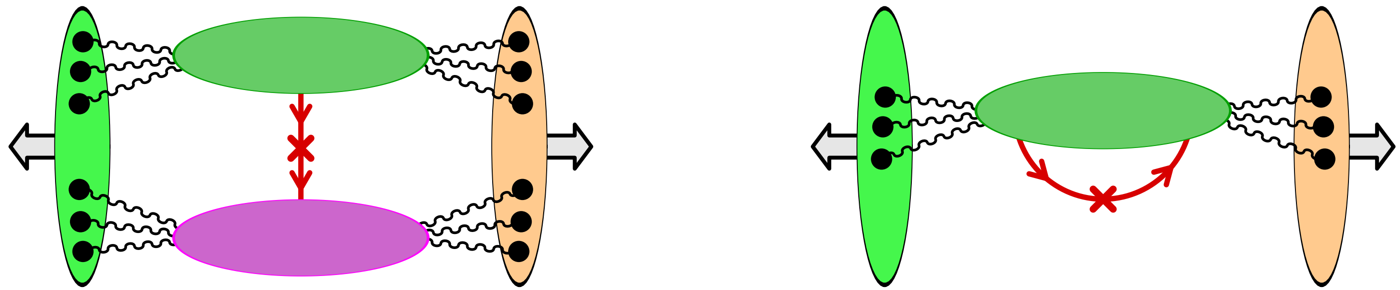
- There are boundary conditions both at $x_0 = -\infty$ and $x_0 = +\infty$ $\triangleright$ not an initial value problem $\triangleright$ hard...

Inclusive quark spectrum

FG, Kajantie, Lappi (2004, 2005)

- One can construct for quarks an operator $\mathcal{C}_{\mathrm{q}}$ that plays the same role as $\mathcal{C}$ for the gluons

- By repeating the same arguments, we find two generic topologies contributing to the inclusive quark spectrum :



(the blobs are sums of cut diagrams)

- The first topology cannot exist because the quark line is not closed on itself

  ▷ the quark spectrum starts at one loop

Quark production at one loop

- At lowest order (one loop), the quark spectrum reads :

$$\frac{d\overline{N}_{\mathrm{q}}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot x}\,\overline{u}(\vec{\boldsymbol{p}})\,(i\,\overrightarrow{\not\partial}_x - m)\,S_{+-}(x,y)\,(i\,\overleftarrow{\not\partial}_y + m)\,u(\vec{\boldsymbol{p}})\,e^{-ip\cdot y}$$

where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

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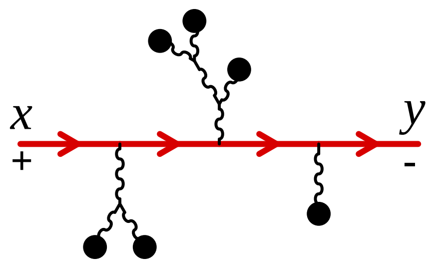
where S_{+-} is the quark propagator (with one endpoint on each side of the cut) to which are attached tree graphs in all the possible ways

- We need to calculate the sum of the following tree diagrams :



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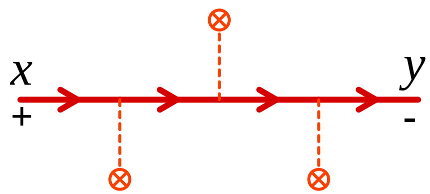
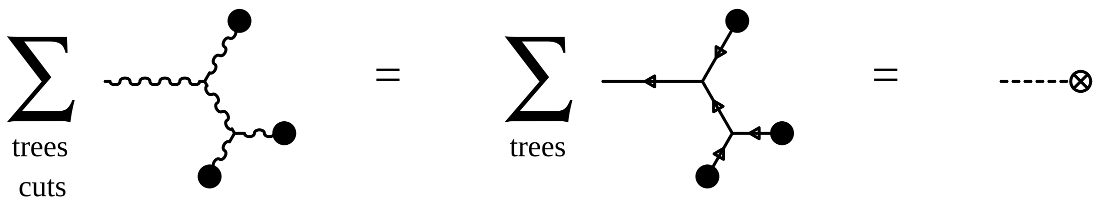


- Perform a resummation of all the sub-diagrams that correspond to the retarded classical solution :



Quark propagator

- The summation of all the classical field insertions can be done by solving a Lippmann-Schwinger equation :

$$S_{\epsilon\epsilon'}(x,y) = S^0_{\epsilon\epsilon'}(x,y) - ig \sum_{\eta=\pm} (-1)^\eta \int d^4z\, S^0_{\epsilon\eta}(x,z) \mathcal{A}_\mu(z) \gamma^\mu S_{\eta\epsilon'}(z,y)$$

- This equation is rather non-trivial to solve in this form, because of the mixing of the 4 components of the propagator. Perform a rotation on the $\pm$ indices :

$$S_{\epsilon\epsilon'} \quad \longrightarrow \quad \boldsymbol{S}_{\alpha\beta} \equiv \sum_{\epsilon,\epsilon'=\pm} U_{\alpha\epsilon} U_{\beta\epsilon'} S_{\epsilon\epsilon'}$$

$$(-1)^\epsilon \delta_{\epsilon\epsilon'} \quad \longrightarrow \quad \boldsymbol{\Sigma}_{\alpha\beta} \equiv \sum_{\epsilon=\pm} U_{\alpha\epsilon} U_{\beta\epsilon} (-1)^\epsilon$$

- A useful choice for the rotation matrix U is $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Quark propagator

- Under this rotation, the matrix propagator and field insertion become :

$$\boldsymbol{S}_{\alpha\beta} = \begin{pmatrix} 0 & S_A \\ S_R & S_D \end{pmatrix} \quad , \qquad \boldsymbol{\Sigma}_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  where $S^0_D(p) = 2\pi(\not{p} + m)\delta(p^2 - m^2)$

- The main simplification comes from the fact that $\boldsymbol{S}^0\boldsymbol{\Sigma}$ is the sum of a diagonal matrix and a nilpotent matrix
- One finds that S_R and S_A do not mix, i.e. they obey equations such as :

$$S_R(x,y) = S^0_R(x,y) - i\,g \int d^4z\, S^0_R(x,z)\mathcal{A}_\mu(z)\gamma^\mu S_R(z,y)$$

- One can solve S_D in terms of S_R and S_A :

$$S_D = S_R * S^{0\ -1}_R * S^0_D * S^{0\ -1}_A * S_A$$

Quark propagator

- In order to go back to S_{+-}, invert the rotation :

$$S_{+-} = \frac{1}{2}\left[S_{A} - S_{R} - S_{D}\right]$$

- At this point, we can rewrite the quark spectrum in terms of retarded and advanced quark propagators in the classical background

- Finally, one can rewrite it in terms of retarded solutions of the Dirac equation on top of the background $\mathcal{A}_\mu(x)$

$$\frac{d\overline{N}_{\rm q}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int\frac{d^3\vec{\boldsymbol{q}}}{(2\pi)^3 2E_q}\left|\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}})\right|^2$$

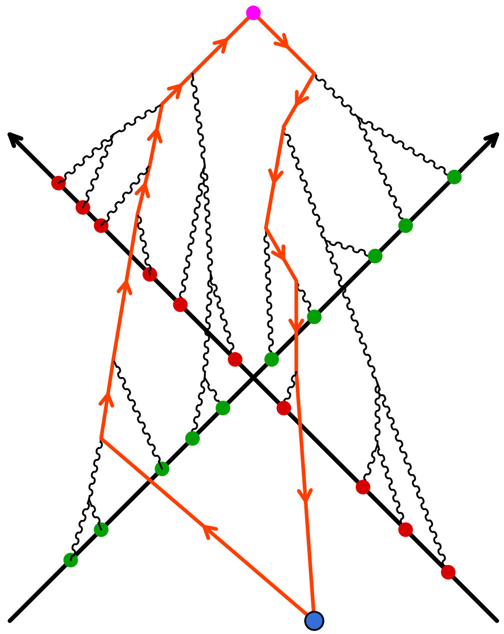
with

$$\mathcal{M}(\vec{\boldsymbol{p}},\vec{\boldsymbol{q}}) = \lim_{x^0\to+\infty}\int d^3\vec{\boldsymbol{x}}\; e^{ip\cdot x}\; u^\dagger(\vec{\boldsymbol{p}})\psi_{\boldsymbol{q}}(x)$$

$$(i\partial\!\!\!/_x - g\mathcal{A\!\!\!/}(x) - m)\psi_{\boldsymbol{q}}(x) = 0\;,\quad \psi_{\boldsymbol{q}}(x^0,\vec{\boldsymbol{x}}) \underset{x^0\to-\infty}{=} v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

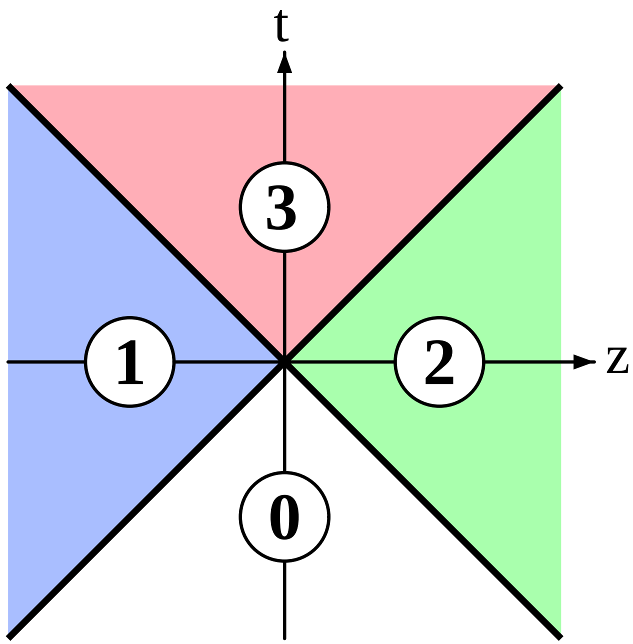
Quark propagator

- This calculation amounts to summing the following diagrams :



Background field

- Space-time structure of the classical color field:

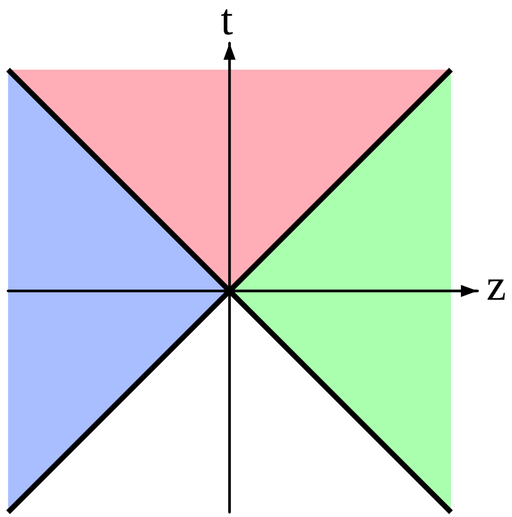


- Region 0: $\mathcal{A}^\mu = 0$
- Region 1: $\mathcal{A}^\pm = 0$, $\mathcal{A}^i = \frac{i}{g} U_1 \nabla_\perp^i U_1^\dagger$
- Region 2: $\mathcal{A}^\pm = 0$, $\mathcal{A}^i = \frac{i}{g} U_2 \nabla_\perp^i U_2^\dagger$
- Region 3: $\mathcal{A}^\mu \neq 0$

- **Notes:**
  - In the region 3, $\mathcal{A}^\mu$ is known only numerically
  - We must solve the Dirac equation numerically as well

Quark propagation

- Propagation through region 0:

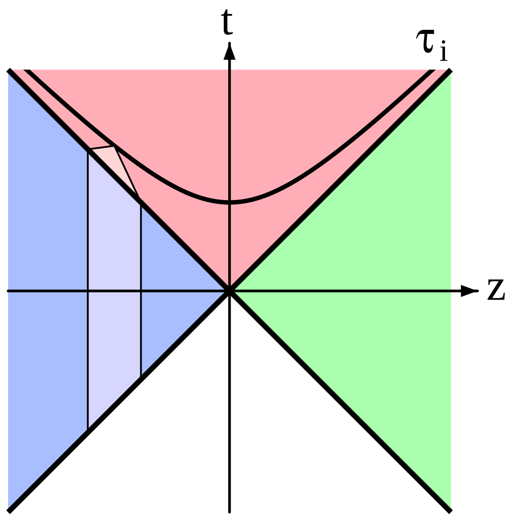


▷ trivial because there is no background field

$$\psi_{\boldsymbol{q}}(x) = v(\vec{\boldsymbol{q}})e^{iq\cdot x}$$

Quark propagation

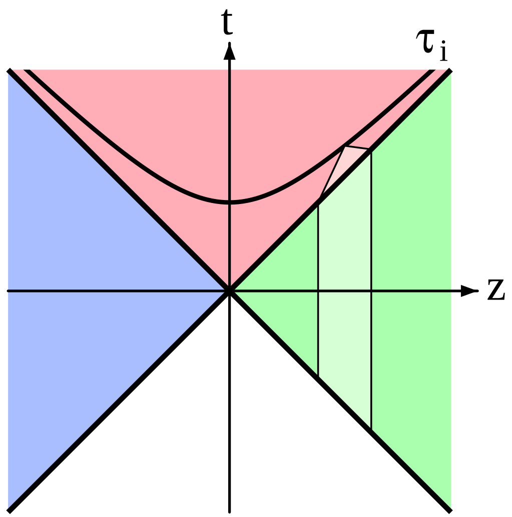
- Propagation through region 1:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},1}(\tau_i)$ can be obtained analytically

Quark propagation

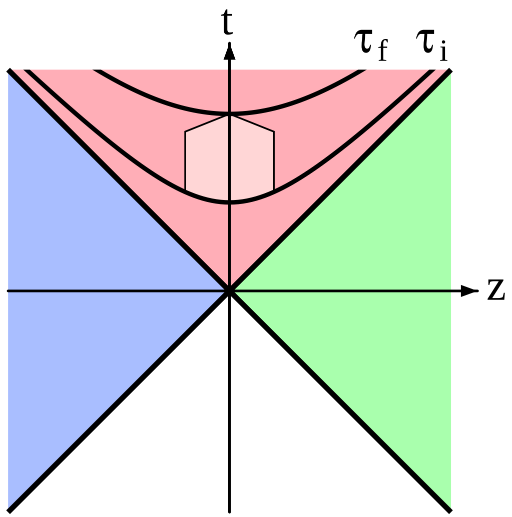
- Propagation through region 2:



  - ▷ Pure gauge background field
    - ▷ $\psi_{\boldsymbol{q},2}(\tau_i)$ can be obtained analytically

Quark propagation

- Propagation through region 3:



▷ One must solve the Dirac equation :

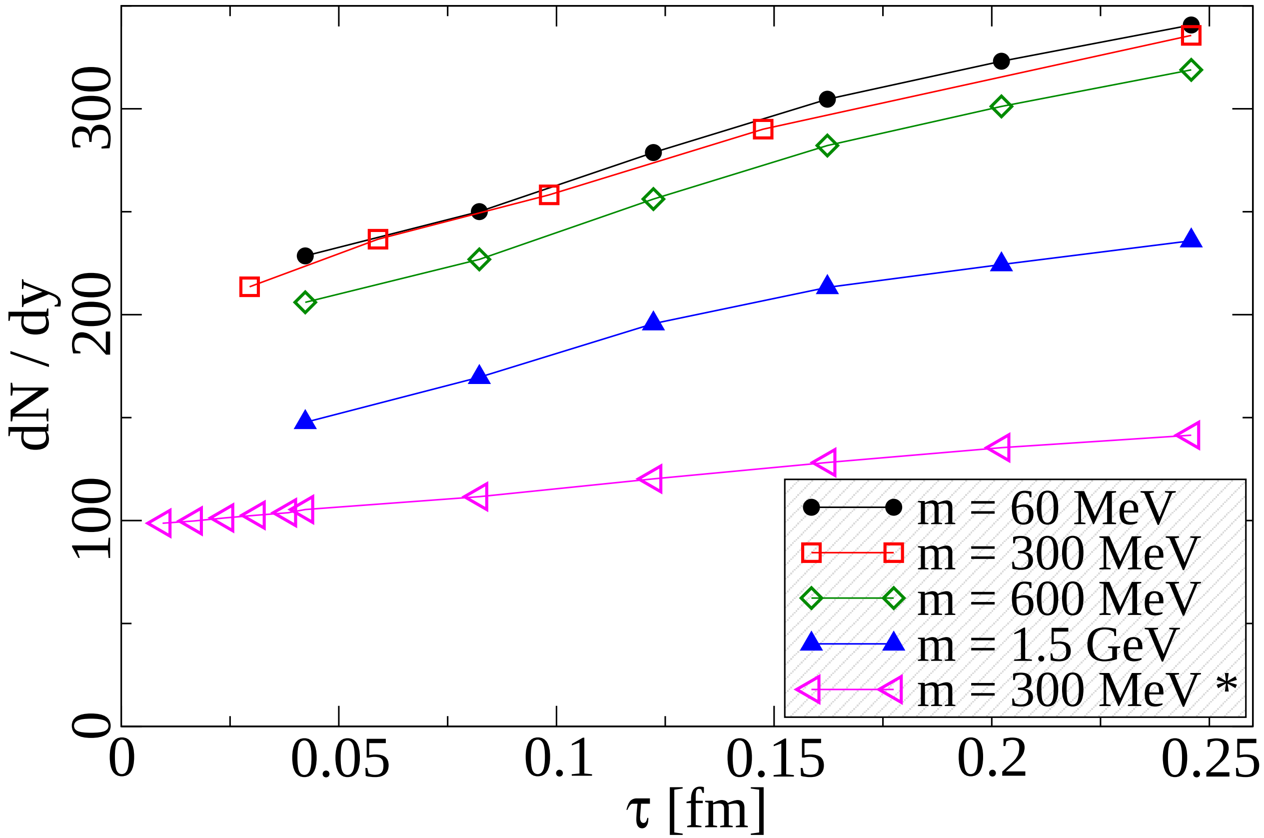
$$\left[i\partial\!\!\!/ - g\mathcal{A}\!\!\!/ - m\right]\psi_{\boldsymbol{q}}(\tau,\eta,\vec{\boldsymbol{x}}_\perp) = 0$$

▷ initial condition: $\psi_{\boldsymbol{q}}(\tau_i) = \psi_{\boldsymbol{q},1}(\tau_i) + \psi_{\boldsymbol{q},2}(\tau_i)$

($\tau_i = 0^+$ in practice)

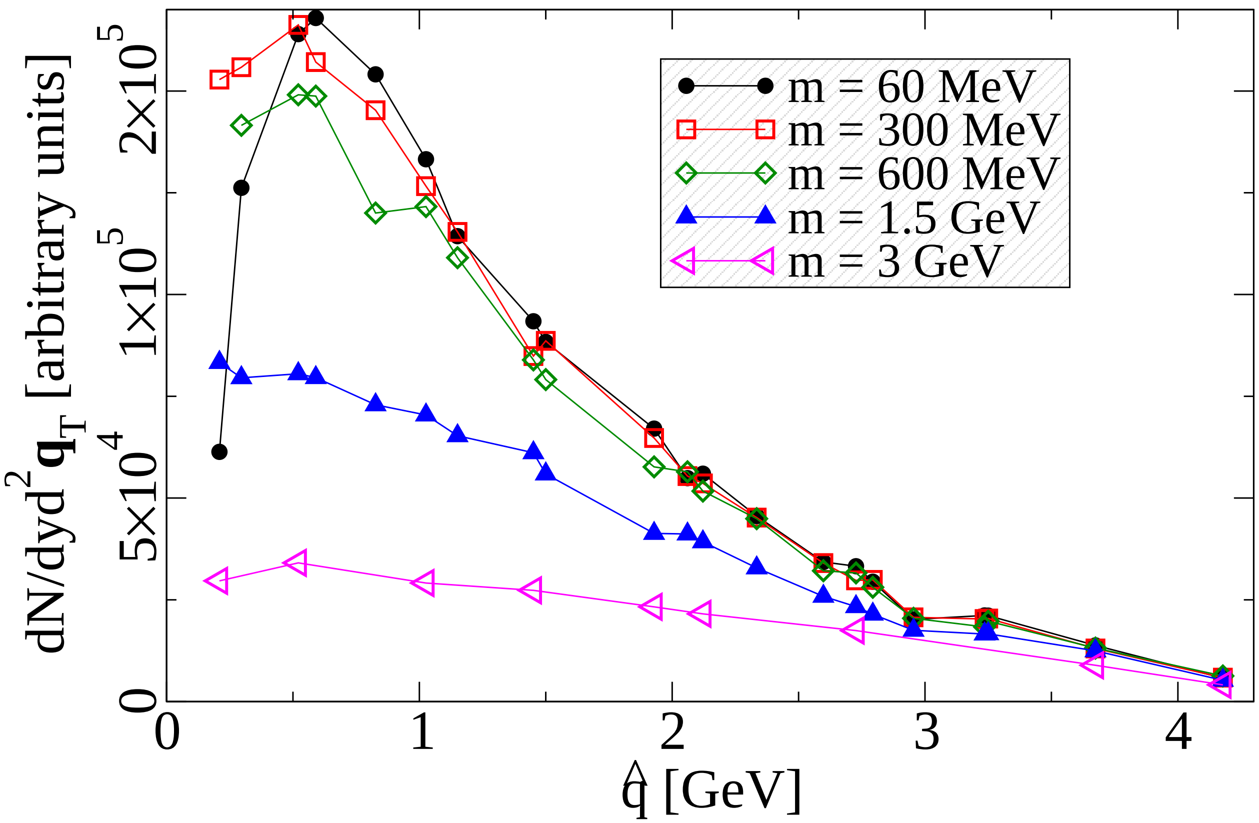
Time dependence

- $g^2\mu = 2$ GeV , (*) $g^2\mu = 1$ GeV :



Spectra for various quark masses

- $g^2\mu = 2$ GeV , $\tau = 0.25$ fm :



Reminder on the LO result

- The LO inclusive gluon spectrum reads :

$$\frac{d\overline{N}_{LO}}{dY\,d^2\vec{\boldsymbol{p}}_\perp} = \frac{1}{16\pi^3}\int_{x,y} e^{ip\cdot(x-y)} \cdots \mathcal{A}_\mu(x)\mathcal{A}_\nu(y)$$

where $\mathcal{A}_\mu(x)$ is the retarded solution of Yang-Mills equations

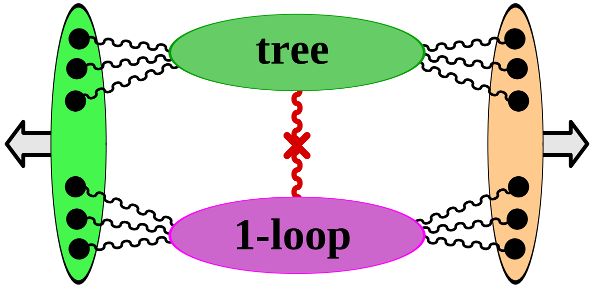
- In the following, it will be useful to see $\overline{N}_{LO}$ as a functional of the classical gauge field $\mathcal{A}_{\rm in}$ on the light-cone
Note : the functional $\overline{N}_{LO}[\mathcal{A}_{\rm in}]$ has no explicit dependence on the sources $\rho_{1,2}$, because there are no sources above the light-cone

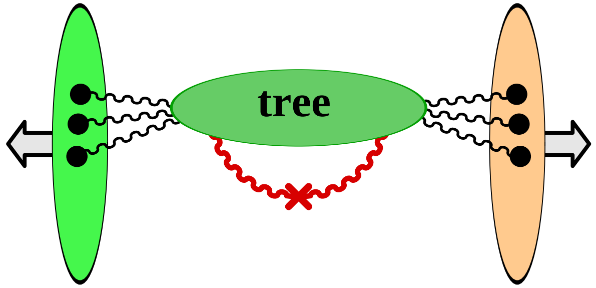
- The dependence on ρ_1 and ρ_2 is hidden in $\mathcal{A}_{\rm in}$. Hence, we can write

$$\overline{N}_{LO} \equiv \overline{N}_{LO}[\mathcal{A}_{\rm in}[\rho_1,\rho_2]]$$

1-loop corrections to $\overline{N}$

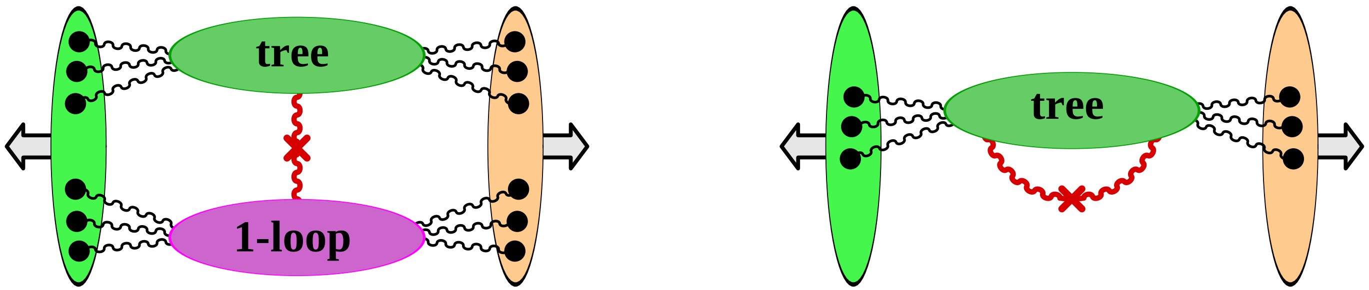
- 1-loop diagrams for $\overline{N}$



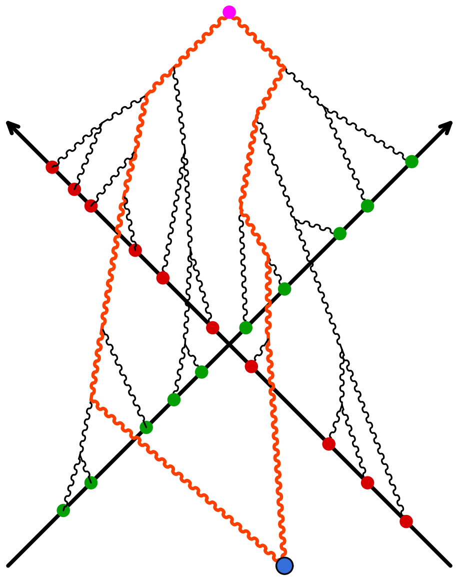


1-loop corrections to $\overline{N}$

- 1-loop diagrams for $\overline{N}$

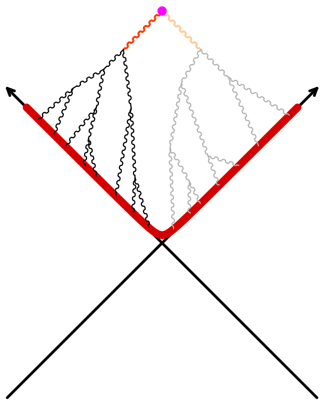


- This involves diagrams such as :



1-loop corrections to $\overline{N}$

- The 1-loop correction to $\overline{N}$ can be written as a perturbation of the initial value problem encountered at LO :



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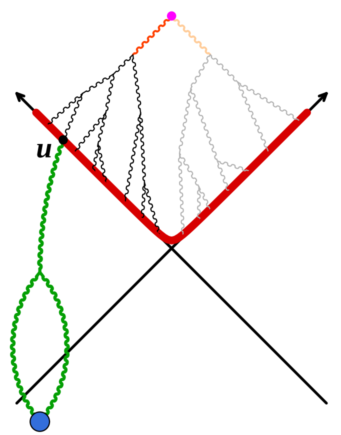

$$\delta\overline{N} = \left[\int_{\vec{\boldsymbol{u}} \in \text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}}) \; \boldsymbol{T}_{\vec{\boldsymbol{u}}} \right] \overline{N}_{LO}$$

  - $\boldsymbol{T}_{\vec{\boldsymbol{u}}}$ is the generator of shifts of the initial condition at the point $\vec{\boldsymbol{u}}$ on the light-cone, i.e. : $\boldsymbol{T}_{\vec{\boldsymbol{u}}} \sim \delta/\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$

1-loop corrections to $\overline{N}$

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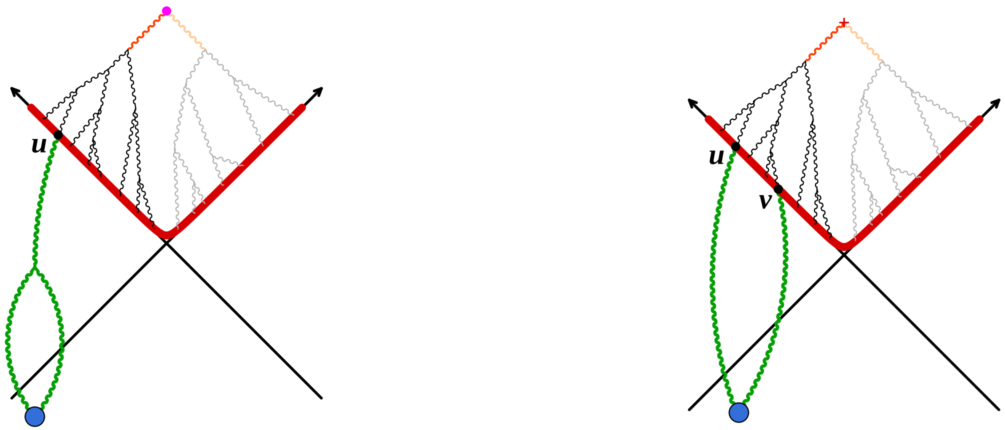
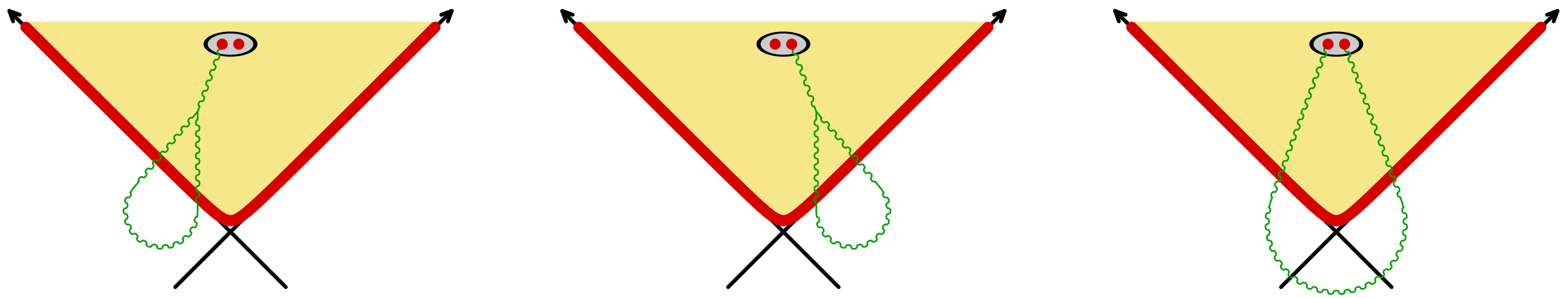


$$\delta\overline{N} = \left[\int\limits_{\vec{\boldsymbol{u}}\,\in\,\text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})\;\boldsymbol{T}_{\vec{\boldsymbol{u}}} + \int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\,\in\,\text{light cone}} \frac{1}{2}\,\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})\;\boldsymbol{T}_{\vec{\boldsymbol{u}}}\,\boldsymbol{T}_{\vec{\boldsymbol{v}}} \right] \overline{N}_{LO}$$

  - $\boldsymbol{T}_{\vec{\boldsymbol{u}}}$ is the generator of shifts of the initial condition at the point $\vec{\boldsymbol{u}}$ on the light-cone, i.e. : $\boldsymbol{T}_{\vec{\boldsymbol{u}}} \sim \delta/\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$
  - $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$ and $\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}})$ are in principle calculable analytically

Sketch of a proof – I



- The first two terms involve :

$$\delta\mathcal{A}(x) \equiv \frac{g}{2}\int d^4z \sum_{\epsilon=\pm} \epsilon\, \boldsymbol{G}_{+\epsilon}(x,z)\boldsymbol{G}_{\epsilon\epsilon}(z,z)$$

- The third term involves $\boldsymbol{G}_{+-}(x,y)$

- The propagators $\boldsymbol{G}_{\pm\pm}$ are propagators in the background $\mathcal{A}$, in the Schwinger-Keldysh formalism. They obey :

$$\begin{cases} \boldsymbol{G}_{+-} = \boldsymbol{G}_R\, {G^0_R}^{-1} G^0_{+-} {G^0_A}^{-1} \boldsymbol{G}_A \\ \boldsymbol{G}_{\pm\pm} = \frac{1}{2}\left[\boldsymbol{G}_R\, {G^0_R}^{-1}(G^0_{+-}+G^0_{-+}){G^0_A}^{-1}\boldsymbol{G}_A \pm (\boldsymbol{G}_R + \boldsymbol{G}_A)\right] \end{cases}$$

$\boldsymbol{G}_{R,A}$ = retarded/advanced propagators in the background $\mathcal{A}$

Sketch of a proof – II

- $\boldsymbol{G}_{++}$ and $\boldsymbol{G}_{--}$ are only needed with equal endpoints
  - ▷ they are both equal to

$$\boldsymbol{G}_{++}(z,z)=\boldsymbol{G}_{--}(z,z)=\frac{1}{2}\left[\boldsymbol{G}_{R}G^{0\;-1}_{R}(G^{0}_{+-}+G^{0}_{-+})G^{0\;-1}_{A}\boldsymbol{G}_{A}\right](z,z)$$

▷ thus, $\delta\mathcal{A}$ can be simplified into :

$$\begin{aligned}\delta\mathcal{A}(x) &= \frac{g}{2}\int d^4z\,\left[\boldsymbol{G}_{++}(x,z)-\boldsymbol{G}_{+-}(x,z)\right]\boldsymbol{G}_{++}(z,z)\\ &= \frac{g}{2}\int d^4z\,\boldsymbol{G}_{R}(x,z)\boldsymbol{G}_{++}(z,z)\end{aligned}$$

- $\boldsymbol{G}_{R}G^{0\;-1}_{R}G^{0}_{+-}G^{0\;-1}_{A}\boldsymbol{G}_{A}$ can be written as :

$$\left[\boldsymbol{G}_{R}G^{0\;-1}_{R}G^{0}_{+-}G^{0\;-1}_{A}\boldsymbol{G}_{A}\right](x,y)=\int\frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}}\;\zeta_{\vec{\boldsymbol{p}}}(x)\zeta^{*}_{\vec{\boldsymbol{p}}}(y)\;,$$

with $\left[\Box_x+m^2+g\mathcal{A}(x)\right]\zeta_{\vec{\boldsymbol{p}}}(x)=0$ and $\lim_{x_0\to-\infty}\zeta_{\vec{\boldsymbol{p}}}(x)=e^{ip\cdot x}$

Sketch of a proof – III

- Green's formulas :

$$\mathcal{A}(x) = \int_\Omega d^4z\, G_R^0(x,z)\Big[j(z) - \frac{g}{2}\mathcal{A}^2(z)\Big] + \int_{\rm LC} d^3\vec{\boldsymbol{u}}\; G_R^0(x,u)\Big[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\Big]\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$$

$$\delta\mathcal{A}(x) = \int_\Omega d^4z\, \boldsymbol{G}_R(x,z)\,\frac{g}{2}\boldsymbol{G}_{++}(z,z) + \int_{\rm LC} d^3\vec{\boldsymbol{u}}\; \boldsymbol{G}_R(x,u)\Big[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\Big]\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})$$

$$\zeta_{\vec{\boldsymbol{p}}}(x) = \int_{\rm LC} d^3\vec{\boldsymbol{u}}\; \boldsymbol{G}_R(x,u)\Big[n\cdot\overrightarrow{\partial}_u - n\cdot\overleftarrow{\partial}_u\Big]\zeta_{\vec{\boldsymbol{p}}\,{\rm in}}(\vec{\boldsymbol{u}})$$

$$\boldsymbol{G}_R(x,y) = G_R^0(x,y) + g\int_\Omega d^4z\, G_R^0(x,z)\mathcal{A}(z)\boldsymbol{G}_R(z,y)$$

Sketch of a proof – IV

- Thanks to the operator

$$a_{\rm in}(\vec{\boldsymbol{u}})\cdot\boldsymbol{T}_{\vec{\boldsymbol{u}}} \equiv a_{\rm in}(\vec{\boldsymbol{u}})\frac{\delta}{\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})} + \Big[(n\cdot\partial_u)a_{\rm in}(\vec{\boldsymbol{u}})\Big]\frac{\delta}{\delta(n\cdot\partial_u)\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})} \quad ,$$

  we can write

$$\zeta_{\vec{\boldsymbol{p}}}(x) = \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \Big[\zeta_{\vec{\boldsymbol{p}}\,{\rm in}}(\vec{\boldsymbol{u}})\cdot\boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]\,\mathcal{A}(x)$$

$$\delta\mathcal{A}(x) = \int_\Omega d^4z\; \boldsymbol{G}_R(x,z)\,\frac{g}{2}\boldsymbol{G}_{++}(z,z) + \int\limits_{\vec{\boldsymbol{u}}\in{\rm LC}} \Big[\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}})\cdot\boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]\,\mathcal{A}(x)$$

  $\triangleright$ from the classical field $\mathcal{A}(x)$, the operator $a_{\rm in}(\vec{\boldsymbol{u}})\cdot\boldsymbol{T}_{\vec{\boldsymbol{u}}}$ builds the fluctuation $a(x)$ whose initial condition on the light-cone is $a_{\rm in}(\vec{\boldsymbol{u}})$

- The 3rd diagram can directly be written as :

$$\int\frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in{\rm LC}} \Big[\Big[\zeta_{\vec{\boldsymbol{p}}\,{\rm in}}(\vec{\boldsymbol{u}})\cdot\boldsymbol{T}_{\vec{\boldsymbol{u}}}\Big]\,\mathcal{A}(x)\Big]\Big[\Big[\zeta^*_{\vec{\boldsymbol{p}}\,{\rm in}}(\vec{\boldsymbol{v}})\cdot\boldsymbol{T}_{\vec{\boldsymbol{v}}}\Big]\,\mathcal{A}(y)\Big]$$

Sketch of a proof – V

- One can finally prove that

$$\int_\Omega d^4z \, \boldsymbol{G}_R(x,z) \, \frac{g}{2} \boldsymbol{G}_{++}(z,z) =$$

$$= \frac{1}{2} \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in \mathrm{LC}} \left[\zeta_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\right]\left[\zeta^*_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{v}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{v}}}\right] \mathcal{A}(x)$$

$$\triangleright \quad \delta\mathcal{A}(x) = \Bigg[\int_{\vec{\boldsymbol{u}}\in \mathrm{LC}} \left[\delta\mathcal{A}_{\mathrm{in}}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\right] + \frac{1}{2} \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \int_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}\in \mathrm{LC}} \left[\zeta_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{u}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{u}}}\right]\left[\zeta^*_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{v}}) \cdot \boldsymbol{T}_{\vec{\boldsymbol{v}}}\right] \Bigg] \mathcal{A}(x)$$

- This leads to the announced formula for $\delta\overline{N}$, with

$$\boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \equiv \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^3 2E_{\boldsymbol{p}}} \zeta_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{u}}) \zeta^*_{\vec{\boldsymbol{p}}\,\mathrm{in}}(\vec{\boldsymbol{v}})$$

Sketch of a proof – VI

- Conjecture : this result can be generalized to any observable that can be written in terms of the gauge field with retarded boundary conditions, $\mathcal{O} \equiv \mathcal{O}[\mathcal{A}]$:

$$\delta\mathcal{O} = \left[\int\limits_{\vec{\boldsymbol{u}} \in \text{light cone}} \delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{u}}) \;\boldsymbol{T}_{\vec{\boldsymbol{u}}} + \int\limits_{\vec{\boldsymbol{u}},\vec{\boldsymbol{v}} \in \text{light cone}} \frac{1}{2}\, \boldsymbol{\Sigma}(\vec{\boldsymbol{u}},\vec{\boldsymbol{v}}) \;\boldsymbol{T}_{\vec{\boldsymbol{u}}}\,\boldsymbol{T}_{\vec{\boldsymbol{v}}} \right] \mathcal{O}_{LO}$$

  ▷ whatever we conclude for the multiplicity from this formula holds true for any such observable

Divergences

- If taken at face value, this 1-loop correction is plagued by several divergences :
  - The two coefficients $\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})$ and $\boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})$ are infinite, because of an unbounded integration over a rapidity variable
  - At late times, $\boldsymbol{T}_{\vec{\boldsymbol{x}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}})$ diverges exponentially,

$$\boldsymbol{T}_{\vec{\boldsymbol{x}}}\mathcal{A}(\tau,\vec{\boldsymbol{y}}) \sim \frac{\delta\mathcal{A}(\tau,\vec{\boldsymbol{y}})}{\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})} \underset{\tau\to+\infty}{\sim} e^{\sqrt{\mu\tau}}$$

because of an instability of the classical solution of Yang-Mills equations under rapidity dependent perturbations (Romatschke, Venugopalan (2005))

$\triangleright$ see lecture III : Weibel instabilities

Initial state factorization

- Why is it plausible ?
  - Reminder :

$$\left[\delta\overline{N}\right]_{\substack{\text{divergent}\\\text{coefficients}}} = \left\{ \int_{\vec{\boldsymbol{x}}} \left[\delta\mathcal{A}_{\rm in}(\vec{\boldsymbol{x}})\right]_{\rm div} \boldsymbol{T}_{\vec{\boldsymbol{x}}} + \frac{1}{2}\int_{\vec{\boldsymbol{x}},\vec{\boldsymbol{y}}} \left[\boldsymbol{\Sigma}(\vec{\boldsymbol{x}},\vec{\boldsymbol{y}})\right]_{\rm div} \boldsymbol{T}_{\vec{\boldsymbol{x}}}\boldsymbol{T}_{\vec{\boldsymbol{y}}} \right\} \overline{N}_{LO}$$

  - Compare with the evolution Hamiltonian :

$$\mathcal{H}[\rho] = \int_{\vec{\boldsymbol{x}}_\perp} \sigma(\vec{\boldsymbol{x}}_\perp)\frac{\delta}{\delta\rho(\vec{\boldsymbol{x}}_\perp)} + \frac{1}{2}\int_{\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta^2}{\delta\rho(\vec{\boldsymbol{x}}_\perp)\delta\rho(\vec{\boldsymbol{y}}_\perp)}$$

- The coefficients σ and χ in the Hamiltonian are well known
  $\triangleright$ one must compute analytically the divergent part of $\delta\mathcal{A}_{\rm in}$ and $\boldsymbol{\Sigma}$