Pre-equilibrium dynamics in heavy ion collisions

I – Parton evolution at small x, Saturation

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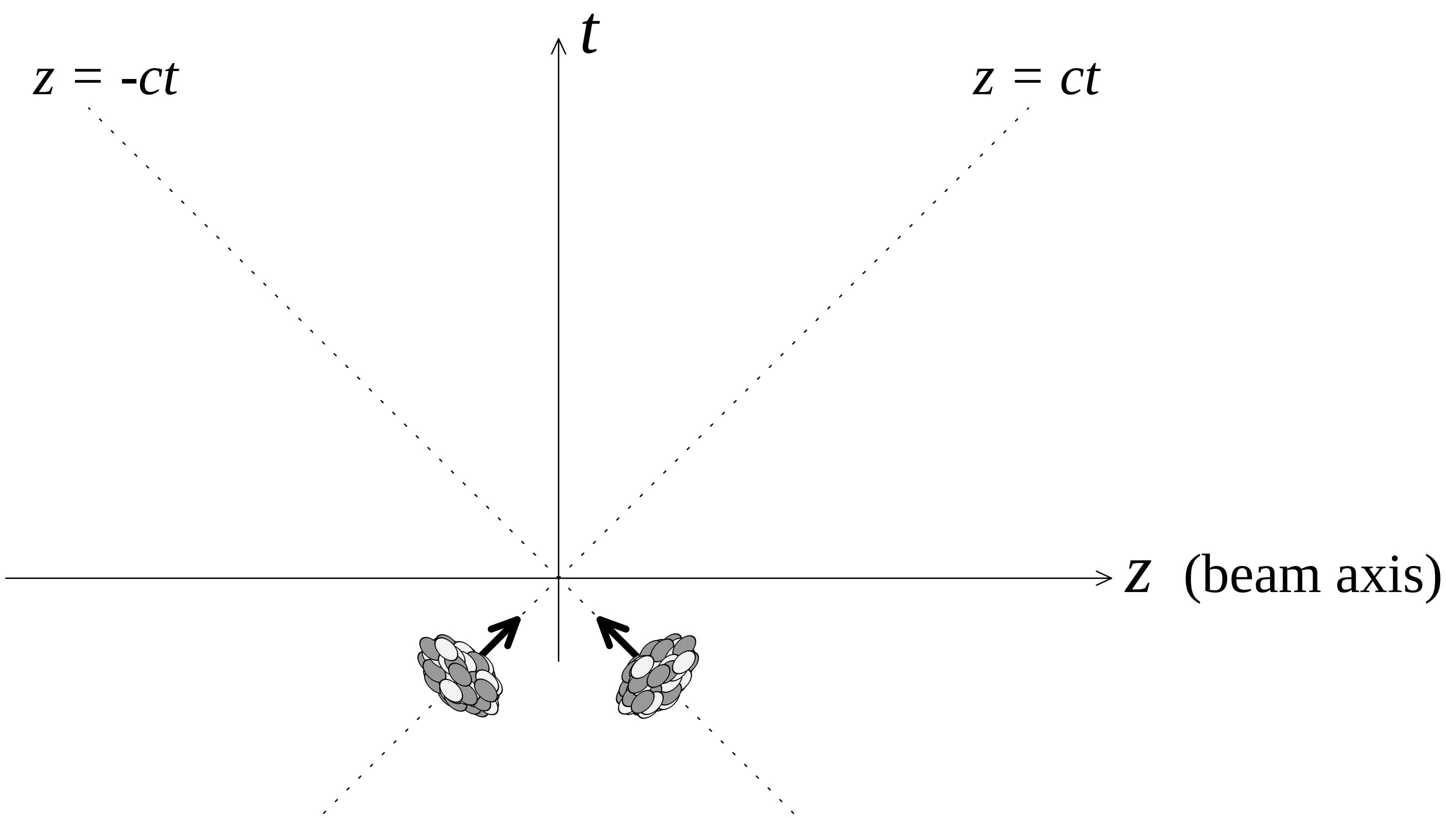
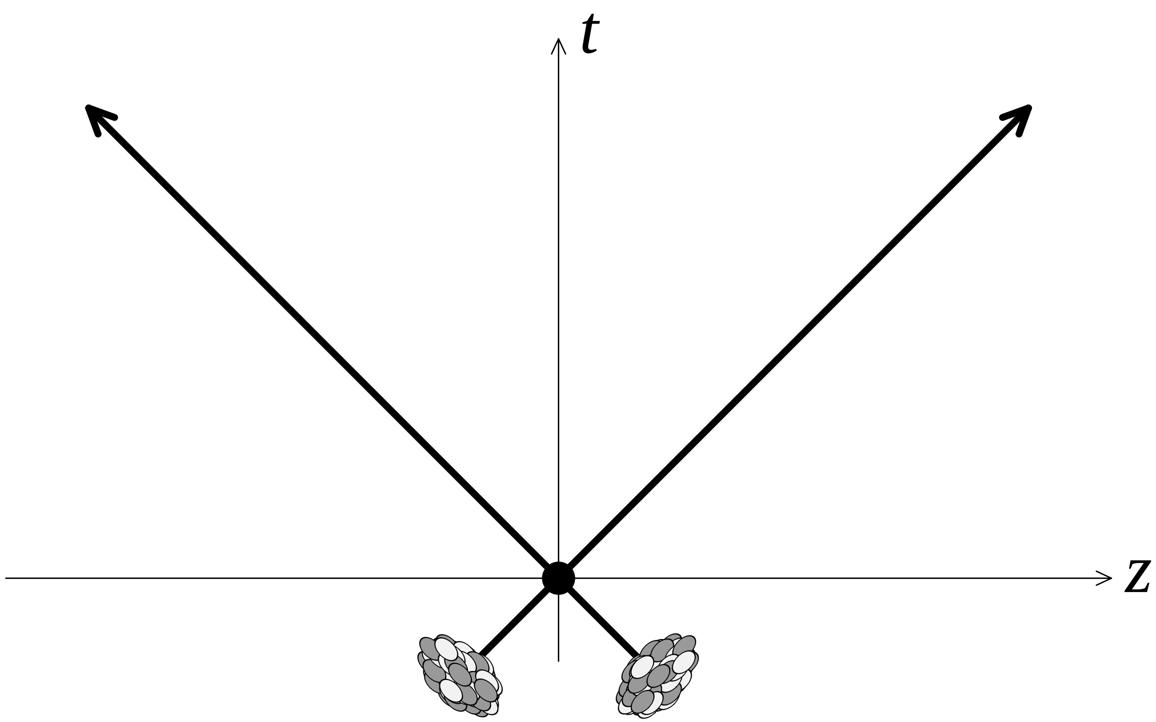
General introduction

Stages of a nucleus-nucleus collision

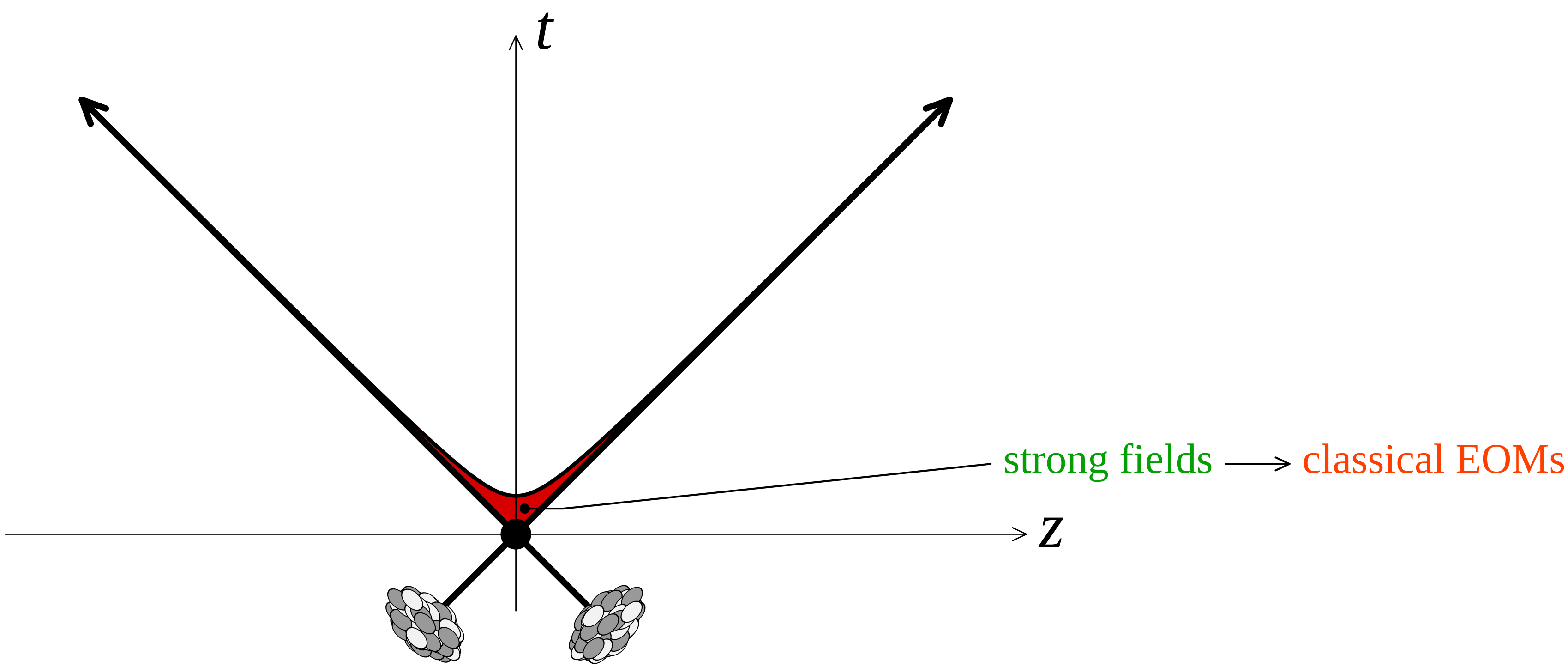


Stages of a nucleus-nucleus collision



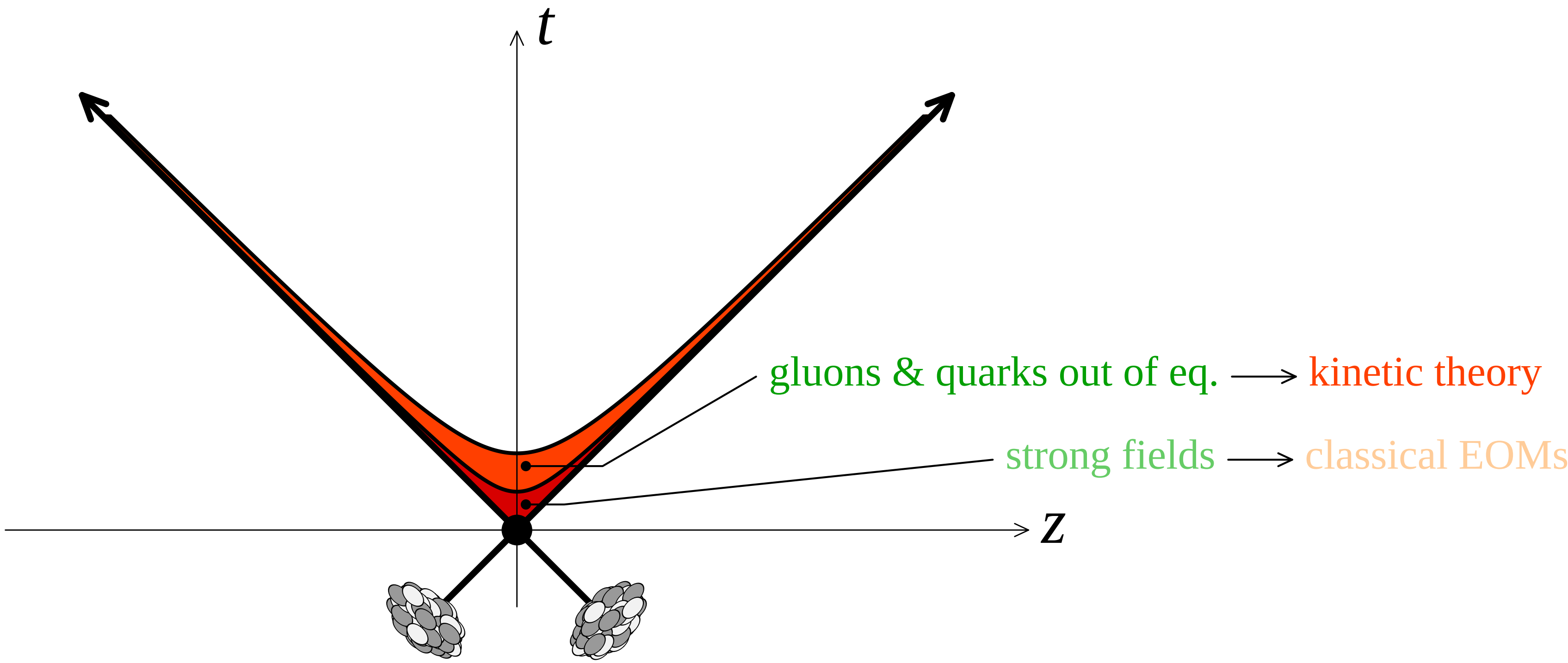
- $\tau \sim 0$ fm/c
- Production of hard particles :
  - jets, direct photons
  - heavy quarks
- calculable with perturbative QCD (leading twist)

Stages of a nucleus-nucleus collision



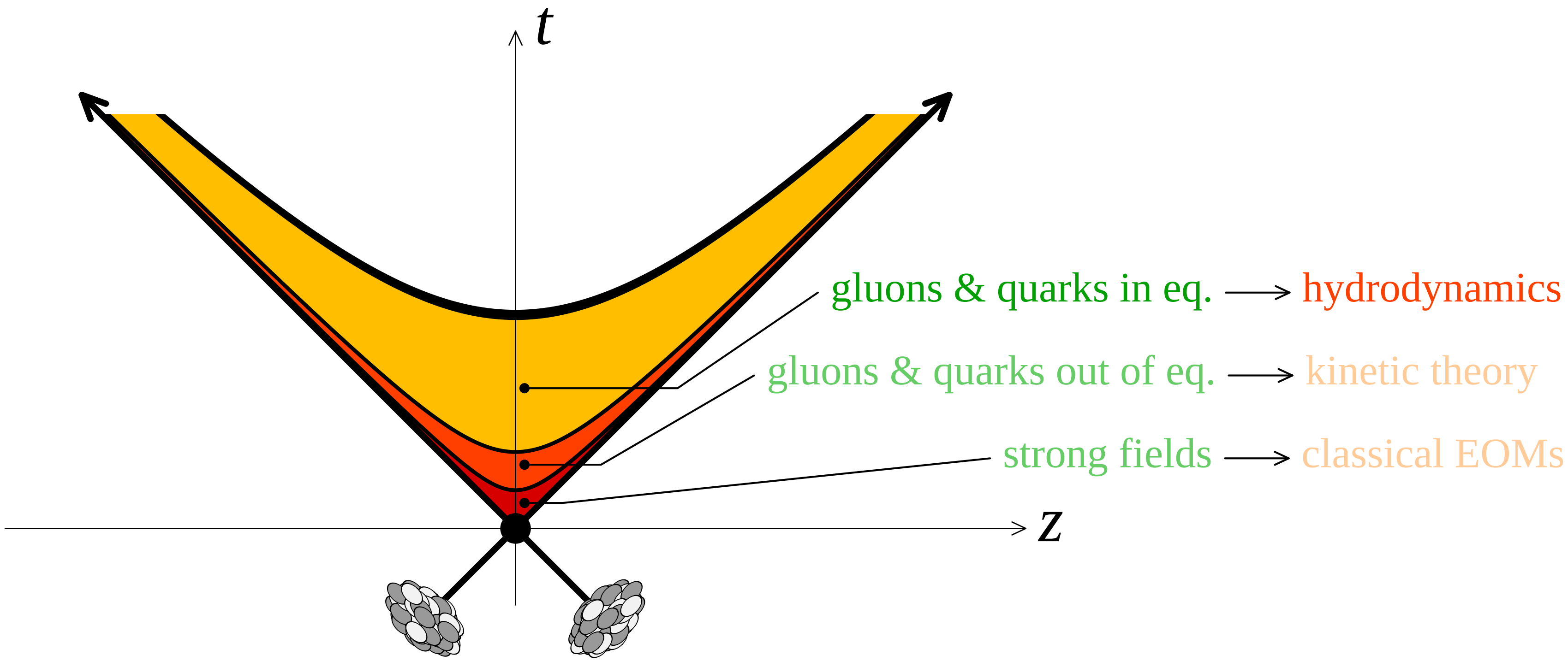
- $\tau \sim 0.2$ fm/c
- Production of semi-hard particles : gluons, light quarks
- relatively small momentum : $p_\perp \lesssim 2$–3 GeV
- make up for most of the multiplicity
- sensitive to the physics of saturation (higher twist)

Stages of a nucleus-nucleus collision



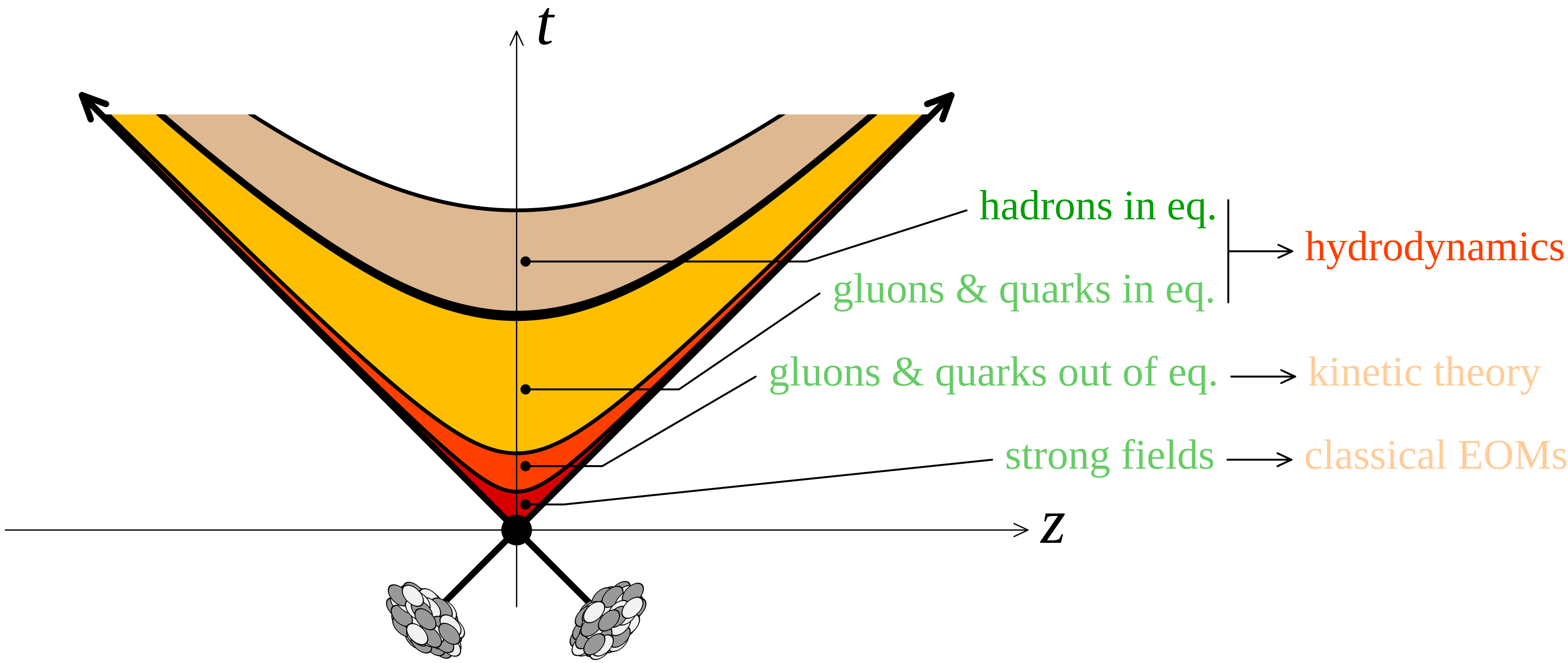
- $\tau \sim 1$–2 fm/c
- Thermalization
  - experiments suggest a fast thermalization
  - but this is still not understood from QCD

Stages of a nucleus-nucleus collision



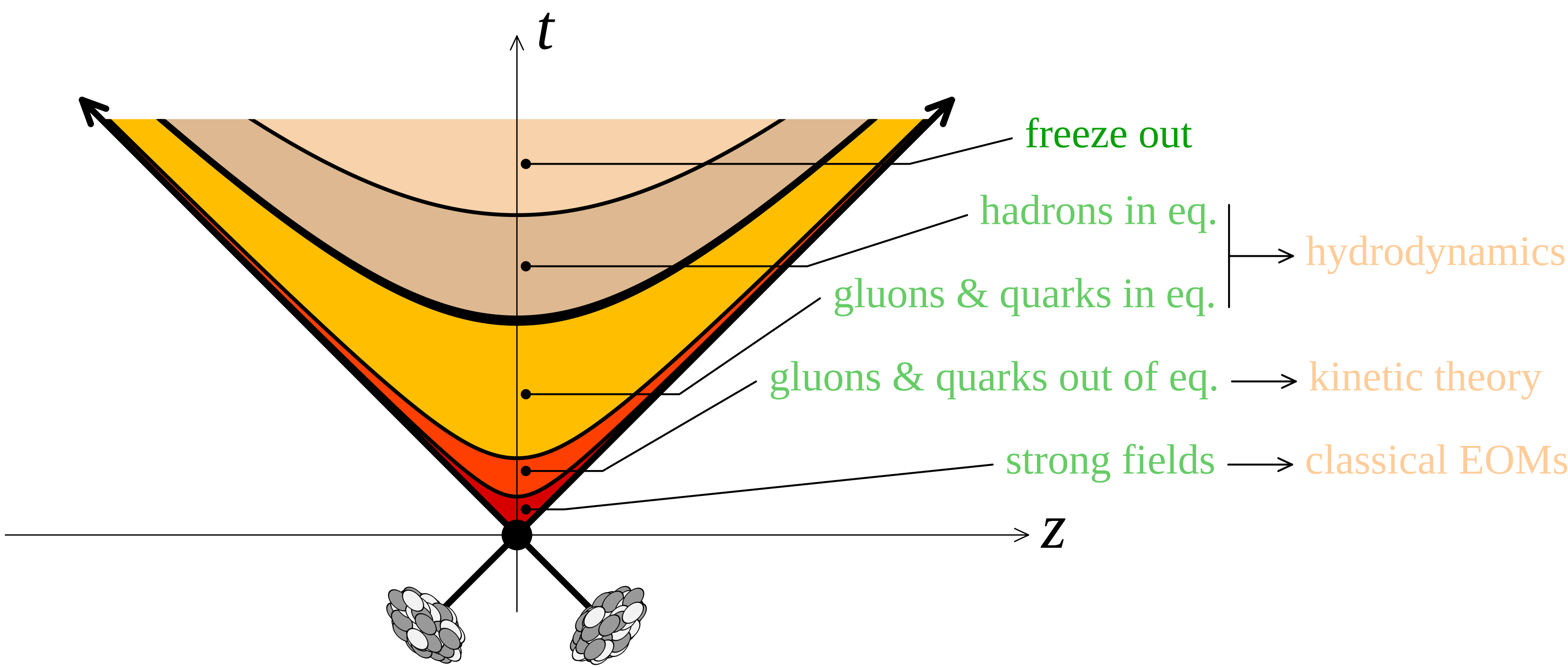
- $2 \leq \tau \lesssim 10$ fm/c
- Quark gluon plasma

Stages of a nucleus-nucleus collision



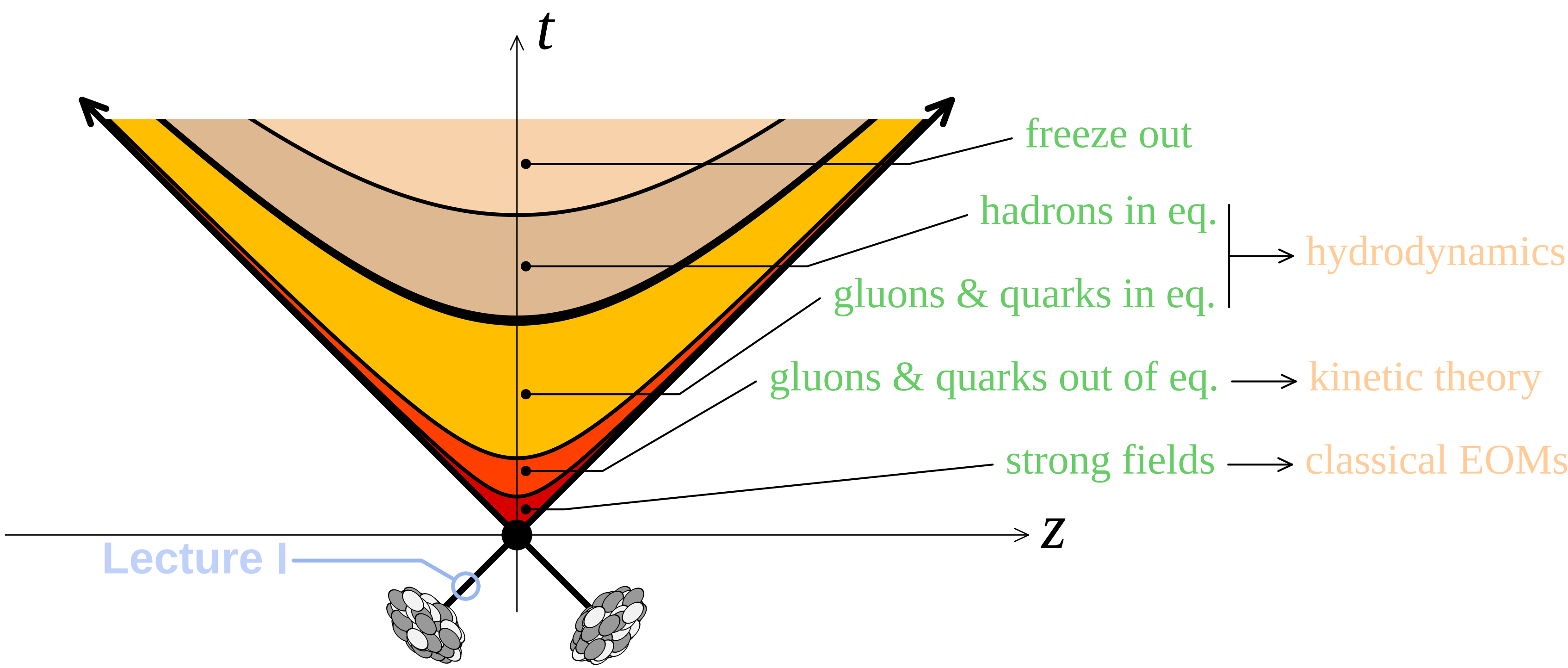
- $10 \lesssim \tau \lesssim 20$ fm/c
- Hot hadron gas

Stages of a nucleus-nucleus collision



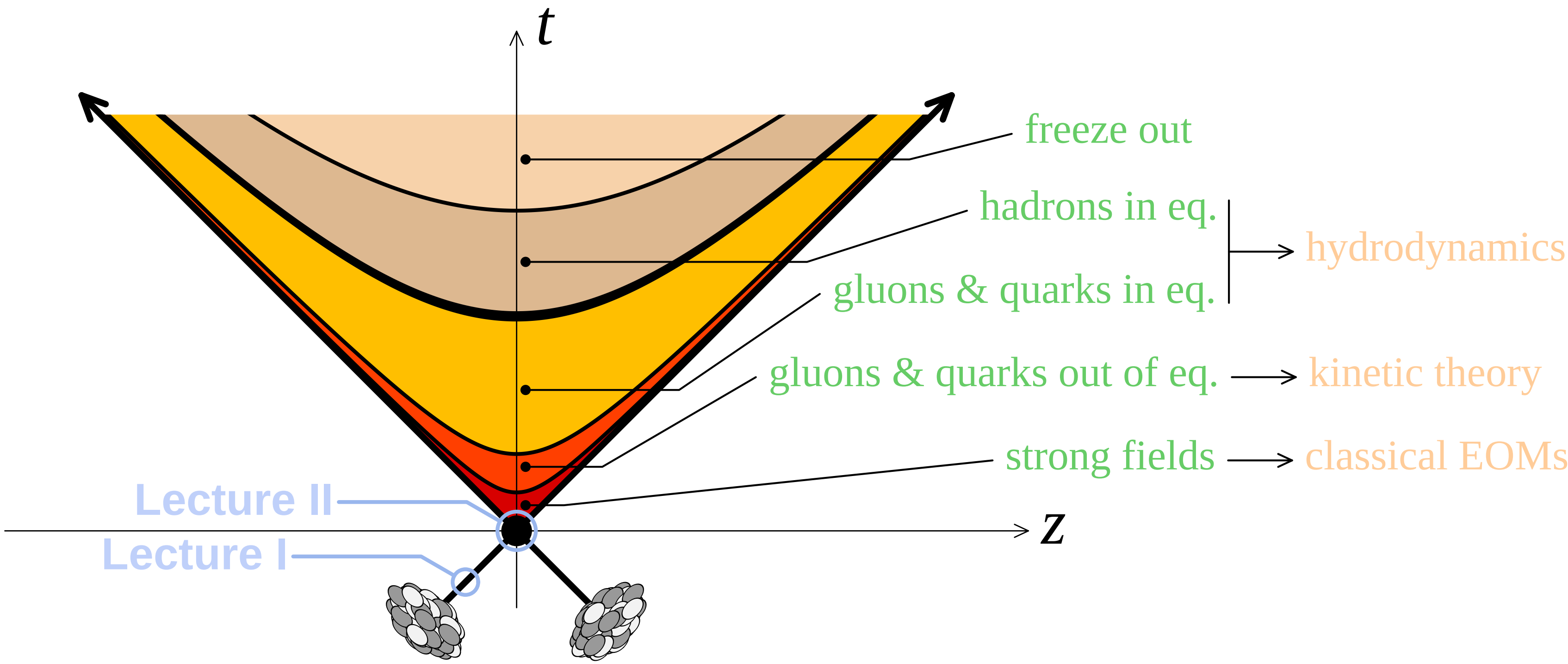
- $\tau \longrightarrow +\infty$
- Chemical freeze-out :
  density too small to have inelastic interactions
- Kinetic freeze-out :
  no more elastic interactions

Stages of a nucleus-nucleus collision



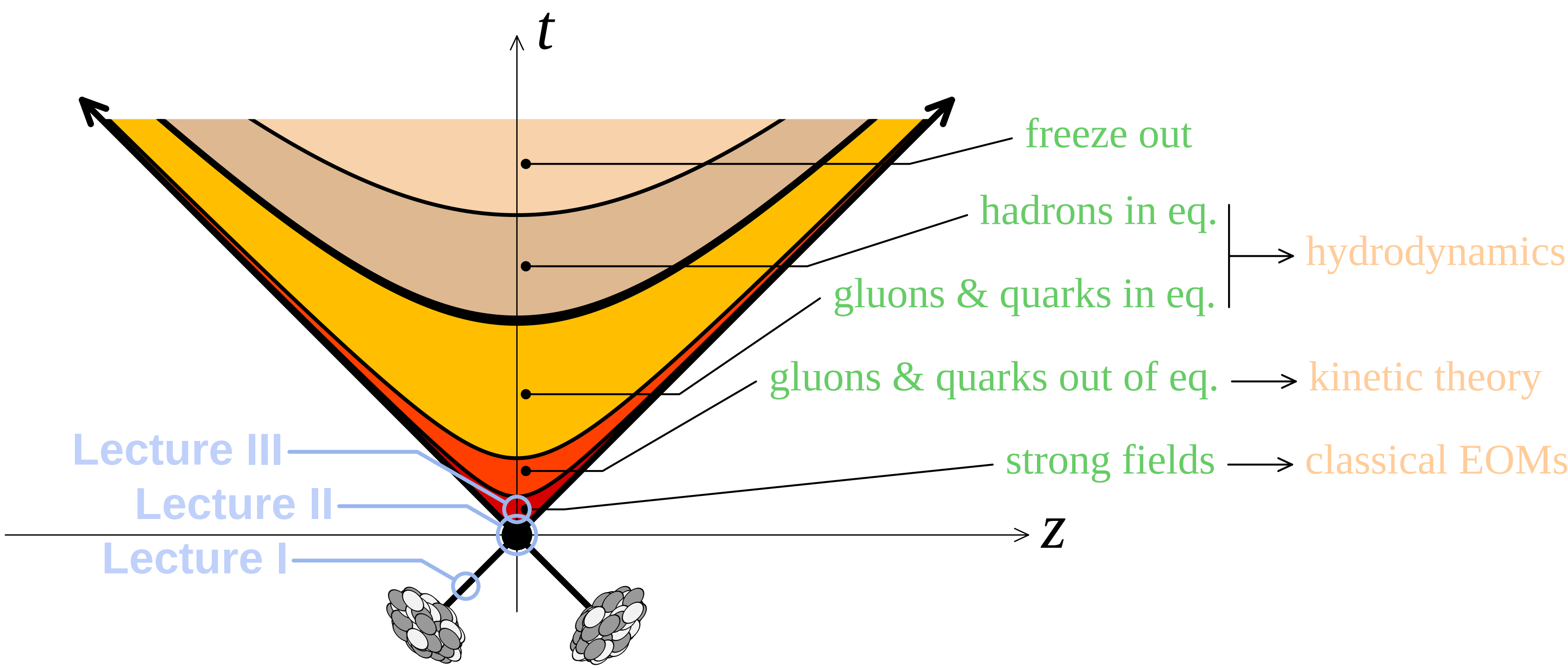
- Lecture I : Parton evolution at small x, Saturation

Stages of a nucleus-nucleus collision



- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production

Stages of a nucleus-nucleus collision



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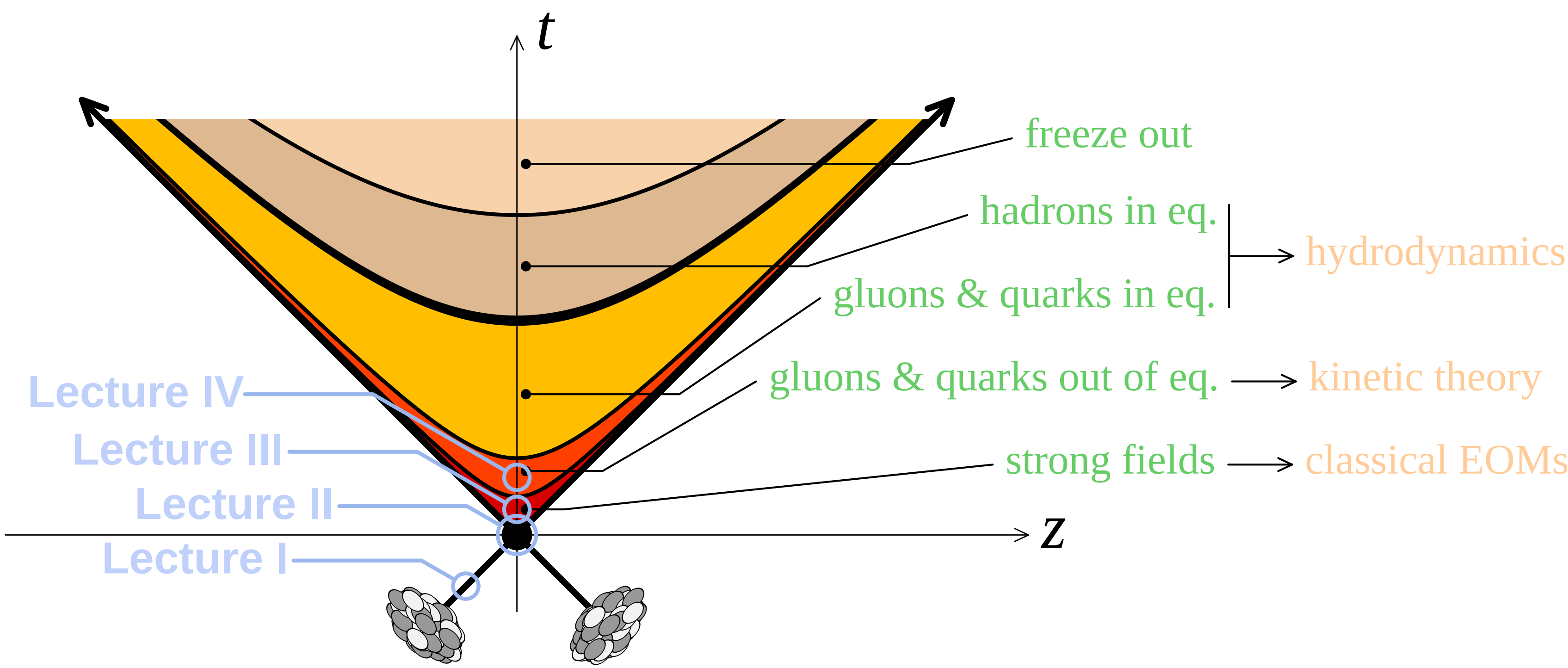
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization

Stages of a nucleus-nucleus collision



- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization
- Lecture IV : Kinetic theory, Near-Equilibrium dynamics

General outline

- Lecture I : Parton evolution at small x, Saturation
- Lecture II : Initial particle production
- Lecture III : Instabilities and thermalization
- Lecture IV : Kinetic theory, Near-Equilibrium dynamics

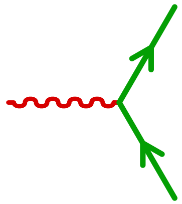
Lecture I : Parton saturation

- Introduction to QCD
- Parton model
- Gluon saturation
- Color Glass Condensate
- Phenomenology of saturation

Introduction to QCD

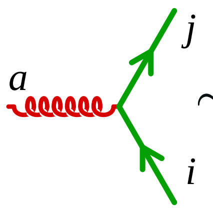
Quarks and gluons

- Electromagnetic interaction : Quantum electrodynamics
  - Matter : electron , interaction carrier : photon
  - Interaction :

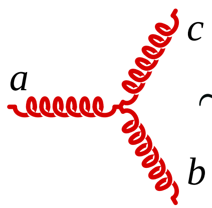


$\sim \; e$ (electric charge of the electron)

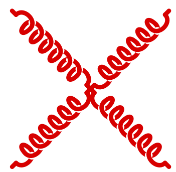
- Strong interaction : Quantum chromo-dynamics
  - Matter : quarks , interaction carriers : gluons
  - Interactions :



$\sim \; g\,(t^a)_{ij}$



$\sim \; g\,(T^a)_{bc}$



  - i, j : colors of the quarks (3 possible values)
  - a, b, c : colors of the gluons (8 possible values)
  - $(t^a)_{ij}$: 3×3 matrix , $(T^a)_{bc}$: 8×8 matrix

QCD Lagrangian

- QCD Lagrangian :

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(i\not{D} - m)\psi$$

  - the gauge field A^μ belongs to $SU(3)$
  - $D^\mu \equiv \partial^\mu - igA^\mu$ is the covariant derivative
  - $F^{\mu\nu} \equiv i[D^\mu, D^\nu]/g = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$

- The Lagrangian is invariant under gauge transformations :

$$A^\mu(x) \rightarrow \Omega(x)A^\mu(x)\Omega^{-1}(x) + \frac{i}{g}\Omega(x)\partial^\mu\Omega^{-1}(x)$$
$$\psi(x) \rightarrow \Omega(x)\psi(x)$$

where $\Omega(x) \in SU(3)$

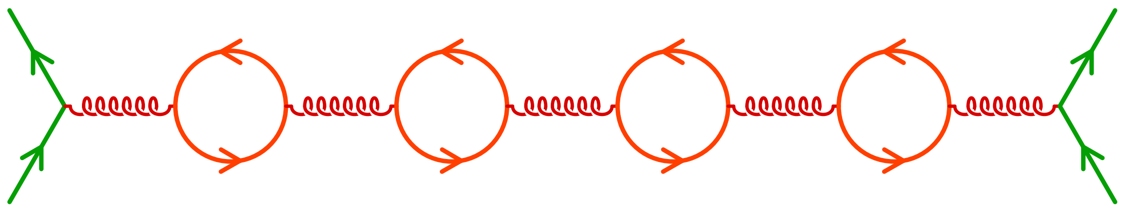
  - Note: the field strength is not invariant but transforms as :

$$F^{\mu\nu}(x) \rightarrow \Omega(x)F^{\mu\nu}(x)\Omega^{-1}(x)$$

Asymptotic freedom

- Running coupling : $\alpha_s = g^2/4\pi$

$$\alpha_s(r) = \frac{2\pi N_c}{(11N_c - 2N_f)\log(1/r\Lambda_{QCD})}$$



- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)

Asymptotic freedom

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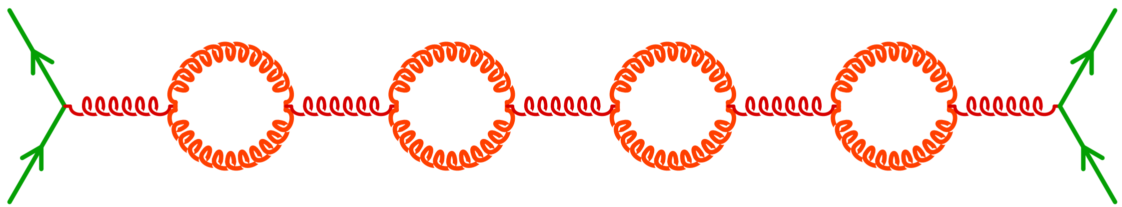
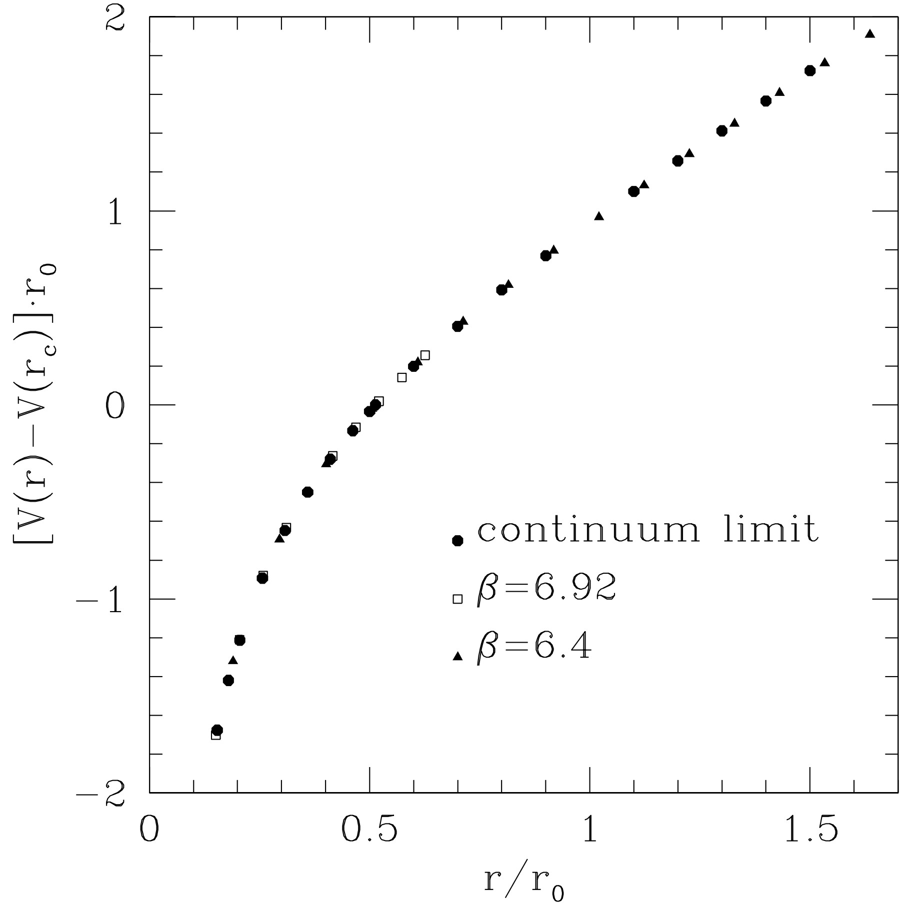


- The effective charge seen at large distance is screened by fermionic fluctuations (as in QED)
- But gluonic vacuum fluctuations produce an anti-screening (because of the non-abelian nature of their interactions)
- As long as $N_f < 11N_c/2 = 16.5$, the gluons win...

Quark confinement



- The quark potential increases linearly with distance
- Color singlet hadrons : no free quarks and gluons in nature

How to test QCD?

- QCD is the fundamental theory of strong interactions among quarks and gluons
- Experiments involve hadrons in their initial and final states, not quarks and gluons
- Hadrons cannot be described perturbatively in QCD
- Scattering amplitudes with time-like on-shell momenta cannot be computed on the lattice

▷ How can we compare theory and experiments?

▷ **Factorization :** separation of short distances (perturbative) and long distance (non perturbative)

Factorization

- At a superficial level, factorization means that :

$$\mathcal{O}_{\rm hadrons} = F \otimes \mathcal{O}_{\rm partons}$$

  - $F =$ parton distribution
  - $\mathcal{O}_{\rm partons} =$ observable at the partonic level (calculable in perturbation theory)

- For this to be useful, F must be universal (i.e. independent of the observable $\mathcal{O}$)

- In order to test QCD experimentally, measure as many observables as possible, and try to find common F's that fit all the data
  Note : at this stage, by looking at only one observable, it is impossible to perform any meaningful test, since it is always possible to adjust F so that it works

Factorization

- Some loop corrections in $\mathcal{O}_{\rm partons}$ are enhanced by large logarithms, e.g.

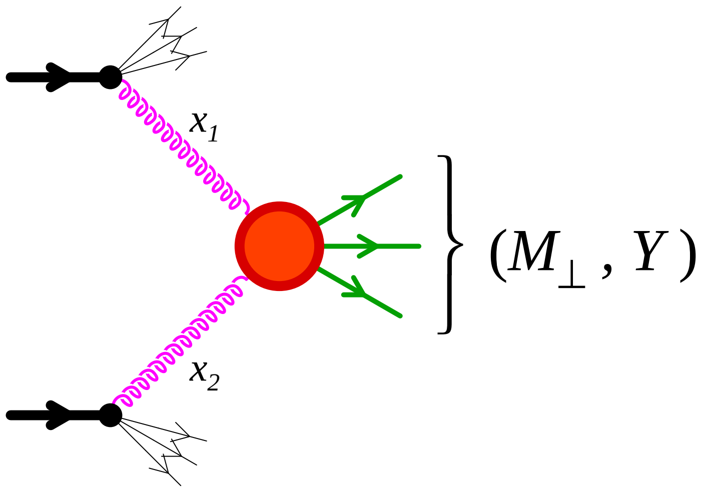
$$\alpha_s \ln\left(\frac{M^2}{m_H^2}\right) \quad , \qquad \alpha_s \ln\left(\frac{s}{M^2}\right) \sim \alpha_s \ln\left(\frac{1}{x}\right)$$

  Note : the log that occurs depends on the details of the kinematics
  - Bjorken limit: $s, M^2 \to +\infty$ with s/M^2 fixed
  - Regge limit: $s \to +\infty$, M^2 fixed
- These logs upset a naive application of perturbation theory when $\alpha_s \ln(\cdot) \sim 1$ $\triangleright$ they must be resummed
- This resummation can be performed analytically
  - the result of the resummation is universal
  - all the leading logs can be absorbed in F
    - $\triangleright$ the factorization formula remains true
    - $\triangleright$ this summation dictates how F evolves with M^2 or x

Factorization

- These logarithms tell us that the relevant parton distributions depend on the resolution scales (in time and in transverse momentum) associated to a given process

- Calculation of some process at LO :

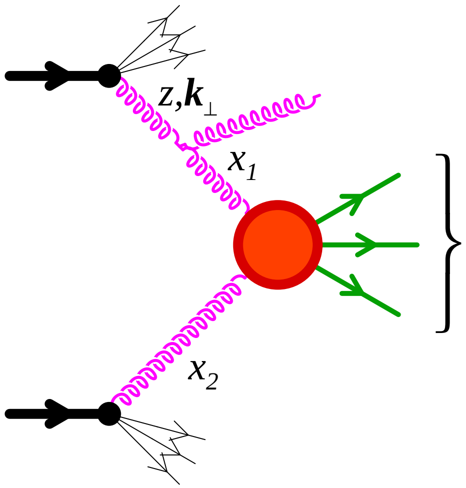


$$\begin{cases} x_1 = M_\perp \, e^{+Y} / \sqrt{s} \\ x_2 = M_\perp \, e^{-Y} / \sqrt{s} \end{cases}$$

Factorization

- These logarithms tell us that the relevant parton distributions depend on the resolution scales (in time and in transverse momentum) associated to a given process

- Radiation of an extra gluon :



$$\Big\} \; (M_\perp \,, Y\,) \quad \Longrightarrow \quad \alpha_s \int_{x_1} \frac{dz}{z} \int^{M_\perp} \frac{d^2\vec{\boldsymbol{k}}_\perp}{\boldsymbol{k}_\perp^2}$$

- Practical consequence : pQCD predicts not only $\mathcal{O}_{\rm partons}$ but also the evolution $\partial_{_M} F$ (or $\partial_x F$)
  - ▷ the only required non-perturbative input is $F(x, M_0)$ or $F(x_0, M)$

Collinear factorization

- Logs of $M_\perp \Longrightarrow$ DGLAP. Important when :
  - $M_\perp \gg \Lambda_{QCD}$, while x_1, x_2 are rather large
- Cross-sections read :

$$\frac{d\sigma}{dY\,d^2\vec{\boldsymbol{P}}_\perp} \propto F(x_1, M_\perp^2)\; F(x_2, M_\perp^2)\; |\mathcal{M}|^2$$

with $x_{1,2} = M_\perp \exp(\pm Y)/\sqrt{s}$

- Note : there are convolutions in x_1 and x_2 if some particles are integrated out in the final state
- The factorization of logarithms has been proven to all orders for sufficiently inclusive quantities (see Collins, Soper, Sterman, 1984–1985)

Kt-factorization

Collins, Ellis (1991), Catani, Ciafaloni, Hautmann (1991)

- Logs of $1/x \Longrightarrow$ BFKL. Important when :
  - $M_\perp$ remains moderate, while x_1 or x_2 (or both) are small
- The BFKL equation is non-local in transverse momentum
  - ▷ it applies to non-integrated gluon distributions $\varphi(x, \vec{\boldsymbol{k}}_\perp)$

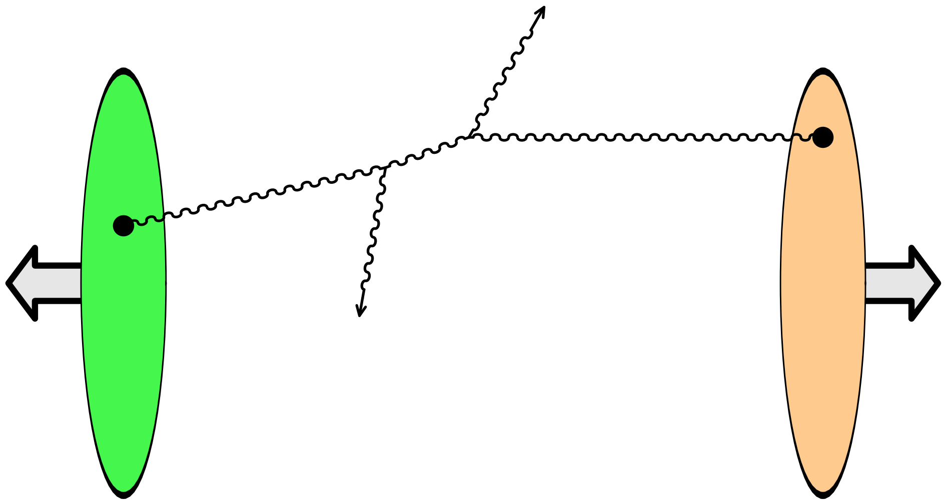
$$xG(x, Q^2) = \int^{Q^2} \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2}\,\varphi(x, \vec{\boldsymbol{k}}_\perp)$$

  - ▷ the matrix element is calculated for (off-shell) gluons with $\vec{\boldsymbol{k}}_\perp \neq \vec{0}$
- In this framework, cross-sections read :

$$\frac{d\sigma}{dY\,d^2\vec{\boldsymbol{P}}_\perp} \propto \int_{\vec{\boldsymbol{k}}_{1\perp},\vec{\boldsymbol{k}}_{2\perp}} \delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}-\vec{\boldsymbol{P}}_\perp)\;\varphi_1(x_1,k_{1\perp})\;\varphi_2(x_2,k_{2\perp})\;\frac{\left|\mathcal{M}\right|^2}{k_{1\perp}^2\,k_{2\perp}^2}$$

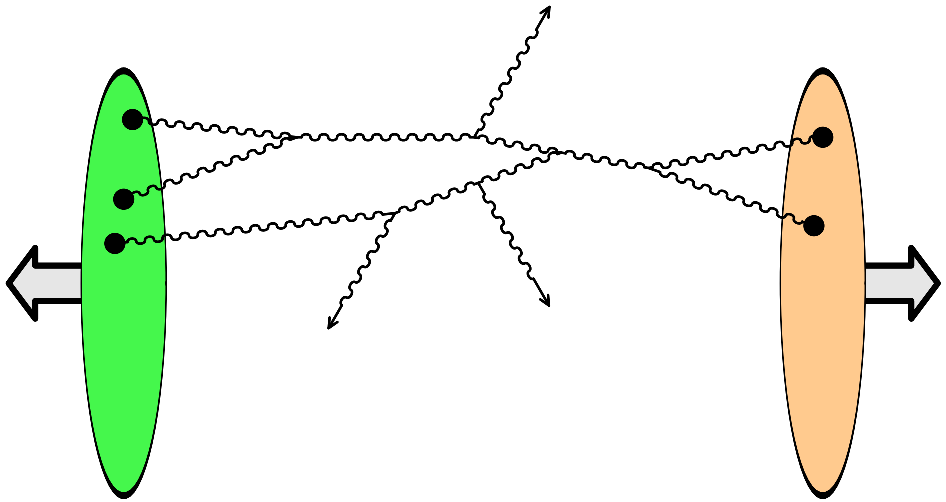
$$(x_{1,2} = M_\perp\, e^{\pm Y}\;/\;\sqrt{s})$$

Multi-parton interactions?



- Collinear or kt-factorization : only one parton in each projectile take part in the process of interest

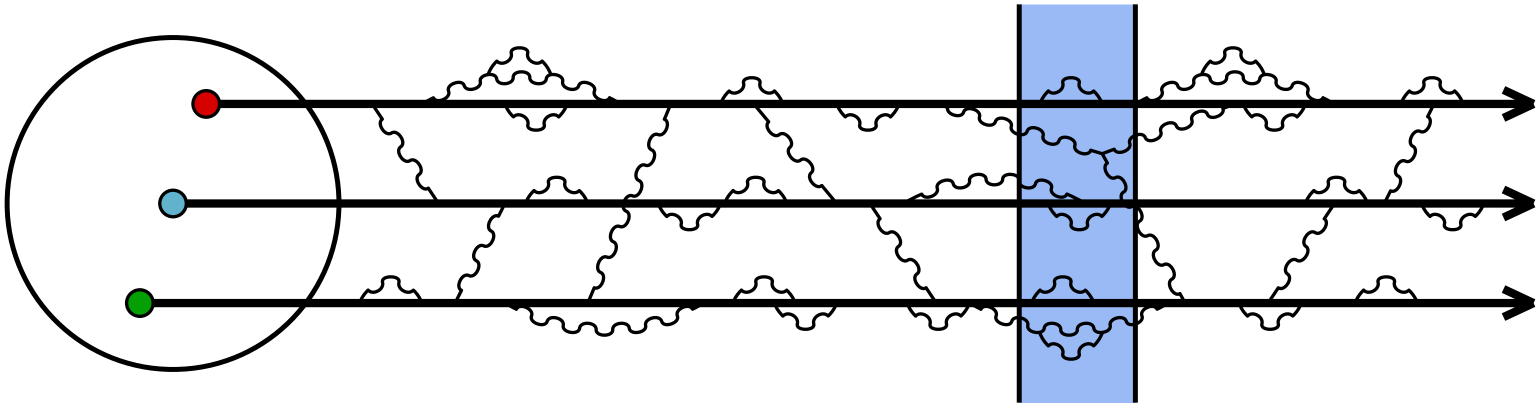
Multi-parton interactions?



- Collinear or kt-factorization : only one parton in each projectile take part in the process of interest
- If multiparton interactions are important : the above forms of factorization cannot work anymore, because the only information they retain about the distribution of partons is their 2-point correlations (i.e. the number of partons)

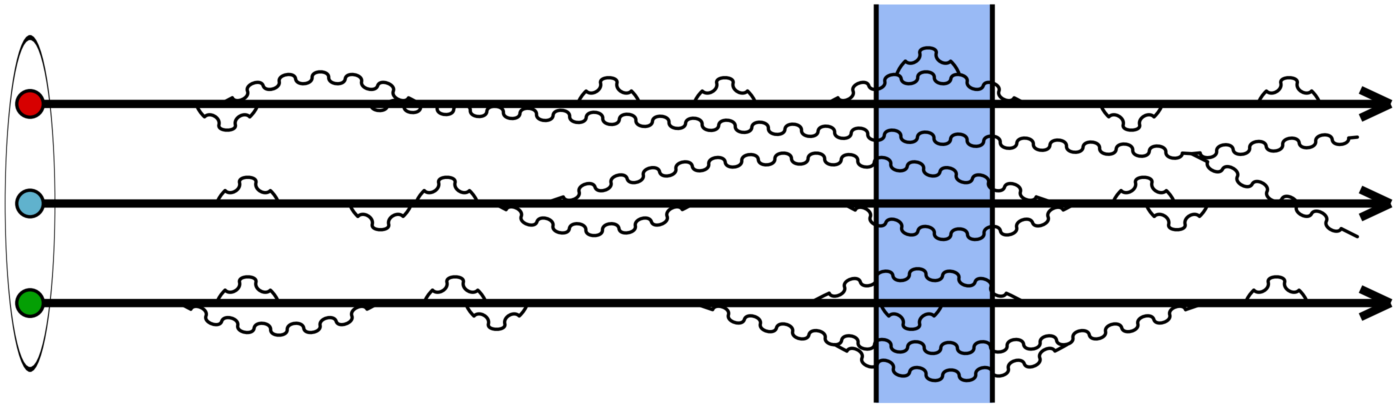
Parton model

Nucleon at low energy



- A nucleon at rest is a very complicated object...
- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe

Nucleon at high energy



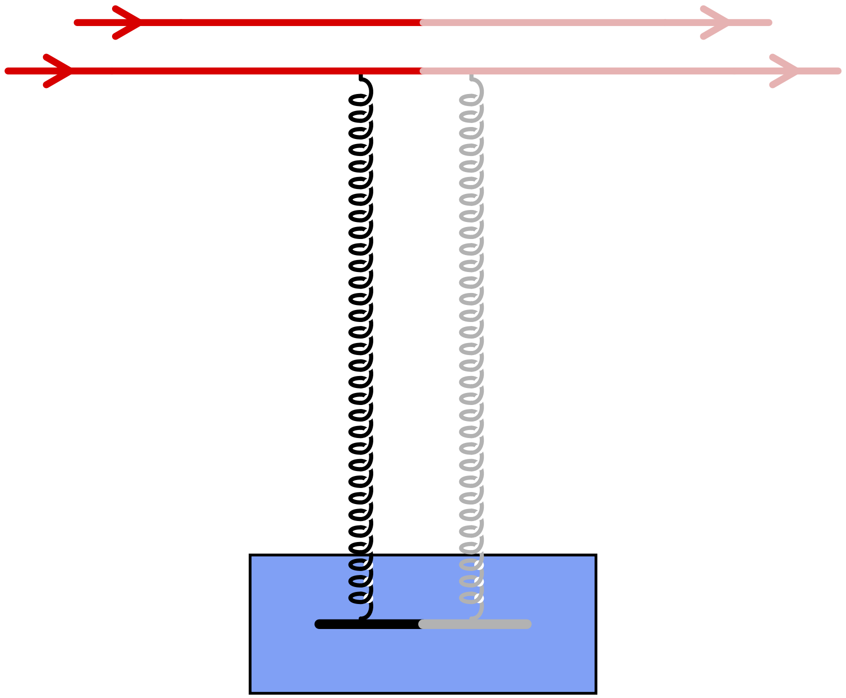
- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe
  ▷ the constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy (it contains more gluons)

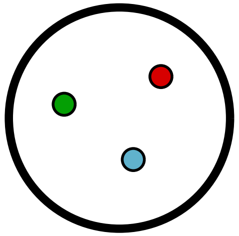
Parton model

- At the time of the interaction, the nucleon can be seen as a collection of free constituents, called partons
- It can be described by non-perturbative parton distributions that depend on the momentum fraction x of the partons
- One can separate the perturbative hard scattering from the non-perturbative distribution functions, because the strong interactions that are responsible for these non-perturbative aspects occur on much larger timescales (factorization)
- All these properties are based only on kinematics and causality, and should remain true in the saturation regime
  - what we use as the “parton distribution” must contain information about multiparton configurations
  - the calculation of the “hard” process is more involved

Gluon saturation

Parton evolution



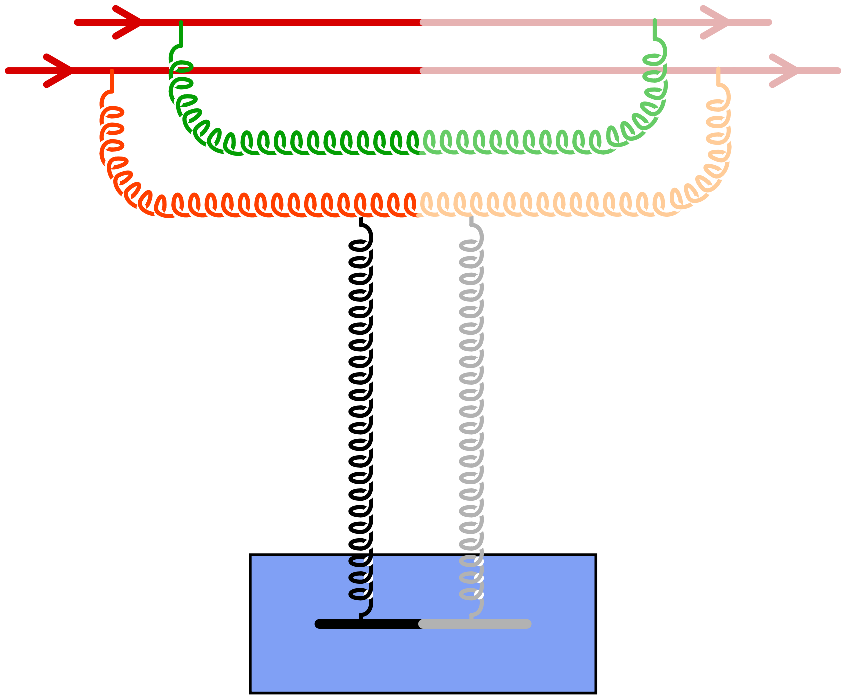


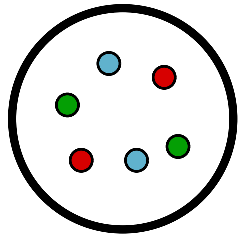
▷ assume that the projectile is big, e.g. a nucleus, and has many valence quarks (only two are represented)

▷ on the contrary, consider a small probe, with few partons

▷ at low energy, only valence quarks are present in the hadron wave function

Parton evolution



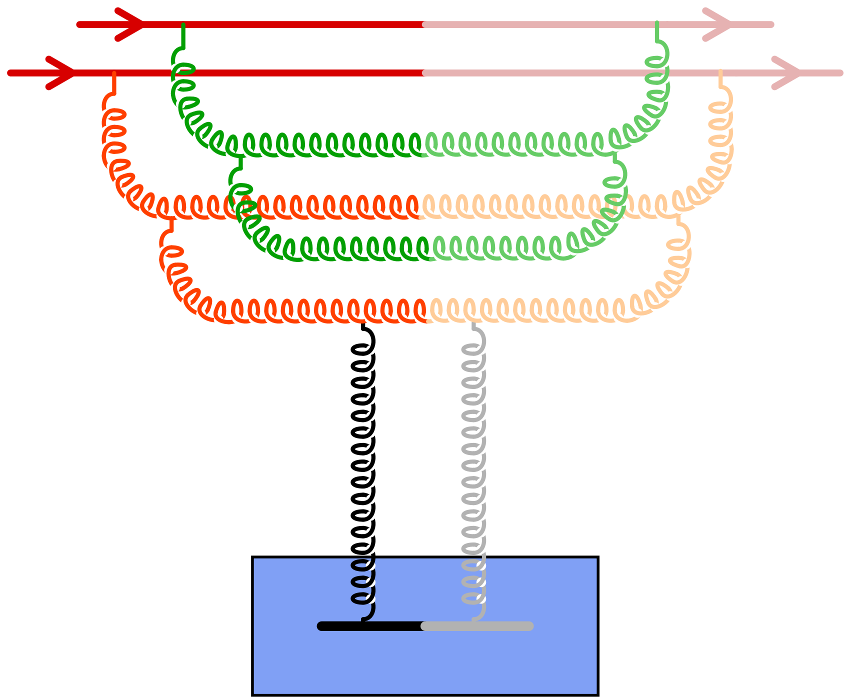


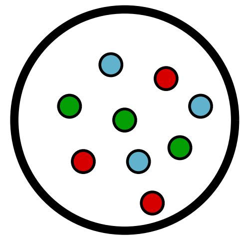
▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

▷ at small-x (i.e. high energy), these logs need to be resummed

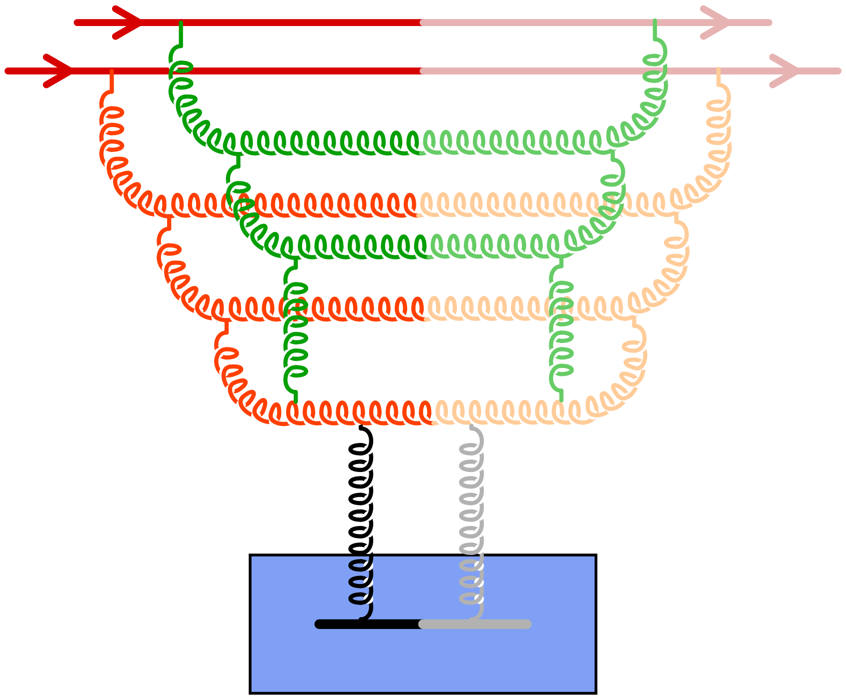
Parton evolution

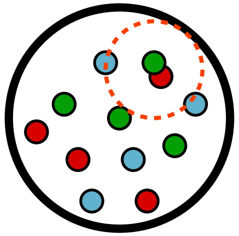




▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step (BFKL)

Parton evolution





▷ eventually, the partons start overlapping in phase-space

▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:
the number of partons created at a given step depends non-linearly on the number of partons present previously

Saturation criterion

Gribov, Levin, Ryskin (1983)

- Number of gluons per unit area:

$$\rho \sim \frac{xG_A(x,Q^2)}{\pi R_A^2}$$

- Recombination cross-section:

$$\sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2}$$

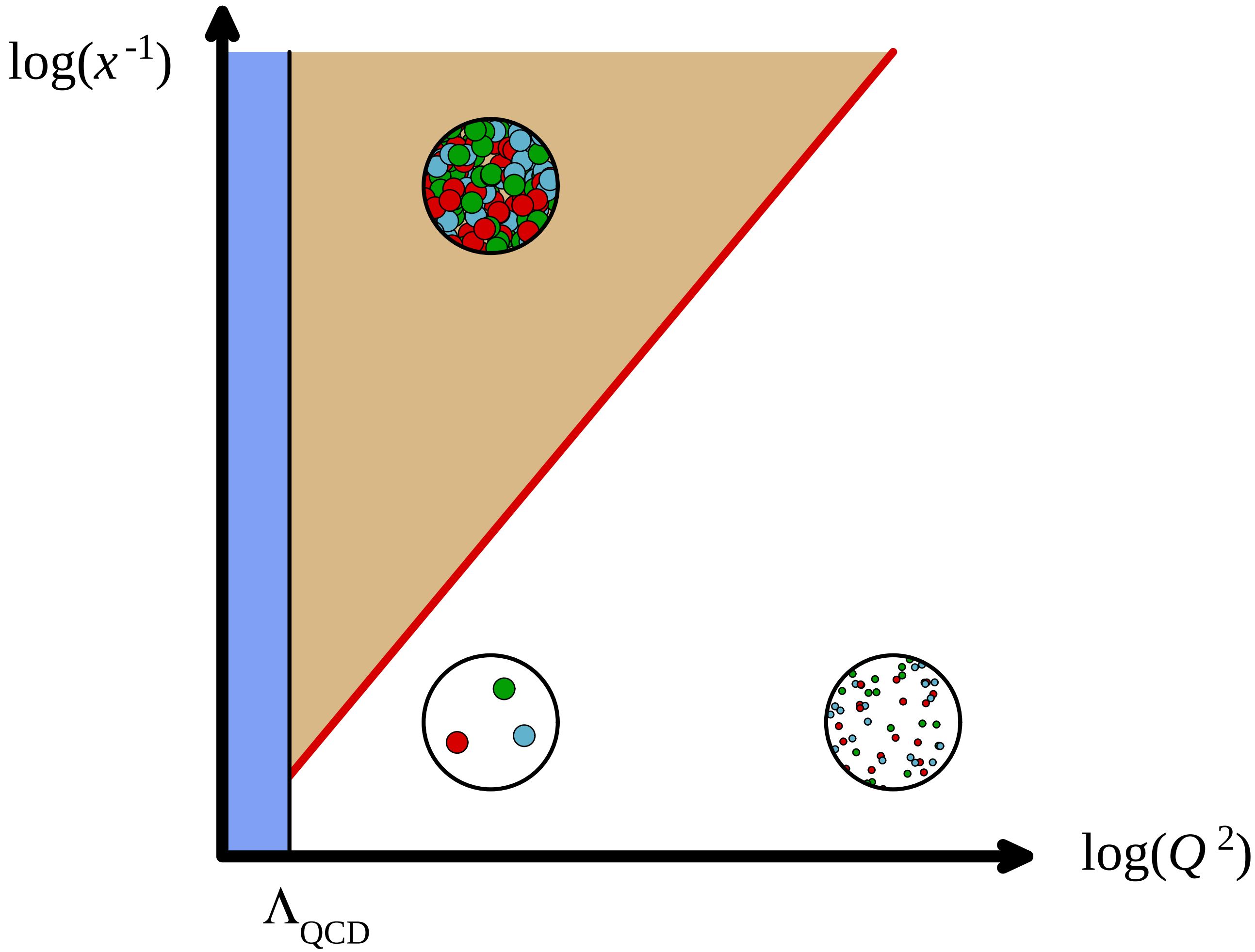
- Recombination happens if $\rho\sigma_{gg\to g} \gtrsim 1$, i.e. $Q^2 \lesssim Q_s^2$, with:

$$Q_s^2 \sim \frac{\alpha_s xG(x,Q_s^2)}{\pi R_A^2} \sim A^{1/3}\frac{1}{x^{0.3}}$$

- At saturation, the phase-space density is:

$$\frac{dN_g}{d^2\vec{\boldsymbol{x}}_\perp d^2\vec{\boldsymbol{p}}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s}$$

Saturation domain



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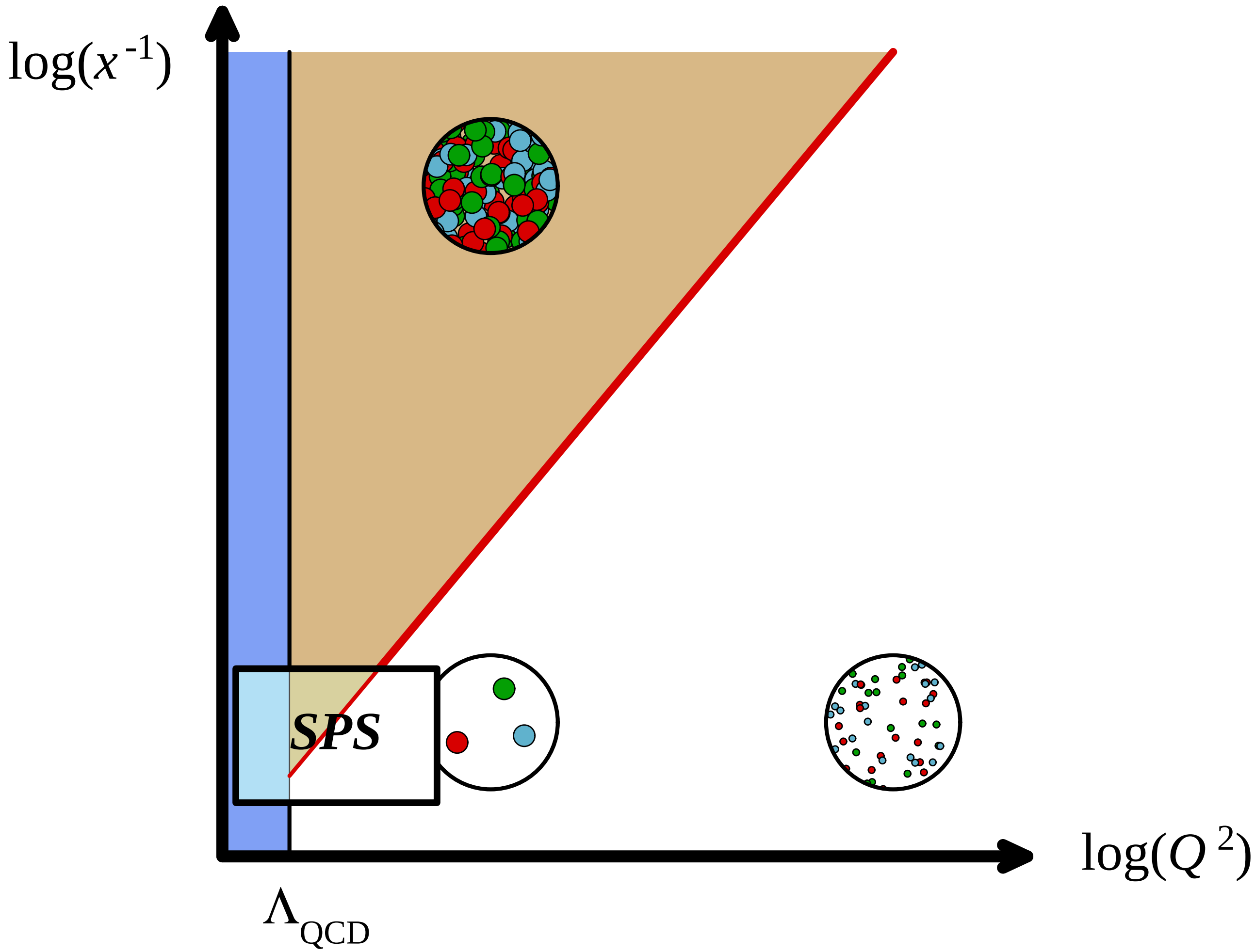

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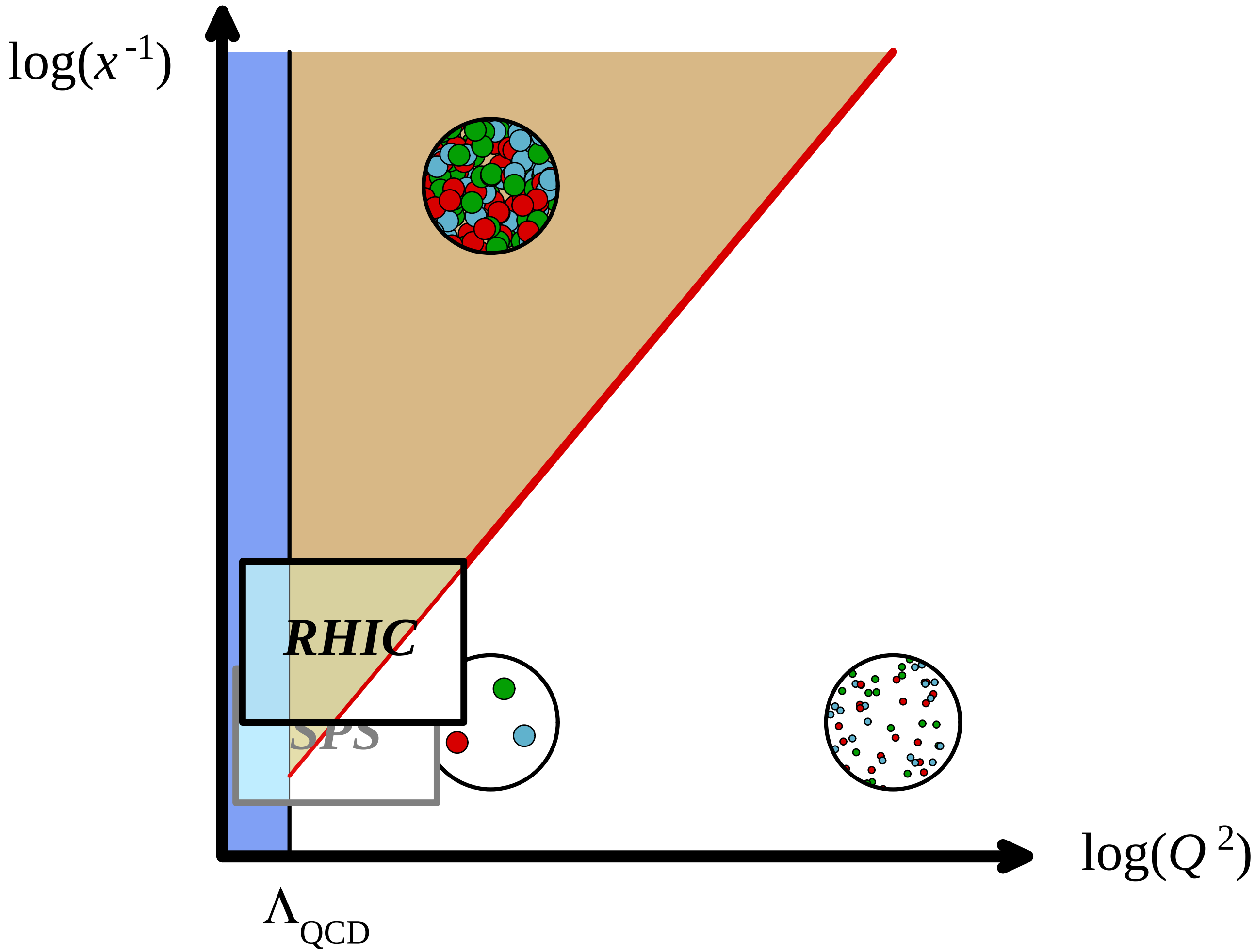

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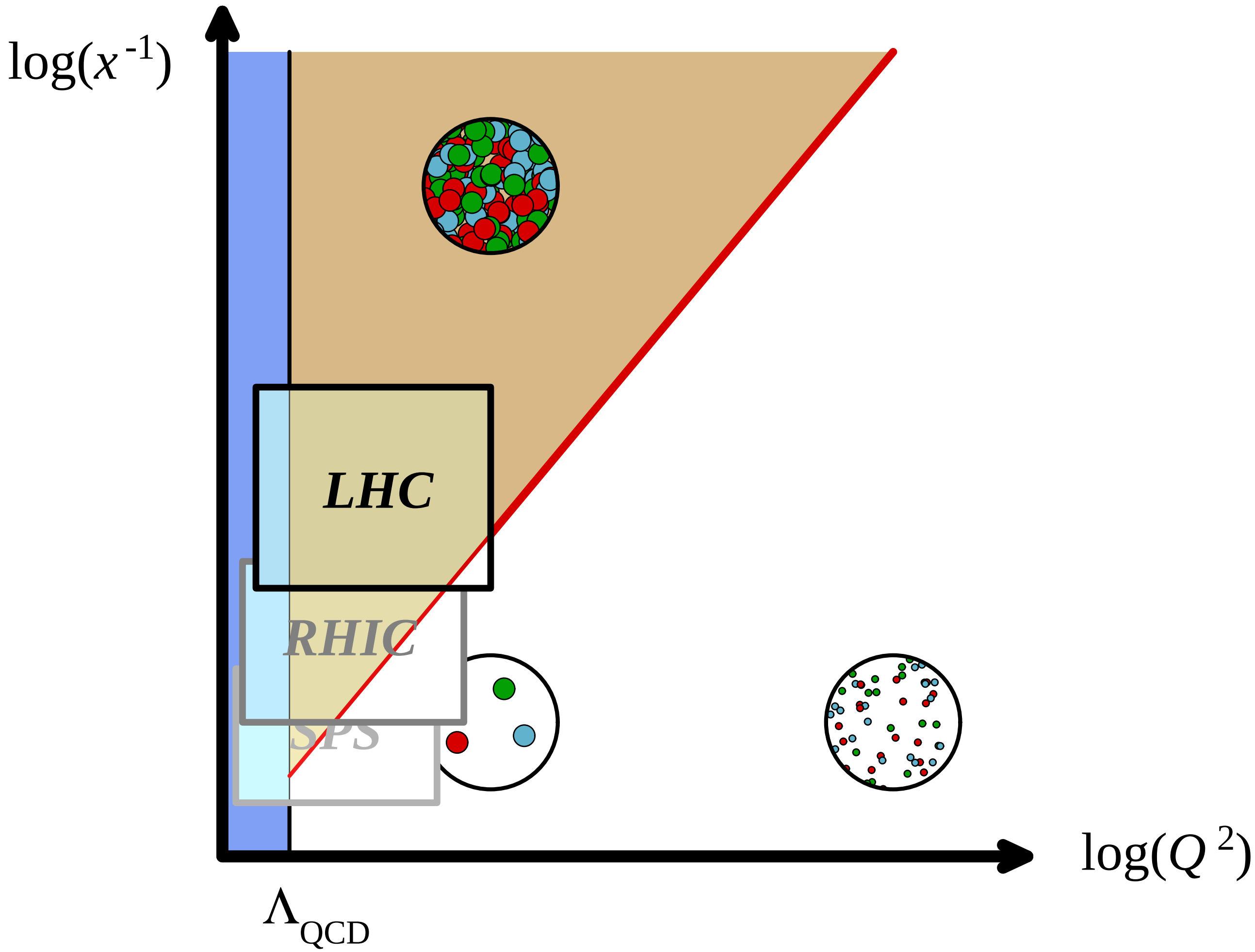

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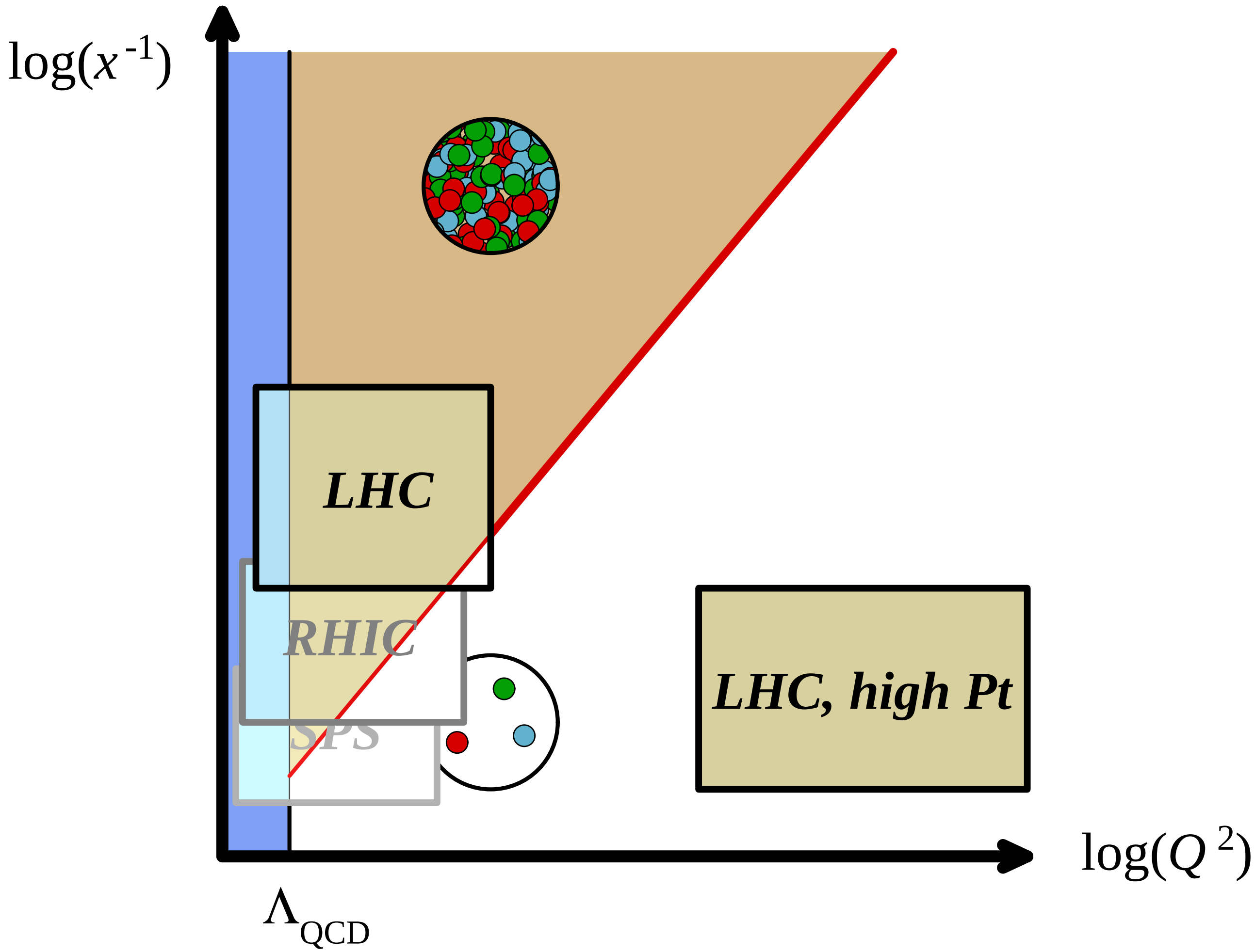
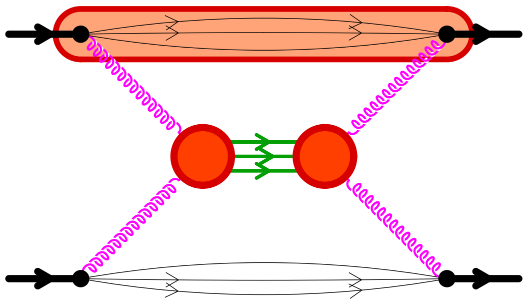


Multiple scatterings

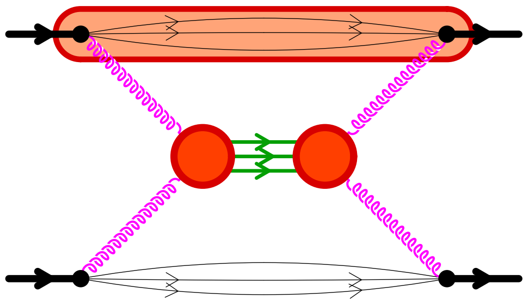
- Single scattering :



▷ 2-point function in the projectile ▷ gluon number

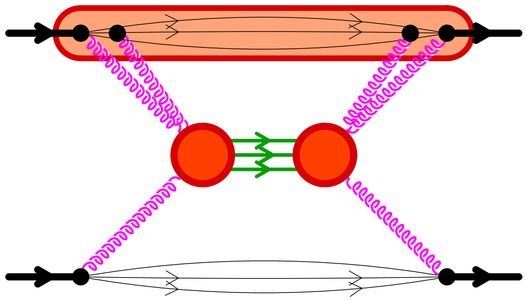
Multiple scatterings

- Single scattering :



▷ 2-point function in the projectile ▷ gluon number

- Multiple scatterings :



▷ 4-point function in the projectile ▷ higher correlation
▷ multiple scatterings in the projectile

Multiple scatterings

- Power counting : rescattering corrections are suppressed by inverse powers of the typical mass scale in the process :

$$\left[\frac{\mu^2}{M_\perp^2}\right]^n$$

- The parameter μ^2 has a factor of α_s, and a factor proportional to the gluon density ▷ rescatterings are important at high density
- Relative order of magnitude :

$$\frac{\text{2 scatterings}}{\text{1 scattering}} \sim \frac{Q_s^2}{M_\perp^2} \quad \text{with} \quad Q_s^2 \sim \alpha_s \frac{xG(x,Q_s^2)}{\pi R^2}$$

- When this ratio becomes ~ 1, all the rescattering corrections become important ▷ one must resum all $\left[Q_s/M_\perp\right]^n$
- These effects are not accounted for in DGLAP or BFKL

Color Glass Condensate

Degrees of freedom

- The fast partons (large x) are frozen by time dilation
  ▷ described as static color sources on the light-cone :

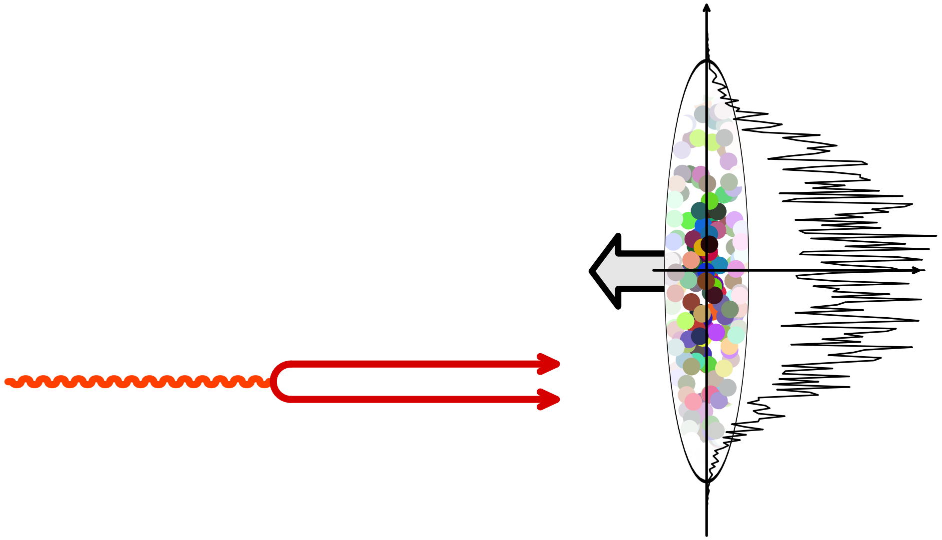
$$J^\mu_a = \delta^{\mu +}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp) \qquad (x^- \equiv (t-z)/\sqrt{2})$$

- Slow partons (small x) are radiated by the fast ones. They have a large occupation number ▷ described by a classical color field A^μ that obeys Yang-Mills's equation:

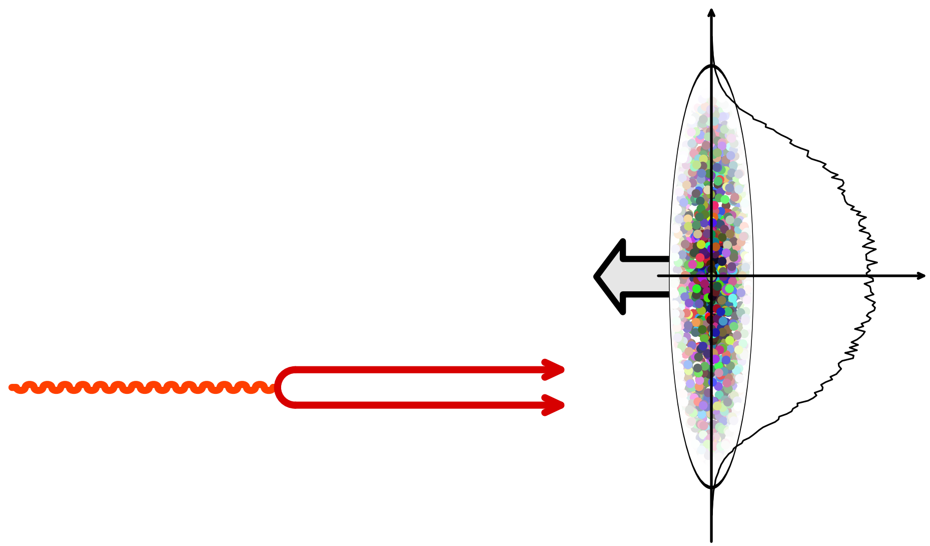
$$\left[D_\nu, F^{\nu\mu}\right]_a = J^\mu_a$$

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with Y the rapidity that separates “soft” and “hard”

Deep Inelastic Scattering



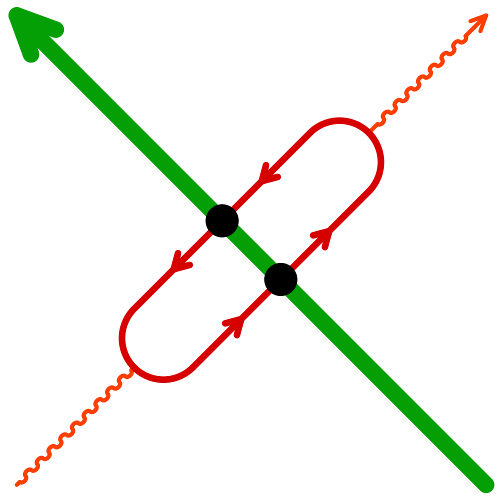
Deep Inelastic Scattering



100 configurations

Deep Inelastic Scattering

- Reactions involving a hadron or nucleus and an “elementary” projectile are fairly straightforward to study
- Example : forward DIS amplitude :



$$\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\rangle = \int [D\rho]\; W_Y[\rho]\; \left[1 - \frac{1}{N_c}\mathrm{tr}(U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp))\right]$$

▷ this formula resums all the $[\alpha_s \ln(1/x)]^m [Q_s/p_\perp]^n$ for the inclusive DIS cross-section

Deep Inelastic Scattering

- $U(\vec{\boldsymbol{x}}_\perp)$ is a Wilson line that represents the scattering at high energy between a quark and the color field of the nucleus (moving in the $-z$ direction) :

$$U(\vec{\boldsymbol{x}}_\perp) \equiv \mathrm{P}_+ \, \exp ig \int_{-\infty}^{+\infty} dz^+ \; A^-(z^+, \vec{\boldsymbol{x}}_\perp)$$

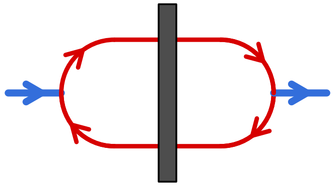
with

$$-\vec{\boldsymbol{\nabla}}_\perp^2 \, A^-(x^+, \vec{\boldsymbol{x}}_\perp) = \delta(x^+)\rho(\vec{\boldsymbol{x}}_\perp)$$

- The scattering of the antiquark is represented by $U^\dagger(\vec{\boldsymbol{y}}_\perp)$

Balitsky-Kovchegov equation

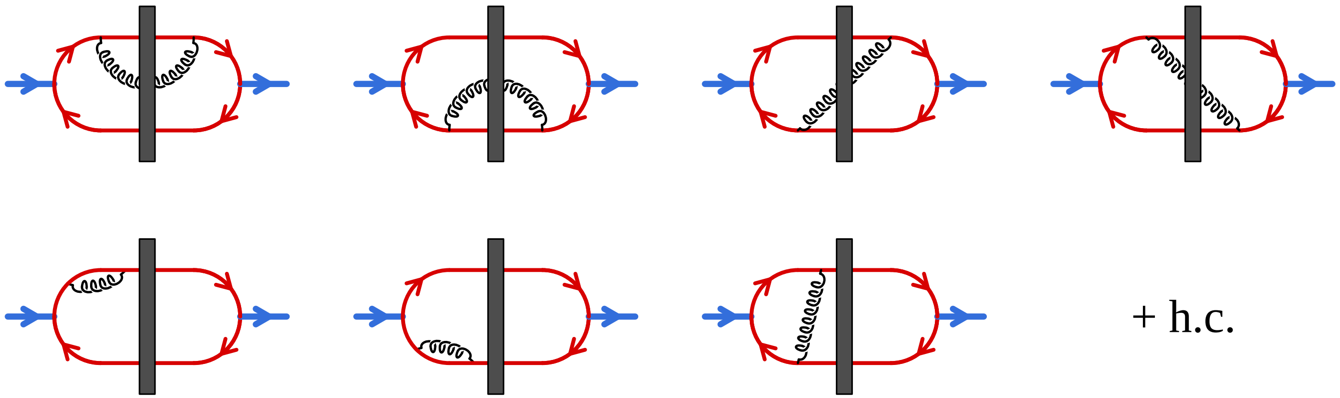
- The bare scattering amplitude of a color singlet $Q\overline{Q}$ dipole can be written as :



$$\propto \mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

Note : this bare dipole amplitude is independent of energy. The energy dependence comes from higher-order corrections

- At one loop, the following diagrams must be evaluated :



$+$ h.c.

Balitsky-Kovchegov equation

- The NLO corrections lead to an equation that drives the dependence of $\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv 1 - \frac{1}{N_c}\mathrm{tr}\,(U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp))$ with respect to $Y \sim \ln(\sqrt{s})$:

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \, \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \underline{- \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp)}\Big\}$$

(Balitsky-Kovchegov equation)

- Both $\boldsymbol{T} = 0$ and $\boldsymbol{T} = 1$ are fixed points of this equation

$\boldsymbol{T} = \epsilon$:	r.h.s. > 0	$\Rightarrow$	$\boldsymbol{T} = 0$ is unstable
$\boldsymbol{T} = 1 - \epsilon$:	r.h.s. > 0	$\Rightarrow$	$\boldsymbol{T} = 1$ is stable

- Without the non-linear term (underlined), one would have the BFKL equation, that has only an unstable fixed point at $\boldsymbol{T} = 0$

JIMWLK equation

- An alternate – equivalent – point of view is to keep the bare Wilson lines, and to say that one boosts the target so that the distribution of its color sources changes

- Evolution equation for $W_Y[\rho]$ (JIMWLK) :

$$\frac{\partial W_Y[\rho]}{\partial Y} = \mathcal{H}[\rho] \; W_Y[\rho]$$

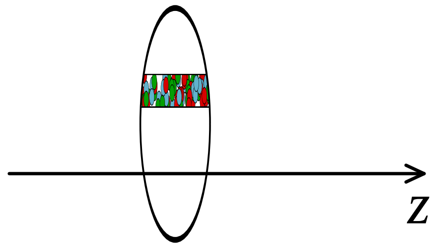
$$\mathcal{H}[\rho] = \int_{\vec{\boldsymbol{x}}_\perp} \sigma(\vec{\boldsymbol{x}}_\perp) \frac{\delta}{\delta\rho(\vec{\boldsymbol{x}}_\perp)} + \frac{1}{2} \int_{\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \frac{\delta^2}{\delta\rho(\vec{\boldsymbol{x}}_\perp)\delta\rho(\vec{\boldsymbol{y}}_\perp)}$$

- σ and χ are non-linear functionals of ρ

- Note : this point of view is more general because it also applies to situations where the observable cannot be written as a certain combination of Wilson lines – e.g. the gluon inclusive spectrum in AA collisions

Initial condition - MV model

- The JIMWLK equation must be completed by an initial condition, given at some moderate x_0
- As with DGLAP, the problem of finding the initial condition is in general non-perturbative
- The McLerran-Venugopalan model is often used as an initial condition at moderate x_0 for a large nucleus :



- partons distributed randomly
- many partons in a small tube
- no correlations at different $\vec{\boldsymbol{x}}_\perp$

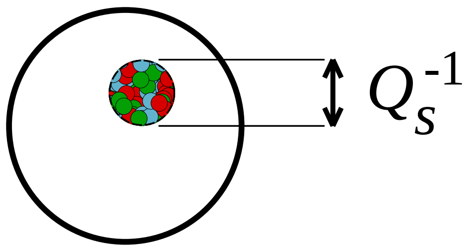
- The MV model assumes that the density of color charges $\rho(\vec{\boldsymbol{x}}_\perp)$ has a Gaussian distribution :

$$W_{x_0}[\rho] = \exp\left[-\int d^2\vec{\boldsymbol{x}}_\perp \frac{\rho_a(\vec{\boldsymbol{x}}_\perp)\rho_a(\vec{\boldsymbol{x}}_\perp)}{2\mu^2(\vec{\boldsymbol{x}}_\perp)}\right]$$

Phenomenology of saturation

Color correlation length

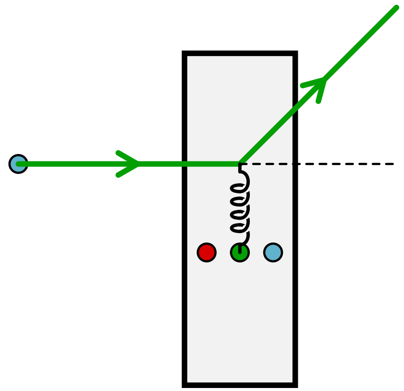
- In a nucleon at low energy, the typical correlation length among color charges is of the order of the nucleon size, i.e. $\Lambda_{QCD}^{-1} \sim 1\ \mathrm{fm}$. This is because the typical color screening distance is Λ_{QCD}^{-1}. At low energy, color screening is due to confinement, and thus non-perturbative
- At high energy (small x), partons are much more densely packed, and it can be shown that color neutralization occurs in fact over distances of the order of $Q_s^{-1} \ll \Lambda_{QCD}^{-1}$



- This implies that all hadrons, and nuclei, behave in the same way at high energy. In this sense, the small x regime described by the CGC is universal

Multiple scatterings

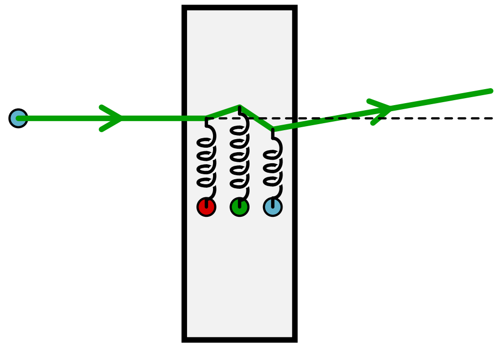
- Single scattering dominates at high $p_\perp$:



  - Differential cross-sections between a parton and a nucleus at high $p_\perp$ should scale like the atomic number A (volume scaling)

Multiple scatterings

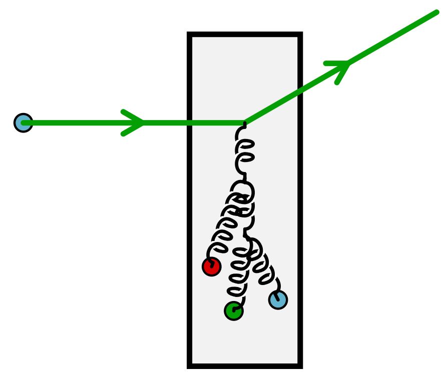
- Multiple scatterings at low $p_\perp$:



  - One of the scatterings "produces" the final state, while the others merely change its momentum
  - Each extra scattering corresponds to a correction $\alpha_s A^{1/3} \Lambda^2 / p_\perp^2$
    ▷ important correction at low $p_\perp$, despite the α_s suppression
  - When this effect is extremal, differential cross-sections at low $p_\perp$ scale like $A^{2/3}$ (area scaling)
  - Multiple scatterings only affect the momentum distribution of the final states, not the yield ▷ the suppression at low $p_\perp$ is compensated by an increase at higher $p_\perp$ (Cronin effect)

Shadowing

- Interactions among the partons in the nuclear target (shadowing) :



  - Modification of the single scattering contribution due to the non-linear interactions of partons inside the target
  - At low x, this effect induces a suppression of the differential cross-section : $d\sigma_{pA}/d^2\vec{\boldsymbol{p}}_\perp \sim A^\alpha$ with $\alpha < 1$

Lecture II : Initial particle production

- Introduction to nucleus-nucleus collisions
- Power counting and bookkeeping
- Inclusive gluon spectrum
- Loop corrections and factorization

Lecture III : Instabilities, thermalization

- Reminder on initial gluon production
- Glasma instabilities
- Instabilities in anisotropic plasmas
- Thermalization ?

Lecture IV : Kinetic theory

- Collisionless kinetic equations
- Boltzmann equation
- Transport coefficients
- From kinetic theory to hydrodynamics

Complements

Light-cone coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$a^{1,2}$ unchanged. Notation : $\vec{\boldsymbol{a}}_\perp \equiv (a^1, a^2)$

- Under a Lorentz boost in the z direction :

$$a^+ \to \Lambda\, a^+ \quad , \quad a^- \to \Lambda^{-1}\, a^- \quad , \quad a^{1,2} \to a^{1,2}$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{\boldsymbol{x}}_\perp \cdot \vec{\boldsymbol{y}}_\perp$$

$$d^4x = dx^+ dx^- d^2\vec{\boldsymbol{x}}_\perp$$

$$\square = 2\partial^+\partial^- - \vec{\boldsymbol{\nabla}}_\perp^2 \qquad \text{Notation :} \quad \partial^+ \equiv \frac{\partial}{\partial x^-} \; , \; \partial^- \equiv \frac{\partial}{\partial x^+}$$

Solution of Yang Mills eq. for 1 nucleus

- The Yang-Mills equations are the classical equations that give the color field induced by a given current. They are the analogue in QCD of the Maxwell equations in electrodynamics :

$$\left[D_\mu, F^{\mu\nu}\right] = J^\nu$$

  where
  - $D^\mu \equiv \partial^\mu - igA^\mu$ is the covariant derivative
  - $F^{\mu\nu} \equiv i[D^\mu, D^\nu]/g = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$

- For a single nucleus moving in the $+z$ direction

$$J^\nu = \delta^{\nu +}\delta(x^-)\rho(\vec{\boldsymbol{x}}_\perp)$$

- This current must be covariantly conserved :

$$\left[D_\nu, J^\nu\right] = 0$$

  ▷ the formula for J^ν may have higher order corrections in ρ

Solution of Yang Mills eq. for 1 nucleus

- Reminder: in QED, we would simply have :

$$\square A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu$$

which in the gauge $\partial_\mu A^\mu = 0$ (Lorenz) simplifies into $\square A^\nu = J^\nu$

- For QCD in the Lorenz gauge, one can first rewrite the Yang-Mills equations as :

$$\square A^\nu - ig\big[A_\mu, F^{\mu\nu} + \partial^\mu A^\nu\big] = J^\nu$$

- It is useful to expand all the quantities in powers of the color density ρ :

$$A^\mu \equiv \sum_{n=0}^{\infty} A^\mu_{(n)} \quad , \quad J^\mu \equiv \sum_{n=0}^{\infty} J^\mu_{(n)}$$

where $A^\mu_{(n)} \sim J^\mu_{(n)} \sim \rho^n$

Solution of Yang Mills eq. for 1 nucleus

- At the order $n = 1$, the equations are simply :

$$\partial_\nu J^\nu_{(1)} = 0 \quad , \quad \square A^\nu_{(1)} = J^\nu_{(1)}$$

and their solution reads

$$J^\nu_{(1)} = \delta^{\nu+}\delta(x^-)\rho(\vec{\boldsymbol{x}}_\perp)$$

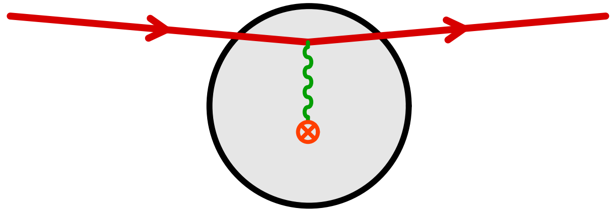
$$A^\nu_{(1)} = \delta^{\nu+}\delta(x^-)\alpha(\vec{\boldsymbol{x}}_\perp) \quad , \quad -\vec{\boldsymbol{\nabla}}_\perp^2\alpha(\vec{\boldsymbol{x}}_\perp) = \rho(\vec{\boldsymbol{x}}_\perp)$$

- At this order, the only non-zero component of $F^{\mu\nu}$ is $F^{i+} = -F^{+i} = \partial^i A^+_{(1)}$
- By writing the equations for the corrections of order ρ^2, we find that all the non-linear terms cancel and that all these corrections are zero
- This feat can be repeated to all orders in ρ $\triangleright$ the complete solution of the non-linear Yang-Mills equations is linear in ρ!

  Note : this result is only valid for Lorenz gauge and for the kind of current we have considered

Eikonal scattering

- Consider the scattering amplitude off an external potential :



$$S_{\beta\alpha} \equiv \left\langle \beta_{\rm out} \middle| \alpha_{\rm in} \right\rangle = \left\langle \beta_{\rm in} \middle| U(+\infty,-\infty) \middle| \alpha_{\rm in} \right\rangle$$

where $U(+\infty,-\infty)$ is the evolution operator from $t=-\infty$ to $t=+\infty$

$$U(+\infty,-\infty) = T\,\exp\left[i\int d^4x\;\mathcal{L}_{\rm int}(\phi_{\rm in}(x))\right]$$

Note : $\mathcal{L}_{\rm int}$ contains the self-interactions of the fields and their interactions with the external potential

- We want to calculate its high energy limit :

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega\to+\infty} \left\langle \beta_{\rm in} \middle| e^{i\omega K^3} U(+\infty,-\infty) e^{-i\omega K^3} \middle| \alpha_{\rm in} \right\rangle$$

where K^3 is the generator of boosts in the $+z$ direction

Eikonal scattering in a nutshell

- In a scattering at high energy, the collision time goes to zero as $s^{-1/2}$
- With scalar interactions, this implies a decrease of the scattering amplitude as $s^{-1/2}$
- With vectorial interactions, this decrease is compensated by the growth of the component J^+ of the vector current

$\triangleright$ the eikonal approximation gives the finite limit of the scattering amplitude in the case of vectorial interactions when $s \to +\infty$

Eikonal limit

- Consider an external vector potential, that couples via $e\,\mathcal{A}_\mu(x)J^\mu(x)$ (J^μ is the Noether current associated to some conserved charge)
- We will assume that the external potential is non-zero only in a finite range in x^+, $x^+ \in [-L,+L]$
- The action of K^3 on states and (scalar) fields is :

$$e^{-i\omega K^3}\big|\vec{\boldsymbol{p}}\cdots{}_{\rm in}\big\rangle = \big|(e^\omega p^+,\vec{\boldsymbol{p}}_\perp)\cdots{}_{\rm in}\big\rangle$$

$$e^{i\omega K^3}\phi_{\rm in}(x)e^{-i\omega K^3} = \phi_{\rm in}(e^{-\omega}x^+,e^\omega x^-,\vec{\boldsymbol{x}}_\perp)$$

- K^3 does not change the ordering in x^+. Hence,

$$e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3} = T_+\exp i\int d^4x\;\mathcal{L}_{\rm int}(e^{i\omega K^3}\phi_{\rm in}(x)e^{-i\omega K^3})$$

where $\mathcal{L}_{\rm int} = \mathcal{L}_{\rm self}(\phi) - e\,\mathcal{A}_\mu J^\mu$

Eikonal limit

- Split the evolution operator $U(+\infty,-\infty)$ into three factors :

$$U(+\infty,-\infty)=U(+\infty,+L)U(+L,-L)U(-L,-\infty)$$

Upon application of K^3, this becomes :

$$e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3}=e^{i\omega K^3}U(+\infty,+L)e^{-i\omega K^3}$$
$$\times e^{i\omega K^3}U(+L,-L)e^{-i\omega K^3}e^{i\omega K^3}U(-L,-\infty)e^{-i\omega K^3}$$

- The external potential $\mathcal{A}_\mu(x)$ is unaffected by K^3
- The components of $J^\mu(x)$ are changed as follows :

$$e^{i\omega K^3}J^i(x)e^{-i\omega K^3}=J^i(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)$$
$$e^{i\omega K^3}J^-(x)e^{-i\omega K^3}=e^{-\omega}\,J^-(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)$$
$$e^{i\omega K^3}J^+(x)e^{-i\omega K^3}=e^{\omega}\,J^+(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)$$

Eikonal limit

- The factors $U(+\infty,+L)$ and $U(-L,-\infty)$ do not contain the external potential. In order to deal with these factors, it is sufficient to change variables : $e^{-\omega}x^+ \to x^+$, $e^{\omega}x^- \to x^-$. This leads to :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+\infty,+L) e^{-i\omega K^3} = U_{\rm self}(+\infty,0)$$

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(-L,-\infty) e^{-i\omega K^3} = U_{\rm self}(0,-\infty)$$

where $U_{\rm self}$ is the same as U, but with the self-interactions only

- For the factor $U(L,-L)$, the change $e^{\omega}x^- \to x^-$ leads to :

$$e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} =$$
$$= T_+ \exp i \int_{-L}^{+L} d^4x \; e^{-\omega} \Big[e\,\mathcal{A}^-(x^+, e^{-\omega}x^-, \vec{\boldsymbol{x}}_\perp) \times e^{\omega} J^+(e^{-\omega}x^+, x^-, \vec{\boldsymbol{x}}_\perp) + \mathcal{O}(1) \Big]$$

Eikonal limit

- Therefore, in the limit $\omega \to +\infty$, we have :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} = \exp\left[ie\int d^2\vec{\boldsymbol{x}}_\perp \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right]$$

with

$$\begin{cases} \chi(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^+ \, \mathcal{A}^-(x^+,0,\vec{\boldsymbol{x}}_\perp) \\ \rho(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^- \, J^+(0,x^-,\vec{\boldsymbol{x}}_\perp) \end{cases}$$

- The high-energy limit of the scattering amplitude is :

$$S^{(\infty)}_{\beta\alpha} = \left\langle\beta_{\rm in}\middle| U_{\rm self}(+\infty,0)\;\exp\left[ie\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right] U_{\rm self}(0,-\infty)\middle|\alpha_{\rm in}\right\rangle$$

  - Only the $-$ component of the vector potential matters
  - The self-interactions and the interactions with the external potential are factorized $\triangleright$ parton model
  - This is an exact result when $s \to +\infty$

Eikonal limit

- The previous formula still contains all the self-interactions of the fields. In order to perform the perturbative expansion, it is convenient to write first :

$$S^{(\infty)}_{\beta\alpha} = \sum_{\gamma,\delta} \big\langle\beta_{\rm in}\big|U_{\rm self}(+\infty,0)\big|\gamma_{\rm in}\big\rangle \times\big\langle\gamma_{\rm in}\big|\exp\Big[ie\int_{\vec{\boldsymbol{x}}_\perp}\chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\Big]\big|\delta_{\rm in}\big\rangle\big\langle\delta_{\rm in}\big|U_{\rm self}(0,-\infty)\big|\alpha_{\rm in}\big\rangle$$

- The factor

$$\sum_{\delta}\big|\delta_{\rm in}\big\rangle\big\langle\delta_{\rm in}\big|U_{\rm self}(0,-\infty)\big|\alpha_{\rm in}\big\rangle$$

is the Fock expansion of the initial state: the state prepared at $x^+ = -\infty$ may have fluctuated into another state before it interacts with the external potential

Eikonal limit

- We need to calculate matrix elements such as $\langle\gamma_{\rm in}|\boldsymbol{F}|\delta_{\rm in}\rangle$, with :

$$\boldsymbol{F} \equiv \exp\, ie \int \chi_a(\vec{\boldsymbol{x}}_\perp)\rho^a(\vec{\boldsymbol{x}}_\perp)$$

  - having QCD in mind, we have reinstated the color indices
  - the contribution of quarks and antiquarks to $\rho^a(\vec{\boldsymbol{x}}_\perp)$ is :

$$\rho^a(\vec{\boldsymbol{x}}_\perp) = t^a_{ij} \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\frac{d^2\vec{\boldsymbol{q}}_\perp}{(2\pi)^2}\Big\{ b^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)b_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp} - d^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)d_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{-i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\Big\}$$

  - Note : one should keep the ordering of the exponential in x^+
  - the contribution of gluons is similar, with a color matrix in the adjoint representation

- The action of $\boldsymbol{F}$ on a state $|\delta_{\rm in}\rangle$ gives a state with the same particle content, the same $+$ components for the momenta, but modified transverse momenta and colors

Light-cone wavefunction

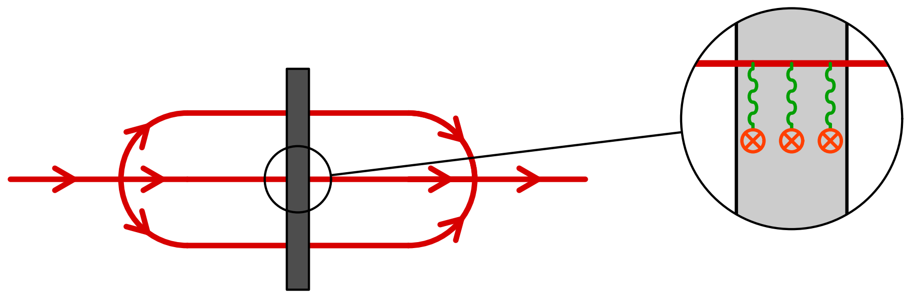
- For each intermediate state $\langle\delta_{\rm in}\big| \equiv \big\langle\{k_i^+,\vec{\boldsymbol{k}}_{i\perp}\}\big|$, define the corresponding light-cone wave function by :

$$\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \equiv \prod_i \int \frac{d^2\vec{\boldsymbol{k}}_{i\perp}}{(2\pi)^2} e^{-i\vec{\boldsymbol{k}}_{i\perp}\cdot\vec{\boldsymbol{x}}_{i\perp}} \big\langle\delta_{\rm in}\big|U_{\rm self}(0,-\infty)\big|\alpha_{\rm in}\big\rangle$$

- Each charged particle going through the external field acquires a phase proportional to its charge (antiparticles get an opposite phase) :

$$\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \longrightarrow \Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \prod_i U_i(\vec{\boldsymbol{x}}_\perp)$$

$$U_i(\vec{\boldsymbol{x}}_\perp) \equiv T_+ \exp\left[ig_i \int dx^+ \; \mathcal{A}_a^-(x^+,0,\vec{\boldsymbol{x}}_\perp)t^a\right]$$



Light-cone wavefunction

- We have seen that the number and the nature of the particles is unchanged under the action of the operator $\boldsymbol{F}$. Moreover, in terms of the transverse coordinates, we simply have

$$\left\langle \gamma_{\rm in} \middle| \boldsymbol{F} \middle| \delta_{\rm in} \right\rangle = \delta_{NN'} \prod_i \left[4\pi k_i^+ \delta(k_i^+ - k_i^{+\prime}) \delta(\vec{\boldsymbol{x}}_{i\perp} - \vec{\boldsymbol{x}}'_{i\perp}) U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right]$$

where $U_R(\vec{\boldsymbol{x}}_\perp)$ is a Wilson line operator, in the representation R appropriate for the particle going through the target

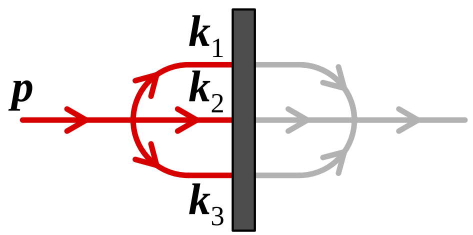
- Therefore, the high energy scattering amplitude can be written as :

$$S_{\beta\alpha}^{(\infty)} = \sum_\delta \int \left[\prod_{i\in\delta} d\Phi_i \right] \Psi^\dagger_{\delta\beta}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \left[\prod_{i\in\delta} U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right] \Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\})$$

- As we shall see shortly, some loop corrections are enhanced by logs of the energy. They must be resummed and drive the energy evolution of the amplitude

Light-cone wave function

- The calculation of $\left\langle\delta_{\rm in}\big|U_{\rm self}(0,-\infty)\big|\alpha_{\rm in}\right\rangle$ is similar to that of scattering amplitudes in the vacuum. The only difference is that the integration over x^+ at each vertex runs only over half of the real axis $[-\infty,0]$
  - In Fourier space, this means that the $-$ component of the momentum is not conserved at the vertices
  - Instead of a δ function, one gets an energy denominator
- Example with a single interaction :

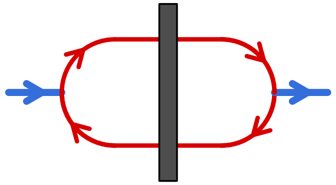


$$\left\langle\vec{\boldsymbol{k}}_1\vec{\boldsymbol{k}}_2\vec{\boldsymbol{k}}_{3\,{\rm in}}\big|U(0,-\infty)\big|\vec{\boldsymbol{p}}_{\rm in}\right\rangle=-ig\int_{-\infty}^{0}d^4x\;e^{i(k_1+k_2+k_3-p)\cdot x}$$

$$=-g\,\frac{(2\pi)^3\delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}+\vec{\boldsymbol{k}}_{3\perp}-\vec{\boldsymbol{p}}_\perp)\delta(k_1^++k_2^++k_3^+-p^+)}{k_1^-+k_2^-+k_3^--p^--i\epsilon}$$

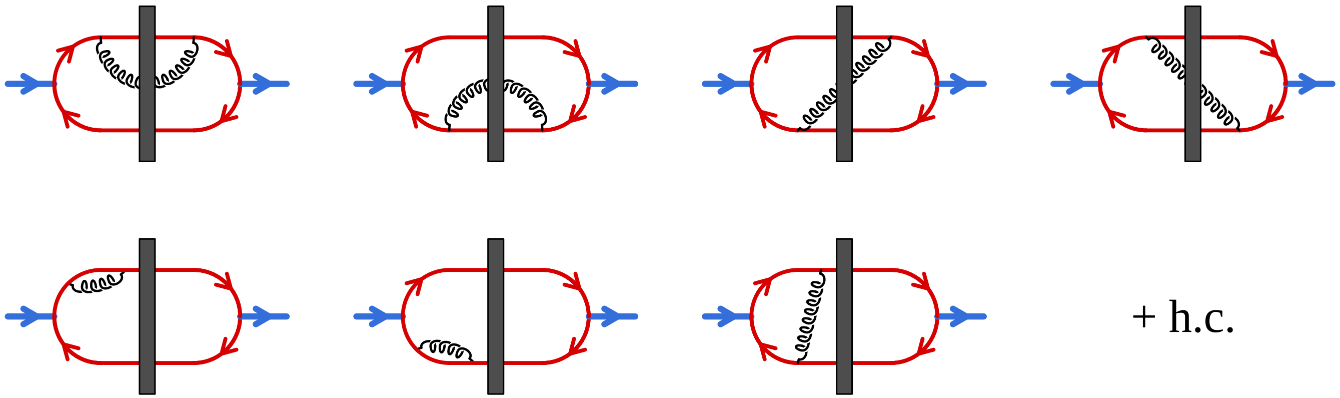
Scattering of a dipole

- Assume that the initial and final states are a color singlet $Q\overline{Q}$ dipole. The bare scattering amplitude can be written as :



$$\propto \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

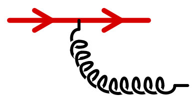
- At one loop, the following diagrams must be evaluated :



$+$ h.c.

Scattering of a dipole

- In the gauge $A^+ = 0$, the emission of a gluon of momentum k by a quark can be written as :



$$= \; 2gt^a \, \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2}$$

- In coordinate space, this reads :

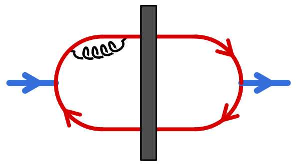
$$\int \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2} \, e^{i\vec{\boldsymbol{k}}_\perp\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)} \, 2gt^a \, \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot \vec{\boldsymbol{k}}_\perp}{k_\perp^2} = \frac{2ig}{2\pi} t^a \frac{\vec{\boldsymbol{\epsilon}}_\lambda \cdot (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- When connecting two gluons, one must use :

$$\sum_\lambda \vec{\boldsymbol{\epsilon}}_\lambda^i \vec{\boldsymbol{\epsilon}}_\lambda^j = -g^{ij}$$

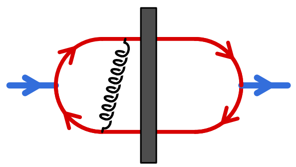
Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole
- Examples :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times -2\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp) t^a\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- Reminder : $t^a t^a = (N_c^2 - 1)/2N_c \equiv C_F$

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F\alpha_s}{\pi^2}\int\frac{dk^+}{k^+}\int d^2\vec{\boldsymbol{z}}_\perp\,\frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

$$\times\left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

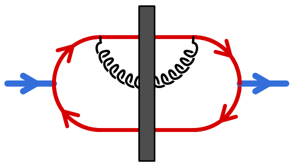
$$\int^{p^+}\frac{dk^+}{k^+}=\ln(p^+)=Y$$

▷ When Y is large, $\alpha_s Y$ may not be small. By differentiating with respect to Y, we will get an evolution equation in Y whose solution resums all the powers $(\alpha_s Y)^n$

- The integral over $\vec{\boldsymbol{z}}_\perp$ is divergent when $\vec{\boldsymbol{z}}_\perp=\vec{\boldsymbol{x}}_\perp$ or $\vec{\boldsymbol{y}}_\perp$

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon
- Example :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

  - $\widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp)$ is a Wilson line in the adjoint representation
- In order to simplify the color structure, first recall that :

$$t^a \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = U(\vec{\boldsymbol{z}}_\perp) t^b U^\dagger(\vec{\boldsymbol{z}}_\perp)$$

- Then use the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Real corrections

- The Wilson lines can be rearranged into :

$$\mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = \frac{1}{2}\mathrm{tr}\left[U^\dagger(\vec{\boldsymbol{z}}_\perp)U(\vec{\boldsymbol{x}}_\perp)\right]\mathrm{tr}\left[U(\vec{\boldsymbol{z}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] - \frac{1}{2N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

  - The term in $1/2N_c$ cancels against a similar term in the virtual contribution
  - All the real terms have the same color structure

- When we sum all the real terms, we generate the same kernel as in the virtual terms :

$$\frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- In order to simplify the notations, let us denote :

$$\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

Evolution equation

- The 1-loop scattering amplitude reads :

$$-\frac{\alpha_s N_c^2 Y}{2\pi^2}\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2} \times\Big\{S(\vec{x}_\perp,\vec{y}_\perp)-S(\vec{x}_\perp,\vec{z}_\perp)S(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

- Reminder: the bare scattering amplitude was :

$$\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 N_c\, S(\vec{x}_\perp,\vec{y}_\perp)$$

- Hence, we have :

$$\frac{\partial S(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y} = -\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2} \times\Big\{S(\vec{x}_\perp,\vec{y}_\perp)-S(\vec{x}_\perp,\vec{z}_\perp)S(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

  - since $S(\vec{x}_\perp,\vec{x}_\perp)=1$, the integral over $\vec{z}_\perp$ is now regular

BFKL equation

Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

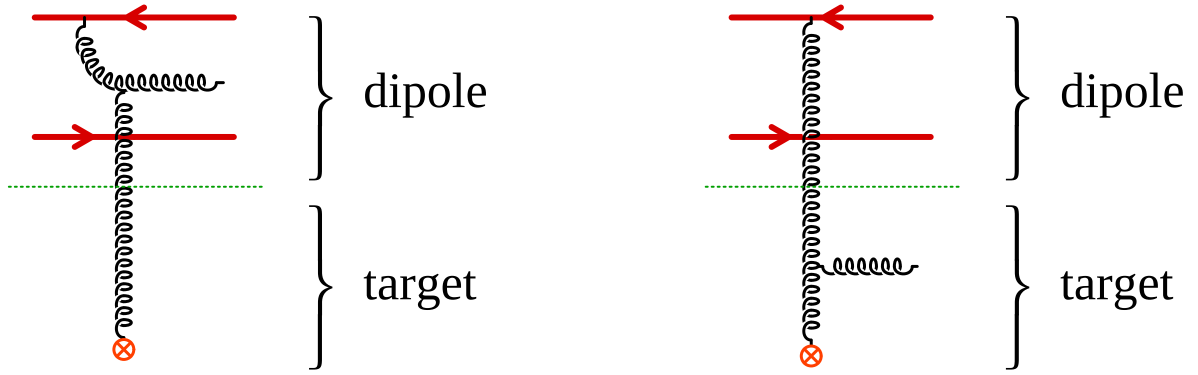
- The BFKL equation can be obtained by linearizing the previous equation

- Write $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp) \equiv 1 - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude $\boldsymbol{T}$ is small. Drop the terms that are non-linear in $\boldsymbol{T}$:

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp \, \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)+\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)-\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\Big\}$$

BFKL equation

- Note : $T(\vec{x}_\perp, \vec{y}_\perp)$ is independent on the frame. In particular, it depends only on the rapidity difference between the dipole and the target
  ▷ in a frame where the dipole is held fixed, the target has to evolve in such a way as to reproduce the Y dependence of T



- The corresponding evolution in the target is the radiation of a gluon

Unitarity problem

- The solution of this equation grows exponentially when $Y \to +\infty$ $\triangleright$ serious unitarity problem...

- In perturbation theory, the forward scattering amplitude between a small dipole and a target made of gluons reads :

$$\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \propto |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp|^2 \, xG(x, |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp|^{-2})$$

where $\quad Y \equiv \ln(1/x)$

- Therefore, the exponential behavior of $\boldsymbol{T}$ implies an increase of the gluon distribution at small x

$$\boldsymbol{T} \sim e^{\lambda Y} \qquad \longleftrightarrow \qquad xG(x, Q^2) \sim \frac{1}{x^\lambda}$$

Non-linear evolution equation

- In fact, the first evolution equation we derived has a bounded solution. The unbounded solutions of BFKL are due to dropping the non-linear term. The full equation reads :

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

$$\times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \underline{- \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp)} \Big\}$$

(Balitsky-Kovchegov equation)

- The r.h.s. vanishes when $\boldsymbol{T}$ reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $\boldsymbol{T} = 0$ and $\boldsymbol{T} = 1$ are fixed points of this equation

$\boldsymbol{T} = \epsilon$: r.h.s. > 0 $\Rightarrow$ $\boldsymbol{T} = 0$ is unstable

$\boldsymbol{T} = 1 - \epsilon$: r.h.s. > 0 $\Rightarrow$ $\boldsymbol{T} = 1$ is stable

Caveats and improvements

- So far, we have studied the scattering amplitude between a color dipole and a "god given" patch of color field. This is too naive to describe any realistic situation
- We need to improve the treatment of the target
- An experimentally measured cross-section is an average over many collisions, and there is no reason why these fields should be the same in different collisions :

$$\boldsymbol{T} \quad \rightarrow \quad \langle \boldsymbol{T} \rangle$$

$\langle \cdots \rangle$ denotes the average over the target configurations, i.e.

$$\langle \cdots \rangle = \int [D\rho] \; W_Y[\rho] \cdots$$

Balitsky hierarchy

- Because of this average over the target configurations, the evolution equation we have derived should be written as :

$$\frac{\partial \left\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right\rangle}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp \, \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\left\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\right\rangle + \left\langle \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\right\rangle - \left\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right\rangle \underline{- \left\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\right\rangle}\Big\}$$

- As one can see, the equation is no longer a closed equation, since the equation for $\langle \boldsymbol{T}\rangle$ depends on a new object, $\langle \boldsymbol{T}\,\boldsymbol{T}\rangle$
- One can derive an evolution equation for $\langle \boldsymbol{T}\,\boldsymbol{T}\rangle$. Its right hand side contains $\langle \boldsymbol{T}\,\boldsymbol{T}\,\boldsymbol{T}\rangle$. And so on...
- There is in fact an infinite hierarchy of nested evolution equations : Balitsky hierarchy

  Note : this hierarchy is equivalent to the JIMWLK equation for $W_Y[\rho]$

Balitsky-Kovchegov equation

- In order to truncate the hierarchy of equations, one may assume the following simplification

$$\langle \boldsymbol{T}\,\boldsymbol{T}\rangle \approx \langle \boldsymbol{T}\rangle \; \langle \boldsymbol{T}\rangle$$

- This approximation gives for $\langle \boldsymbol{T}\rangle$ the same evolution equation as the one we had for a fixed configuration of the target

Analogy with reaction-diffusion

Munier, Peschanski (2003,2004)

- Assume rotation invariance, and define :

$$N(Y,k_\perp) \equiv 2\pi \int d^2\vec{\boldsymbol{x}}_\perp\; e^{i\vec{\boldsymbol{k}}_\perp\cdot\vec{\boldsymbol{x}}_\perp}\;\frac{\left<\boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right>_Y}{x_\perp^2}$$

- From the Balitsky-Kovchegov equation for $\left<\boldsymbol{T}\right>_Y$, we obtain the following equation for N :

$$\frac{\partial N(Y,k_\perp)}{\partial Y} = \frac{\alpha_s N_c}{\pi}\Big[\chi(-\partial_L)N(Y,k_\perp) - N^2(Y,k_\perp)\Big]$$

with

$$L \equiv \ln(k_\perp^2/k_0^2)$$

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

Analogy with reaction-diffusion

- Expand the function $\chi(\gamma)$ to second order around its minimum $\gamma = 1/2$
- Introduce new variables :

$$t \sim Y$$
$$z \sim L + \frac{\alpha_s N_c}{2\pi}\chi''(1/2)\, Y$$

- The equation for N becomes :

$$\partial_t N = \partial_z^2 N + N - N^2$$

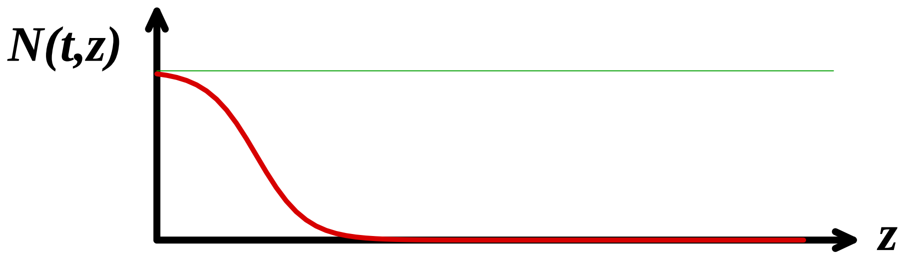
(known as the Fisher-Kolmogorov-Petrov-Piscounov equation)

Analogy with reaction-diffusion

- Interpretation : this equation is typical for all the diffusive systems in which a reaction $A \longleftrightarrow A + A$ takes place
  - $\partial_z^2 N$: diffusion term (the quantity under consideration can hop from a site to the neighboring sites)
  - $+N$: gain term corresponding to $A \to A + A$
  - $-N^2$: loss term corresponding to $A + A \to A$
- Note : this equation has two fixed points :
  - $N = 0$: unstable
  - $N = 1$: stable
- The stable fixed point at $N = 1$ exists only if one keeps the loss term. In other words, one would not have it from the BFKL equation

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

$$N(t,z) \underset{t\to+\infty}{\sim} N(z - m_\beta(t))$$

with $\quad m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1) \quad$ if $\quad \beta > 1$

▷ universal front velocity

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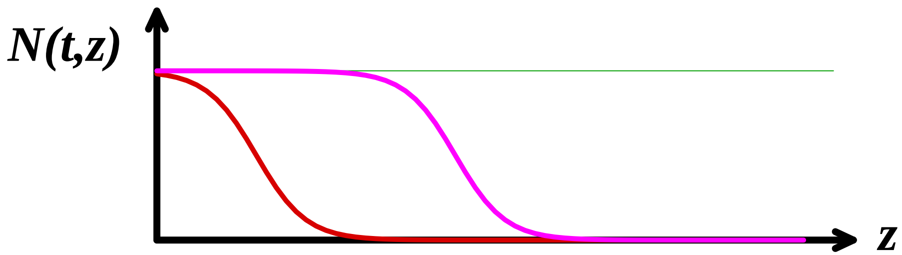

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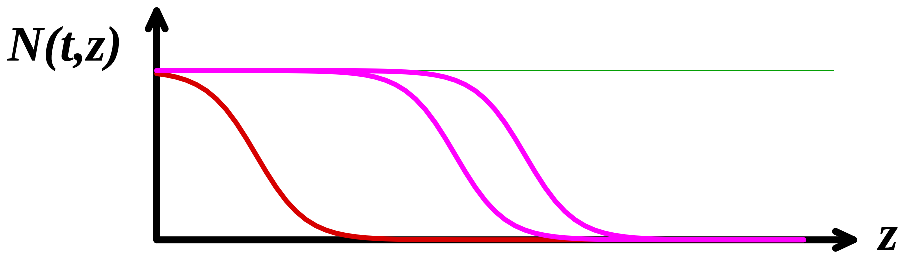

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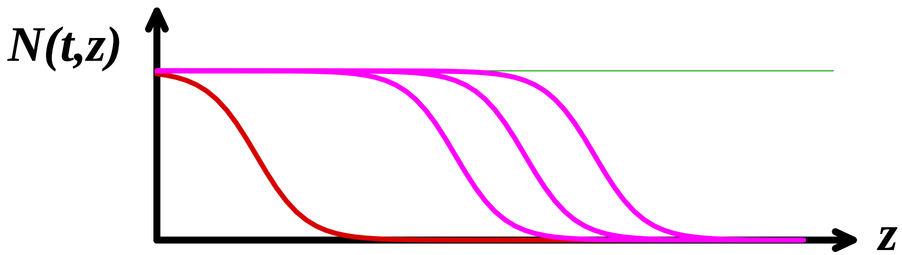

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▷ universal front velocity

Geometrical scaling in DIS

Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)
Munier, Peschanski (2003)

- In QCD, the initial condition is of the required form, with $\beta > 1$
  ▷ front velocity independent of the initial condition

- Going back to the original variables, one gets :

$$N(Y, k_\perp) = N\left(k_\perp / Q_s(Y)\right)$$

with

$$Q_s^2(Y) = k_0^2 \; Y^{-\delta} \; e^{\lambda Y}$$

- Going from $N(Y, k_\perp)$ to $\left<\boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right>_Y$, we obtain :

$$\left<\boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right>_Y = T(Q_s(Y) x_\perp)$$

Geometrical scaling in DIS

- The total $\gamma^* p$ cross-section, measured in Deep Inelastic Scattering, can be written in terms of N:

$$\sigma^{\rm tot}_{\gamma^* p}(Y, Q^2) = 2\pi R^2 \int d^2\vec{\boldsymbol{x}}_\perp \int_0^1 dz \, \big|\psi(z, x_\perp, Q^2)\big|^2 \, \langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\rangle_Y$$

  - The photon wavefunction ψ is calculable in QED. It depends on the dipole size $x_\perp$ only via

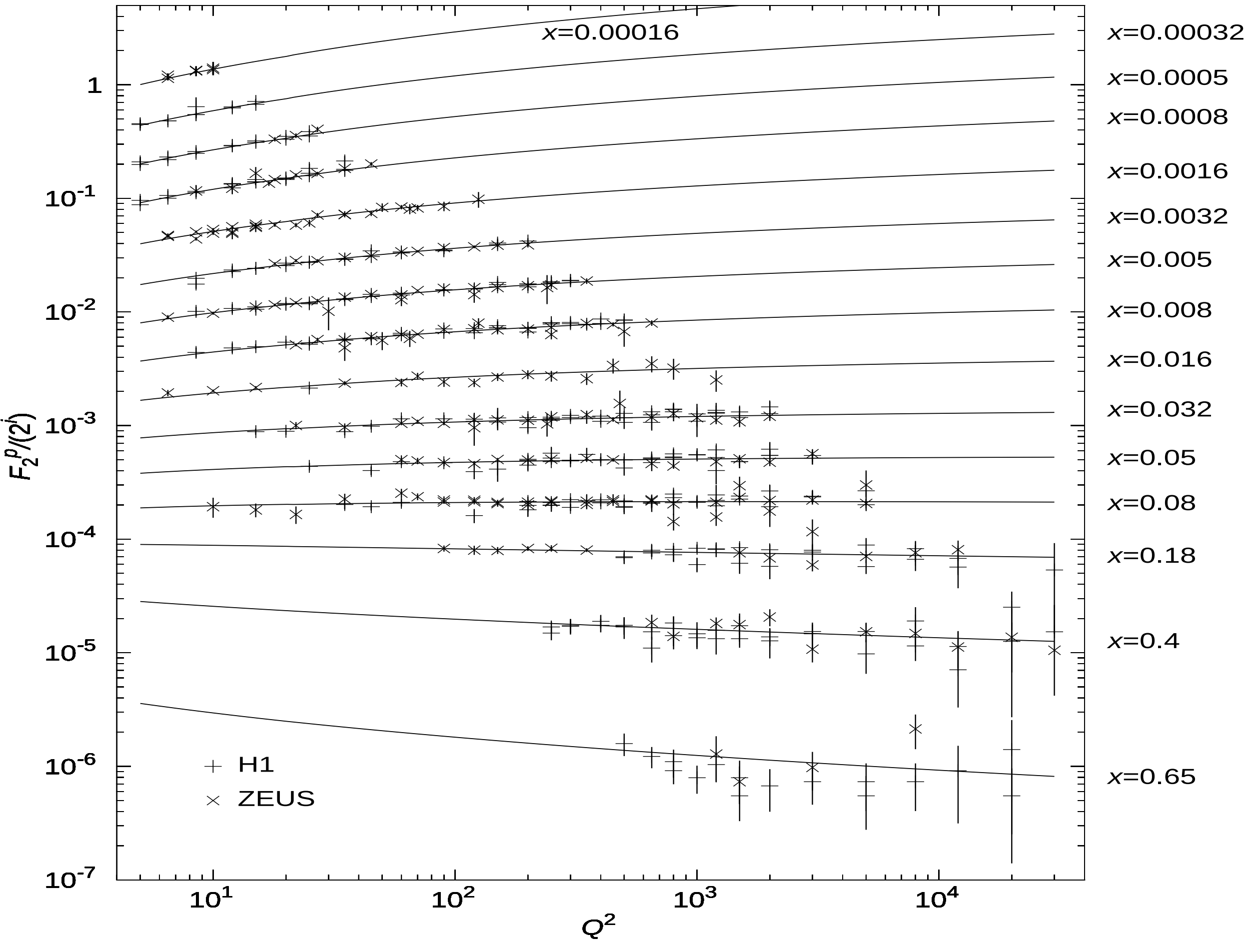
$$\big|\psi(z, x_\perp, Q^2)\big|^2 = f(\overline{Q}_f x_\perp)$$

with $\overline{Q}_f^2 \equiv m_f^2 + Q^2 z^2 (1 - z^2)$

- If one neglects the quark masses, the scaling properties of $\langle \boldsymbol{T}\rangle_Y$ imply that $\sigma_{\gamma^* p}$ depends only on the ratio $Q^2/Q_s^2(Y)$, rather than on Q^2 and Y separately

Geometrical scaling in DIS

- HERA data as a function of Q^2 and x :



Geometrical scaling in DIS

Stasto, Golec-Biernat, Kwiecinski (2000)