High energy hadronic interactions in QCD and applications to heavy ion collisions

III – QCD on the light-cone

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General outline

- Lecture I : Introduction and phenomenology
- Lecture II : Lessons from Deep Inelastic Scattering
- Lecture III : QCD on the light-cone
- Lecture IV : Saturation and the Color Glass Condensate
- Lecture V : Calculating observables in the CGC

Lecture III : QCD on the light-cone

- Light-cone coordinates - Infinite Momentum Frame
- Poincaré algebra on the light-cone - Galilean sub-algebra
- Canonical quantization on the light-cone
- Scattering by an external potential
- Light-cone QCD

Motivation

- The Operator Product Expansion provides a rigorous way of justifying the parton model in the case of the Deep Inelastic Scattering reaction, in the Bjorken limit ($Q^2 \to +\infty$, $x =$constant)
- Unfortunately, for reactions with a more involved kinematics, there is usually no region of phase-space in which the OPE provides useful results
- Here, we aim at finding a framework which, although less rigorous than the OPE, can be used in more diverse situations
- QCD on the light-cone is a formulation of QCD in which the main ideas of the parton model appear in a transparent way

Light-cone coordinates

- Light-cone coordinates are defined by choosing a privileged axis (generally the z axis) along which particles have a large momentum. Then, for any 4-vector a^μ, one defines :

$$a^+ \equiv \frac{a^0 + a^3}{\sqrt{2}} \quad , \quad a^- \equiv \frac{a^0 - a^3}{\sqrt{2}}$$

$a^{1,2}$ unchanged. Notation : $\vec{\boldsymbol{a}}_\perp \equiv (a^1, a^2)$

which can be inverted by

$$a^0 = \frac{a^+ + a^-}{\sqrt{2}} \quad , \quad a^3 = \frac{a^+ - a^-}{\sqrt{2}}$$

- Some useful formulas :

$$x \cdot y = x^+ y^- + x^- y^+ - \vec{\boldsymbol{x}}_\perp \cdot \vec{\boldsymbol{y}}_\perp$$

$$d^4x = dx^+ dx^- d^2\vec{\boldsymbol{x}}_\perp$$

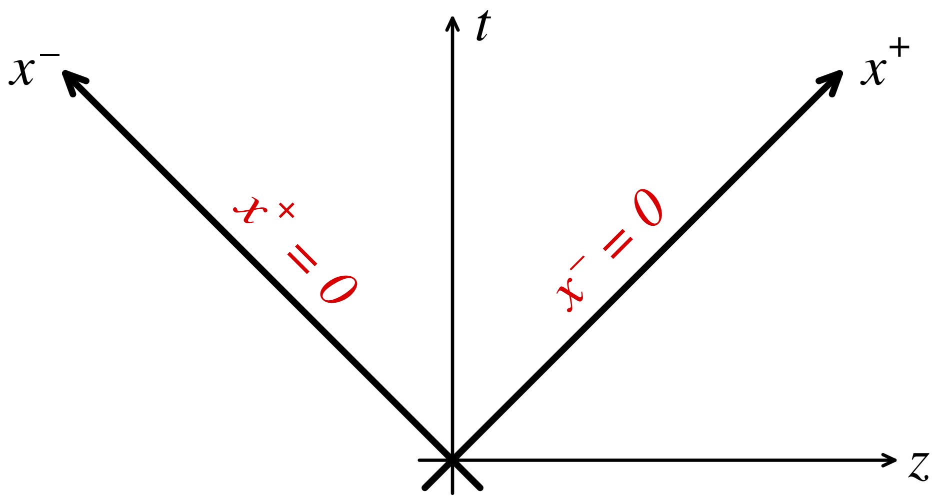
$$\square = 2\partial^+\partial^- - \vec{\boldsymbol{\nabla}}_\perp^2$$

Notation : $\partial^+ \equiv \frac{\partial}{\partial x^-}$, $\partial^- \equiv \frac{\partial}{\partial x^+}$

Metric tensor

- Remarks :
  - The Dalembertian is bilinear in the derivatives ∂^+, ∂^-
  - The metric tensor is non diagonal :

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$



Definition of the Poincaré group

- The Poincaré group is the 10-dimensional group of transformations that contains :
  - 4 translations : P^α
  - 3 spatial rotations : $J^i = \frac{1}{2}\epsilon^{ijk}M^{jk}$
  - 3 Lorentz boosts : $K^i = M^{i0}$
- Note: it is a subgroup of the 15-dimensional conformal group, i.e. the group of transformations that preserve the relation $dx_\alpha dx^\alpha = 0$. In addition to the transformations of the Poincaré group, the conformal group contains 1 dilatation and 4 non-linear conformal transformations
- The commutation relations among the generators are :

$$\left[P^\alpha, P^\beta\right] = 0$$

$$\left[M^{\alpha\beta}, P^\delta\right] = i\left(g^{\beta\delta}P^\alpha - g^{\alpha\delta}P^\beta\right)$$

$$\left[M^{\alpha\beta}, M^{\delta\gamma}\right] = i\left(g^{\alpha\gamma}M^{\beta\delta} + g^{\beta\delta}M^{\alpha\gamma} - g^{\alpha\delta}M^{\beta\gamma} - g^{\beta\gamma}M^{\beta\delta}\right)$$

Light-cone Poincaré algebra

- In light-cone coordinates, the previous commutation relations should be unchanged, provided we use the transformed $g^{\mu\nu}$ (they have a geometrical meaning, and do not depend on the system of coordinates)

- In order to obtain the transformed generators, notice that :

if $x^\alpha \equiv (x^0, x^1, x^2, x^3)$ and $x^\mu \equiv (x^+, x^1, x^2, x^-)$,

$$x^\mu = C^\mu{}_\alpha x^\alpha \quad , \quad \text{with } C^\mu{}_\alpha \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

- This also applies to tensors :

$$T^{\mu_1 \cdots \mu_n} = C^{\mu_1}{}_{\alpha_1} \cdots C^{\mu_n}{}_{\alpha_n} \, T^{\alpha_1 \cdots \alpha_n}$$

Light-cone Poincaré algebra

- Light-cone Poincaré generators :

$$P^\mu = \begin{pmatrix} P^+ \\ P^1 \\ P^2 \\ P^- \end{pmatrix} \quad , \quad M^{\mu\nu} = \begin{pmatrix} 0 & -S^1 & -S^2 & K^3 \\ S^1 & 0 & J^3 & B^1 \\ S^2 & -J^3 & 0 & B^2 \\ -K^3 & -B^1 & -B^2 & 0 \end{pmatrix}$$

with

$$\begin{cases} B^1 \equiv \dfrac{K^1 + J^2}{\sqrt{2}} \quad , \quad B^2 \equiv \dfrac{K^2 - J^1}{\sqrt{2}} \\ S^1 \equiv \dfrac{K^1 - J^2}{\sqrt{2}} \quad , \quad S^2 \equiv \dfrac{K^2 + J^1}{\sqrt{2}} \end{cases}$$

Kinematic transformations

- 6 of the light-cone Poincaré generators leave $x^+ = \text{const}$:
  - $J^3, P^{1,2}$: do not touch t, nor z
  - P^+ : increases t and decreases z by the same amount
  - $B^{1,2}$: not so obvious...
- $B^1 \propto K^1 + J^2$
  - K^1 : infinitesimal boost in the x direction

$$t' = t + \omega x$$
$$x' = \omega t + x$$

  - J^2 : infinitesimal rotation around the y axis

$$x' = x + \omega z$$
$$z' = -\omega x + z$$

  - Hence, $t' + z' = t + z$, and x^+ is left unchanged

Galilean sub-algebra

- P^- generates translations in the x^+ direction
- The set $\{J^3, P^+, P^{1,2}, B^{1,2}, P^-\}$ generates a 7-dimensional sub-algebra of the Poincaré algebra :

$$\left[P^+, P^-\right] = \left[P^+, P^j\right] = \left[P^+, J^3\right] = \left[P^+, B^j\right] = 0$$
$$\left[P^-, P^j\right] = \left[P^-, J^3\right] = 0 \, , \quad \left[P^-, B^j\right] = iP^j$$
$$\left[J^3, P^j\right] = i\epsilon^{jk} P^k \, , \quad \left[J^3, B^j\right] = i\epsilon^{jk} B^k \, , \quad \left[P^k, B^j\right] = i\delta^{jk} P^+$$

- This sub-algebra is isomorphic to the algebra of Galilean transformations in 2 dimensional quantum mechanics :

P^+	$\longleftrightarrow$	mass
P^-	$\longleftrightarrow$	Hamiltonian (the “time” is x^+)
J^3	$\longleftrightarrow$	rotation in the x, y plane
$P^{1,2}$	$\longleftrightarrow$	translations in the x, y plane
$B^{1,2}$	$\longleftrightarrow$	Galilean boosts in the x, y plane

Galilean sub-algebra

- Most of these commutation relations are obvious. Those involving the Galilean boosts B^j need some extra explanations...
- In non-relativistic 2-dimensional quantum mechanics, Galilean boosts are the transformations that change the velocity v^j

$$v^j \longrightarrow v^j + \delta v^j$$

- The commutation relations involving B^i with the other generators can be understood from the way the other quantities transform under the Galilean boosts :

$$M \longrightarrow M \qquad \Rightarrow \qquad \left[P^+, B^j\right] = 0$$

$$E \longrightarrow E + p^j \delta v^j \qquad \Rightarrow \qquad \left[P^-, B^j\right] = iP^j$$

$$p^k \longrightarrow p^k + \delta^{jk} M \delta v^j \qquad \Rightarrow \qquad \left[P^k, B^j\right] = i\delta^{jk} P^+$$

$$J \longrightarrow J + \epsilon^{jk} M x^k \delta v^j \qquad \Rightarrow \qquad \left[J^3, B^j\right] = i\epsilon^{jk} B^k$$

Galilean sub-algebra

Susskind (1968)

- The existence of this sub-algebra is responsible for the non-relativistic features of quantum field theories on the light-cone. For instance, the Hamiltonian of a free particle reads :

$$P^- = \frac{\vec{\boldsymbol{P}}_\perp^2}{2P^+}$$

- For a particle moving at the speed of light in the $+z$ direction, $x^- = 0$, while x^+ increases as the particle moves on its trajectory. Therefore, it is legitimate to interpret x^+ as the "time". And P^-, which generates translations in x^+, is indeed a Hamiltonian

Action of K-

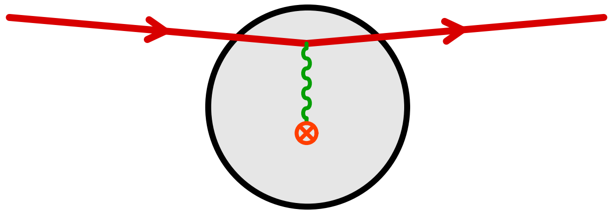
- The action of boosts in the direction of x^- is also quite remarkable :

$$
\begin{aligned}
e^{i\omega K^-}\; P^-\; e^{-i\omega K^-} &= e^{-\omega}\, P^- \\
e^{i\omega K^-}\; P^+\; e^{-i\omega K^-} &= e^{+\omega}\, P^+ \\
e^{i\omega K^-}\; P^j\; e^{-i\omega K^-} &= P^j \\
e^{i\omega K^-}\; J^3\; e^{-i\omega K^-} &= J^3 \\
e^{i\omega K^-}\; B^j\; e^{-i\omega K^-} &= e^{+\omega}\, B^j \\
e^{i\omega K^-}\; S^j\; e^{-i\omega K^-} &= e^{-\omega}\, S^j
\end{aligned}
$$

  - Simple rescaling of the various operators. This is another hint that the light-cone framework might be simpler in order to study processes involving very fast particles
  - These relations will play an essential role when we discuss the eikonal approximation

Motivation

- In order to show how the parton picture emerges in this formulation, let us study a scattering process by an external potential
- Our goal is to calculate the scattering amplitude of some state going through the external potential, in the limit where the particles approach the speed of light



- More precisely, we want to calculate :

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega\to+\infty} \langle\beta_{\rm in}|e^{i\omega K^-}U(+\infty,-\infty)e^{-i\omega K^-}|\alpha_{\rm in}\rangle$$

- In order to study this limit, we will need a perturbation theory "ordered in x^+". As we shall see, this is provided by "light-cone quantization"

Model

- In order to keep the discussion elementary, we first consider a scalar field theory, whose Lagrangian density is :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3$$

- The conjugate momentum of ϕ is :

$$\Pi_\phi = \frac{\delta \mathcal{L}}{\delta \partial^- \phi} = \partial^+ \phi$$

- The Hamiltonian density reads :

$$\mathcal{H} = \Pi_\phi \partial^- \phi - \mathcal{L} = \frac{1}{2}\left(\vec{\nabla}_\perp \phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3$$

Interaction picture

- From the Heisenberg field ϕ, one defines the field $\phi_{\rm in}$ of the interaction picture by :

$$\phi(x) \equiv U(-\infty, x^+)\phi_{\rm in}(x)U(x^+, -\infty)$$

where $\quad U(x^+, -\infty) \equiv T_+ \exp i \int_{-\infty}^{x^+} \mathcal{L}_{\rm int}(\phi_{\rm in}(x))\; d^4x$

- $\phi_{\rm in}$ is a free field if ϕ obeys the full equation of motion :

$$(\square + m^2)\phi(x) - \frac{\mathcal{L}_{\rm int}(\phi)}{\delta\phi(x)} = U(-\infty, x^+)\left[(\square + m^2)\phi_{\rm in}(x)\right]U(x^+, -\infty)$$

- Being a free field, $\phi_{\rm in}$ can be decomposed as :

$$\phi_{\rm in}(x) \equiv \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\left[e^{-ip\cdot x}a_{\rm in}(p) + e^{ip\cdot x}a^\dagger_{\rm in}(p)\right]$$

where implicitly $\quad p^- \equiv (\vec{\boldsymbol{p}}_\perp^2 + m^2)/2p^+$

Free Hamiltonian

- The conjugate momentum of $\phi_{\rm in}$ is given by :

$$\Pi_{\phi_{\rm in}}(x) = \partial^+\phi_{\rm in}(x) = -i\int \frac{dp^+}{4\pi}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\left[e^{-ip\cdot x}a_{\rm in}(p) - e^{ip\cdot x}a^\dagger_{\rm in}(p)\right]$$

- The free Hamiltonian is :

$$\begin{aligned} H_{\rm free} &= \int dx^- d^2\vec{\boldsymbol{x}}_\perp \left[\frac{1}{2}\left(\vec{\boldsymbol{\nabla}}_\perp\phi_{\rm in}\right)^2 + \frac{1}{2}m^2\phi_{\rm in}^2\right] \\ &= \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\; p^-\;\frac{1}{2}\left[a_{\rm in}(p)a^\dagger_{\rm in}(p) + a^\dagger_{\rm in}(p)a_{\rm in}(p)\right] \end{aligned}$$

- After normal ordering, this reads :

$$H_{\rm free} = \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\; p^-\; a^\dagger_{\rm in}(p)a_{\rm in}(p)$$

Commutation relations

- For $a_{\rm in}^\dagger(p)$ and $a_{\rm in}(p)$ to have the proper interpretation as operators that create or destroy a quantum of energy p^-, we must have :

$$\begin{cases} \left[H_{\rm free}, a_{\rm in}(p)\right] = -p^-\, a_{\rm in}(p) \\ \left[H_{\rm free}, a_{\rm in}^\dagger(p)\right] = p^-\, a_{\rm in}^\dagger(p) \end{cases}$$

- These relations will hold provided we have :

$$\left[a_{\rm in}(p), a_{\rm in}^\dagger(q)\right] = (2\pi)^3\, 2p^+\, \delta(p^+ - q^+)\delta(\vec{\boldsymbol p}_\perp - \vec{\boldsymbol q}_\perp)$$

- From this follows the equal-x^+ canonical commutator :

$$\left[\phi_{\rm in}(x), \Pi_{\phi_{\rm in}}(y)\right]_{x^+=y^+} = \frac{i}{2}\delta(x^- - y^-)\delta(\vec{\boldsymbol x}_\perp - \vec{\boldsymbol y}_\perp)$$

Scattering theory

- Start from transition amplitudes defined as :

$$S_{\beta\alpha} \equiv \left\langle \beta_{\rm in} \big| U(+\infty, -\infty) \big| \alpha_{\rm in} \right\rangle$$

- The states $\big|\alpha_{\rm in}\big\rangle$ and $\big|\beta_{\rm in}\big\rangle$ are obtained by acting with $a^\dagger_{\rm in}$ on the vacuum state $\big|0_{\rm in}\big\rangle$:

$$\big|\vec{\boldsymbol{p}}_1 \cdots \vec{\boldsymbol{p}}_{m\,{\rm in}}\big\rangle = \left[\prod_{i=1}^{m} a^\dagger_{\rm in}(p_i)\right] \big|0_{\rm in}\big\rangle$$

- The relation between $\phi_{\rm in}$ and $a_{\rm in}, a^\dagger_{\rm in}$ can be inverted as :

$$a^\dagger_{\rm in}(p) = -i \int dx^- d^2\vec{\boldsymbol{x}}_\perp \; e^{-ip\cdot x} (\partial^+ + ip^+) \phi_{\rm in}(x)$$

Note : this expression is in fact independent of x^+. One can therefore chose x^+ at will in this equation

Scattering theory

- One can use the freedom to chose x^+ in the previous relation to take $x^+ = -\infty$ for a field in the initial state and $x^+ = +\infty$ for a field in the final state. This choice enables us to write the S-matrix element as the expectation value of an x^+-ordered product of fields :

$$
\begin{aligned}
S_{\beta\alpha} = & \prod_{i\in\alpha}\Big[-i\int dx_i^- d^2\vec{\boldsymbol{x}}_{i\perp} e^{-ip_i\cdot x_i}(\partial_i^+ + ip_i^+)\Big] \\
& \times \prod_{j\in\beta}\Big[i\int dy_j^- d^2\vec{\boldsymbol{y}}_{j\perp} e^{iq_j\cdot y_j}(\partial_j^+ - iq_j^+)\Big] \\
& \times \big\langle 0_{\rm in}\big| T_+\Big\{\prod_{j\in\beta}\phi_{\rm in}(y_j^+=+\infty, y_j^-, \vec{\boldsymbol{y}}_{j\perp})\prod_{i\in\alpha}\phi_{\rm in}(x_i^+=-\infty, x_i^-, \vec{\boldsymbol{x}}_{i\perp}) \\
& \times \exp i\int_{-\infty}^{+\infty} d^4x\, \mathcal{L}_{\rm int}(\phi_{\rm in}(x))\Big\}\big|0_{\rm in}\big\rangle
\end{aligned}
$$

Perturbation theory

- The previous correlator contains only the free field $\phi_{\rm in}$. The effects of the interactions are obtained by expanding the x^+-ordered exponential to the desired order
  ▷ this naturally leads to an x^+-ordered perturbation theory
- This can be done more systematically by introducing a generating functional for these correlators :

$$Z[j] \equiv \big<0_{\rm in}\big|T_+ \exp i\int_{-\infty}^{+\infty} d^4x\,\Big[\mathcal{L}_{\rm int}(\phi_{\rm in}(x)) + j(x)\phi_{\rm in}(x)\big|0_{\rm in}\big>\Big]$$

such that

$$\big<0_{\rm in}\big|T_+\phi_{\rm in}(x_1)\cdots\phi_{\rm in}(x_n)\exp i\int_{-\infty}^{+\infty} d^4x\;\mathcal{L}_{\rm int}(\phi_{\rm in}(x))\big|0_{\rm in}\big>$$

$$= \frac{\delta}{i\delta j(x_1)}\cdots\frac{\delta}{i\delta j(x_n)}Z[j]\Big|_{j=0}$$

Perturbation theory

- By using standard methods, one obtains the following closed formula for $Z[j]$:

$$Z[j] = \exp i \int d^4x \, \mathcal{L}_{\rm int}\left(\frac{\delta}{i\delta j(x)}\right) \; \exp -\frac{1}{2}\int d^4x d^4y \, j(x) j(y) \, G_0(x,y)$$

where $G_0(x-y)$ is the free x^+-ordered propagator, defined as :

$$G_0(x,y) \equiv \big\langle 0_{\rm in}\big| T_+ \phi_{\rm in}(x)\phi_{\rm in}(y)\big|0_{\rm in}\big\rangle$$

- Since the $\phi_{\rm in}$ is a free field, this propagator can be calculated explicitly :

$$G_0(x,y) = \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\left[\theta(x^+ - y^+)e^{-ip\cdot(x-y)} + \theta(y^+ - x^+)e^{ip\cdot(x-y)}\right]$$

Note : the x^+-ordered propagator is in fact identical to the x^0-ordered propagator

Perturbation theory

- Contribution of the external lines :
  - Incoming particles :

$$-i\int dx^- d^2\vec{\boldsymbol{x}}_\perp\; e^{-ip\cdot x}(\partial_x^+ + ip^+)G_0(x,y)_{x^+=-\infty} = e^{-ip\cdot y}$$

  - Outgoing particles :

$$i\int dx^- d^2\vec{\boldsymbol{x}}_\perp\; e^{ip\cdot x}(\partial_x^+ - ip^+)G_0(x,y)_{x^+=+\infty} = e^{ip\cdot y}$$

- Feynman rules :
  - Consider only amputated diagrams
  - At each vertex $-ig\int d^4x$
  - $G_0(x,y)$ for each internal line
  - A factor $e^{-ip\cdot x}$ for incoming particles, and $e^{ip\cdot x}$ for outgoing particles

▷ identical to the usual Feynman rules in momentum space

Noether's currents

- Energy-momentum tensor :

$$T^{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} = (\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\mathcal{L}$$

  - It obeys :

$$\partial_\nu T^{\mu\nu} = 0 \quad , \quad P^\mu = \int dx^- d^2\vec{\boldsymbol{x}}_\perp \, T^{\mu +}$$

  - The time components of the currents associated to P^+, P^j, P^- are :

$$\begin{cases} T^{++} = (\partial^+\phi)^2 \\ T^{j+} = (\partial^j\phi)(\partial^+\phi) \\ T^{-+} = (\partial^-\phi)(\partial^+\phi) - \mathcal{L} = \mathcal{H} \end{cases}$$

Noether's currents

- Angular-momentum tensor :

$$J^{\mu\nu\rho} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}$$

$$\partial_\rho J^{\mu\nu\rho} = 0 \quad , \quad M^{\mu\nu} = \int dx^- d^2\vec{\boldsymbol{x}}_\perp \; J^{\mu\nu+}$$

- For instance, the boost operator in the $-$ direction reads :

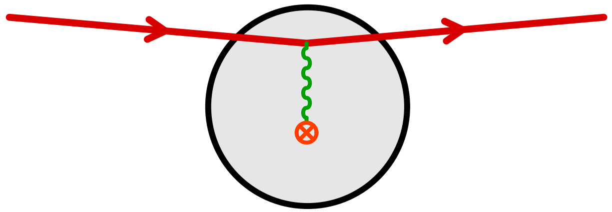
$$\begin{aligned} K^- &= M^{-+} = \int dx^- d^2\vec{\boldsymbol{x}}_\perp \; J^{-++} \\ &= \frac{i}{2}\int \frac{dp^+}{4\pi}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\left\{\left[\frac{\partial a^\dagger_{\rm in}(p)}{\partial p^+}\right] a_{\rm in}(p) - a^\dagger_{\rm in}(p)\left[\frac{\partial a_{\rm in}(p)}{\partial p^+}\right]\right\} \end{aligned}$$

and it obeys :

$$\left[K^-, a^\dagger_{\rm in}(p)\right] = ip^+ \frac{\partial a^\dagger_{\rm in}(p)}{\partial p^+}$$

Model

- Let us now go back to the problem of the scattering off an external potential at high energy



- More precisely, we want to calculate :

$$S^{(\infty)}_{\beta\alpha} \equiv \lim_{\omega\to+\infty} \left\langle\beta_{\rm in}\middle| e^{i\omega K^-} U(+\infty,-\infty) e^{-i\omega K^-} \middle|\alpha_{\rm in}\right\rangle$$

- We will consider two different potentials :
  - Scalar potential : $\lambda\,\mathcal{A}(x)\phi(x)\phi^*(x)$
  - Vector potential : $e\,\mathcal{A}_\mu(x)J^\mu(x)$

- We will assume that the external potential is non-zero only on a finite range in x^+, $x^+\in[-L,+L]$

Model

- In order to have a consistent model, we must consider a complex scalar field, with a $U(1)$ symmetry :

$$\mathcal{L} \equiv (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2 \phi\phi^* - \frac{g}{2}(\phi\phi^*)^2$$

- Conjugate momenta :

$$\Pi_\phi = \partial^+ \phi^* \quad , \qquad \Pi_{\phi^*} = \partial^+ \phi$$

- The Lagrangian is invariant under a U(1) symmetry :

$$\phi(x) \to e^{i\alpha} \phi(x) \quad , \qquad \phi^*(x) \to e^{-i\alpha} \phi^*(x)$$

The associated Noether's current is :

$$J^\mu = i\,[\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi]$$

Eikonal limit

- We have already seen that :

$$\left[K^-, a_{\rm in}^\dagger(p)\right] = ip^+ \frac{\partial a_{\rm in}^\dagger(p)}{\partial p^+}$$

- This relation implies :

$$e^{-i\omega K^-} a_{\rm in}^\dagger(q) e^{i\omega K^-} = a_{\rm in}^\dagger(e^\omega q^+, e^{-\omega} q^-, \boldsymbol{\vec{q}}_\perp)$$

$$e^{-i\omega K^-} \big|\boldsymbol{\vec{p}} \cdots {\rm in}\big\rangle = \big|(e^\omega p^+, \boldsymbol{\vec{p}}_\perp) \cdots {\rm in}\big\rangle$$

$$e^{i\omega K^-} \phi_{\rm in}(x) e^{-i\omega K^-} = \phi_{\rm in}(e^{-\omega} x^+, e^\omega x^-, \boldsymbol{\vec{x}}_\perp)$$

- Since the boost K^- does not change the ordering in x^+ :

$$e^{i\omega K^-} U(+\infty, -\infty) e^{-i\omega K^-} = T_+ \exp i \int d^4x\, \mathcal{L}_{\rm int}(e^{i\omega K^-} \phi_{\rm in}(x) e^{-i\omega K^-})$$

where $\mathcal{L}_{\rm int}$ includes all the interaction terms :

$$\mathcal{L}_{\rm int}(\phi) = -\frac{g}{2}(\phi\phi^*)^2 - \lambda\, \mathcal{A}\phi\phi^* - e\, \mathcal{A}_\mu J^\mu$$

Eikonal limit

- It is useful to split the S matrix $U(+\infty,-\infty)$ into three factors :

$$U(+\infty,-\infty) = U(+\infty,+L)U(+L,-L)U(-L,-\infty)$$

Upton application of K^-, this becomes :

$$e^{i\omega K^-}U(+\infty,-\infty)e^{-i\omega K^-} = e^{i\omega K^-}U(+\infty,+L)e^{-i\omega K^-} \times e^{i\omega K^-}U(+L,-L)e^{-i\omega K^-}e^{i\omega K^-}U(-L,-\infty)e^{-i\omega K^-}$$

- The external potentials $\mathcal{A}(x)$ and $\mathcal{A}_\mu(x)$ are unaffected by the action of K^-
- The components of $J^\mu(x)$ are changed as follows :

$$e^{i\omega K^-}J^i(x)e^{-i\omega K^-} = J^i(e^{-\omega}x^+, e^\omega x^-, \vec{\boldsymbol{x}}_\perp)$$
$$e^{i\omega K^-}J^-(x)e^{-i\omega K^-} = e^{-\omega}\, J^-(e^{-\omega}x^+, e^\omega x^-, \vec{\boldsymbol{x}}_\perp)$$
$$e^{i\omega K^-}J^+(x)e^{-i\omega K^-} = e^{\omega}\, J^+(e^{-\omega}x^+, e^\omega x^-, \vec{\boldsymbol{x}}_\perp)$$

Eikonal limit

- The factors $U(+\infty,+L)$ and $U(-L,-\infty)$ do not contain the external potential. In order to deal with these factors, it is sufficient to perform the change of variables : $e^{-\omega}x^+ \to x^+$, $e^{\omega}x^- \to x^-$. This leads to :

$$\lim_{\omega\to+\infty} e^{i\omega K^-}U(+\infty,+L)e^{-i\omega K^-} = U_0(+\infty,0)$$

$$\lim_{\omega\to+\infty} e^{i\omega K^-}U(-L,-\infty)e^{-i\omega K^-} = U_0(0,-\infty)$$

where U_0 is the same as U, but with only the self-interactions of the fields, $g(\phi\phi^*)^2$

- For the factor $U(L,-L)$, the $e^{\omega}x^- \to x^-$ change leads to :

$$e^{i\omega K^-}U(+L,-L)e^{-i\omega K^-} = \\ = T_+\exp i\int_{-L}^{+L} d^4x\, e^{-\omega}\Big[e\,\mathcal{A}^-(x^+,e^{-\omega}x^-,\vec{\boldsymbol{x}}_\perp) \\ \times e^{\omega}J^+(e^{-\omega}x^+,x^-,\vec{\boldsymbol{x}}_\perp)+\mathcal{O}(1)\Big]$$

Eikonal limit

- Therefore, in the limit $\omega \to +\infty$, we have :

$$\lim_{\omega\to+\infty} e^{i\omega K^-} U(+L,-L) e^{-i\omega K^-} = \exp\left[ie \int d^2\vec{\boldsymbol{x}}_\perp \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right]$$

with
$$\begin{cases} \chi(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^+ \; \mathcal{A}^-(x^+, 0, \vec{\boldsymbol{x}}_\perp) \\ \rho(\vec{\boldsymbol{x}}_\perp) \equiv \int dx^- \; J^+(0, x^-, \vec{\boldsymbol{x}}_\perp) \end{cases}$$

- The high-energy limit of the scattering amplitude is :

$$S^{(\infty)}_{\beta\alpha} = \langle\beta_{\rm in}\big|U_0(+\infty,0)\;\exp\left[ie\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\right]U_0(0,-\infty)\big|\alpha_{\rm in}\rangle$$

  - Only the $-$ component of the vector potential matters
  - The self-interactions and the interactions with the external potential are factorized ▷ parton model
  - Still not completely trivial, because ρ is an operator

Perturbative expansion

- The previous formula still contains all the self-interactions of the field ϕ. In order to perform the perturbative expansion, it is convenient to write first :

$$S^{(\infty)}_{\beta\alpha} = \sum_{\gamma,\delta} \big\langle \beta_{\rm in} \big| U_0(+\infty,0) \big| \gamma_{\rm in} \big\rangle \times \big\langle \gamma_{\rm in} \big| \exp\Big[ie\int_{\vec{\boldsymbol{x}}_\perp} \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)\Big] \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

- The factor

$$\sum_{\delta} \big| \delta_{\rm in} \big\rangle \big\langle \delta_{\rm in} \big| U(0,-\infty) \big| \alpha_{\rm in} \big\rangle$$

is the Fock expansion of the initial state: the state prepared at $x^+ = -\infty$ may have fluctuated into another state before it interacts with the external potential

Perturbative expansion

- Denote $\boldsymbol{F} \equiv \exp ie\int \chi(\vec{\boldsymbol{x}}_\perp)\rho(\vec{\boldsymbol{x}}_\perp)$. We need to calculate matrix elements such as $\big\langle\gamma_{\rm in}\big|\boldsymbol{F}\big|\delta_{\rm in}\big\rangle$
- The operator $\rho(\vec{\boldsymbol{x}}_\perp)$ reads :

$$\rho(\vec{\boldsymbol{x}}_\perp) = \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\frac{d^2\vec{\boldsymbol{q}}_\perp}{(2\pi)^2}\Big\{b^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp)b_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp)e^{i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp} - d^\dagger_{\rm in}(p^+,\vec{\boldsymbol{p}}_\perp)d_{\rm in}(p^+,\vec{\boldsymbol{q}}_\perp)e^{-i(\vec{\boldsymbol{p}}_\perp-\vec{\boldsymbol{q}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\Big\}$$

- Therefore :

$$\boldsymbol{F}\, b^\dagger_{\rm in}(k) = \left[\int_{\vec{\boldsymbol{p}}_\perp} b^\dagger_{\rm in}(k^+,\vec{\boldsymbol{p}}_\perp)\int_{\vec{\boldsymbol{x}}_\perp} e^{i(\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{p}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}e^{ie\chi(\vec{\boldsymbol{x}}_\perp)}\right]\boldsymbol{F}$$

$$\boldsymbol{F}\, d^\dagger_{\rm in}(k) = \left[\int_{\vec{\boldsymbol{p}}_\perp} d^\dagger_{\rm in}(k^+,\vec{\boldsymbol{p}}_\perp)\int_{\vec{\boldsymbol{x}}_\perp} e^{i(\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{p}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}e^{-ie\chi(\vec{\boldsymbol{x}}_\perp)}\right]\boldsymbol{F}$$

Perturbative expansion

- Consider a state :

$$\left|\delta_{\rm in}\right\rangle \equiv b^\dagger_{\rm in}(k)\cdots\left|0_{\rm in}\right\rangle$$

- $\boldsymbol{F}$ changes only the transverse momenta in a state:

$$\boldsymbol{F}\,\left|\delta_{\rm in}\right\rangle = \boldsymbol{F}\;b^\dagger_{\rm in}(k)\cdots\left|0_{\rm in}\right\rangle$$

Perturbative expansion

- Consider a state :

$$\left|\delta_{\rm in}\right\rangle \equiv b^\dagger_{\rm in}(k)\cdots\left|0_{\rm in}\right\rangle$$

- $\boldsymbol{F}$ changes only the transverse momenta in a state:

$$\boldsymbol{F}\left|\delta_{\rm in}\right\rangle = \left[\int_{\vec{\boldsymbol{p}}_\perp} b^\dagger_{\rm in}(k^+,\vec{\boldsymbol{p}}_\perp)\int_{\vec{\boldsymbol{x}}_\perp} e^{i(\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{p}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\, e^{ie\chi(\vec{\boldsymbol{x}}_\perp)}\right]\cdots\boldsymbol{F}\left|0_{\rm in}\right\rangle$$

Perturbative expansion

- Consider a state :

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Perturbative expansion

- Consider a state :

$$\left|\delta_{\rm in}\right\rangle \equiv b^\dagger_{\rm in}(k)\cdots\left|0_{\rm in}\right\rangle$$

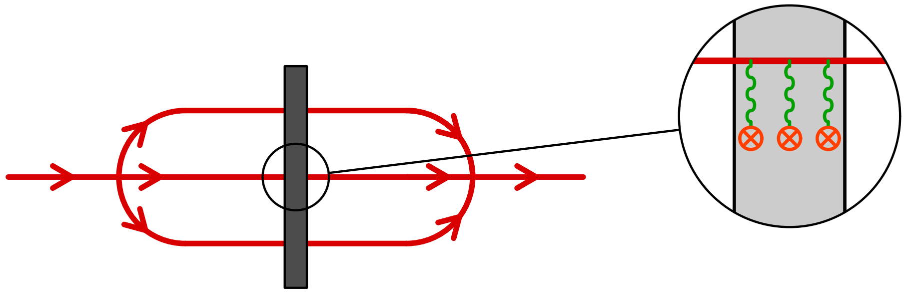
- $\boldsymbol{F}$ changes only the transverse momenta in a state:

$$\boldsymbol{F}\left|\delta_{\rm in}\right\rangle = \left[\int_{\vec{\boldsymbol{p}}_\perp}\int_{\vec{\boldsymbol{x}}_\perp} e^{i(\vec{\boldsymbol{k}}_\perp-\vec{\boldsymbol{p}}_\perp)\cdot\vec{\boldsymbol{x}}_\perp}\, e^{ie\chi(\vec{\boldsymbol{x}}_\perp)}\right]\cdots\left|(k^+,\vec{\boldsymbol{p}}_\perp)\cdots{}_{\rm in}\right\rangle$$

- The projection on $\left\langle\gamma_{\rm in}\right|$ is non zero only if this state contains the same number of particles and antiparticles as $\left|\delta_{\rm in}\right\rangle$

Perturbative expansion

- The action of $\boldsymbol{F}$ on $|\delta_{\rm in}\rangle$ changes the transverse momenta of the particles contained in the state, but not their nature and number :



- Let us define the light-cone wave function of the incoming state by :

$$\psi(\vec{\boldsymbol{x}}_{1\perp}\cdots\vec{\boldsymbol{x}}_{n\perp}) \equiv \prod_i \int \frac{d^2\vec{\boldsymbol{k}}_{i\perp}}{(2\pi)^2} e^{-i\vec{\boldsymbol{k}}_{i\perp}\cdot\vec{\boldsymbol{x}}_{i\perp}} \langle \vec{\boldsymbol{k}}_1 \cdots \vec{\boldsymbol{k}}_{n\,{\rm in}} \big| U(0,-\infty) \big| \alpha_{\rm in}\rangle$$

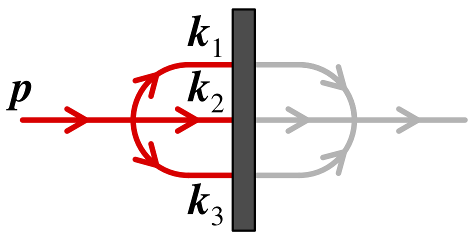
- Each charged particle going through the external field acquires a phase proportional to its charge (antiparticles get an opposite phase) :

$$\psi(\vec{\boldsymbol{x}}_{1\perp}\cdots\vec{\boldsymbol{x}}_{n\perp}) \longrightarrow \psi(\vec{\boldsymbol{x}}_{1\perp}\cdots\vec{\boldsymbol{x}}_{n\perp}) \prod_i e^{ie_i\,\chi(\vec{\boldsymbol{x}}_{i\perp})}$$

![cea logo]

Perturbative expansion

- The calculation of $\left\langle\delta_{\rm in}\middle|U_0(0,-\infty)\middle|\alpha_{\rm in}\right\rangle$ is similar to that of scattering amplitudes in the vacuum. The only difference is that the integration over x^+ at each vertex runs only over half of the real axis $[-\infty, 0]$
  - In Fourier space, this means that the $-$ component of the momentum is not conserved at the vertices
  - Instead of a δ function, one gets an energy denominator
- Example with a single interaction :



$$\left\langle\vec{\boldsymbol{k}}_1\vec{\boldsymbol{k}}_2\vec{\boldsymbol{k}}_{3\,{\rm in}}\middle|U(0,-\infty)\middle|\vec{\boldsymbol{p}}_{\rm in}\right\rangle = -ig\int_{-\infty}^{0} d^4x\; e^{i(k_1+k_2+k_3-p)\cdot x}$$

$$= -g\,\frac{(2\pi)^3\delta(\vec{\boldsymbol{k}}_{1\perp}+\vec{\boldsymbol{k}}_{2\perp}+\vec{\boldsymbol{k}}_{3\perp}-\vec{\boldsymbol{p}}_\perp)\delta(k_1^++k_2^++k_3^+-p^+)}{k_1^-+k_2^-+k_3^--p^--i\epsilon}$$

Perturbative expansion

- More generally, one should expand the evolution operator $U_0(0,-\infty)$ to the desired order and insert $\mathbf{1}=\sum_\gamma \big|\gamma_{\rm in}\big\rangle\big\langle\gamma_{\rm in}\big|$ between the successive interactions :

$$\big\langle\delta_{\rm in}\big|U_0(0,-\infty)\big|\alpha_{\rm in}\big\rangle = \sum_{n=0}^{+\infty}\int_{-\infty}^{0} d^4x_1 \int_{-\infty}^{x_1^+} d^4x_2 \cdots \int_{-\infty}^{x_{n-1}^+} d^4x_n$$

$$\times \sum_{\gamma_1\cdots\gamma_{n-1}} \big\langle\delta_{\rm in}\big|i\mathcal{L}^0_{\rm int}(\phi_{\rm in}(x_1))\big|\gamma_{1\,\rm in}\big\rangle\cdots\big\langle\gamma_{n-1\,\rm in}\big|i\mathcal{L}^0_{\rm int}(\phi_{\rm in}(x_n))\big|\alpha_{\rm in}\big\rangle$$

- In practice $\sum_\gamma$ is an integral over the phase-space of the (on-shell) particles of the intermediate state. Conservation laws may restrict what is allowed in the intermediate state
- The elementary factors are given by :

$$\big\langle\gamma_{i-1\,\rm in}\big|i\mathcal{L}^0_{\rm int}(\phi_{\rm in}(x_i))\big|\gamma_{i\,\rm in}\big\rangle \propto -ig\; e^{i(\sum_{a\in\gamma_i} p_a - \sum_{b\in\gamma_{i-1}} p_b)\cdot x_i}$$

Basics of QCD

- We are now going to extend the previous formalism to the case of QCD, in order to highlight some peculiarities of QCD on the light-cone
- Reminder: the QCD Lagrangian is :

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) + \overline{\psi}(i\not{D} - m)\psi$$

  - the gauge field A^μ belongs to $SU(3)$
  - $D^\mu \equiv \partial^\mu - igA^\mu$ is the covariant derivative
  - $F^{\mu\nu} \equiv i[D^\mu, D^\nu]/g = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]$
- The classical equations of motion are :

$$\begin{cases} (i\not{D} - m)\psi = 0 \\ \left[D_\mu, F^{\mu\nu}\right]_a = -gJ_a^\nu = -g\overline{\psi}\gamma^\nu t^a\psi \end{cases}$$

  - Note that $\left[D_\nu, J^\nu\right] = 0$ (covariant current conservation)

Independent field components

- We will work in the light-cone gauge $A^+ = 0$
- In QCD, it turns out that some of the field components have a vanishing conjugate momentum ▷ they should be treated as constraints
- For the spinors, let us introduce two orthogonal projectors :

$$\mathcal{P}_+ \equiv \frac{1}{2}\gamma^-\gamma^+ \quad , \quad \mathcal{P}_- \equiv \frac{1}{2}\gamma^+\gamma^-$$

and decompose the spinor ψ as follows :

$$\psi = \psi_+ + \psi_- \qquad \text{with} \quad \psi_+ \equiv \mathcal{P}_+\psi \quad , \quad \psi_- \equiv \mathcal{P}_-\psi$$

- Using $A^+ = 0$, we can rewrite the matter part as :

$$\mathcal{L}_\psi = i\sqrt{2}\psi_+^\dagger D^-\psi_+ + i\sqrt{2}\psi_-^\dagger \partial^+\psi_- - \frac{m}{\sqrt{2}}\left\{\psi_+^\dagger\gamma^-\psi_- + \psi_-^\dagger\gamma^+\psi_+\right\}$$
$$- \frac{i}{\sqrt{2}}\left\{\psi_+^\dagger\gamma^-(\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp)\psi_- + \psi_-^\dagger\gamma^+(\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp)\psi_+\right\}$$

Independent field components

- The gauge part of the Lagrangian can be decomposed as :

$$\mathcal{L}_A = -\frac{1}{4} F_{ij}^a F_a^{ij} - F_a^{+}{}_i F_a^{-i} - \frac{1}{2} F_a^{+-} F_a^{-+}$$

with

$$\begin{cases} F^{+-} = -F^{-+} = \partial^+ A^- \quad , \quad F^{+i} = -F^{i+} = \partial^+ A^i \\ F^{-i} = -F^{i-} = \partial^- A^i - \partial^i A^- - ig[A^-, A^i] \\ F^{ij} = \partial^i A^j - \partial^j A^i - ig[A^i, A^j] \end{cases}$$

- The fields A^- and ψ_- have a vanishing conjugate momentum :

$$\Pi_{A_a^-} = \frac{\delta \mathcal{L}}{\delta(\partial^- A_a^-)} = 0 \quad , \quad \Pi_{\psi_-} = \frac{\delta \mathcal{L}}{\delta(\partial^- \psi_-)} = 0$$

▷ this means that the Poisson brackets involving these fields are zero, and that the standard canonical quantization procedure is bound to fail for these fields

Independent field components

- There is no such problem with A^i and ψ_+ :

$$\Pi_{A^i_a} = \partial^+ A^i_a \quad , \quad \Pi_{\psi_+} = i\sqrt{2}\psi_+^\dagger$$

▷ these fields can be quantized by the usual method

- By multiplying it by γ^+ and γ^-, the Dirac equation can be split into :

$$\left\{ \begin{aligned} \partial^+ \psi_- &= -\frac{i}{2}\big[-i\boldsymbol{\gamma}_\perp \cdot \boldsymbol{D}_\perp + m\big]\gamma^+\psi_+ \\ D^-\psi_+ &= -\frac{i}{2}\big[-i\boldsymbol{\gamma}_\perp \cdot \boldsymbol{D}_\perp + m\big]\gamma^-\psi_- \end{aligned} \right.$$

  - The first equation does not contain any time derivative ∂^-. It is therefore local in time, and can be seen as a constraint that we can use to express ψ_- in terms of ψ_+ and the gauge fields at the same time x^+
  - The second equation contains time derivatives, and is thus the dynamical evolution equation for ψ_+

Independent field components

- Similarly, the Yang-Mills equations can be divided into :

$$\begin{cases} (\partial^+)^2 A_a^- = g J_a^+ - \left[D_i, \partial^+ A^i\right]_a \\ \left[\partial^+, F^{-i}\right] + \left[D^-, \partial^+ A^i\right] + \left[D_j, F^{ji}\right] = -g J^i \end{cases}$$

  - The first equation is local in time. It is a constraint that can be used to express A_a^- in terms of the other fields at the same time
  - The second equation describes the dynamical evolution of the field A_a^i
- Note: the absence of time derivatives of the fields ψ_- and A^- in the equations of motion is equivalent to the fact that their conjugate momenta are zero
- The strategy for quantizing such a theory is to solve the constraints and rewrite the Lagrangian in terms of the dynamical fields only. Then, one can proceed with the usual canonical quantization procedure, by writing commutation relations at equal x^+ for the dynamical fields

Solution of the constraints

- Formally, the solutions of the constraints read :

$$\begin{cases} \psi_- = -\frac{i}{2\partial^+}\left[-i\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp + m\right]\gamma^+\psi_+ \\ A_a^- = \frac{1}{(\partial^+)^2}\left[gJ_a^+ - \left[D_i, \partial^+ A^i\right]_a\right] \end{cases}$$

(they depend on g)

- It is then a simple (but tedious...) exercise to obtain the Lagrangian in terms of dynamical fields only :

$$\begin{aligned} \mathcal{L} = -\frac{1}{4}F_{ij}^a F_a^{ij} + \left(\partial^+ A_a^i\right)\left(\partial^- A_a^i\right) \\ + \frac{1}{2}\left([D_i, \partial^+ A^i]_a - g\,J_a^+\right)\frac{1}{(\partial^+)^2}\left([D_i, \partial^+ A^i]_a - g\,J_a^+\right) \\ + i\sqrt{2}\,\psi_+^\dagger \partial^- \psi_+ + \frac{i}{\sqrt{2}}\psi_+^\dagger (m - i\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp)\frac{1}{\partial^+}(m + i\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp)\psi_+ \end{aligned}$$

- Note the additional couplings, with $1/\partial^+$ or $1/(\partial^+)^2$

Noether's currents

- Energy-momentum tensor :

$$T^{\mu\nu} = i\overline{\psi}\partial^\mu\gamma^\nu\psi + (\partial^\mu A_\rho)F^{\nu\rho} - g^{\mu\nu}\mathcal{L}$$

  - It obeys :

$$\partial_\nu T^{\mu\nu} = 0 \quad , \quad P^\mu = \int dx^- d^2\vec{\boldsymbol{x}}_\perp \, T^{\mu+}$$

  - The time components of the currents associated to P^+, P^j are :

$$\begin{cases} T^{++} = i\sqrt{2}\psi_+^\dagger\partial^+\psi_+ + (\partial^+ A_a^k)(\partial^+ A_a^k) \\ T^{j+} = i\sqrt{2}\psi_+^\dagger\partial^j\psi_+ + (\partial^j A_a^k)(\partial^+ A_a^k) \end{cases}$$

  - Note that these currents depend only on the dynamical fields. Hence, they do not contain explicit factors of g
  - As we shall see, this is a property of the generators of all the kinematical Poincaré transformations

Noether's currents

- Angular-momentum tensor :

$$J^{\mu\nu\rho} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho} + \frac{i}{8}\overline{\psi}(\gamma^\rho[\gamma^\mu,\gamma^\nu] + [\gamma^\mu,\gamma^\nu]\gamma^\rho)\psi + F^{\rho\mu}A^\nu - F^{\rho\nu}A^\mu$$

  - It obeys :

$$\partial_\rho J^{\mu\nu\rho} = 0 \quad , \quad M^{\mu\nu} = \int dx^- d^2\vec{\boldsymbol{x}}_\perp \, J^{\mu\nu+}$$

  - The time components of the currents associated to J^3, B^j are :

$$\begin{cases} J^{ij+} = x^i T^{j+} - x^j T^{i+} \\ \qquad + \frac{i}{2\sqrt{2}}\psi_+^\dagger[\gamma^i,\gamma^j]\psi_+ + (\partial^+ A_a^i)A_a^j - (\partial^+ A_a^j)A_a^i \\ J^{j++} = x^j T^{++} - x^+ T^{j+} \end{cases}$$

  - Again, they depend only on the dynamical fields

QCD light-cone Hamiltonian

- The Hamiltonian density is obtained in the usual way :

$$\begin{aligned}\mathcal{H} = T^{-+} &= \Pi_{\psi_+}\partial^-\psi_+ + \Pi_{A_a^i}\partial^- A_a^i - \mathcal{L} \\ &= \frac{1}{4}F_{ij}^a F_a^{ij} \\ &- \frac{1}{2}\left(\left[D_i,\partial^+ A^i\right]_a - g\boldsymbol{J}_a^+\right)\frac{1}{(\partial^+)^2}\left(\left[D_i,\partial^+ A^i\right]_a - g\boldsymbol{J}_a^+\right) \\ &- \frac{i}{\sqrt{2}}\psi_+^\dagger(m - i\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp)\frac{1}{\partial^+}(m + i\boldsymbol{\gamma}_\perp\cdot\boldsymbol{D}_\perp)\psi_+\end{aligned}$$

Reminder :

$$\boldsymbol{J}_a^+ = \sqrt{2}\psi_+^\dagger t^a \psi_+$$

Quantization

- The strategy is the same as for scalar fields :
  - Write a Fourier representation of the dynamical fields in the interaction picture, introducing creation and annihilation operators
  - Express the Hamiltonian in terms of these operators
  - Find the commutation relations between the $a_{\rm in}$ and $a_{\rm in}^\dagger$ so that they have the proper interpretation
  - Check that this leads to the expected canonical commutation relations between the fields and their conjugate momenta

- One peculiarity of light-cone quantization is that an equal-x^+ surface is light-like (contrary to the equal-x^0 surfaces in ordinary quantization, which are space-like). Since causality imposes that commutators of local operators separated by a space-like interval vanish, its translation in this formalism will be slightly different

Quantization

- Write the free gauge field operator as (with $p^- = \vec{\boldsymbol{p}}_\perp^2/2p^+$):

$$A^i_{a\ \mathrm{in}}(x) = \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\sum_\lambda \epsilon^i_\lambda(p)\left[a^\lambda_{a\ \mathrm{in}}(p)e^{-ip\cdot x} + a^{\lambda\dagger}_{a\ \mathrm{in}}(p)e^{ip\cdot x}\right]$$

- The free light-cone gluon Hamiltonian reads :

$$H_A^{\mathrm{free}} = \int dx^- d^2\vec{\boldsymbol{x}}_\perp \Big[\frac{1}{4}(\partial_i A^a_{j\ \mathrm{in}} - \partial_j A^a_{i\ \mathrm{in}})(\partial^i A^j_{a\ \mathrm{in}} - \partial^j A^i_{a\ \mathrm{in}})$$
$$- \frac{1}{2}(\partial_i A^i_{a\ \mathrm{in}})\frac{1}{(\partial^+)^2}(\partial_j A^j_{a\ \mathrm{in}})\Big]$$
$$= \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\frac{p^-}{2}\sum_\lambda \left[a^\lambda_{a\ \mathrm{in}}(p)a^{\lambda\dagger}_{a\ \mathrm{in}}(p) + a^{\lambda\dagger}_{a\ \mathrm{in}}(p)a^\lambda_{a\ \mathrm{in}}(p)\right]$$

  - As usual, it should be normal ordered in order to have a vanishing expectation value in the vacuum :

$$H_A^{\mathrm{free}} = \int \frac{dp^+}{4\pi p^+}\frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2}\, p^- \sum_\lambda a^{\lambda\dagger}_{a\ \mathrm{in}}(p)a^\lambda_{a\ \mathrm{in}}(p)$$

Quantization

- For the interpretation of $a^{\lambda\dagger}_{a\ \rm in}$ and $a^{\lambda}_{a\ \rm in}$ as creation and annihilation operators to hold, we need to have :

$$\left[H^{\rm free}_A, a^{\lambda\dagger}_{a\ \rm in}(p)\right] = p^- a^{\lambda\dagger}_{a\ \rm in}(p) \quad , \quad \left[H^{\rm free}_A, a^{\lambda}_{a\ \rm in}(p)\right] = -p^- a^{\lambda}_{a\ \rm in}(p)$$

- This can be achieved if we have :

$$\left[a^{\lambda}_{a\ \rm in}(p), a^{\lambda'}_{b\ \rm in}(q)\right] = 0$$
$$\left[a^{\lambda\dagger}_{a\ \rm in}(p), a^{\lambda'\dagger}_{b\ \rm in}(q)\right] = 0$$
$$\left[a^{\lambda}_{a\ \rm in}(p), a^{\lambda'\dagger}_{b\ \rm in}(q)\right] = \delta_{ab}\delta^{\lambda\lambda'} 2p^+ (2\pi)^3 \delta(p^+ - q^+)\delta(\vec{\boldsymbol{p}}_\perp - \vec{\boldsymbol{q}}_\perp)$$

- Then, one can use these relations in order to get the canonical commutation relation :

$$\left[A^i_{a\ \rm in}(x), \Pi_{A^j_{b\ \rm in}}(y)\right]_{x^+=y^+} = \left[A^i_{a\ \rm in}(x), \partial^+ A^j_{b\ \rm in}(y)\right]_{x^+=y^+}$$
$$= \frac{i}{2}\delta_{ab}\delta^{ij}\delta(x^- - y^-)\delta(\vec{\boldsymbol{x}}_\perp - \vec{\boldsymbol{y}}_\perp)$$

Quantization

- In order to quantize the fermion field $\psi_{+\text{in}}$, notice first that the subspace spanned by $\psi_{+\text{in}} = \mathcal{P}_+\psi_{\text{in}}$ is only 2-dimensional
  - Therefore, we need only two elementary spinors, $w_{+1/2}$ and $w_{-1/2}$, in order to decompose any $\psi_{+\ \text{in}}$

- We can choose them normalized as follows :

$$\begin{cases} w_r^\dagger(p) w_s(p) = 2p^+ \delta_{rs} \\ \sum\limits_{s=\pm\frac{1}{2}} w_s(p) w_s^\dagger(p) = 2p^+ \mathcal{P}_+ \end{cases}$$

- Reminder: in ordinary quantum field theory, the spinor ψ has four independent components, and one need 4 elementary spinors to perform the decomposition: $u_s(p)$ and $v_s(p)$

Quantization

- One can then write $\psi_{+\text{in}}(x)$ as (with $p^- = (\vec{\boldsymbol{p}}_\perp^2 + m^2)/2p^+$):

$$\psi_{+\text{in}}(x) = \frac{1}{2^{1/4}} \int \frac{dp^+}{4\pi p^+} \frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2} \sum_s \left[w_s b_{s\ \text{in}}(p) e^{-ip\cdot x} + w_{-s} d^\dagger_{s\ \text{in}}(p) e^{ip\cdot x} \right]$$

- The free light-cone quark Hamiltonian reads :

$$H_\psi^{\text{free}} = -\frac{i}{\sqrt{2}} \int dx^- d^2\vec{\boldsymbol{x}}_\perp \psi^\dagger_{+\text{in}} (m - i\boldsymbol{\gamma}_\perp \cdot \boldsymbol{\partial}_\perp) \frac{1}{\partial^+} (m + i\boldsymbol{\gamma}_\perp \cdot \boldsymbol{\partial}_\perp) \psi_{+\text{in}}$$

$$= \int \frac{dp^+}{4\pi p^+} \frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2} p^- \sum_{s=\pm\frac{1}{2}} \left[b^\dagger_{s\ \text{in}}(p) b_{s\ \text{in}}(p) - d_{s\ \text{in}}(p) d^\dagger_{s\ \text{in}}(p) \right]$$

  - After normal ordering, it reads :

$$H_\psi^{\text{free}} = \int \frac{dp^+}{4\pi p^+} \frac{d^2\vec{\boldsymbol{p}}_\perp}{(2\pi)^2} p^- \sum_s \left[b^\dagger_{s\ \text{in}}(p) b_{s\ \text{in}}(p) + d^\dagger_{s\ \text{in}}(p) d_{s\ \text{in}}(p) \right]$$

(of course, we need anti-commutation relations for this to work)

Quantization

- In order to have the expected interpretation of $d_{s\ \mathrm{in}}, d^\dagger_{s\ \mathrm{in}}, b_{s\ \mathrm{in}}, b^\dagger_{s\ \mathrm{in}}$, we need :

$$\left[H_\psi^{\mathrm{free}}, b^\dagger_{s\ \mathrm{in}}(p)\right] = p^- b^\dagger_{s\ \mathrm{in}}(p) \quad , \quad \left[H_\psi^{\mathrm{free}}, b_{s\ \mathrm{in}}(p)\right] = -p^- b_{s\ \mathrm{in}}(p)$$
$$\left[H_\psi^{\mathrm{free}}, d^\dagger_{s\ \mathrm{in}}(p)\right] = p^- d^\dagger_{s\ \mathrm{in}}(p) \quad , \quad \left[H_\psi^{\mathrm{free}}, d_{s\ \mathrm{in}}(p)\right] = -p^- d_{s\ \mathrm{in}}(p)$$

- This will be realized with the following choice :

$$\left\{b_{r\ \mathrm{in}}(p), b_{s\ \mathrm{in}}(q)\right\} = \left\{b^\dagger_{r\ \mathrm{in}}(p), b^\dagger_{s\ \mathrm{in}}(q)\right\} = 0$$
$$\left\{d_{r\ \mathrm{in}}(p), d_{s\ \mathrm{in}}(q)\right\} = \left\{d^\dagger_{r\ \mathrm{in}}(p), d^\dagger_{s\ \mathrm{in}}(q)\right\} = 0$$
$$\left\{b_{r\ \mathrm{in}}(p), b^\dagger_{s\ \mathrm{in}}(q)\right\} = \left\{d_{r\ \mathrm{in}}(p), d^\dagger_{s\ \mathrm{in}}(q)\right\} = 2p^+ \delta_{rs} (2\pi)^3 \delta(p^+ - q^+) \delta(\boldsymbol{\vec{p}}_\perp - \boldsymbol{\vec{q}}_\perp)$$

- From these relations, we obtain the expected canonical anti-commutation relations :

$$\begin{cases} \left\{\psi_{+\mathrm{in}}(x), \psi_{+\mathrm{in}}(y)\right\}_{x^+=y^+} = \left\{\psi^\dagger_{+\mathrm{in}}(x), \psi^\dagger_{+\mathrm{in}}(y)\right\}_{x^+=y^+} = 0 \\ \left\{\psi_{+\mathrm{in}}(y), \Pi_{\psi_{+\mathrm{in}}}(x)\right\}_{x^+=y^+} = i\mathcal{P}_+ \delta(x^- - y^-) \delta(\boldsymbol{\vec{x}}_\perp - \boldsymbol{\vec{y}}_\perp) \end{cases}$$

Perturbative expansion

- The method for calculating the Fock state expansion is the same as for scalar fields
  - Expand the evolution operator to the desired order
  - Insert a complete sum of states between successive interactions
  - The ordered integrations over the x^+ variables generate energy denominators
  - The integrations over the other space-time variables generate delta functions
- Note that there are additional couplings, coming from the terms in $1/\partial^+$ and $1/(\partial^+)^2$. They are due to the instantaneous exchange of the fields ψ_- and A^-, which have been eliminated by solving the constraints

Lecture IV : Saturation and CGC

- BFKL equation
- Saturation of parton distributions
- Balitsky-Kovchegov equation
- Color Glass Condensate - JIMWLK
- Analogies with reaction-diffusion processes
- Pomeron loops

Lecture V : Calculating observables

- Field theory coupled to time-dependent sources
- Generating function for the probabilities
- Average particle multiplicity
- Numerical methods for nucleus-nucleus collisions
  - Gluon production
  - Quark production