

# Cours de physique théorique de Saclay

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Vendredi 03/02/2023, 10:00

Orme des Merisiers Salle Claude Itzykson, Bât. 774

## Rough paths for physicists

# Michel Bauer

Rough path theory emerged in the 1990's motivated by practical and conceptual questions. The general context is that of controlled differential equations, for example  $dU_t = f(U_t)dX_t$  where  $X_t$  is a known property of a system, changing with time, and  $U_t$  is an unknown whose evolution is to be computed via the equation. The emphasis is on the regularity of  $X_t$  as a function of time, and its consequences on that of  $U_t$ , making the necessary assumptions on  $f$ . If  $X_t$  is smooth, we are on familiar grounds, but what if  $X_t$ , though continuous, has little regularity? In that case, it is the rule rather than the exception that Euler's discretization  $U_{t_{n+1}} = U_{t_n} + f(U_{t_n})(X_{t_{n+1}} - X_{t_n})$  does not converge when the mesh goes to 0. Approaching  $X_t$  more and more closely by smooth functions  $X_t^{(n)}$ , and seeing if the corresponding  $U_t^{(n)}$ s get close to something, has its own difficulties. What is the right notion of "close"? Does  $\lim_{n \rightarrow \infty} U_t^{(n)}$ , when it exists, depend on the approximation sequence  $X_t^{(n)}$  and how? Rough path theory gives an answer to these questions in a form suggestive of deep relations to physics (power counting, counterterms, etc) that I will try to explain. The short answer is that depending on the regularity of  $X_t$ , making unambiguous sense of the equation  $dU_t = f(U_t)dX_t$  requires to supplement  $X_t$  with new data which roughly (!) speaking replace some undefined integrals by a formal, yet concrete, mathematical object: a rough path structure.

Plan:

1. Motivations. The sewing lemma and Young integrals.
2. Rough paths, combinatorics and regularity. A waltz with Brownian motion.
3. Controlled rough paths, rough integrals and rough differential equations. Rough integrals versus stochastic integrals, the Itô stochastic area.
4. Thermalization of a particle in a magnetic field. Outlook.

Course website: <https://courses.ipht.fr/?q=en/node/309>

Videoconference and in person in Salle Itzykson, IPhT

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