High energy scattering in QCD

II – Parton evolution at small x, Saturation

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General outline

- Lecture I : Parton model, Bjorken scaling, Scaling violations
- Lecture II : Parton evolution at small x, Saturation
- Lecture III : Hadron-hadron collisions in the CGC framework

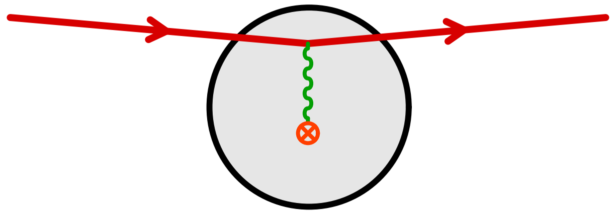
Lecture II

- Eikonal scattering
- BFKL equation
- Saturation of parton distributions
- Balitsky-Kovchegov equation
- Color Glass Condensate - JIMWLK
- Analogies with reaction-diffusion processes

Eikonal scattering

Goal

- Consider the scattering amplitude off an external potential :



$$S_{\beta\alpha} \equiv \langle \beta_{\text{out}} | \alpha_{\text{in}} \rangle = \langle \beta_{\text{in}} | U(+\infty, -\infty) | \alpha_{\text{in}} \rangle$$

where $U(+\infty, -\infty)$ is the evolution operator from $t = -\infty$ to $t = +\infty$

$$U(+\infty, -\infty) = T \, \exp\left[i \int d^4x \, \mathcal{L}_{\text{int}}(\phi_{\text{in}}(x)) \right]$$

Note : $\mathcal{L}_{\text{int}}$ contains the self-interactions of the fields and their interactions with the external potential

- We want to calculate its high energy limit (eikonal limit):

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega \to +\infty} \langle \beta_{\text{in}} | e^{i\omega K^3} U(+\infty, -\infty) e^{-i\omega K^3} | \alpha_{\text{in}} \rangle$$

where K^3 is the generator of boosts in the $+z$ direction

Eikonal scattering in a nutshell

- In a scattering at high energy, the collision time goes to zero as $s^{-1/2}$
- With scalar interactions, this implies a decrease of the scattering amplitude as $s^{-1/2}$
- With vectorial interactions, this decrease is compensated by the growth of the component J^+ of the vector current

$\triangleright$ the eikonal approximation gives the finite limit of the scattering amplitude in the case of vectorial interactions when $s \to +\infty$

Light-cone coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$a^{1,2}$ unchanged. Notation : $\vec{\boldsymbol{a}}_\perp \equiv (a^1, a^2)$

which can be inverted by

$$a^0 = \frac{a^+ + a^-}{\sqrt{2}} \quad , \quad a^3 = \frac{a^+ - a^-}{\sqrt{2}}$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{\boldsymbol{x}}_\perp \cdot \vec{\boldsymbol{y}}_\perp$$

$$d^4x = dx^+ dx^- d^2\vec{\boldsymbol{x}}_\perp$$

$$\square = 2\partial^+\partial^- - \vec{\boldsymbol{\nabla}}_\perp^2 \qquad \text{Notation : } \partial^+ \equiv \frac{\partial}{\partial x^-} \, , \, \partial^- \equiv \frac{\partial}{\partial x^+}$$

Eikonal limit

- Consider an external vector potential, that couples via $e\,\mathcal{A}_\mu(x)J^\mu(x)$ (J^μ is the Noether current associated to some conserved charge)
- We will assume that the external potential is non-zero only in a finite range in x^+, $x^+ \in [-L,+L]$
- The action of K^3 on states and operators is :

$$e^{-i\omega K^3} a^\dagger_{\rm in}(q) e^{i\omega K^3} = a^\dagger_{\rm in}(e^\omega q^+, e^{-\omega} q^-, \vec{\boldsymbol{q}}_\perp)$$
$$e^{-i\omega K^3} \big|\vec{\boldsymbol{p}} \cdots {\rm in}\big\rangle = \big|(e^\omega p^+, \vec{\boldsymbol{p}}_\perp) \cdots {\rm in}\big\rangle$$
$$e^{i\omega K^3} \phi_{\rm in}(x) e^{-i\omega K^3} = \phi_{\rm in}(e^{-\omega} x^+, e^\omega x^-, \vec{\boldsymbol{x}}_\perp)$$

- K^3 does not change the ordering in x^+. Hence,

$$e^{i\omega K^3} U(+\infty,-\infty) e^{-i\omega K^3} = T_+ \exp i \int d^4x\, \mathcal{L}_{\rm int}(e^{i\omega K^3}\phi_{\rm in}(x)e^{-i\omega K^3})$$

where $\mathcal{L}_{\rm int} = \mathcal{L}_{\rm self}(\phi) - e\,\mathcal{A}_\mu J^\mu$

Eikonal limit

- Split the S matrix $U(+\infty,-\infty)$ into three factors :

$$U(+\infty,-\infty)=U(+\infty,+L)U(+L,-L)U(-L,-\infty)$$

Upon application of K^3, this becomes :

$$\begin{aligned}e^{i\omega K^3}U(+\infty,-\infty)e^{-i\omega K^3} &= e^{i\omega K^3}U(+\infty,+L)e^{-i\omega K^3}\\ &\times e^{i\omega K^3}U(+L,-L)e^{-i\omega K^3}e^{i\omega K^3}U(-L,-\infty)e^{-i\omega K^3}\end{aligned}$$

- The external potential $\mathcal{A}_\mu(x)$ is unaffected by K^3
- The components of $J^\mu(x)$ are changed as follows :

$$\begin{aligned}e^{i\omega K^3}J^i(x)e^{-i\omega K^3} &= J^i(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)\\ e^{i\omega K^3}J^-(x)e^{-i\omega K^3} &= e^{-\omega}\,J^-(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)\\ e^{i\omega K^3}J^+(x)e^{-i\omega K^3} &= e^{\omega}\,J^+(e^{-\omega}x^+,e^{\omega}x^-,\vec{\boldsymbol{x}}_\perp)\end{aligned}$$

Eikonal limit

- The factors $U(+\infty,+L)$ and $U(-L,-\infty)$ do not contain the external potential. In order to deal with these factors, it is sufficient to change variables : $e^{-\omega}x^+ \to x^+$, $e^{\omega}x^- \to x^-$. This leads to :

$$\lim_{\omega\to+\infty} e^{i\omega K^3}U(+\infty,+L)e^{-i\omega K^3} = U_0(+\infty,0)$$

$$\lim_{\omega\to+\infty} e^{i\omega K^3}U(-L,-\infty)e^{-i\omega K^3} = U_0(0,-\infty)$$

where U_0 is the same as U, but with the self-interactions only

- For the factor $U(L,-L)$, the change $e^{\omega}x^- \to x^-$ leads to :

$$e^{i\omega K^3}U(+L,-L)e^{-i\omega K^3} = \\ = T_+ \exp i\int_{-L}^{+L} d^4x\; e^{-\omega}\Big[e\,\mathcal{A}^-(x^+,e^{-\omega}x^-,\vec{\boldsymbol{x}}_\perp) \\ \times e^{\omega}J^+(e^{-\omega}x^+,x^-,\vec{\boldsymbol{x}}_\perp)+\mathcal{O}(1)\Big]$$

Eikonal limit

- Therefore, in the limit $\omega \to +\infty$, we have :

$$\lim_{\omega\to+\infty} e^{i\omega K^3} U(+L,-L) e^{-i\omega K^3} = \exp\left[ie\int d^2\vec{\boldsymbol{x}}_\perp \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right]$$

with

$$\begin{cases} \chi(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^+ \ \mathcal{A}^-(x^+,0,\vec{\boldsymbol{x}}_\perp) \\ \rho(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^- \ J^+(0,x^-,\vec{\boldsymbol{x}}_\perp) \end{cases}$$

- The high-energy limit of the scattering amplitude is :

$$S^{(\infty)}_{\beta\alpha} = \left\langle \beta_{\rm in} \middle| U_0(+\infty,0)\ \exp\left[ie\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right] U_0(0,-\infty) \middle| \alpha_{\rm in}\right\rangle$$

  - Only the $-$ component of the vector potential matters
  - The self-interactions and the interactions with the external potential are factorized in U_0 $\triangleright$ parton model
  - This is an exact result

Eikonal limit

- The previous formula still contains all the self-interactions of the fields. In order to perform the perturbative expansion, it is convenient to write first :

$$S^{(\infty)}_{\beta\alpha} = \sum_{\gamma,\delta} \big\langle \beta_{\rm in} \big| U_0(+\infty,0) \big| \gamma_{\rm in} \big\rangle \times \big\langle \gamma_{\rm in} \big| \exp\left[ie \int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp) \right] \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

- The factor

$$\sum_{\delta} \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

is the Fock expansion of the initial state: the state prepared at $x^+ = -\infty$ may have fluctuated into another state before it interacts with the external potential

Eikonal limit

- We need to calculate matrix elements such as $\left\langle\gamma_{\rm in}\middle|\boldsymbol{F}\middle|\delta_{\rm in}\right\rangle$, with :

$$\boldsymbol{F}\equiv \exp\, ie\int \chi_a(\vec{\boldsymbol{x}}_\perp)\rho^a(\vec{\boldsymbol{x}}_\perp)$$

  - having QCD in mind, we have reinstated the color indices
  - the contribution of quarks and antiquarks to $\rho^a(\vec{\boldsymbol{x}}_\perp)$ is :

$$\rho^a(\vec{\boldsymbol{x}}_\perp)=t^a_{ij}\int\frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\frac{d^2\vec{\boldsymbol{q}}_\perp}{(2\pi)^2}\Big\{b^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)b_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}$$
$$-d^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp;i)d_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp;j)e^{-i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\Big\}$$

  - Note : one should keep the ordering of the exponential in x^+
  - the contribution of gluons is similar, with a color matrix in the adjoint representation

- The action of $\boldsymbol{F}$ on a state $\left|\delta_{\rm in}\right\rangle$ gives a state with the same particle content, the same $+$ components for the momenta, but modified transverse momenta

Eikonal limit

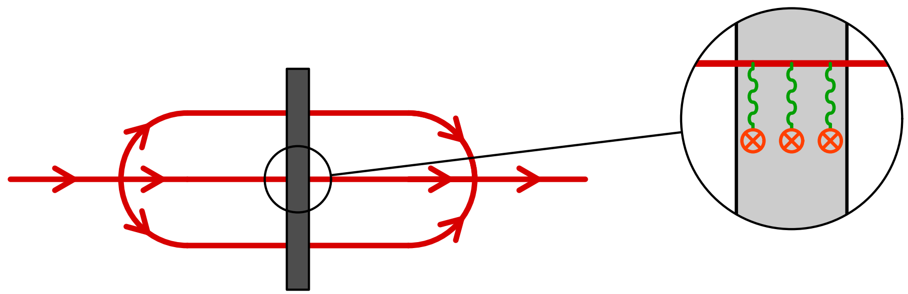
- For each intermediate state $\langle\delta_{\rm in}\big| \equiv \big\langle\{k_i^+,\vec{\boldsymbol{k}}_{i\perp}\}\big|$, define the corresponding light-cone wave function by :

$$\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \equiv \prod_i \int \frac{d^2\vec{\boldsymbol{k}}_{i\perp}}{(2\pi)^2} e^{-i\vec{\boldsymbol{k}}_{i\perp}\cdot\vec{\boldsymbol{x}}_{i\perp}} \big\langle\delta_{\rm in}\big|U(0,-\infty)\big|\alpha_{\rm in}\big\rangle$$

- Each charged particle going through the external field acquires a phase proportional to its charge (antiparticles get an opposite phase) :

$$\Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \longrightarrow \Psi_{\delta\alpha}(\{k_i^+,\vec{\boldsymbol{x}}_{i\perp}\}) \prod_i U_i(\vec{\boldsymbol{x}}_\perp)$$

$$U_i(\vec{\boldsymbol{x}}_\perp) \equiv T_+ \exp\left[ig_i \int dx^+ \; \mathcal{A}_a^-(x^+,0,\vec{\boldsymbol{x}}_\perp)t^a\right]$$



Eikonal limit

- We have seen that the number and the nature of the particles is unchanged under the action of the operator $\boldsymbol{F}$. Moreover, in terms of the transverse coordinates, we simply have

$$\left\langle \gamma_{\rm in} \middle| \boldsymbol{F} \middle| \delta_{\rm in} \right\rangle = \delta_{NN'} \prod_i \left[4\pi k_i^+ \delta(k_i^+ - k_i^{+\prime}) \delta(\vec{\boldsymbol{x}}_{i\perp} - \vec{\boldsymbol{x}}'_{i\perp}) U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right]$$

where $U_R(\vec{\boldsymbol{x}}_\perp)$ is a Wilson line operator, in the representation R appropriate for the particle going through the target

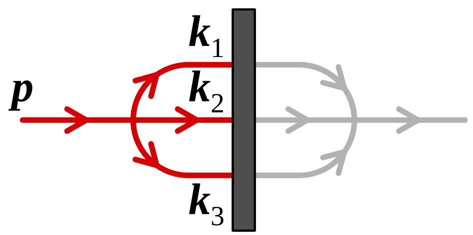
- Therefore, the high energy scattering amplitude can be written as :

$$S_{\beta\alpha}^{(\infty)} = \sum_\delta \int \left[\prod_{i\in\delta} d\Phi_i \right] \Psi_{\delta\beta}^\dagger(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\}) \left[\prod_{i\in\delta} U_{R_i}(\vec{\boldsymbol{x}}_{i\perp}) \right] \Psi_{\delta\alpha}(\{k_i^+, \vec{\boldsymbol{x}}_{i\perp}\})$$

Light-cone wave function

- The calculation of $\left\langle\delta_{\rm in}\middle|U_0(0,-\infty)\middle|\alpha_{\rm in}\right\rangle$ is similar to that of scattering amplitudes in the vacuum. The only difference is that the integration over x^+ at each vertex runs only over half of the real axis $[-\infty, 0]$
  - In Fourier space, this means that the $-$ component of the momentum is not conserved at the vertices
  - Instead of a δ function, one gets an energy denominator

- Example with a single interaction :



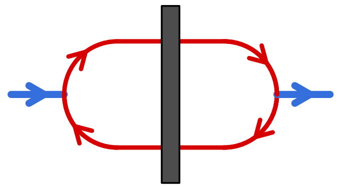
$$\left\langle\vec{\boldsymbol{k}}_1\vec{\boldsymbol{k}}_2\vec{\boldsymbol{k}}_{3\,\rm in}\middle|U(0,-\infty)\middle|\vec{\boldsymbol{p}}_{\rm in}\right\rangle = -ig\int_{-\infty}^{0} d^4x\; e^{i(k_1+k_2+k_3-p)\cdot x}$$

$$= -g\,\frac{(2\pi)^3\delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}+\vec{\boldsymbol{k}}_{3\perp}-\vec{\boldsymbol{p}}_{\perp})\delta(k_1^++k_2^++k_3^+-p^+)}{k_1^-+k_2^-+k_3^--p^--i\epsilon}$$

BFKL equation (and a bit more)

Scattering of a dipole

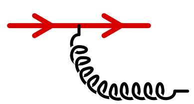
- Assume that the initial and final states α and β are a color singlet $Q\overline{Q}$ dipole. The simplest Fock state that contributes to their wave function is a $Q\overline{Q}$ pair, and the bare scattering amplitude can be written as :



$$\propto \Psi^{(0)*}_{ij}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\Psi^{(0)}_{kl}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)U_{ik}(\vec{\boldsymbol{x}}_\perp)U^\dagger_{lj}(\vec{\boldsymbol{y}}_\perp)$$

$$\propto \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

- It turns out that 1-loop corrections to this contribution are enhanced by $\alpha_s \log(p^+)$, which can be large when the quark or antiquark has a large p^+
- In the gauge $A^+=0$, the emission of a gluon of momentum k by a quark can be written as :



$$= 2gt^a\,\frac{\vec{\boldsymbol{\epsilon}}_\lambda\cdot\vec{\boldsymbol{k}}_\perp}{k_\perp^2}$$

Scattering of a dipole

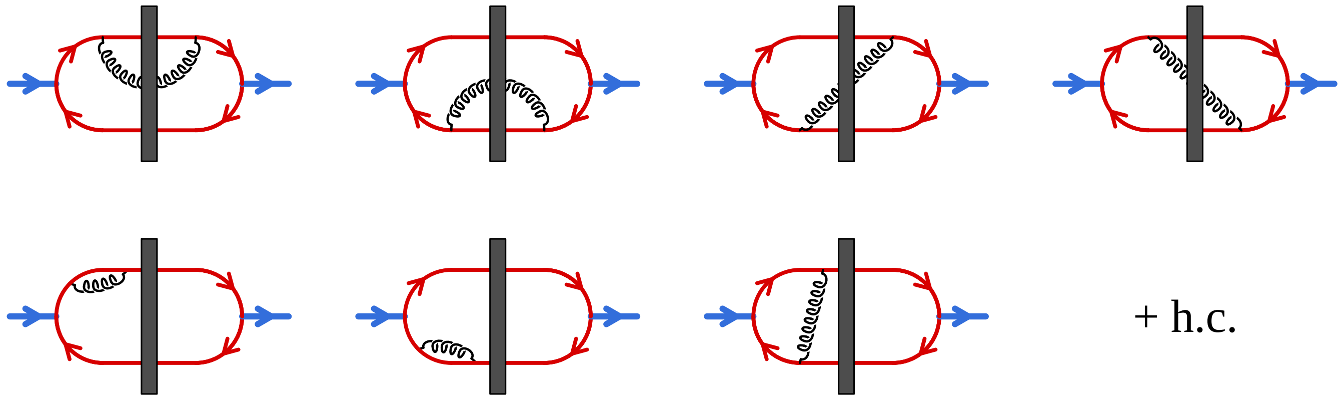
Eikonal scattering
BFKL equation (and a bit more)
- Scattering of a dipole
- Virtual corrections
- Real corrections
- Evolution equation
- BFKL equation
- Unitarity problem
- Parton evolution

Parton saturation
Balitsky-Kovchegov equation
Color Glass Condensate
Reaction-diffusion processes

- In coordinate space, this reads :

$$\int \frac{d^2\vec{\boldsymbol{k}}_\perp}{(2\pi)^2}\, e^{i\vec{\boldsymbol{k}}_\perp\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}\, 2gt^a\, \frac{\vec{\boldsymbol{\epsilon}}_\lambda\cdot\vec{\boldsymbol{k}}_\perp}{k_\perp^2} = \frac{2ig}{2\pi}t^a\,\frac{\vec{\boldsymbol{\epsilon}}_\lambda\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

- The following diagrams must be evaluated :



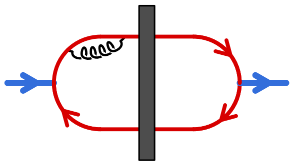
+ h.c.

- When connecting two gluons, one must use :

$$\sum_\lambda \vec{\boldsymbol{\epsilon}}_\lambda^i\vec{\boldsymbol{\epsilon}}_\lambda^j = -g^{ij}$$

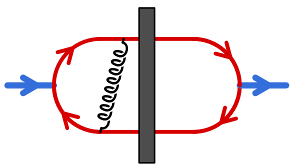
Virtual corrections

- Consider first the loop corrections inside the wavefunction of the incoming or outgoing dipole
- Examples :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a t^a U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times -2\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)t^a\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$

- Reminder : $t^a t^a = (N_c^2-1)/2N_c \equiv C_F$

Virtual corrections

- The sum of all virtual corrections is :

$$-\frac{C_F \alpha_s}{\pi^2} \int \frac{dk^+}{k^+} \int d^2\vec{z}_\perp \, \frac{(\vec{x}_\perp - \vec{y}_\perp)^2}{(\vec{x}_\perp - \vec{z}_\perp)^2 (\vec{y}_\perp - \vec{z}_\perp)^2}$$

$$\times \left| \Psi^{(0)}(\vec{x}_\perp, \vec{y}_\perp) \right|^2 \mathrm{tr} \left[U(\vec{x}_\perp) U^\dagger(\vec{y}_\perp) \right]$$

- The integral over k^+ is divergent. It should have an upper bound at p^+ :

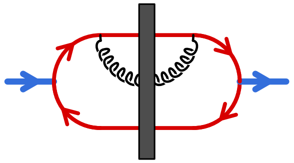
$$\int^{p^+} \frac{dk^+}{k^+} = \ln(p^+) = Y$$

$\triangleright$ When Y is large, $\alpha_s Y$ may not be small. By differentiating with respect to Y, we will get an evolution equation in Y whose solution resums all the powers $(\alpha_s Y)^n$

- The integral over $\vec{z}_\perp$ is divergent when $\vec{z}_\perp = \vec{x}_\perp$ or $\vec{y}_\perp$

Real corrections

- There are also real corrections, for which the state that interacts with the target has an extra gluon
- Example :



$$= \left|\Psi^{(0)}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)\right|^2 \mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

$$\times 4\alpha_s \int \frac{dk^+}{k^+} \int \frac{d^2\vec{\boldsymbol{z}}_\perp}{(2\pi)^2} \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

  - $\widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp)$ is a Wilson line in the adjoint representation
- In order to simplify the color structure, first notice that :

$$t^a \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = U(\vec{\boldsymbol{z}}_\perp) t^b U^\dagger(\vec{\boldsymbol{z}}_\perp)$$

- Then use the $SU(N_c)$ Fierz identity :

$$t^b_{ij} t^b_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2N_c}\delta_{ij}\delta kl$$

Real corrections

- The Wilson lines can be rearranged into :

$$\mathrm{tr}\left[t^a U(\vec{\boldsymbol{x}}_\perp) t^b U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] \widetilde{U}_{ab}(\vec{\boldsymbol{z}}_\perp) = \frac{1}{2}\mathrm{tr}\left[U^\dagger(\vec{\boldsymbol{z}}_\perp)U(\vec{\boldsymbol{x}}_\perp)\right]\mathrm{tr}\left[U(\vec{\boldsymbol{z}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right] - \frac{1}{2N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

  - The term in $1/2N_c$ cancels against a similar term in the virtual contribution
  - All the real terms have the same color structure

- When we sum all the real terms, we generate the same kernel as in the virtual terms :

$$\frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

- In order to simplify the notations, let us denote :

$$\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv \frac{1}{N_c}\mathrm{tr}\left[U(\vec{\boldsymbol{x}}_\perp)U^\dagger(\vec{\boldsymbol{y}}_\perp)\right]$$

Evolution equation

- The 1-loop scattering amplitude reads :

$$-\frac{\alpha_s N_c^2 Y}{2\pi^2}\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 \int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{S(\vec{x}_\perp,\vec{y}_\perp)-S(\vec{x}_\perp,\vec{z}_\perp)S(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

- Reminder: the bare scattering amplitude was :

$$\left|\Psi^{(0)}(\vec{x}_\perp,\vec{y}_\perp)\right|^2 N_c\, S(\vec{x}_\perp,\vec{y}_\perp)$$

- Hence, we have :

$$\frac{\partial S(\vec{x}_\perp,\vec{y}_\perp)}{\partial Y} = -\frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{z}_\perp \; \frac{(\vec{x}_\perp-\vec{y}_\perp)^2}{(\vec{x}_\perp-\vec{z}_\perp)^2(\vec{y}_\perp-\vec{z}_\perp)^2}$$
$$\times\Big\{S(\vec{x}_\perp,\vec{y}_\perp)-S(\vec{x}_\perp,\vec{z}_\perp)S(\vec{z}_\perp,\vec{y}_\perp)\Big\}$$

  - since $S(\vec{x}_\perp,\vec{x}_\perp)=1$, the integral over $\vec{z}_\perp$ is now regular

BFKL equation

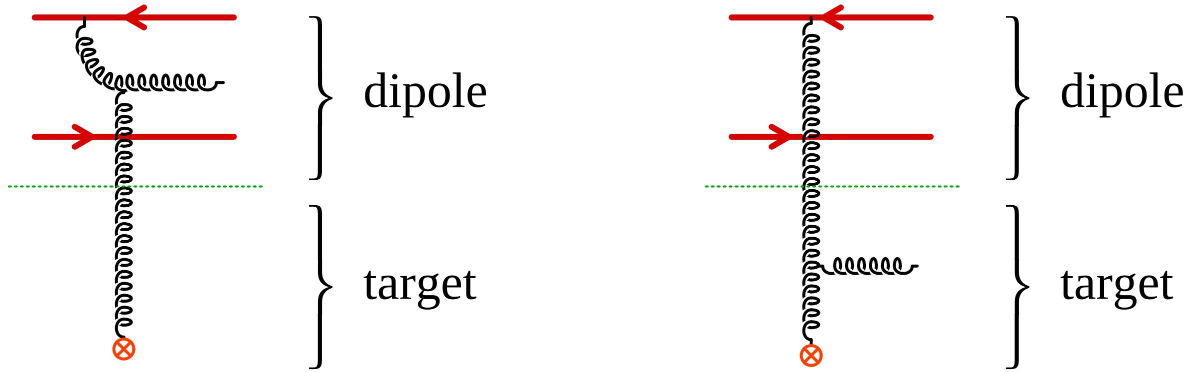
Kuraev, Lipatov, Fadin (1977), Balitsky, Lipatov (1978)

- Actually, we've got more than we need : we must simplify this equation in order to obtain the BFKL equation...
- Write $\boldsymbol{S}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \equiv 1 - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)$ and assume that we are in the dilute regime, so that the scattering amplitude $\boldsymbol{T}$ is small. Drop the terms that are non-linear in $\boldsymbol{T}$:

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \, \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$
$$\times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \Big\}$$

BFKL equation

- Note : $\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)$ is independent on the frame. In particular, it depends only on the rapidity difference between the dipole and the target
  ▷ in a frame where the dipole is held fixed, the target has to evolve in such a way as to reproduce the Y dependence of $\boldsymbol{T}$



- The corresponding evolution in the target is the radiation of a gluon

Unitarity problem

- The solution of this equation grows exponentially when $Y \to +\infty$ $\quad \triangleright$ serious unitarity problem...
- In perturbation theory, the forward scattering amplitude between a small dipole and a target made of gluons reads :

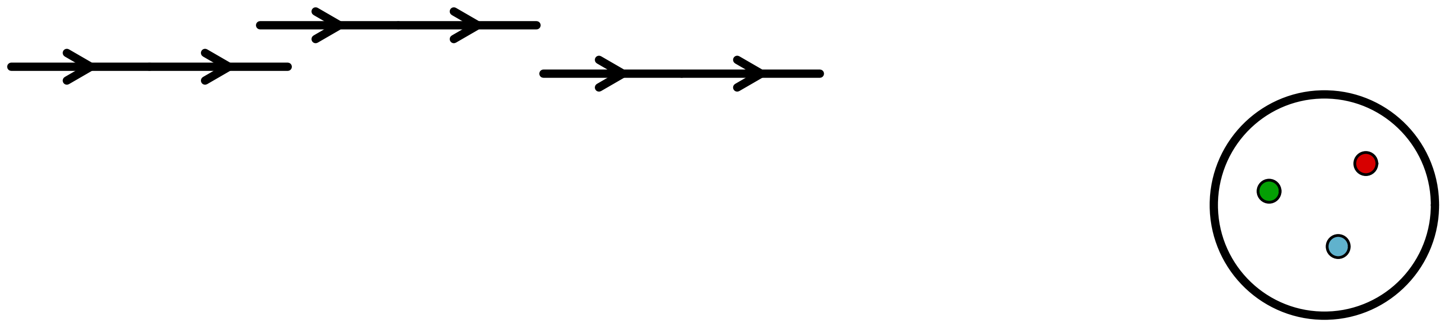
$$\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \propto |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp|^2 \, xG(x, |\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp|^{-2})$$

where $\quad Y \equiv \ln(1/x)$

- Therefore, the exponential behavior of $\boldsymbol{T}$ implies an increase of the gluon distribution at small x

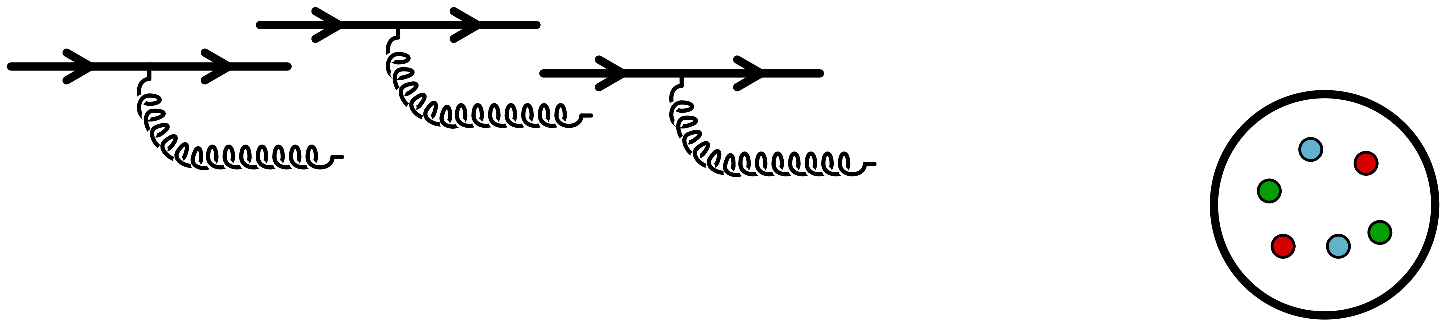
$$\boldsymbol{T} \sim e^{\omega Y} \qquad \longleftrightarrow \qquad xG(x, Q^2) \sim \frac{1}{x^\delta}$$

Parton evolution under boosts



▷ at low energy, only valence quarks are present in the hadron wave function

Parton evolution under boosts

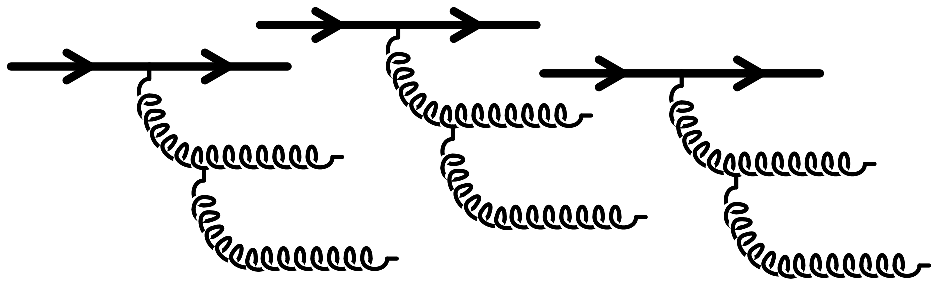


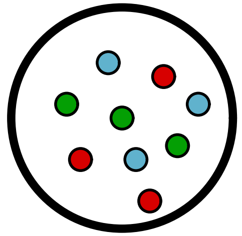
▷ when energy increases, new partons are emitted

▷ the emission probability is $\alpha_s \int \frac{dx}{x} \sim \alpha_s \ln(\frac{1}{x})$, with x the longitudinal momentum fraction of the gluon

▷ at small-x (i.e. high energy), these logs need to be resummed

Parton evolution under boosts

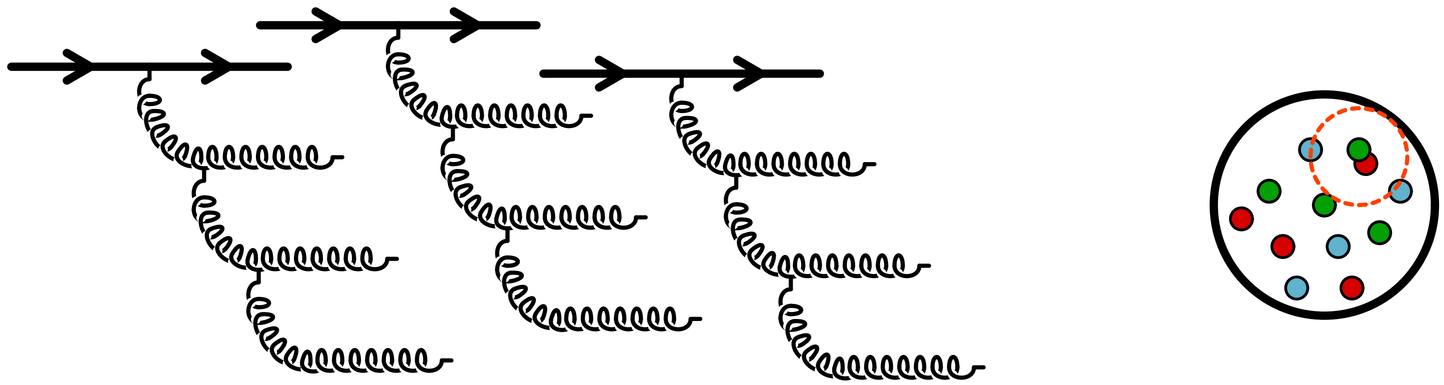




▷ as long as the density of constituents remains small, the evolution is linear: the number of partons produced at a given step is proportional to the number of partons at the previous step

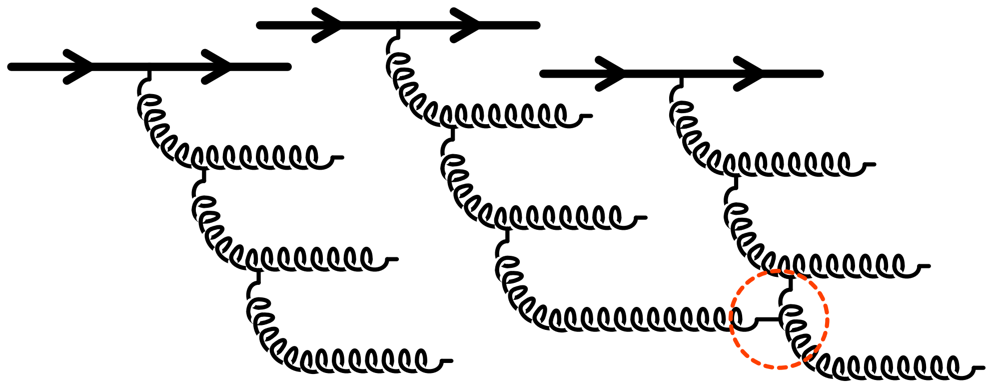
Parton saturation

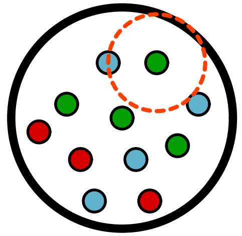
Parton recombination



▷ eventually, the partons start overlapping in phase-space

Parton recombination





▷ parton recombination becomes favorable

▷ after this point, the evolution is non-linear:

the number of partons created at a given step depends non-linearly on the number of partons present previously

Balitsky (1996), Kovchegov (1996,2000)

Jalilian-Marian, Kovner, Leonidov, Weigert (1997,1999)

Iancu, Leonidov, McLerran (2001)

Saturation criterion

Gribov, Levin, Ryskin (1983), Mueller, Qiu (1986)

- Number of partons per unit area:

$$\rho \sim \frac{xG(x,Q^2)}{\pi R^2}$$

- Recombination cross-section:

$$\sigma_{gg\to g} \sim \frac{\alpha_s}{Q^2}$$

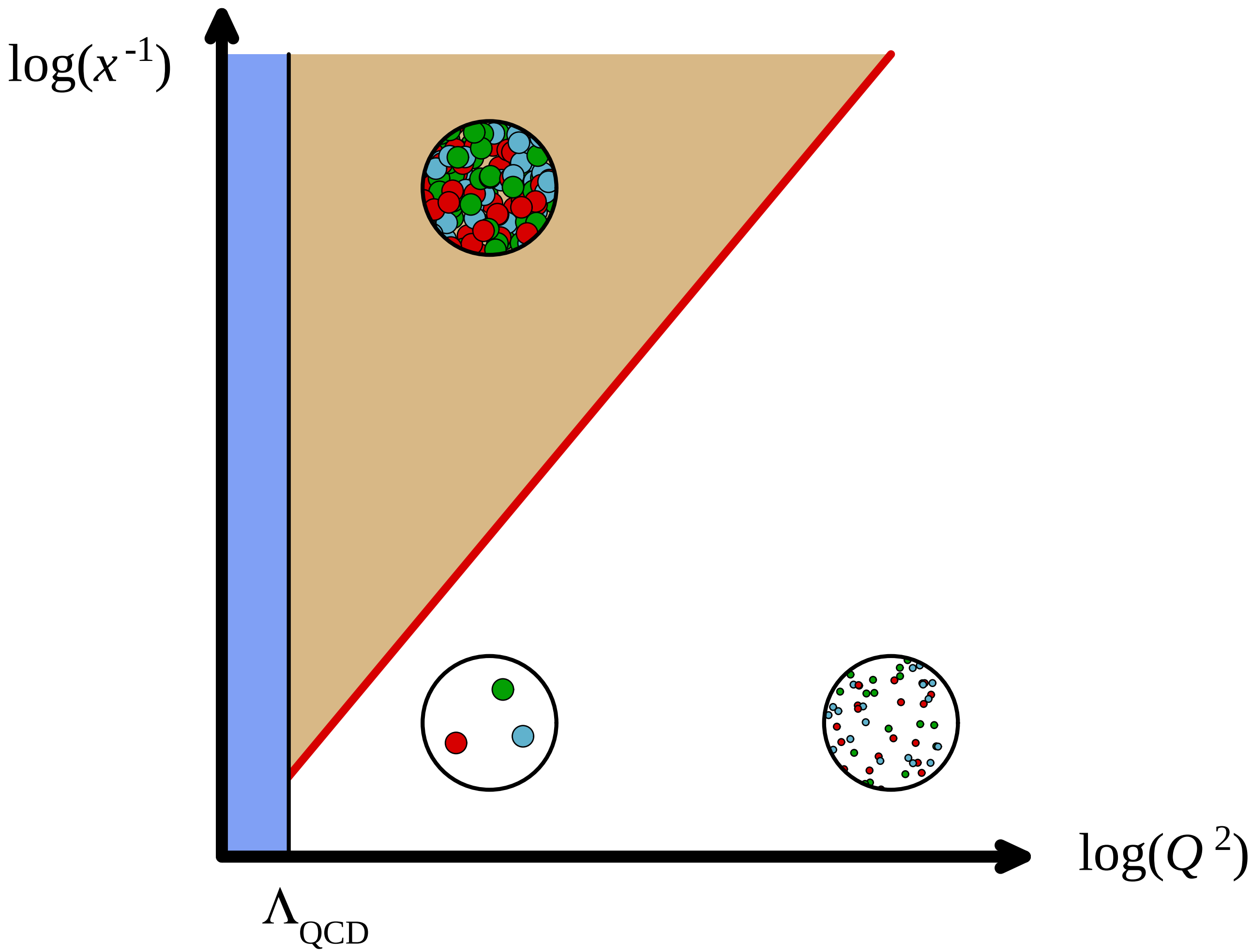
- Recombination if $\rho\sigma_{gg\to g} \gtrsim 1$, or $Q^2 \lesssim Q_s^2$, with:

$$Q_s^2 \sim \frac{\alpha_s xG(x,Q_s^2)}{\pi R^2} \quad \sim A^{1/3}\frac{1}{x^{0.3}}$$

- At saturation, the gluon phase-space density is:

$$\frac{dN_g}{d^2\vec{\boldsymbol{x}}_\perp d^2\vec{\boldsymbol{p}}_\perp} \sim \frac{\rho}{Q^2} \sim \frac{1}{\alpha_s}$$

Saturation domain



- Boundary defined by $Q^2 = Q_s^2(x)$

Balitsky-Kovchegov equation

Non-linear evolution equation

- In fact, the first evolution equation we derived has a bounded solution. The BFKL equation has unbounded solutions because it is an approximation in which a term quadratic in $\boldsymbol{T}$ has been neglected. The full equation reads :

$$\frac{\partial \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp)}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2} \int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{z}}_\perp)^2 (\vec{\boldsymbol{y}}_\perp - \vec{\boldsymbol{z}}_\perp)^2}$$

$$\times \Big\{ \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp) + \boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) - \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp, \vec{\boldsymbol{y}}_\perp) \Big\}$$

  (Balitsky-Kovchegov equation)

- The r.h.s. vanishes when $\boldsymbol{T}$ reaches 1, and the growth stops. The non-linear term lets both dipoles interact after the splitting of the original dipole

- Both $\boldsymbol{T} = 0$ and $\boldsymbol{T} = 1$ are fixed points of this equation
  - $\boldsymbol{T} = \epsilon$: r.h.s.> 0 $\Rightarrow$ $\boldsymbol{T} = 0$ is unstable
  - $\boldsymbol{T} = 1 - \epsilon$: r.h.s.> 0 $\Rightarrow$ $\boldsymbol{T} = 1$ is stable

Caveats

- So far, we have studied the scattering amplitude between a color dipole and a “god given” patch of color field. This is too crude to describe any realistic situation

- One can describe Deep Inelastic Scattering as an interaction between a dipole and the proton, but for that we need to improve the treatment of the target

- At high energy, the duration of the interaction between the dipole and the proton is short. Therefore, it is legitimate to treat the proton as a frozen configuration of color fields. But an experimentally measured cross-section is an average over many collisions, and there is no reason why these fields should be the same in different collisions :

$$\boldsymbol{T} \quad \longrightarrow \quad \langle \boldsymbol{T} \rangle$$

Balitsky hierarchy

- Because of this average over the target configurations, the evolution equation we have derived should be written as :

$$\frac{\partial \left<\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right>}{\partial Y} = \frac{\alpha_s N_c}{2\pi^2}\int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{y}}_\perp)^2}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2}$$
$$\times\Big\{\left<\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\right>+\left<\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\right>-\left<\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\right>-\left<\boldsymbol{T}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{z}}_\perp)\boldsymbol{T}(\vec{\boldsymbol{z}}_\perp,\vec{\boldsymbol{y}}_\perp)\right>\Big\}$$

- As one can see, the equation is no longer a closed equation, since the equation for $\left<\boldsymbol{T}\right>$ depends on a new object, $\left<\boldsymbol{T}\,\boldsymbol{T}\right>$
- One can derive an evolution equation for $\left<\boldsymbol{T}\,\boldsymbol{T}\right>$. Its right hand side contains objects with six Wilson lines
  - Unlike what happened previously, this combination of six Wilson lines simplifies into dipolar operators only in the large N_c limit
  - There is in fact an infinite hierarchy of nested evolution equations, whose generic structure is

$$\frac{\partial \left<(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\right>}{\partial Y} = \int\cdots\left<(\boldsymbol{U}\boldsymbol{U}^\dagger)^n\right>\oplus\left<(\boldsymbol{U}\boldsymbol{U}^\dagger)^{n+1}\right>$$

Balitsky-Kovchegov equation

- If one performs the large N_c approximation on all the equations of the Balitsky hierarchy, they can be rewritten in terms of the dipole operator $\boldsymbol{T} \equiv \mathrm{tr}(UU^\dagger)$ only. But they still contain averages like $\langle \boldsymbol{T}^n \rangle$

- In order to truncate the hierarchy of equations, one may assume that

$$\langle \boldsymbol{T}\,\boldsymbol{T} \rangle \approx \langle \boldsymbol{T} \rangle \; \langle \boldsymbol{T} \rangle$$

- This approximation gives for $\langle \boldsymbol{T} \rangle$ the same evolution equation as the one we had for a fixed configuration of the target

- Moreover, it was shown by Janik that if the initial condition is factorized :

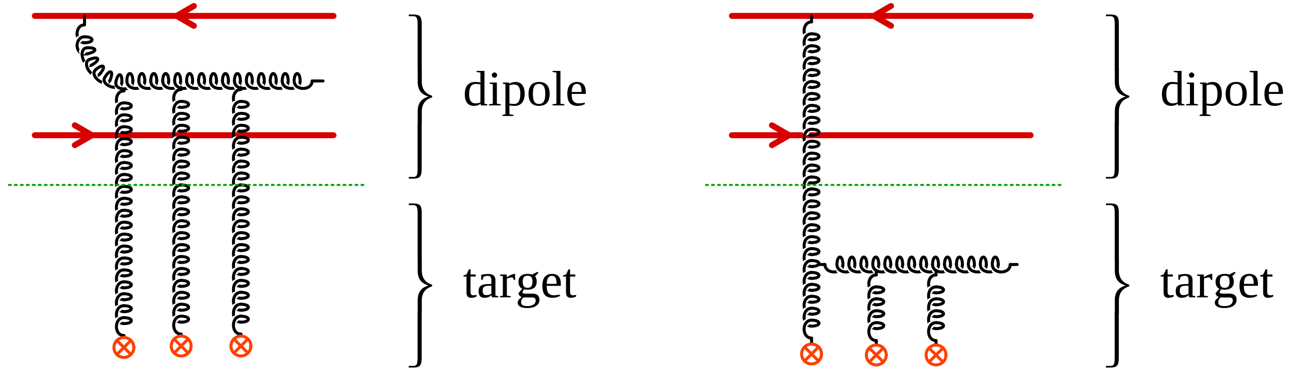
$$\langle \boldsymbol{T}_1 \cdots \boldsymbol{T}_n \rangle_{Y_0} = \langle \boldsymbol{T}_1 \rangle_{Y_0} \cdots \langle \boldsymbol{T}_n \rangle_{Y_0}$$

then the solution remains factorized at all $Y > Y_0$

Color Glass Condensate

Introduction

- One may view the Color Glass Condensate as a description centered on the target of the physics contained in Balitsky's hierarchy
- In this "target-centric" description, we need to describe how the distribution of color fields in the target changes with rapidity
- In the non-linear regime, the gluon radiation in the target must be corrected by rescatterings in the field of the target :



Degrees of freedom and their interplay

McLerran, Venugopalan (1994)
Iancu, Leonidov, McLerran (2001)

- Small-x modes have a large occupation number
  - ▷ they are described by a classical color field A^μ

- The classical field obeys Yang-Mills's equation:

$$[D_\nu, F^{\nu\mu}]_a = J_a^\mu$$

- The source term J_a^μ comes from the faster partons. The large-x modes, slowed down by time dilation, are described as frozen color sources ρ_a. Hence :

$$J_a^\mu = \delta^{\mu+}\delta(x^-)\rho_a(\vec{\boldsymbol{x}}_\perp)$$

Semantics

McLerran (mid 2000)

- Color : pretty much obvious...
- Glass : the system has degrees of freedom whose timescale is much larger than the typical timescales for interaction processes. Moreover, these degrees of freedom are stochastic variables, like in "spin glasses" for instance
- Condensate : the soft degrees of freedom are as densely packed as they can (the density remains finite, of order α_s^{-1}, due to the interactions between gluons)

Target average

- The color sources ρ_a are random, and described by a distribution functional $W_Y[\rho]$, with $Y \equiv \ln(1/x_0)$, x_0 being the frontier between "small-x" and "large-x"

- The averaged dipole operator $\langle \boldsymbol{T} \rangle$ studied in the Balitsky-Kovechegov approach can be written as :

$$\langle \boldsymbol{T}(\vec{\boldsymbol{x}}_\perp, \vec{\boldsymbol{y}}_\perp) \rangle = \int [D\rho] \; W_Y[\rho] \; \left[1 - \frac{1}{N_c} \mathrm{tr}(U(\vec{\boldsymbol{x}}_\perp) U^\dagger(\vec{\boldsymbol{y}}_\perp)) \right]$$

- Since in this description, all the evolution is placed inside the target, Y must in fact be the rapidity difference between the projectile and the target

- The Y dependence of $\langle \boldsymbol{T} \rangle$ will have to come from the Y dependence of $W_Y[\rho]$

JIMWLK evolution equation

Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner

- The distribution $W_Y[\rho]$ evolves with Y (more modes are included in W as x_0 decreases)
- In a high density environment, the newly created gluons can interact with all the sources already present
- The evolution is governed by a functional diffusion equation:

$$\frac{\partial W_Y[\rho]}{\partial Y} = \frac{1}{2}\int_{\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp} \frac{\delta}{\delta\rho_a(\vec{\boldsymbol{x}}_\perp)}\chi_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp)\frac{\delta}{\delta\rho_b(\vec{\boldsymbol{y}}_\perp)} W_Y[\rho]$$

with

$$\chi_{ab}(\vec{\boldsymbol{x}}_\perp,\vec{\boldsymbol{y}}_\perp) \equiv \frac{\alpha_s}{4\pi^3}\int d^2\vec{\boldsymbol{z}}_\perp \frac{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)\cdot(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)}{(\vec{\boldsymbol{x}}_\perp-\vec{\boldsymbol{z}}_\perp)^2(\vec{\boldsymbol{y}}_\perp-\vec{\boldsymbol{z}}_\perp)^2} \times\Big[\Big(1-\widetilde{U}^\dagger(\vec{\boldsymbol{x}}_\perp)\widetilde{U}(\vec{\boldsymbol{z}}_\perp)\Big)\Big(1-\widetilde{U}^\dagger(\vec{\boldsymbol{z}}_\perp)\widetilde{U}(\vec{\boldsymbol{y}}_\perp)\Big)\Big]_{ab}$$

  - $\widetilde{U}$ is a Wilson line in the adjoint representation, constructed from the gauge field A^+ such that $\boldsymbol{\nabla}_\perp^2 A^+ = -\rho$

JIMWLK evolution equation

- Sketch of a proof : exploit the frame independence in order to write :

$$\left<\mathcal{O}\right>_Y = \int [D\rho]\;\; W_0[\rho]\; \mathcal{O}_Y[\rho] = \int [D\rho]\;\; W_Y[\rho]\; \mathcal{O}_0[\rho]$$

- The first formula leads to

$$\frac{\partial \left<\mathcal{O}\right>_Y}{\partial Y} = \int [D\rho]\;\; W_0[\rho]\; \frac{\partial \mathcal{O}_Y[\rho]}{\partial Y}$$

  - The derivative under the integral is determined by a method similar to the derivation of the Balitsky-Kovchegov equation, by attaching one extra gluon to the operator $\mathcal{O}_Y[\rho]$ in all the possible ways
  - As pointed out by Mueller (2001), $\partial\mathcal{O}_Y[\rho]/\partial Y$ can be written as the action of an Hamiltonian on $\mathcal{O}_Y[\rho]$:

$$\frac{\partial \mathcal{O}_Y[\rho]}{\partial Y} = \mathcal{H}\left[\frac{\delta}{\delta\rho}\right]\;\; \mathcal{O}_Y[\rho]$$

JIMWLK evolution equation

- Then, one can write formally :

$$\mathcal{O}_Y[\rho] = \mathcal{U}(Y)\ \mathcal{O}_0[\rho]$$

with $\quad d\mathcal{U}(Y)/dY = \mathcal{H}\,\mathcal{U}(Y)$ and $\mathcal{U}(0) = 1$

- From there, we get :

$$\langle\mathcal{O}\rangle_Y = \int [D\rho]\ W_0[\rho]\,\mathcal{U}(Y)\ \mathcal{O}_0[\rho] = \int [D\rho]\ \left[\mathcal{U}^\dagger(Y)\ W_0[\rho]\right]\ \mathcal{O}_0[\rho]$$

and we are led to identify :

$$W_Y[\rho] = \mathcal{U}^\dagger(Y)\ W_0[\rho]$$

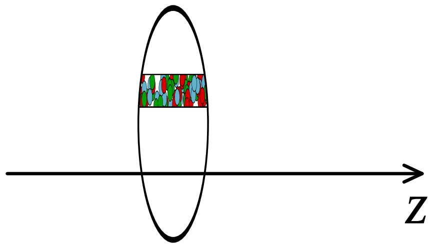
- And finally :

$$\frac{\partial W_Y[\rho]}{\partial Y} = \left[\frac{d\mathcal{U}^\dagger(Y)}{dY}\mathcal{U}(Y)\right]\ \mathcal{U}^\dagger(Y)\ W_0[\rho] = \mathcal{H}_{JIMWLK}\ W_Y[\rho]$$

with $\quad \mathcal{H}_{JIMWLK} = \left[d\mathcal{U}^\dagger(Y)/dY\right]\ \mathcal{U}(Y)$

Initial condition - MV model

- The JIMWLK equation must be completed by an initial condition, given at some moderate x_0
- As with DGLAP, the problem of finding the initial condition is in general non-perturbative
- The McLerran-Venugopalan model is often used as an initial condition at moderate x_0 for a large nucleus :



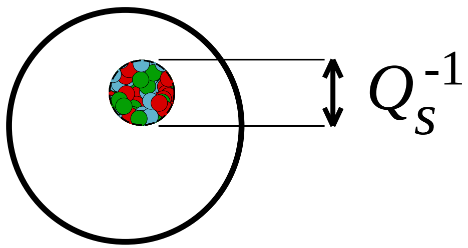
- partons distributed randomly
- many partons in a small tube
- no correlations at different $\vec{x}_\perp$

- The MV model assumes that the density of color charges $\rho(\vec{x}_\perp)$ has a Gaussian distribution :

$$W_{x_0}[\rho] = \exp\left[-\int d^2\vec{x}_\perp \frac{\rho_a(\vec{x}_\perp)\rho_a(\vec{x}_\perp)}{2\mu^2(\vec{x}_\perp)}\right]$$

Color correlation length

- In a nucleon at low energy, the typical correlation length among color charges is of the order of the nucleon size, i.e. $\Lambda_{QCD}^{-1} \sim 1\ \mathrm{fm}$. This is because the typical color screening distance is Λ_{QCD}^{-1}. At low energy, color screening is due to confinement, and thus non-perturbative
- At high energy (small x), partons are much more densely packed, and it can be shown that color neutralization occurs in fact over distances of the order of $Q_s^{-1} \ll \Lambda_{QCD}^{-1}$



- This implies that all hadrons, and nuclei, behave in the same way at high energy. In this sense, the small x regime described by the CGC is universal

Reaction-diffusion processes

Analogy with reaction-diffusion

Munier, Peschanski (2003,2004)

- Assume rotation invariance, and define :

$$N(Y,k_\perp) \equiv 2\pi \int d^2\vec{\boldsymbol{x}}_\perp \; e^{i\vec{\boldsymbol{k}}_\perp\cdot\vec{\boldsymbol{x}}_\perp} \; \frac{\left<\boldsymbol{T}(0,\vec{\boldsymbol{x}}_\perp)\right>_Y}{x_\perp^2}$$

- From the Balitsky-Kovchegov equation for $\left<\boldsymbol{T}\right>_Y$, we obtain the following equation for N :

$$\frac{\partial N(Y,k_\perp)}{\partial Y} = \frac{\alpha_s N_c}{\pi}\Big[\chi(-\partial_L)N(Y,k_\perp) - N^2(Y,k_\perp)\Big]$$

with

$$L \equiv \ln(k^2/k_0^2)$$

$$\chi(\gamma) \equiv 2\psi(1) - \psi(\gamma) - \psi(1-\gamma)$$

Analogy with reaction-diffusion

- Expand the function $\chi(\gamma)$ to second order around its minimum $\gamma = 1/2$
- Introduce new variables :

$$t \sim Y$$
$$z \sim L + \frac{\alpha_s N_c}{2\pi} \chi''(1/2)\, Y$$

- The equation for N becomes :

$$\partial_t N = \partial_z^2 N + N - N^2$$

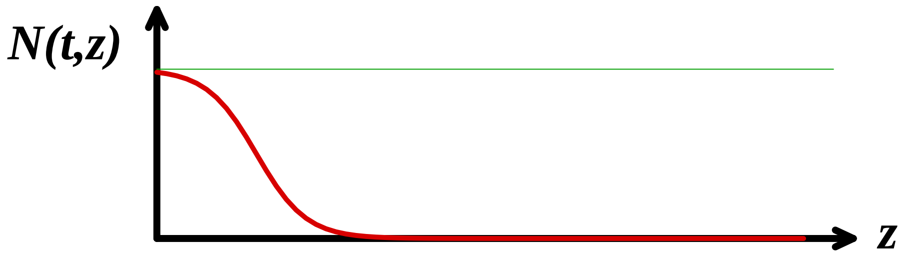
(known as the Fisher-Kolmogorov-Petrov-Piscounov equation)

Analogy with reaction-diffusion

- Interpretation : this equation is typical for all the diffusive systems in which a reaction $A \longleftrightarrow A + A$ takes place
  - $\partial_z^2 N$: diffusion term (the quantity under consideration can hop from a site to the neighboring sites)
  - $+N$: gain term corresponding to $A \to A + A$
  - $-N^2$: loss term corresponding to $A + A \to A$
- Note : this equation has two fixed points :
  - $N = 0$: unstable
  - $N = 1$: stable
- The stable fixed point at $N = 1$ exists only if one keeps the loss term. In other words, one would not have it from the BFKL equation

Traveling waves

- Assume an initial condition $N(t_0, z)$ that goes smoothly from 1 at $z = -\infty$ to 0 at $z = +\infty$, and behaves like $\exp(-\beta z)$ when $z \gg 1$



- The solution of the F-KPP equation is known to behave like a traveling wave at asymptotic times (Bramson, 1983) :

$$N(t, z) \underset{t \to +\infty}{\sim} N(z - m_\beta(t))$$

with

- $m_\beta(t) = (\beta + \beta^{-1})t + \mathcal{O}(1)$ if $\beta < 1$
- $m_\beta(t) = 2t - \ln(t)/2 + \mathcal{O}(1)$ for $\beta = 1$
- $m_\beta(t) = 2t - 3\ln(t)/2 + \mathcal{O}(1)$ if $\beta > 1$

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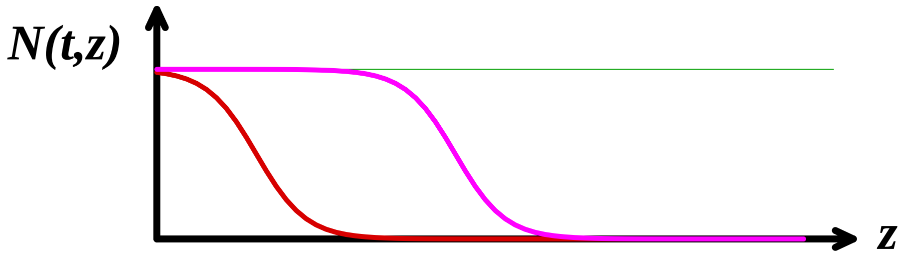

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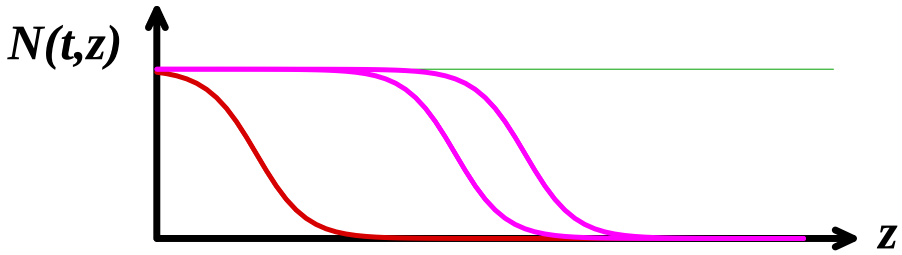

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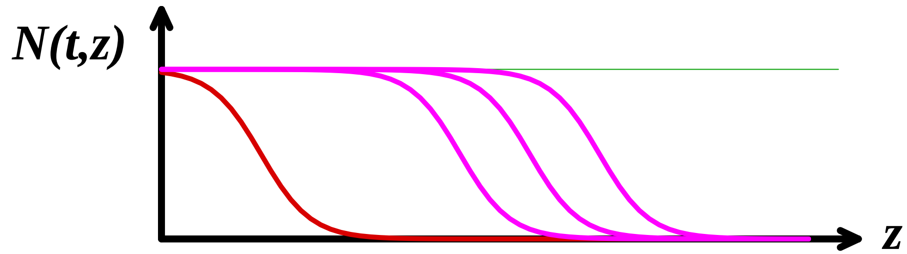

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Geometrical scaling in DIS

Iancu, Itakura, McLerran (2002)
Mueller, Triantafyllopoulos (2002)
Munier, Peschanski (2003)

- In QCD, the initial condition is of the required form, with $\beta > 1$
  $\triangleright$ front velocity independent of the initial condition

- Going back to the original variables, one gets :

$$N(Y, k_\perp) = N\left(k_\perp / Q_s(Y)\right)$$

with

$$Q_s^2(Y) = k_0^2 \; Y^{-\frac{3}{2(1-\bar{\gamma})}} \; e^{\overline{\alpha}_s \chi''(\frac{1}{2})(\frac{1}{2}-\bar{\gamma})Y}$$

- Going from $N(Y, k_\perp)$ to $\left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right\rangle_Y$, we obtain :

$$\left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right\rangle_Y = T(Q_s(Y) x_\perp)$$

Geometrical scaling in DIS

- The $\gamma^* p$ cross-section, measured in Deep Inelastic Scattering, can be written in terms of N:

$$\sigma_{\gamma^* p}(Y, Q^2) = 2\pi R^2 \int d^2\vec{\boldsymbol{x}}_\perp \int_0^1 dz \left|\psi(z, x_\perp, Q^2)\right|^2 \left\langle \boldsymbol{T}(0, \vec{\boldsymbol{x}}_\perp)\right\rangle_Y$$

  - The photon wavefunction ψ is calculable in QED :

$$\left|\psi_T(z, x_\perp, Q^2)\right|^2 = \frac{3\alpha_{\rm em}}{2\pi^2} \sum_f e_f^2 \Big\{ [z^2 + (1-z)^2]\, \overline{Q}_f^2\, K_1^2(\overline{Q}_f x_\perp) + m_f^2\, K_0^2(\overline{Q}_f x_\perp) \Big\}$$

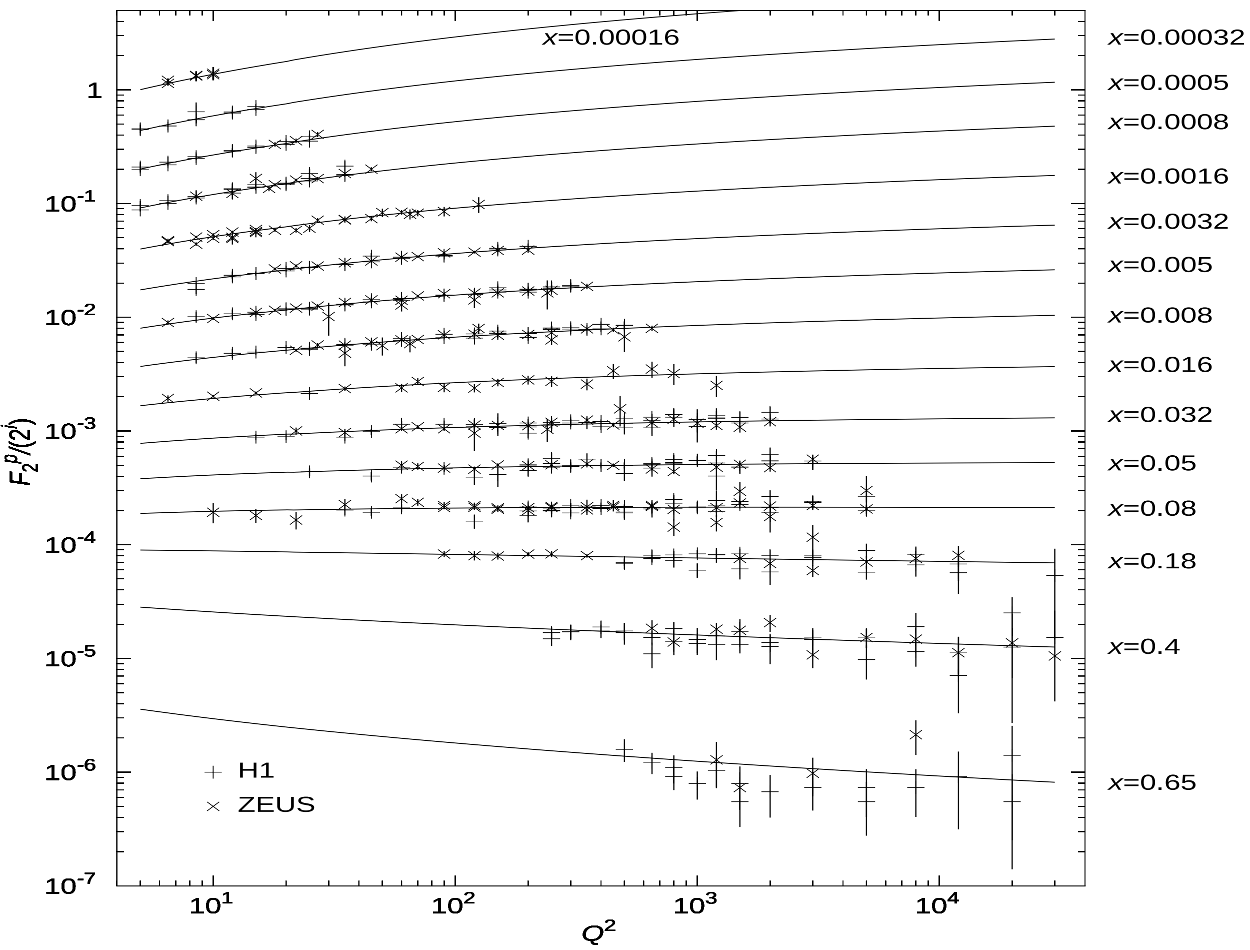
$$\left|\psi_L(z, x_\perp, Q^2)\right|^2 = \frac{3\alpha_{\rm em}}{2\pi^2} \sum_f e_f^2 \Big\{ 4\, Q^2\, z^2 (1-z)^2\, K_0^2(\overline{Q}_f x_\perp) \Big\}$$

with $\overline{Q}_f^2 \equiv m_f^2 + Q^2 z^2 (1 - z^2)$

- If one neglects the quark masses, the scaling properties of $\langle \boldsymbol{T} \rangle_Y$ imply that $\sigma_{\gamma^* p}$ depends only on the ratio $Q^2/Q_s^2(Y)$, rather than on Q^2 and Y separately

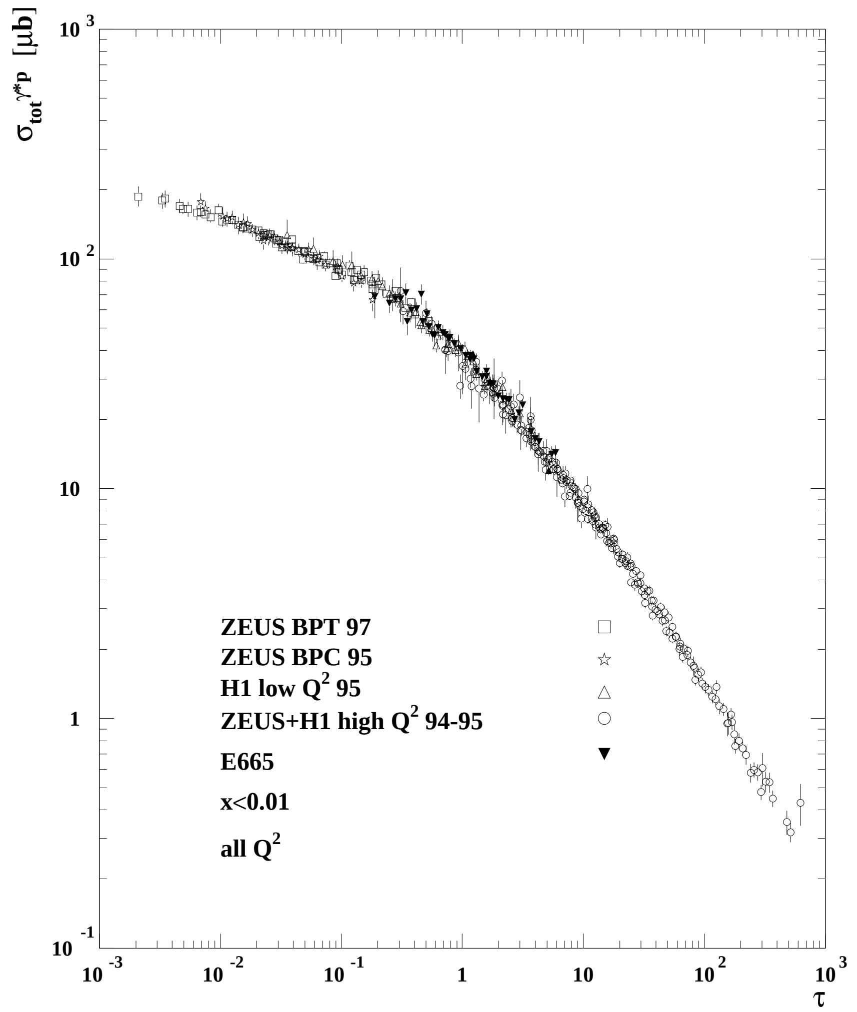
Geometrical scaling in DIS

- HERA data as a function of Q^2 and x :



Geometrical scaling in DIS

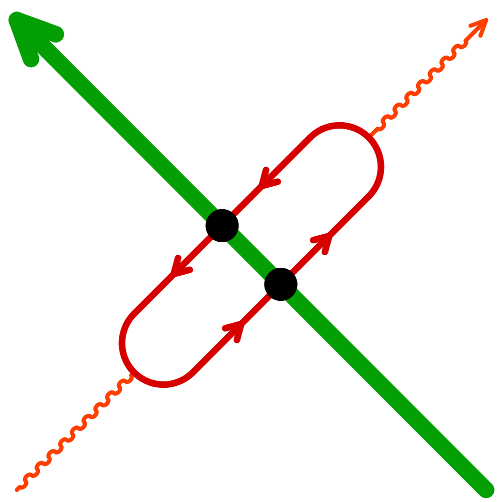
Stasto, Golec-Biernat, Kwiecinski (2000)



Motivation for Lecture III

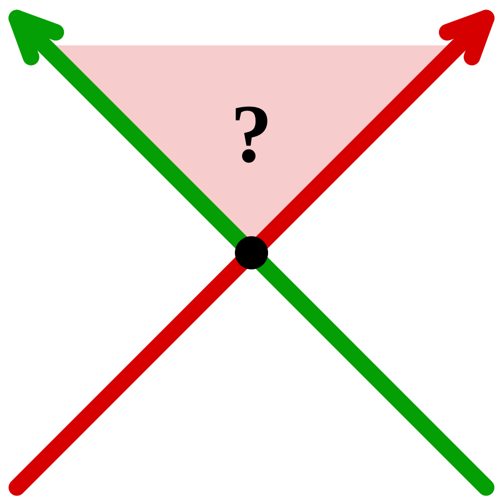
Hadron-hadron collisions ?

- In DIS, only one of the two projectiles is described by the CGC, while the virtual photon is elementary :



Hadron-hadron collisions ?

- In hadron-hadron collisions, the two projectiles must be described by the CGC :



  - What happens when the two color fields meet ?
  - Can we compute observables for two saturated objects ?
  - Can we include all the multiple scattering corrections ?
  - Can we sum all the large logs of $1/x_{1,2}$?

Lecture III

- Introduction
- Field theories with sources
- Power counting and bookkeeping
- Inclusive gluon spectrum
- Inclusive quark spectrum
- Loop corrections