High energy scattering in QCD

I – Parton model, Bjorken scaling, Scaling violations

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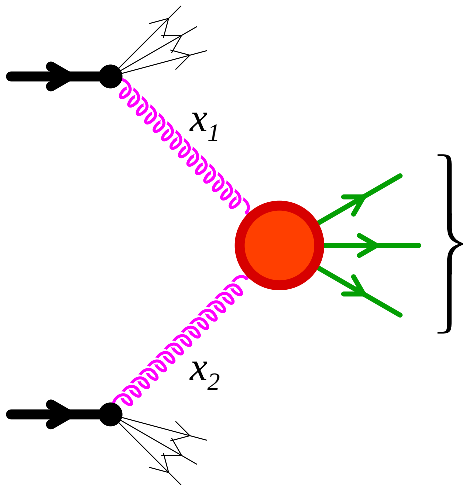
General introduction

Infrared and collinear divergences

- Calculation of some process at LO :



$$(M_\perp\,,\,Y) \qquad \begin{cases} x_1 = M_\perp\, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp\, e^{-Y}/\sqrt{s} \end{cases}$$

Infrared and collinear divergences

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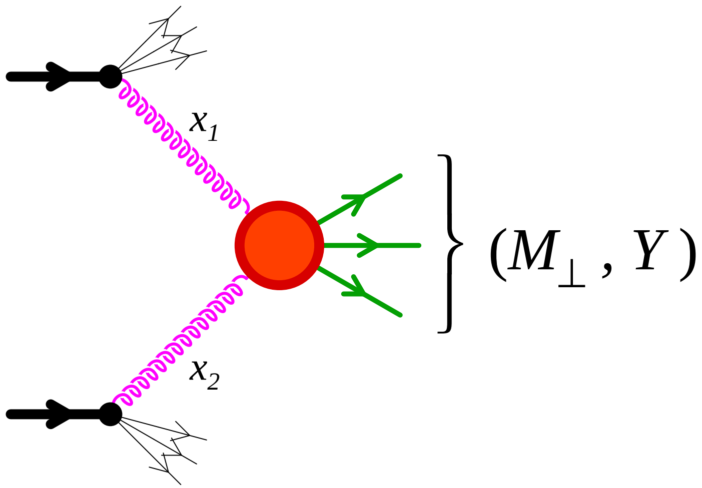
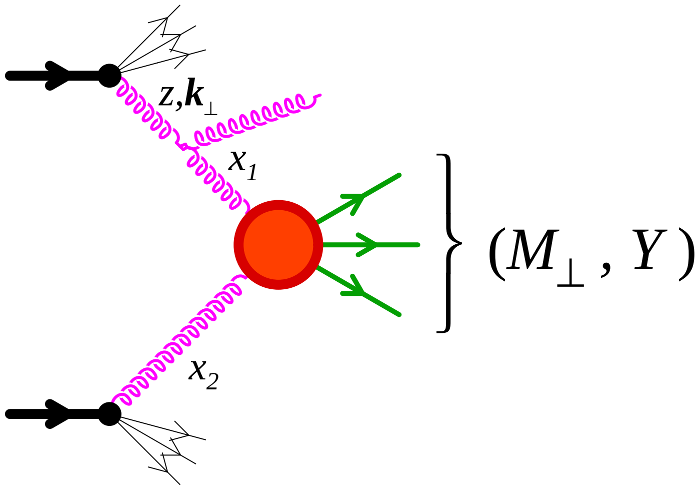


$$\begin{cases} x_1 = M_\perp \, e^{+Y}/\sqrt{s} \\ x_2 = M_\perp \, e^{-Y}/\sqrt{s} \end{cases}$$

- Radiation of an extra gluon :



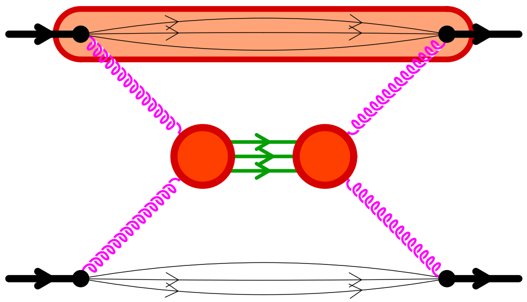
$$\Longrightarrow \quad \alpha_s \int_{x_1} \frac{dz}{z} \int^{M_\perp} \frac{d^2\vec{\boldsymbol{k}}_\perp}{\boldsymbol{k}_\perp^2}$$

Infrared and collinear divergences

- Large logs : $\log(M_\perp)$ or $\log(1/x_1)$, under certain conditions
  ▷ these logs can compensate the additional α_s, and void the naive application of perturbation theory
  ▷ resummations are necessary
- Logs of $M_\perp \Longrightarrow$ DGLAP. Important when :
  - $M_\perp \gg \Lambda_{QCD}$
  - x_1, x_2 are rather large
- Logs of $1/x \Longrightarrow$ BFKL. Important when :
  - $M_\perp$ remains moderate
  - x_1 or x_2 (or both) are small
- Physical interpretation :
  - The physical process can resolve the gluon splitting if $M_\perp \gg k_\perp$
  - If $x_1 \ll 1$, the gluon that initiates the process is likely to result from bremsstrahlung from another parent gluon

Multiple scatterings

- Single scattering :



▷ 2-point function in the projectile ▷ gluon number

Multiple scatterings

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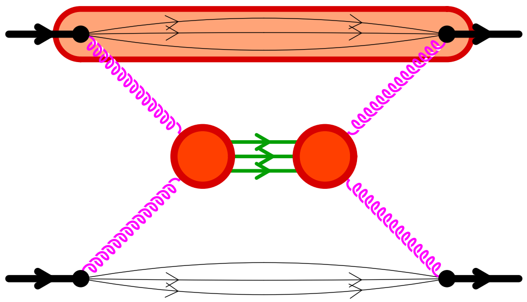
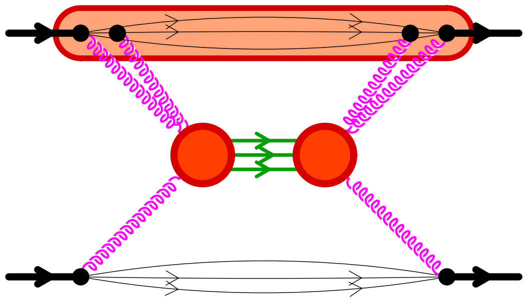


▷ 2-point function in the projectile ▷ gluon number

- Multiple scatterings :



▷ 4-point function in the projectile ▷ higher correlation
▷ multiple scatterings in the projectile

Multiple scatterings

- Power counting : rescattering corrections are suppresssed by inverse powers of the typical mass scale in the process :

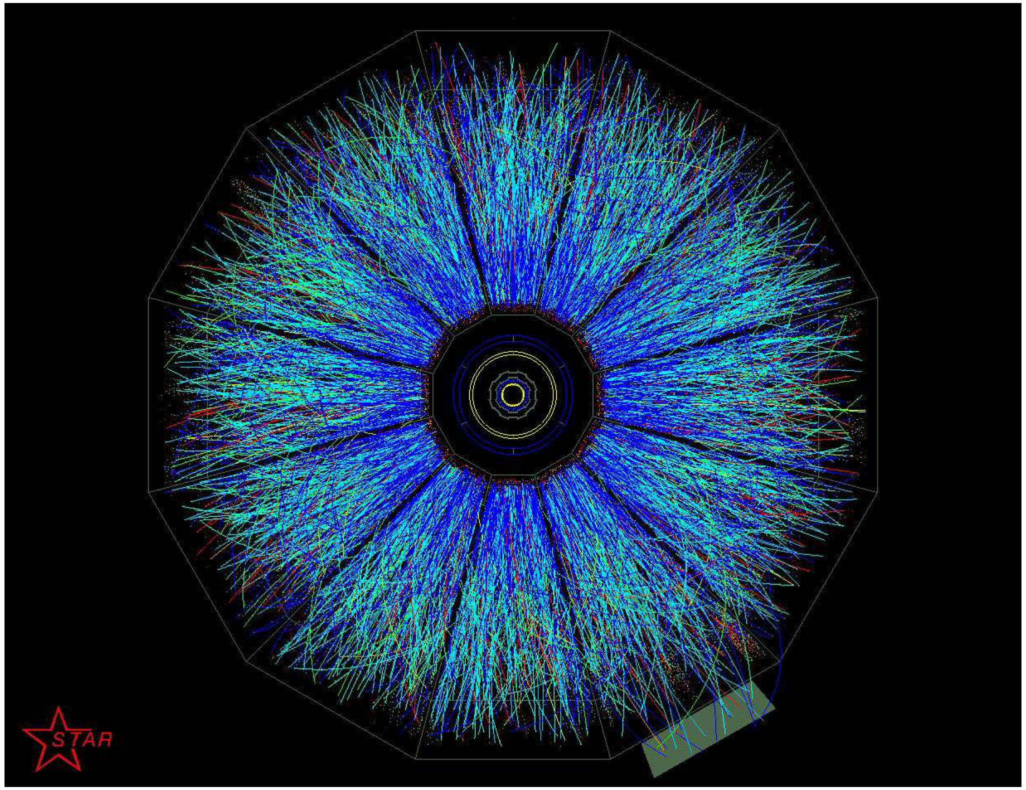
$$\left[\frac{\mu^2}{M_\perp^2}\right]^n$$

- The parameter μ^2 has a factor of α_s, and a factor proportional to the gluon density $\triangleright$ rescatterings are important at high density
- Relative order of magnitude :

$$\frac{\text{2 scatterings}}{\text{1 scattering}} \sim \frac{Q_s^2}{M_\perp^2} \quad \text{with} \quad Q_s^2 \sim \alpha_s \frac{xG(x, Q_s^2)}{\pi R^2}$$

- When this ratio becomes ~ 1, all the rescattering corrections become important
- These effects are not accounted for in DGLAP or BFKL

Heavy Ion Collisions



- 99% of the multiplicity below $p_\perp \sim 2$ GeV
- Q_s^2 might be as large as 10 GeV^2 at the LHC ($\sqrt{s} = 5.5$ TeV)

  ▷ both the logs of $1/x$ and the multiple scatterings are important

Goals

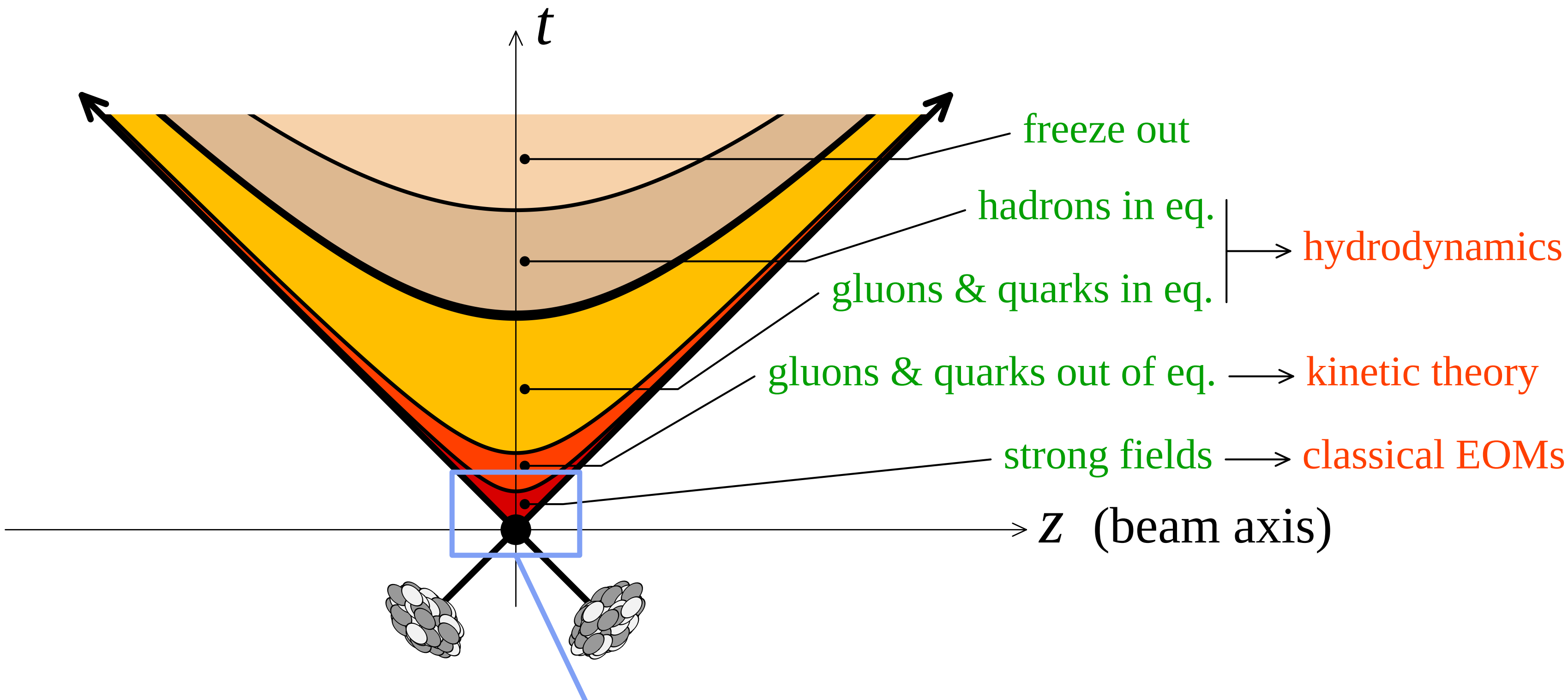
- Develop a framework for resumming all the $[\alpha_s \ln(1/x)]^m [Q_s/M_\perp]^n$ corrections

- Generalize the concept of "parton distribution"
  - Due to the high density of partons, observables depend on higher correlations (beyond the usual parton distributions, which are 2-point correlation functions)

- These distributions should be universal, with non-perturbative information relegated into the initial condition of some evolution equation

- Develop techniques for describing the early stages of heavy ion collisions in this framework

Goals



- calculate the initial production of semi-hard particles
- prepare the stage for kinetic theory or hydrodynamics

General outline

- Lecture I : Parton model, Bjorken scaling, Scaling violations
- Lecture II : Parton evolution at small x, Saturation
- Lecture III : Hadron-hadron collisions in the CGC framework

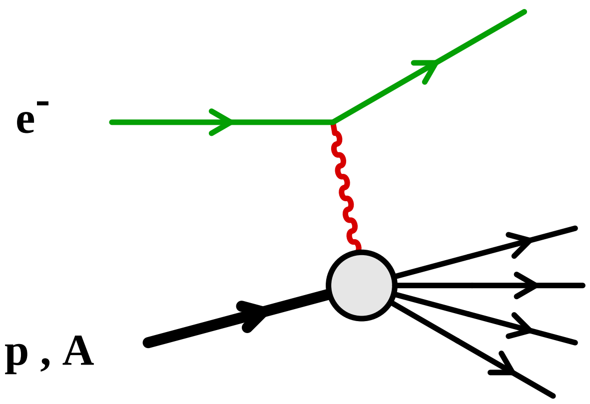
Lecture I

- General introduction
- Kinematics of Deep Inelastic Scattering
- Structure functions
- Experimental facts
- Naive parton model
- Light-cone behavior of a free field theory
- Scaling violations, DGLAP equation
- Factorization

Kinematics of DIS

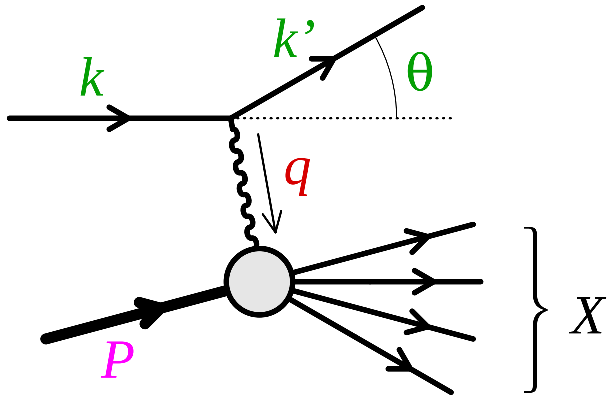
Introduction to DIS

- Basic idea : smash a well known probe on a nucleon or nucleus in order to try to figure out what is inside...
- Photons are very well suited for that purpose because their interactions are well understood
- Deep Inelastic Scattering : collision between an electron and a nucleon or nucleus, by exchange of a virtual photon



- Variant : collision with a neutrino, by exchange of $Z^0, W^{\pm}$

Kinematical variables



- Note : the virtual photon is space-like: $q^2 \le 0$
- Other invariants of the reaction :

$$\begin{aligned} \nu &\equiv P \cdot q \\ s &\equiv (P+k)^2 \\ M_X^2 &\equiv (P+q)^2 = m_N^2 + 2\nu + q^2 \end{aligned}$$

- One uses commonly : $Q^2 \equiv -q^2$ and $x \equiv Q^2/2\nu$
- In general $M_X^2 \ge m_N^2$, and we have : $0 \le x \le 1$
  ($x = 1$ corresponds to the case of elastic scattering)

DIS cross-section

- The simplest cross-section is the inclusive cross-section, obtained by measuring the momentum of the scattered electron and summing over all the hadronic final states X

$$E' \frac{d\sigma_{e^-N}}{d^3\vec{\boldsymbol{k}}'} = \sum_{\text{states } X} E' \frac{d\sigma_{e^-N\to e^-X}}{d^3\vec{\boldsymbol{k}}'}$$

$$E' \frac{d\sigma_{e^-N\to e^-X}}{d^3\vec{\boldsymbol{k}}'} = \int \frac{[d\Phi_X]}{32\pi^3(s-m_N^2)} (2\pi)^4 \delta(P+k-k'-P_X) \left\langle |\mathcal{M}_X|^2 \right\rangle_{\text{spin}}$$

$$\mathcal{M}_X = \frac{ie}{q^2} \left[\overline{u}(\vec{\boldsymbol{k}}')\gamma^\mu u(\vec{\boldsymbol{k}}) \right] \left\langle X \middle| J_\mu(0) \middle| N(P) \right\rangle$$

- In the amplitude squared appears the leptonic tensor :

$$\begin{aligned} L^{\mu\nu} &\equiv \left\langle \overline{u}(\vec{\boldsymbol{k}}')\gamma^\mu u(\vec{\boldsymbol{k}})\overline{u}(\vec{\boldsymbol{k}})\gamma^\nu u(\vec{\boldsymbol{k}}') \right\rangle_{\text{spin}} \\ &= 2(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu}\, k\cdot k') \end{aligned}$$

(the electron mass has been neglected)

DIS cross-section

- The inclusive cross-section can be written as :

$$E' \frac{d\sigma_{e-N}}{d^3\vec{\boldsymbol{k}}'} = \frac{1}{32\pi^3(s - m_N^2)} \frac{e^2}{q^4} 4\pi L^{\mu\nu} W_{\mu\nu}$$

where $W_{\mu\nu}$ is the hadronic tensor, defined as:

$$4\pi W_{\mu\nu} \equiv \sum_{\text{states } X} \int [d\Phi_X] (2\pi)^4 \delta(P + q - P_X)$$

$$\times \Big\langle \langle N(P) | J_\nu^\dagger(0) | X \rangle \langle X | J_\mu(0) | N(P) \rangle \Big\rangle_{\text{spin}}$$

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- $W_{\mu\nu}$ contains all the informations about the properties of the nucleon under consideration that are relevant to the interaction with the photon
- This object cannot be calculated perturbatively
- It obeys: $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$ (conservation of e.m. current)

Structure functions

- For a (spin-averaged) nucleon, the most general form of $W_{\mu\nu}$ is:

$$W_{\mu\nu} = -W_1 g_{\mu\nu} + W_2 \frac{P_\mu P_\nu}{m_N^2} + W_3 \epsilon_{\mu\nu\rho\sigma} \frac{P^\rho q^\sigma}{m_N^2} + W_4 \frac{q_\mu q_\nu}{m_N^2} + W_5 \frac{P_\mu q_\nu}{m_N^2} + W_6 \frac{q_\mu P_\nu}{m_N^2}$$

  - $W_3 = 0$ for parity conserving currents (like e.m. currents)
  - $W_{\mu\nu} = W_{\nu\mu}$ from parity and time-reversal symmetry hence $W_5 = W_6$
  - From the Ward identities $q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$, one gets:

$$W_5 = -W_2 \frac{P \cdot q}{q^2}$$

$$W_4 = W_1 \frac{m_N^2}{q^2} + W_2 \frac{(P \cdot q)^2}{q^4}$$

Structure functions

- Therefore, for interactions with a photon, we have:

$$W_{\mu\nu} = -W_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{W_2}{m_N^2} \left(P_\mu - q_\mu \frac{P \cdot q}{q^2} \right) \left(P_\nu - q_\nu \frac{P \cdot q}{q^2} \right)$$

- And the DIS cross-section in the nucleon rest frame reads:

$$\frac{d\sigma_{e-N}}{dE' d\Omega} = \frac{\alpha_{\rm em}^2}{4 m_N E^2 \sin^4(\theta/2)} \left[2 \sin^2(\theta/2) W_1 + \cos^2(\theta/2) W_2 \right]$$

where Ω is the solid angle of the scattered electron

- It is customary to define slightly rescaled structure functions:

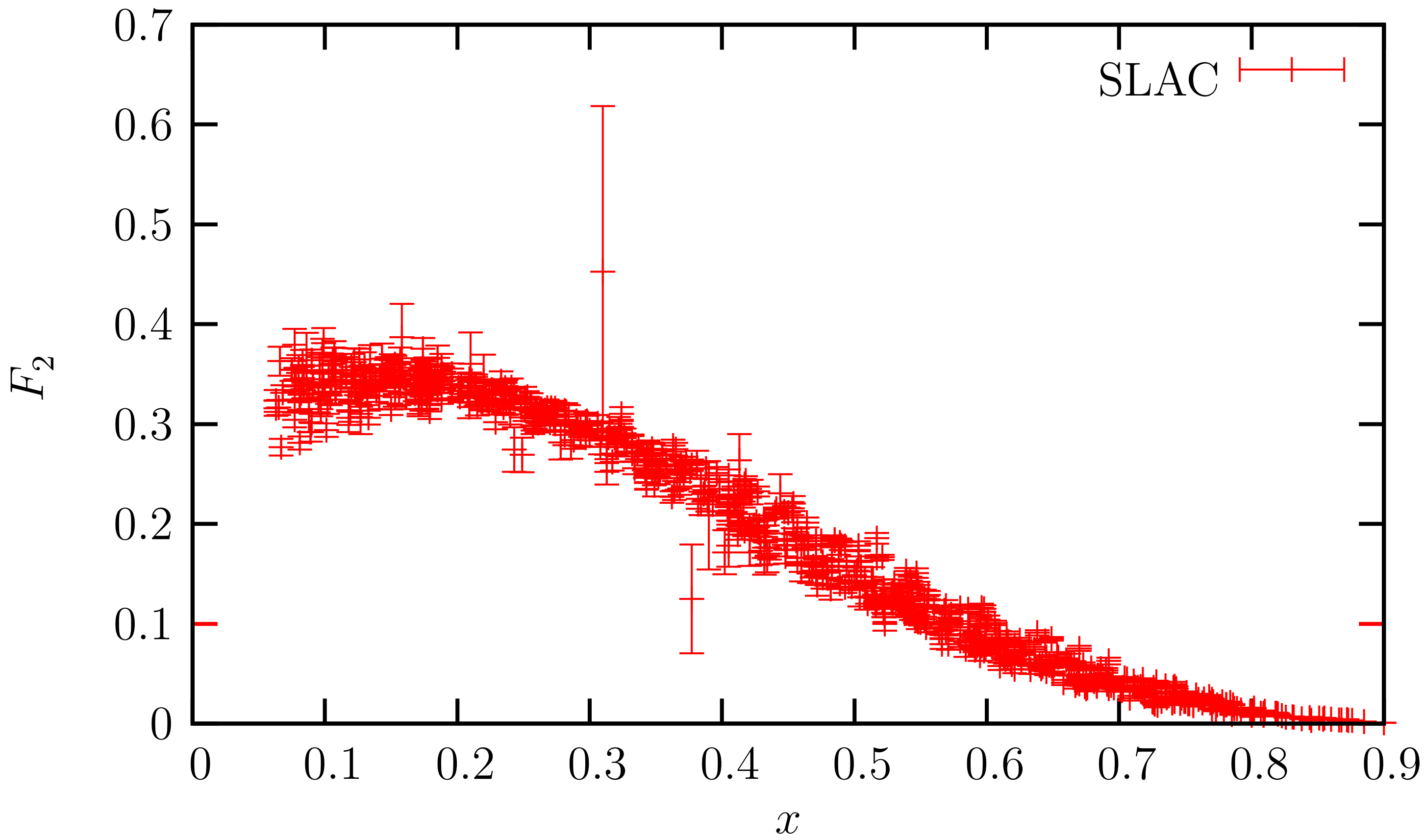
$$F_1 \equiv W_1 \quad , \quad F_2 \equiv \frac{\nu}{m_N^2} W_2$$

- Note: F_1 is proportional to the interaction cross-section between the nucleon and a transverse photon

Experimental facts

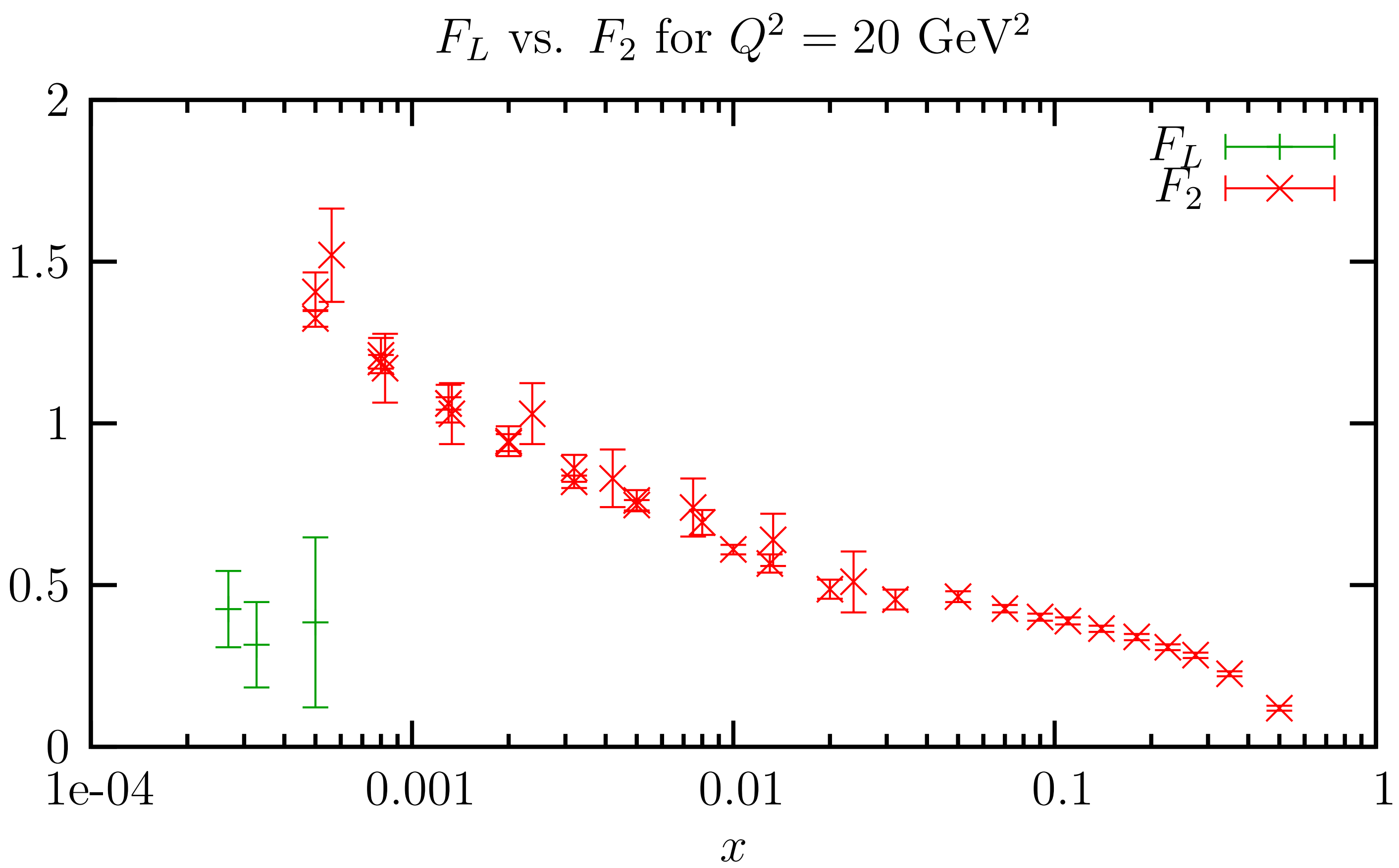
Bjorken scaling

- Bjorken scaling : F_2 depends very weakly on Q^2



Longitudinal F

- $F_L \equiv F_2 - 2xF_1$ is quite smaller than F_2 :



Naive parton model

Analogy with the e- mu- cross-section

- In terms of F_1 and F_2, the DIS cross-section reads:

$$\frac{d\sigma_{e-N}}{dE'd\Omega} = \frac{\alpha_{\rm em}^2}{4m_N E^2 \sin^4\frac{\theta}{2}}\left[2F_1 \sin^2\frac{\theta}{2} + \frac{m_N^2}{\nu} F_2 \cos^2\frac{\theta}{2}\right]$$

- It is instructive to compare it to the $e^-\mu^-$ cross-section:

$$\frac{d\sigma_{e^-\mu^-}}{dE'd\Omega} = \frac{\alpha_{\rm em}^2 \delta(1-x)}{4m_\mu E^2 \sin^4\frac{\theta}{2}}\left[\sin^2\frac{\theta}{2} + \frac{m_\mu^2}{\nu} \cos^2\frac{\theta}{2}\right]$$

  - If the constituents of the nucleon that interact in the DIS process were spin $1/2$ point-like particles, we would have:

$$2F_1 = \frac{m_N}{m_c}\delta(1-x_c) \quad , \quad F_2 = \frac{m_c}{m_N}\delta(1-x_c)$$

    where m_c is some effective mass for the constituent (comparable to m_N because it is trapped inside the nucleon) and $x_c \equiv Q^2/2q \cdot p_c$ with p_c^μ the momentum of the constituent

Analogy with the e- mu- cross-section

- If $p_c^\mu = x_{_F} P^\mu$, then $x_c = x/x_{_F}$, and:

$$2F_1 \sim \delta(x - x_{_F}) \quad , \quad F_2 \sim \delta(x - x_{_F})$$

- The structure functions F_1 and F_2 would therefore not depend on Q^2, but only on x
- Conclusion : Bjorken scaling could be explained if the constituents of the nucleon that are probed in DIS are spin $1/2$ point-like particles

  The variable x measured in DIS would have to be identified with the fraction of momentum carried by the struck constituent

Naive parton model

- The historical parton model describes the nucleon as a collection of point-like fermions, called partons
- A parton of type i, carrying the fraction x_F of the nucleon momentum, gives the following contribution to the hadronic tensor :

$$4\pi W_i^{\mu\nu} = \int \frac{d^4\boldsymbol{p'}}{(2\pi)^4} 2\pi\delta(\boldsymbol{p'^2})(2\pi)^4\delta(\boldsymbol{x_F P + q - p'})$$
$$\times \Big\langle \langle x_F P | J^{\mu\dagger}(0) | \boldsymbol{p'} \rangle \langle \boldsymbol{p'} | J^{\nu}(0) | x_F P \rangle \Big\rangle_{\rm spin}$$

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$$
\begin{aligned}
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$$\times \mathbf{2}e_i^2\,(\boldsymbol{x_F^2 P^\mu P^\nu} + \boldsymbol{x_F}(\boldsymbol{P^\mu q^\nu} + \boldsymbol{q^\mu P^\nu}) - \boldsymbol{x_F g^{\mu\nu}}\,\boldsymbol{P}\cdot\boldsymbol{q})$$

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- If there are $f_i(x_F)dx_F$ partons of type i with a momentum fraction between x_F and $x_F + dx_F$, we have

$$W^{\mu\nu} = \sum_i \int_0^1 \frac{dx_F}{x_F} f_i(x_F)\, W_i^{\mu\nu}$$

- One obtains the following structure functions :

$$F_1 = \frac{1}{2} \sum_i e_i^2 f_i(x) \quad , \quad F_2 = 2xF_1$$

Naive parton model

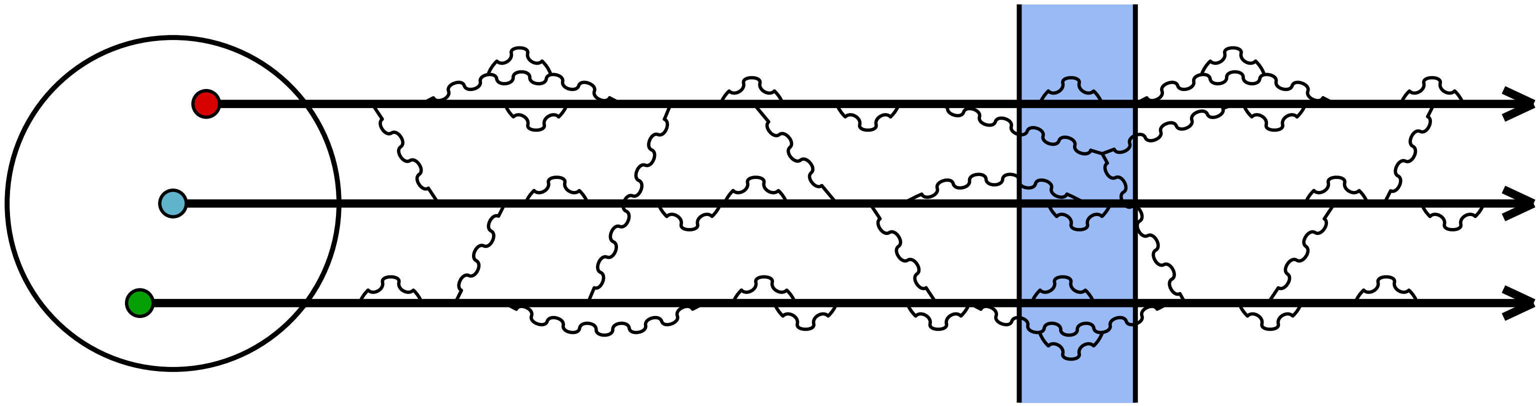
- This model provides an explicit realization of Bjorken scaling
- The relation $F_2 = 2xF_1$ implies that the cross-section between a longitudinally polarized photon and the nucleon is suppressed compared to that of a transverse photon
  - The observation of this property provides further support of the fact that the relevant constituents are spin $1/2$ fermions
  - If the partons were spin 0 particles, we would have

$$W_i^{\mu\nu} \propto (2x_{_F}P^\mu + q^\mu)(2x_{_F}P^\nu + q^\nu)$$

and it is easy to check that this leads to $F_1 = 0$ ($\sigma_{\rm transverse} = 0$)

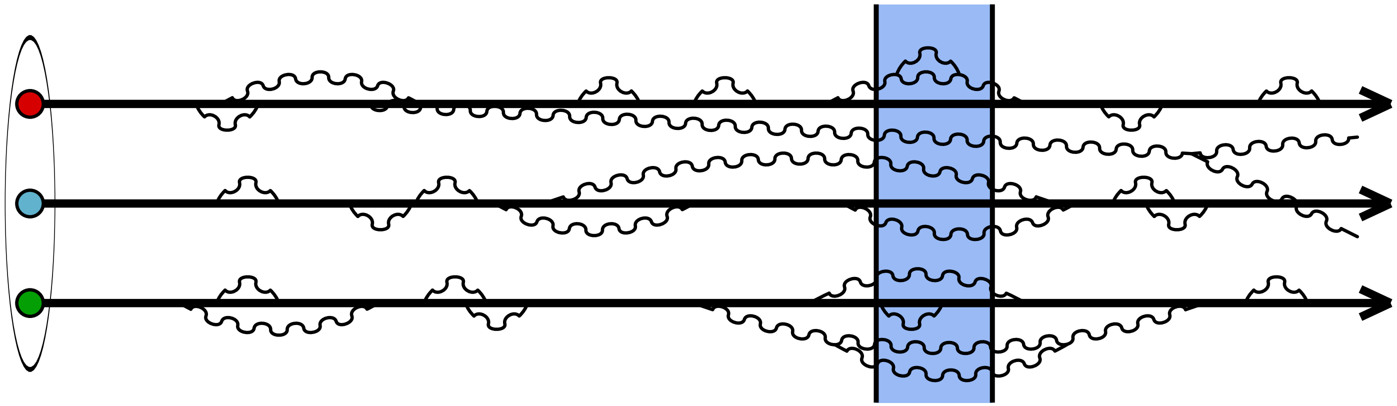
- Caveats and puzzles :
  - The parton model assumes that partons are free inside the nucleon. How can this be true in a strongly bound state ?
  - One would like to have a field theoretical description of what is going on, including the effect of interactions, quantum fluctuations, etc...

Field theory point of view



- A nucleon at rest is a very complicated object...
- Contains fluctuations at all space-time scales smaller than its own size
- Only the fluctuations that are longer lived than the external probe participate in the interaction process
- The only role of short lived fluctuations is to renormalize the masses and couplings
- Interactions are very complicated if the constituents of the nucleon have a non trivial dynamics over time-scales comparable to those of the probe

Field theory point of view



- Dilation of all internal time-scales for a high energy nucleon
- Interactions among constituents now take place over time-scales that are longer than the characteristic time-scale of the probe
  ▷ the constituents behave as if they were free
- Many fluctuations live long enough to be seen by the probe. The nucleon appears denser at high energy (it contains more gluons)

What would we learn ?

- The field theory that describes the interactions among partons should be able to explain the evolution with x of the parton distributions, since it comes from bremsstrahlung

- This field theory should also describe the evolution with Q^2 (i.e. the deviations from Bjorken scaling), which is due to the fact that the probe resolves more quantum fluctuations when Q^2 increases

- For the picture to be predictive, one should be able to prove from first principles the factorization of hadronic cross-section into a hard process (calculable?) and the parton distributions (not calculable?)

Bjorken scaling from field theory

Kinematics of the Bjorken limit

- Bjorken limit : $Q^2, \nu \to +\infty \; , \quad x = \text{constant}$
- Go to a frame where the photon momentum is :

$$q^\mu = \frac{1}{m_N}\left(\nu, 0, 0, \sqrt{\nu^2 + m_N^2 Q^2}\right)$$

- Therefore :

$$q^+ \equiv \frac{q^0 + q^3}{\sqrt{2}} \sim \frac{\nu}{m_N} \to +\infty$$

$$q^- \equiv \frac{q^0 - q^3}{\sqrt{2}} \sim m_N x \to \text{constant}$$

- Since $q \cdot y = q^+ y^- + q^- y^+ - \vec{q}_\perp \cdot \vec{y}_\perp$, the integration over y^μ is dominated by :

$$y^- \sim \frac{m_N}{\nu} \to 0 \quad , \quad y^+ \sim (m_N x)^{-1}$$

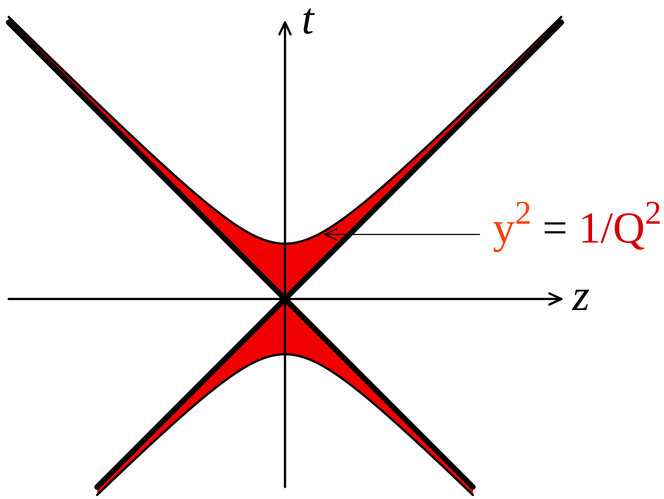
- Hence : $y^2 \le 2 y^+ y^- \sim 1/Q^2 \to 0$

Kinematics of the Bjorken limit

- $W_{\mu\nu}$ can be rewritten in terms of the commutator $[J_\mu^\dagger(y), J_\nu(0)]$. Thus, $y^2 \geq 0$ (causality). Therefore, the Bjorken limit is dominated by :

$$0 \leq y^2 \lesssim \frac{1}{Q^2} \to 0$$

i.e. by points very close to (and above) the light-cone

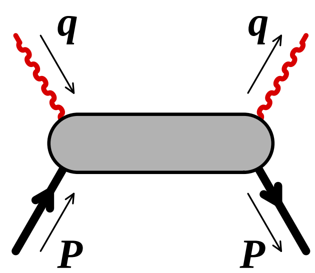


- Note : in this limit, the components of y^μ are not small

Time ordered correlator of currents

- Consider a time-ordered product of currents :

$$4\pi T_{\mu\nu} \equiv i \int d^4y e^{iq\cdot y} \left\langle \left\langle N(P) \middle| T(J^\dagger_\mu(y) J_\nu(0)) \middle| N(P) \right\rangle \right\rangle_{\text{spin}}$$



- $T_{\mu\nu}$ is a forward Compton amplitude

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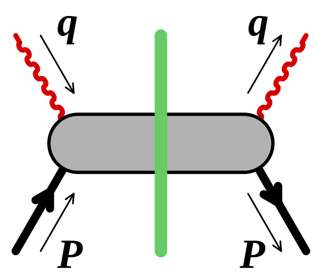

- $T_{\mu\nu}$ is a forward Compton amplitude
- $W_{\mu\nu} = 2\,\mathrm{Im}\, T_{\mu\nu}$
- At fixed Q^2, $T_{\mu\nu}(\nu, Q^2)$ is analytic in ν, with cuts on the real axis starting at $\pm Q^2/2$
  - the branch point at $\nu = Q^2/2$ comes from $(P+q)^2 \geq m_N^2$
  - $T_{\mu\nu}$ is symmetric under $(\mu \leftrightarrow \nu, q \leftrightarrow -q)$
- $T_{\mu\nu}$ has a tensor decomposition similar to $W_{\mu\nu}$, with structure functions T_1 and T_2 :

$$T_{\mu\nu} = -T_1 \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) + \frac{T_2}{P\cdot q} \left(P_\mu - q_\mu \frac{P\cdot q}{q^2} \right) \left(P_\nu - q_\nu \frac{P\cdot q}{q^2} \right)$$

▷ F_r is related to the discontinuity of T_r across the cut

Operator Product Expansion

- Consider a correlator $\left\langle \mathcal{A}(0)\mathcal{B}(x)\phi(x_1)\cdots\phi(x_n)\right\rangle$ where $\mathcal{A}$ and $\mathcal{B}$ are two local operators, possibly composite
- When $|x|\to 0$, this function is usually singular because products of operators at the same point are ill-defined
- These singularities do not depend on the nature and localization of the other fields $\phi(x_i)$
- One can obtain them from an expansion of the form

$$\mathcal{A}(0)\mathcal{B}(x) \underset{|x|\to 0}{=} \sum_i C_i(x)\ \mathcal{O}_i(0)$$

  - the $\mathcal{O}_i(0)$ are local operators with the quantum numbers of $\mathcal{AB}$
  - the $C_i(x)$ are numbers that contain the singular behavior
- When $|x|\to 0$, $C_i(x)$ behaves as

$$C_i(x) \underset{|x|\to 0}{\sim} |x|^{\mathrm{d}(\mathcal{O}_i)-\mathrm{d}(\mathcal{A})-\mathrm{d}(\mathcal{B})} \qquad \text{(up to logs)}$$

▷ only the operators with a low mass dimension matter

Operator Product Expansion of T(JJ)

- The local operators that may appear in the OPE of $T(J_\mu^\dagger(y)J_\nu(0))$ can be classified according to the representation of the Lorentz group to which they belong.

  Denote them $\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}$, where s is the "spin" of the operator, and the index i labels the various operators having the same structure.

  The OPE takes the form :

$$\sum_{s,i} C_{\mu_1\cdots\mu_s}^{s,i}(y)\ \mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}(0)$$

- The Wilson coefficients of these operators must have the following structure :

$$C_{\mu_1\cdots\mu_s}^{s,i}(y) \equiv y_{\mu_1}\cdots y_{\mu_s}\ C_{s,i}(y^2)$$

- The expectation values in the nucleon state are of the form :

$$\Big\langle\big\langle N(P)\big|\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}(0)\big|N(P)\big\rangle\Big\rangle_{\rm spin} = \left[P^{\mu_1}\cdots P^{\mu_s} + \text{trace terms}\right]\big\langle\mathcal{O}_{s,i}\big\rangle$$

Power counting and 'twist'

- Let $d_{s,i}$ be the mass dimension of the operator $\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}$
- Then, the dimension of $C_{s,i}(y^2)$ is $6+s-d_{s,i}$
  ▷ this function scales as $(y^2)^{(\boldsymbol{d}_{s,i}-s-6)/2}$ (up to logs)
- In a standard OPE, where $y_\mu \to 0$, the factor $y_{\mu_1}\cdots y_{\mu_s}$ would bring an extra $|y|^s$ to this scaling behavior, making the coefficient of $\mathcal{O}_{s,i}^{\mu_1\cdots\mu_s}$ scale as $|y|^{d_{s,i}-6}$, and high-dimension operators would be suppressed
- But in the Bjorken limit, the components of y_μ do not go to zero, and therefore the factor $y_{\mu_1}\cdots y_{\mu_s}$ should not be counted. In this case, it is the difference $d_{s,i}-s$ (called the "twist") that controls the scaling behavior of the coefficient
- The leading behavior of $T(J_\mu^\dagger(y)J_\nu(0))$ is controlled by the operators having the smallest twist. There is an infinity of them, because the dimension $\boldsymbol{d}_{s,i}$ can be compensated by a higher spin

Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s,i} \langle \mathcal{O}_{s,i} \rangle \int d^4y\, e^{iq\cdot y}\, C_{s,i}(y^2)\, (P\cdot y)^s$$

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Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s,i} \langle \mathcal{O}_{s,i} \rangle \left(-\boldsymbol{i} P_\mu \frac{\boldsymbol{\partial}}{\boldsymbol{\partial q_\mu}} \right)^s \int \boldsymbol{d^4y}\; e^{\boldsymbol{iq\cdot y}}\; C_{s,i}(y^2)$$

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Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_{s,i} \left\langle \mathcal{O}_{s,i} \right\rangle \; (-2iP\cdot q)^s \;\; \widetilde{C}^{(s)}_{s,i}(-q_\mu q^\mu)$$

Operator Product Expansion of T(JJ)

- Going back to the OPE of the structure functions T_1 and T_2, we can write generically :

$$\sum_s \boldsymbol{x}^{-s} \sum_i \left\langle \mathcal{O}_{s,i} \right\rangle \underbrace{(-i)^s \, \boldsymbol{Q}^{2s} \, \widetilde{C}^{(s)}_{s,i}(Q^2)}_{D_{s,i}(Q^2)}$$

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- Note: from their definitions, T_1 and T_2 differ by a power of P. Having the same dimension, they differ in fact by a factor x :

$$T_1(x,Q^2) = \sum_s x^{-s} \sum_i \left\langle \mathcal{O}_{s,i} \right\rangle D_{1;s,i}(Q^2)$$

$$T_2(x,Q^2) = \sum_s x^{1-s} \sum_i \left\langle \mathcal{O}_{s,i} \right\rangle D_{2;s,i}(Q^2)$$

  - Since all the powers of x and Q^2 have been counted explicitly, $D_{1;s,i}$ and $D_{2;s,i}$ can only differ by constant factors and logs
  - Note : T_1 is even and T_2 is odd in x $\triangleright$ s is even in this sum

Operator Product Expansion of T(JJ)

- The coefficient function $C_{s,i}(y^2)$ behaves like $y^{\boldsymbol{d}_{s,i}-s-6}$
  - Its Fourier transform $\widetilde{C}_{s,i}(Q^2)$ scales as $Q^{2+s-\boldsymbol{d}_{s,i}}$
  - So does $D_{r;s,i}(Q^2) \propto Q^{2s}\widetilde{C}^{(s)}_{s,i}(Q^2)$
- Therefore, if the leading twist operators correspond to $\boldsymbol{d}_{s,i} - s = 2$, we get Bjorken scaling automatically
- The coefficients $D_{r;s,i}(Q^2)$ are calculable in perturbation theory, and do not depend on the target
- The matrix elements $\left<\mathcal{O}_{s,i}\right>$ are non perturbative, and contain all the information about the target
- At this stage, the predictive power of this approach is limited to scaling properties, because we do not know the target dependent factors $\left<\mathcal{O}_{s,i}\right>$

Moments of F1 and F2

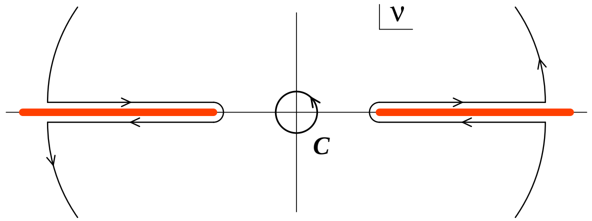
- The OPE provides a Taylor expansion of $T_{1,2}$ in powers of x^{-1} (all the x dependence is in the factor x^{-s}) :

$$T_r = \sum_{\text{even } s} t_r(s, Q^2)\, x^{a_r - s} = \sum_{\text{even } s} t_r(s, Q^2) \left(\frac{2}{Q^2}\right)^{s-a_r} \nu^{s-a_r}$$

with $a_1 = 0, a_2 = 1$. From this, we get (for s even) :

$$t_r(s, Q^2) = \frac{1}{2\pi i}\left(\frac{Q^2}{2}\right)^{s-a_r} \int_{\mathcal{C}} \frac{d\nu}{\nu}\nu^{a_r-s}\, T_r(\nu, Q^2)$$

- Do the integration by wrapping the contour around the cuts, and use the relation between F_r and the discontinuity of T_r across the cut :



$$t_r(s, Q^2) = \frac{2}{\pi}\int_0^1 \frac{dx}{x}\, x^{s-a_r}\, F_r(x, Q^2)$$

▷ the OPE gives the x-moments of the DIS structure functions

Bare Wilson coefficients

- Now, let us assume that the underlying field theory of strong interactions has spin $1/2$ fermions (quarks) and vector bosons (gluons). The operators with the lowest twist are (dimension $s+2$ and spin s, hence twist 2) :

$$\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s} \equiv \overline{\psi}_f \gamma^{\{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_s\}}\psi_f$$
$$\mathcal{O}_{s,g}^{\mu_1\cdots\mu_s} \equiv F_\alpha{}^{\{\mu_1}\partial^{\mu_2}\cdots\partial^{\mu_{s-1}}F^{\mu_s\}\alpha}$$

  where the brackets $\{\cdots\}$ denote a symmetrization of the indices $\mu_1\cdots\mu_s$ and a subtraction of the trace terms on those indices

- In order to compute the Wilson coefficients, one can exploit the fact that they do not depend on the target:

  $\triangleright$ consider an elementary target (single fermion or vector boson) for which everything is calculable (including the $\langle\mathcal{O}_{s,i}\rangle$)

Bare Wilson coefficients

- Consider a quark state of a given flavor f and spin σ. At lowest order, one has :

$$\big\langle f,\sigma\big|\mathcal{O}_{s,f'}^{\mu_1\cdots\mu_s}\big|f,\sigma\big\rangle = \delta_{ff'}\overline{u}_\sigma(P)\gamma^{\{\mu_1}u_\sigma(P)P^{\mu_2}\cdots P^{\mu_s\}}$$
$$\big\langle f,\sigma\big|\mathcal{O}_{s,g}^{\mu_1\cdots\mu_s}\big|f,\sigma\big\rangle = 0$$

- Averaging over the spin of the quark, and comparing with $P^{\mu_1}\cdots P^{\mu_s}\big\langle\mathcal{O}_{s,i}\big\rangle$, leads to :

$$\big\langle\mathcal{O}_{s,f'}\big\rangle_f = \delta_{ff'} \quad , \qquad \big\langle\mathcal{O}_{s,g}\big\rangle_f = 0$$

- On the other hand, one can calculate directly the expectation value of the current-current correlator in this quark state. This is simply done by taking the parton model results for $F_{1,2}$ and using dispersion relations to get $T_{1,2}$. For s even :

$$t_1(s,Q^2) = \frac{1}{\pi}\,e_f^2 \quad , \qquad t_2(s,Q^2) = \frac{2}{\pi}\,e_f^2$$

Bare Wilson coefficients

- Therefore, the bare coefficient functions are :

$$D_{1;s,f}(Q^2) = \frac{1}{\pi}\, e_f^2 \quad , \qquad D_{2;s,f}(Q^2) = \frac{2}{\pi}\, e_f^2$$

- Repeating the same steps with a vector boson state gives :

$$D_{1;s,g}(Q^2) = D_{2;s,g}(Q^2) = 0$$

if the vector bosons are assumed to be electrically neutral

- Going back to a nucleon target, it is convenient to define singlet quark distributions from their moments :

$$\int_0^1 \frac{dx}{x}\, x^s \left[f_f(x) + f_{\bar{f}}(x)\right] = \langle \mathcal{O}_{s,f} \rangle$$

so that :

$$F_1(x) = \frac{1}{2} \sum_f e_f^2 \left[f_f(x) + f_{\bar{f}}(x)\right] \quad , \qquad F_2(x) = 2x F_1(x)$$

Bare Wilson coefficients

- Note : by considering the Deep Inelastic Scattering of a neutrino on the same target, one would access the non singlet quark distribution. By repeating the same arguments, one would obtain :

$$\int_0^1 \frac{dx}{x}\, x^s \left[f_f(x) - f_{\bar{f}}(x)\right] = \langle \mathcal{O}_{s,f} \rangle$$

for s odd

Learnings from free field theory

- Despite the fact that the result is embarrassingly similar to what we obtained in a much simpler way in the naive parton model, this exercise has taught us several things :

- Bjorken scaling can be derived from first principles in a field theory of free fermions (somewhat disturbing given that these fermions are constituents of a strongly bound state)
- We now have an operatorial definition of the distribution $f_i(x)$ (not calculable perturbatively however)
- More importantly, the experimental observation of Bjorken scaling is telling us that the field theory of strong interactions must become a free theory in the limit $Q^2 \to +\infty$
  ▷ asymptotic freedom
- As shown by Gross, Wilczek, Politzer in 1973, non-abelian gauge theories with a reasonable number of fermionic fields (like QCD with 6 flavors of quarks) have this property

Scaling violations, DGLAP equation

Introduction

- In the previous discussion, we have implicitly assumed that there is no scale dependence in the moments $\langle \mathcal{O}_{s,i} \rangle$ of the distribution functions

- In fact, they depend on the renormalization scale μ^2
  $\triangleright$ the distribution functions are scale dependent as well

- The structure functions F_1 and F_2 being cross-sections, they cannot depend on the renormalization scale μ^2
  $\triangleright$ there should also be a μ^2 dependence in the coefficient functions, in order to compensate the μ^2 dependence from $\langle \mathcal{O}_{s,i} \rangle$

- The Wilson coefficients will be some trivial power of Q^2 imposed by their dimension (that alone would imply Bjorken scaling), times a function of the ratio Q^2/μ^2. This corrective factor will violate Bjorken scaling

Callan-Symanzik equation

- Consider the following correlators :

$$G_{JJ}(x) \equiv \langle T(J(x)J(0))\rangle \quad , \qquad G_{s,i}(0) \equiv \langle \mathcal{O}_{s,i}(0)\rangle$$

$$G_{JJ}(x) = \sum_{s,i} C_{s,i}(x) G_{s,i}(0)$$

- The Callan-Symanzik equations for G_{JJ} and $G_{s,i}$ are :

$$\left[\mu\partial_\mu + \beta\partial_g + 2\gamma_J\right] G_{JJ} = 0$$

$$\left[\left(\mu\partial_\mu + \beta\partial_g\right)\delta_{ij} + \gamma_{s,ij}\right] G_{s,j} = 0$$

where β is the beta function, γ_J the anomalous dimension of the current J (in fact $\gamma_J = 0$ for conserved currents), and $\gamma_{s,ij}$ the matrix of anomalous dimensions for the $\mathcal{O}_{s,i}$ (the operator mixing is limited to operators with the same Lorentz structure)

- By combining the previous equations, one gets :

$$\left[\left(\mu\partial_\mu + \beta\partial_g\right)\delta_{ij} - \gamma_{s;ji}\right] C_{s,j} = 0$$

Solution of the CS equation

- The dimensionless coefficients $D_{r;s,i}(Q,\mu,g)$ are in fact functions $D_{r;s,i}(Q/\mu,g)$. Under rescalings of Q, they obey :

$$\left[\left(-Q\partial_Q+\beta(g)\partial_g\right)\delta_{ij}-\gamma_{s,ji}(g)\right]D_{r;s,j}(Q/\mu,g)=0$$

- In order to solve this equation, let us first introduce the running coupling $\overline{g}(Q,g)$ such that :

$$\ln(Q/Q_0)=\int_g^{\overline{g}(Q,g)}\frac{dg'}{\beta(g')}$$

(this is equivalent to $Q\partial_Q\overline{g}(Q,g)=\beta(\overline{g}(Q,g))$ and $\overline{g}(Q_0,g)=g$)

- Any function $F(\overline{g}(Q,g))$ is a solution of

$$\left[-Q\partial_Q+\beta(g)\partial_g\right]F=0$$

- We also have

$$\left[-Q\partial_Q+\beta(g)\partial_g\right]e^{-\int_{Q_0}^{Q}\frac{dM}{M}\gamma(\overline{g}(M,g))}=\left[e^{-\int_{Q_0}^{Q}\frac{dM}{M}\gamma(\overline{g}(M,g))}\right]\gamma(g)$$

Solution of the CS equation

- Therefore, the Wilson coefficients at scale Q can be expressed in terms of the Wilson coefficients at scale Q_0 by :

$$D_{r;s,i}(Q/\mu,g) = D_{r;s,j}(Q_0/\mu,\overline{g}(Q,g)) \left[e^{-\int_{Q_0}^{Q} \frac{dM}{M} \gamma_s(\overline{g}(M,g))} \right]_{ji}$$

- If the underlying theory is asymptotically free, like QCD, then at large Q the coupling is small and we can approximate :

$$\gamma_{s,ij}(\overline{g}) = \overline{g}^2 A_{ij}(s) \quad , \qquad \overline{g}^2(Q,g) = \frac{8\pi^2}{\beta_0 \ln(Q/\Lambda_{QCD})}$$

where the $A_{ij}(s)$ are given by a 1-loop perturbative calculation

- Finally, the solution can be rewritten as :

$$D_{r;s,i}(Q/\mu,g) = D_{r;s,j}(Q_0/\mu,\overline{g}(Q,g)) \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})} \right)^{-\frac{8\pi^2}{\beta_0} A(s)} \right]_{ji}$$

Scaling violations in F1 and F2

- The moments of the structure function F_1 at scale Q^2 read :

$$\int_0^1 \frac{dx}{x}\, x^s\, F_1(x,Q^2) = \sum_{i,f} \frac{e_f^2}{2} \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})} \right)^{-\frac{8\pi^2}{\beta_0}A(s)} \right]_{fi} \langle \mathcal{O}_{s,i} \rangle_{Q_0}$$

- F_1 keeps the same form $F_1(x,Q^2) = \frac{1}{2}\sum_f e_f^2\, [f_f + f_{\bar{f}}]$, provided we define singlet quark distributions by:

$$\int_0^1 \frac{dx}{x}\, x^s \left[f_f(x,Q^2) + f_{\bar{f}}(x,Q^2) \right] \equiv \sum_i \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})} \right)^{-\frac{8\pi^2}{\beta_0}A(s)} \right]_{fi} \langle \mathcal{O}_{s,i} \rangle_{Q_0}$$

  - The quark distribution is now Q^2 dependent
  - It depends on the expectation value of operators involving gluons

- Scaling violations at LO preserve the Callan-Gross relation :

$$F_2(x,Q^2) = 2x F_1(x,Q^2)$$

Probabilistic interpretation

- In order to make the interpretation of the Q dependence more transparent, let us introduce as well a gluon distribution, even though it is not probed directly in DIS :

$$\int_0^1 \frac{dx}{x}\, x^s\, f_g(x,Q^2) \equiv \sum_i \left[\left(\frac{\ln(Q/\Lambda_{QCD})}{\ln(Q_0/\Lambda_{QCD})}\right)^{-\frac{8\pi^2}{\beta_0}A(s)}\right]_{gi} \langle \mathcal{O}_{s,i}\rangle_{Q_0}$$

- The derivative of the moments of the parton distributions with respect to $\ln(Q^2)$ is $(\boldsymbol{f}_f \equiv f_f + f_{\bar{f}}\ ,\ \boldsymbol{f}_g \equiv f_g)$:

$$Q^2 \frac{\partial \boldsymbol{f}_i(s,Q^2)}{\partial Q^2} = -\frac{\overline{g}^2(Q,g)}{2} A_{ji}(s)\, \boldsymbol{f}_j(s,Q^2)$$

- In order to go further, we need the following result :

$$A(s)\boldsymbol{f}(s) = \int_0^1 \frac{dx}{x}\, x^s \int_x^1 \frac{dy}{y}\, A(x/y)\, \boldsymbol{f}(y)$$

Probabilistic interpretation

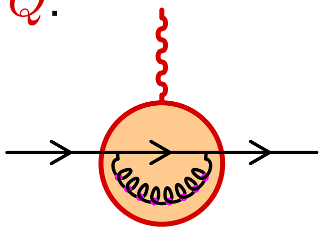
- Define the splitting functions P_{ij} from their moments :

$$\int_0^1 \frac{dx}{x}\, x^s\, P_{ij}(x) \equiv -4\pi^2 A_{ij}(s)$$

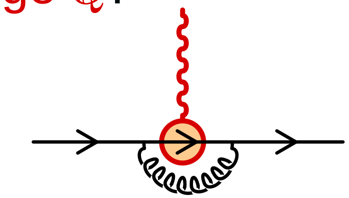
- Therefore, one has the following evolution equation for $f_i(x, Q^2)$ (DGLAP) :

$$Q^2 \frac{\partial f_i(x,Q^2)}{\partial Q^2} = \frac{\overline{g}^2(Q,g)}{8\pi^2} \int_x^1 \frac{dy}{y}\, P_{ji}(x/y) f_j(y, Q^2)$$

- Interpretation : the resolution of the γ^* changes with Q
  - Low Q:



  - Large Q:



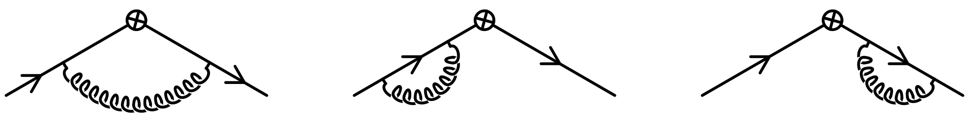
  - $\overline{g}^2 P_{ji}(z)$ describes the splitting $j \rightarrow i$, where the daughter parton takes the fraction z of the momentum of the original parton

Anomalous dimensions

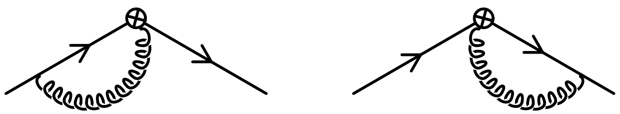
- The anomalous dimension of an operator $\mathcal{O}$ is given by :

$$\gamma_{\mathcal{O}} = \frac{\mu}{Z_{\mathcal{O}}}\frac{\partial Z_{\mathcal{O}}}{\partial\mu} \quad , \qquad \text{where} \quad \mathcal{O}_{\text{renormalized}} = Z_{\mathcal{O}}{}^{-1}\mathcal{O}_{\text{bare}}$$

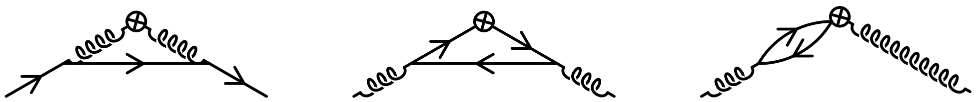
- At 1-loop, the operator $\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s}$ has the following corrections :



- Moreover, to ensure gauge invariance, the operator $\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s}$ should be defined as : $\mathcal{O}_{s,f}^{\mu_1\cdots\mu_s} \equiv \overline{\psi}_f\gamma^{\{\mu_1}D^{\mu_2}\cdots D^{\mu_s\}}\psi_f$
Therefore, one has also the following 1-loop diagrams :



- For s even, there are mixings between $\mathcal{O}_{s,f}$ and $\mathcal{O}_{s,g}$



(for s odd, $\mathcal{O}_{s,g}$ is a total derivative that does not play any role)

Anomalous dimensions

- At 1-loop, the coefficients $A_{ij}(s)$ in the anomalous dimensions are :

$$A_{gg}(s) = \frac{1}{2\pi^2}\left\{3\left[\frac{1}{12} - \frac{1}{s(s-1)} - \frac{1}{(s+1)(s+2)} + \sum_{j=2}^{s}\frac{1}{j}\right] + \frac{N_f}{6}\right\}$$

$$A_{gf}(s) = -\frac{1}{4\pi^2}\left\{\frac{1}{s+2} + \frac{2}{s(s+1)(s+2)}\right\}$$

$$A_{fg}(s) = -\frac{1}{3\pi^2}\left\{\frac{1}{s+1} + \frac{2}{s(s-1)}\right\}$$

$$A_{ff'}(s) = \frac{1}{6\pi^2}\left\{1 - \frac{2}{s(s+1)} + 4\sum_{j=2}^{s}\frac{1}{j}\right\}\delta_{ff'}$$

- Since $A_{gf}(s)$ is flavor independent, the non-singlet linear combinations ($\sum_f a_f \mathcal{O}_{s,f}$ with $\sum_f a_f = 0$) are eigenvectors of the matrix of anomalous dimensions, with an eigenvalue $A_{ff}(s)$
These linear combinations do not mix with the remaining two operators, $\sum_f \mathcal{O}_{s,f}$ and $\mathcal{O}_{s,g}$, through renormalization

Valence sum rules (s=1)

- In the case of $s = 1$, the anomalous dimension of the non-singlet quark operators is

$$A_{ff}(s = 1) = 0$$

- Going back to the evolution equation for the moments of quark distributions, this means that we have :

$$\frac{\partial}{\partial Q^2} \left\{ \int_0^1 dx \sum_f a_f \left[f_f(x, Q^2) + f_{\bar{f}}(x, Q^2) \right] \right\} = 0$$

for any linear combination such that $\sum_f a_f = 0$

- For instance, for a nucleon, this implies that the number of $u + \overline{u}$ quarks minus the number of $d + \overline{d}$ quarks is independent of Q^2
- Interpretation : the production of extra quarks by $g \to q\bar{q}$ produces quarks of all flavors in equal numbers

Momentum sum rule (s=2)

- In the singlet sector, the matrix of anomalous dimensions for $s = 2$ reads :

$$\begin{pmatrix} A_{ff}(2) & A_{fg}(2) \\ N_f A_{gf}(2) & A_{gg}(2) \end{pmatrix} = \frac{1}{\pi^2} \begin{pmatrix} \frac{4}{9} & -\frac{4}{9} \\ -\frac{N_f}{12} & \frac{N_f}{12} \end{pmatrix}$$

- This matrix has a vanishing determinant, which means that a linear combination of the flavor singlet operators is not renormalized : $\mathcal{O}^{\mu\nu}_{2,g} + \sum_f \mathcal{O}^{\mu\nu}_{2,f}$

- This leads also to a sum rule :

$$\frac{\partial}{\partial Q^2} \left\{ \int_0^1 dx\, x \left[\sum_f \left[f_f(x,Q^2) + f_{\bar{f}}(x,Q^2) \right] + f_g(x,Q^2) \right] \right\} = 0$$

- Interpretation : the total longitudinal momentum of the target is conserved, and the momentum that goes into the newly produced gluons must be taken from the quarks

Practical strategy

- Due to the non-perturbative nature of the parton distributions at a given fixed scale Q, it does not make sense to try to predict the value of F_r at a given Q out of nothing
- Instead,
  - fit the parton distributions from the measurement of F_r at a moderately low scale Q_0
  - using DGLAP, evolve them to a higher scale Q
  - predict the values of the structure functions F_r at the scale Q
  - compare with DIS measurements
- This approach can be systematically improved by going to higher order, both for the hard subprocess, and for the splitting functions and beta function
- Current state of the art :
  - NNLO program fully implemented (3-loop splitting functions : Moch, Vermaseren, Vogt (2004))

HERA results for F2

Kinematics of DIS

Experimental facts

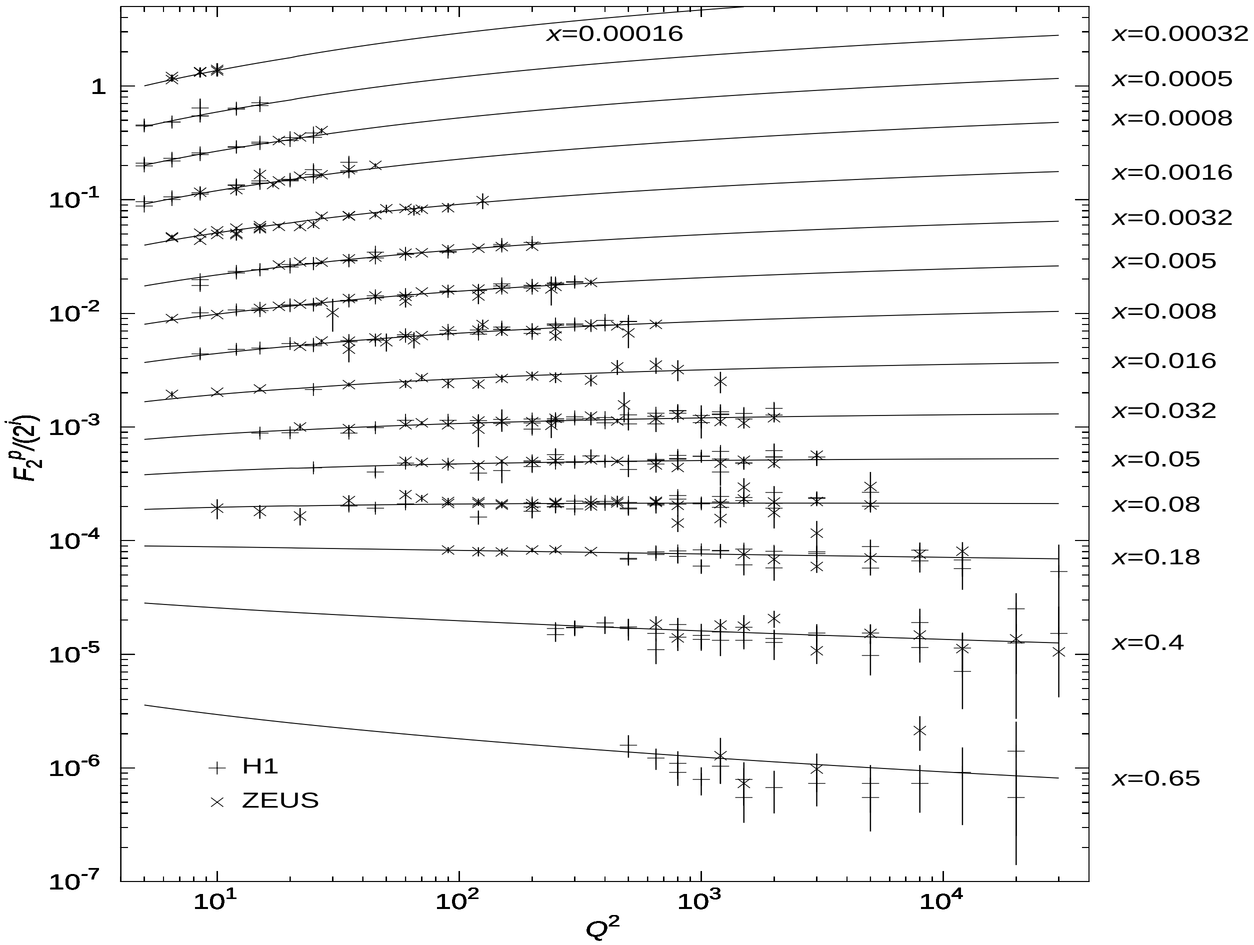
Naive parton model

Bjorken scaling from field theory

Scaling violations
- Callan-Symanzik equation
- Solution of the CS equation
- Scaling violations
- Probabilistic interpretation
- Anomalous dimensions
- Valence sum rules
- Momentum sum rule
- Practical strategy
- HERA results for F2

Factorization

- HERA results and NLO DGLAP fit :



Factorization

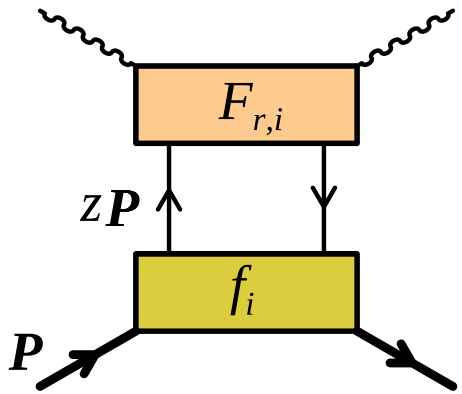
Factorization in DIS

- The DIS structure functions can be written as :

$$F_r(x,Q^2) = \sum_i \int_x^1 dz\, f_i(z,Q^2) F_{r,i}(x/z,Q^2) + \mathcal{O}\left(\frac{m_N^2}{Q^2}\right)$$

  - $F_{r,i}$ is the structure function for a target parton i (at leading order, it is non-zero only for quarks)
  - x/z is the Bjorken-x variable for the system $\gamma^* i$

- Schematically, one can represent this factorization as :

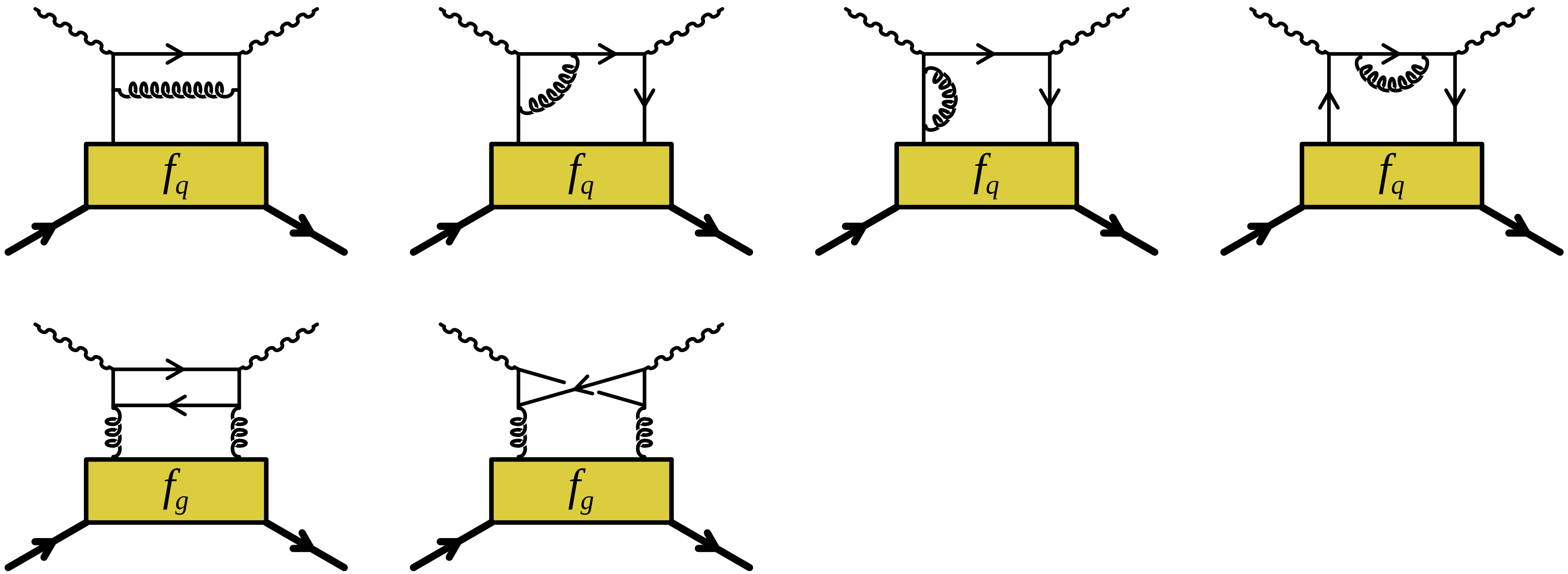


Factorization in DIS

- In perturbation theory, the terms included by the RG evolution correspond to factors of g^2 enhanced by large logarithms :

$$g^2 \, \ln\left(Q^2/\mu^2\right) \qquad \text{where } \mu^2 \text{ is some soft cutoff}$$

- The logs are due to collinear divergences in loop corrections to $F_{r,i}$. The first power of $g^2 \ln(Q^2/\mu^2)$ comes from :

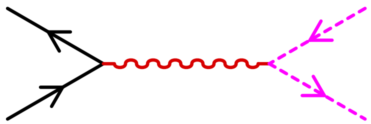


Factorization in DIS - Beyond LO

- For DIS, the procedure for going to NLO is straightforward and dictated by the OPE approach. One needs the following quantities at NLO :
  - coefficient functions
  - beta function
  - anomalous dimensions (or splitting functions)
- Changes compared to LO :
  - The Callan-Gross relation does not hold anymore
  - There are various ways to define parton distributions: they are not directly measurable, and one should regard them as an intermediate device to relate various measurable cross-sections. The hard scattering part of the factorization formula must be changed accordingly
  - Some parton sum rules may get modified at NLO

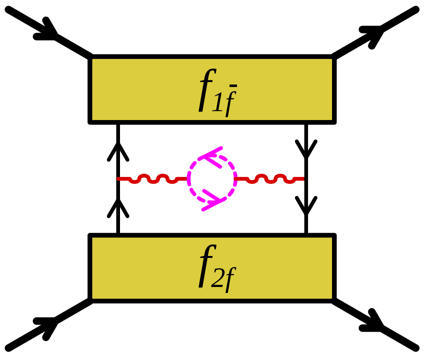
Factorization in Drell-Yan

- The Drell-Yan process is a reaction between two hadrons in which a virtual photon is produced, that later decays into a lepton-antilepton pair
- At the parton level, the simplest process responsible for this reaction is a $q\bar{q} \to \gamma^*$ annihilation :



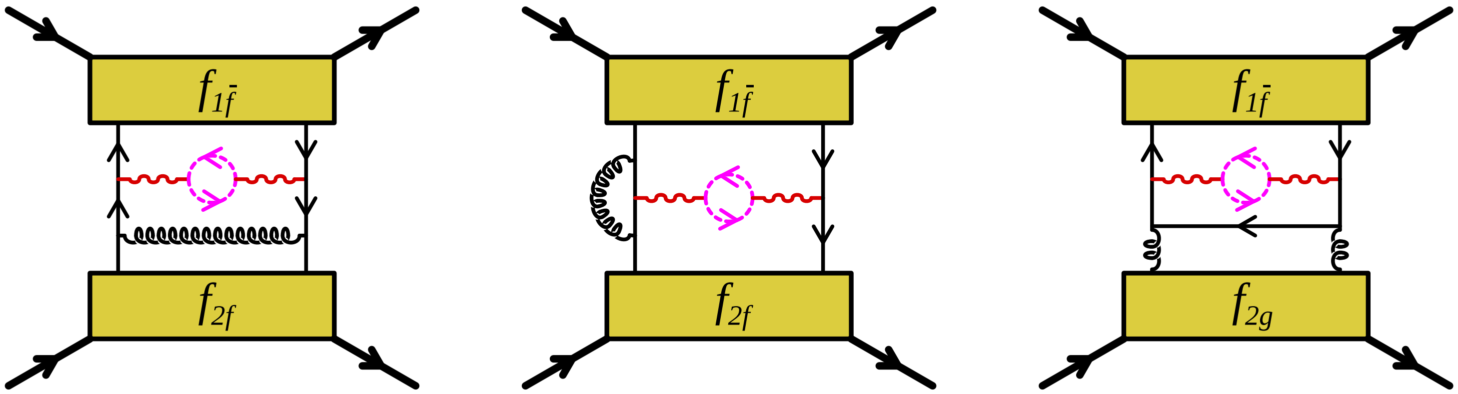
- The cross-section in the naive parton model reads :

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9Q^4} \sum_f e_f^2 \int_0^1 dx_1\, dx_2\, x_1 x_2\, \delta(x_1 x_2 - Q^2/s) \times \Big[f_{1f}(x_1) f_{2\bar{f}}(x_2) + f_{1\bar{f}}(x_1) f_{2f}(x_2) \Big]$$



Factorization in Drell-Yan

- Sample of loop diagrams with leading-log contributions :



- At LO, the naive parton model Drell-Yan formula remains true after resummation of all the leading log corrections, modulo the replacement $f_{if}(x_i) \to f_{if}(x_i, Q^2)$, with the same distribution functions as in DIS :

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{9Q^4} \sum_f e_f^2 \int_0^1 dx_1\, dx_2\, x_1 x_2\, \delta(x_1 x_2 - Q^2/s)$$
$$\times \left[f_{1f}(x_1, Q^2) f_{2\bar{f}}(x_2, Q^2) + f_{1\bar{f}}(x_1, Q^2) f_{2f}(x_2, Q^2) \right]$$

Collinear factorization

- Factorization is the possibility to resum all the powers $[g^2 \ln(Q^2/\mu^2)]^n$ into universal parton distributions
  - The neglected contributions are suppressed by powers of $1/Q$
  - The hard subprocess is infrared safe
- The "bare" parton distributions are turned into Q-dependent distributions, that obey the DGLAP equation
- The universality of the parton distributions confers to QCD a much stronger predictive power, since the distributions measured in DIS can be used to predict other processes
- Interactions due to soft gluons in the final state cancel when one sums over degenerate final states (KLN)
- Crucial for factorization is the large difference between the short and long timescales : at high energy, internal hadronic timescales get dilated while the duration of the interaction goes to zero because of Lorentz contraction

Separation of timescales

- Consider a massless parton of longitudinal momentum p splitting into two partons of longitudinal momenta zp and $(1-z)p$ and transverse momenta $+\vec{\boldsymbol{k}}_\perp$ and $-\vec{\boldsymbol{k}}_\perp$. Their energies are :

$$E_0 = p \quad , \quad E_1 \approx |z|p + \frac{\vec{\boldsymbol{k}}_\perp^2}{2|z|p} \quad , \quad E_2 \approx |1-z|p + \frac{\vec{\boldsymbol{k}}_\perp^2}{2|1-z|p}$$

- The lifetime of this fluctuation is given by :

$$\tau_{\text{fluct}}^{-1} \sim E_1 + E_2 - E_0 = (|z| + |1-z| - 1)p + \frac{\vec{\boldsymbol{k}}_\perp^2}{2p}\left(\frac{1}{|z|} + \frac{1}{|1-z|}\right)$$

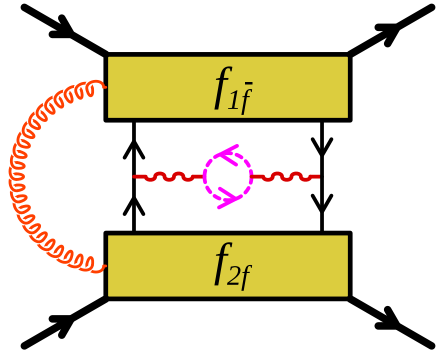
- If $z < 0$ or $z > 1$, this fluctuation is very short-lived
- If $0 < z < 1$, $|z| + |1-z| = 1$, and the lifetime becomes :

$$\tau_{\text{fluct}} \sim 2z(1-z)p/\vec{\boldsymbol{k}}_\perp^2$$

- This must be compared with the interaction time of the virtual photon : $\tau_{\text{int}} \sim p/Q^2$. For the collinear contributions: $\vec{\boldsymbol{k}}_\perp^2 \ll Q^2$, hence $\tau_{\text{int}} \ll \tau_{\text{fluct}}$

Initial state interactions

- A major complication in processes with two incoming hadrons, like Drell-Yan, is the possibility that the two hadrons may be connected by soft gluons before the collision :



- This could have the disastrous effect of making the parton distributions of a hadron non-universal
- Such interactions can be seen as the interactions of one projectile with the Coulomb field of the other projectile
- For very high energy projectiles, Lorentz contraction implies that the field strength $F_{\mu\nu}$ is localized on a sheet perpendicular to the trajectory. Therefore, it cannot affect the contents of the other hadron before the collision

Final state : infrared safety

- Infrared divergences cancel when one sums over all the possible final states (Kinoshita-Lee-Nauenberg theorem)
- One can see such a cross-section as the sum of cuts through a forward scattering amplitude. Each individual cut is a divergent contribution, but the sum of all the cuts is finite
- Completely inclusive final states are not the only ones to be infrared safe. Consider the following weighted cross-section :

$$\sigma_S \equiv \int [d\Phi_n]\,\frac{d\sigma}{d\Phi_n}\,S_n(p_1,\cdots,p_n)$$

  - Such a final state is infrared safe if the function S_n gives the same weight to configurations that differ by a soft gluon, or that are identical up to the collinear splitting of a hard parton
  - Indeed, all the cuts through a potentially dangerous loop correction in the forward amplitude have the same weight, and the KLN cancellation works in the same manner as in the completely inclusive case

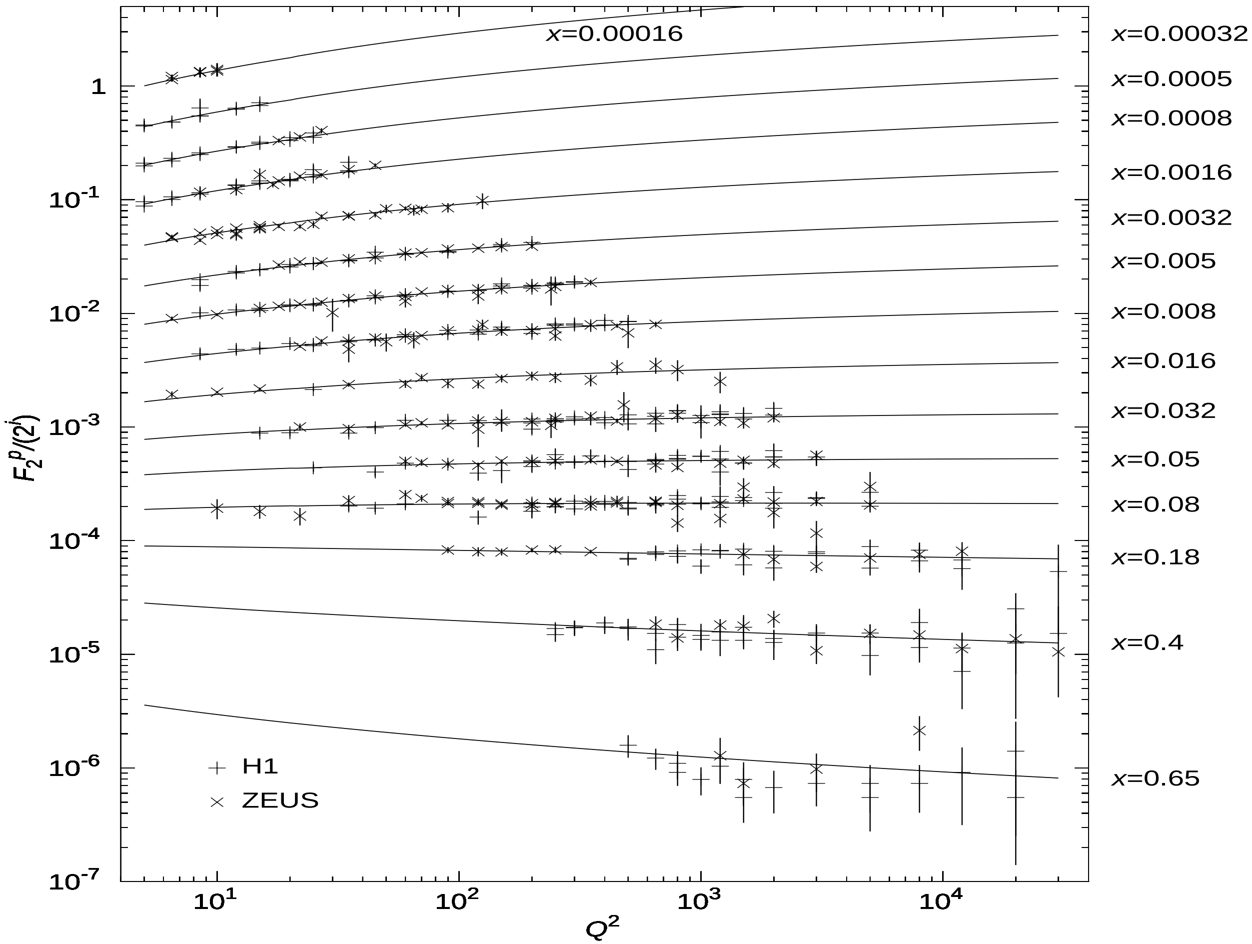
Final state : inclusive hadrons

- When considering a specific hadron in the final state, one needs a fragmentation function $D_{H/i}(z,\mu^2)$, which represent the probability to obtain the hadron H from the parton i with a momentum fraction z
- Again, such a probabilistic description is possible thanks to the incoherence of the hadronization process with respect to the hard scattering :
  - The process of hadronization occurs over timescales which are large compared to that of hard processes
  - Moreover, the hadronization of a particular parton does not depend on the other hard partons produced in the event
- The resummation of leading logarithms leads to a scale dependence of the fragmentation functions, which obey a DGLAP equation

Motivation for Lecture II

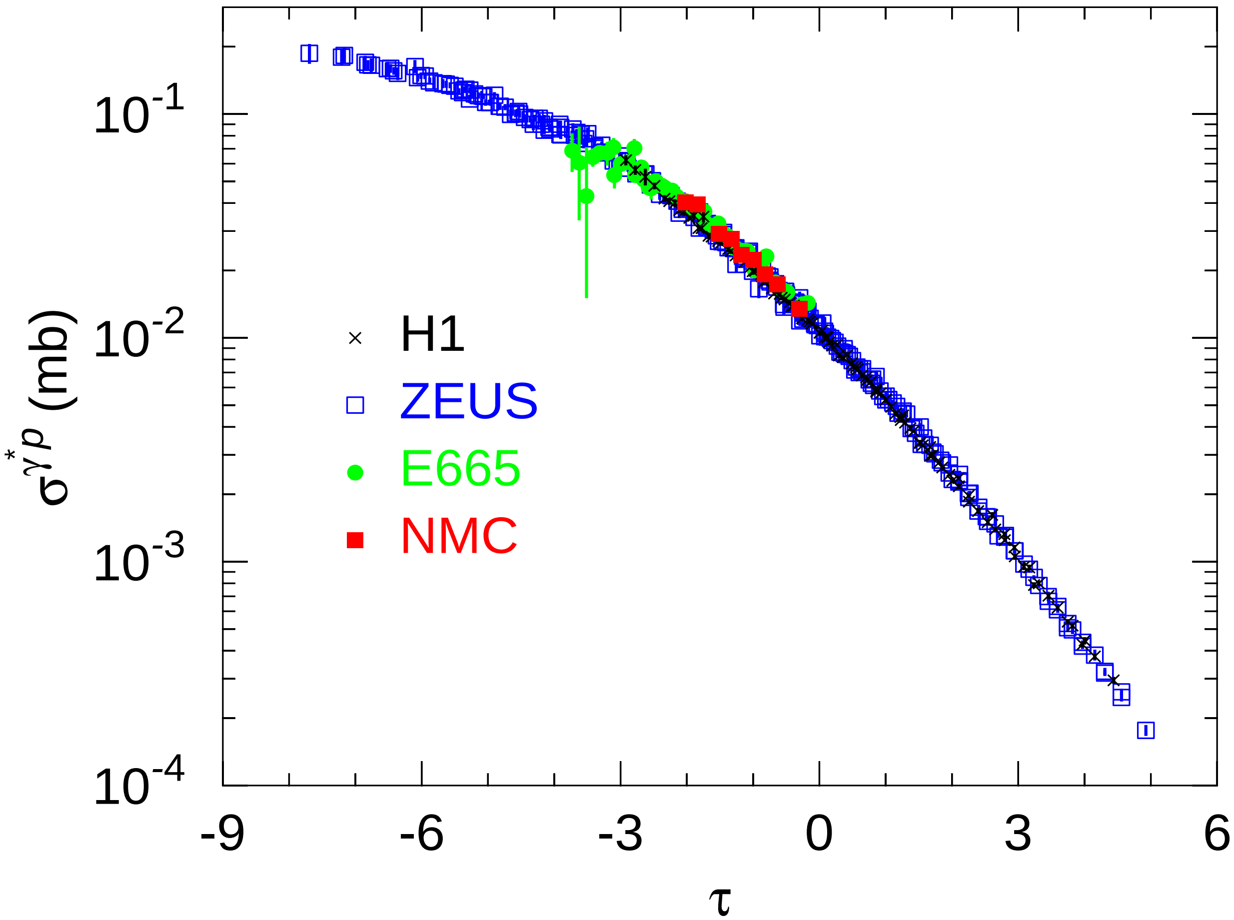
HERA results for F2

- HERA results and NLO DGLAP fit :



Same data displayed differently...

- Small x data ($x \le 10^{-2}$) displayed against $\tau = x^{0.32}\,Q^2$:



Lecture II

- Eikonal scattering
- BFKL equation
- Saturation of parton distributions
- Balitsky-Kovchegov equation
- Color Glass Condensate - JIMWLK
- Analogies with reaction-diffusion processes