One-dimensional disordered Bose-Fermi mixtures in optical lattices

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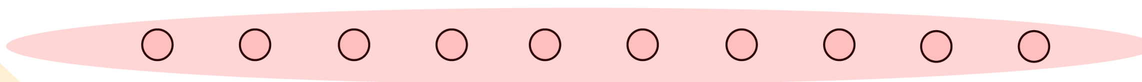
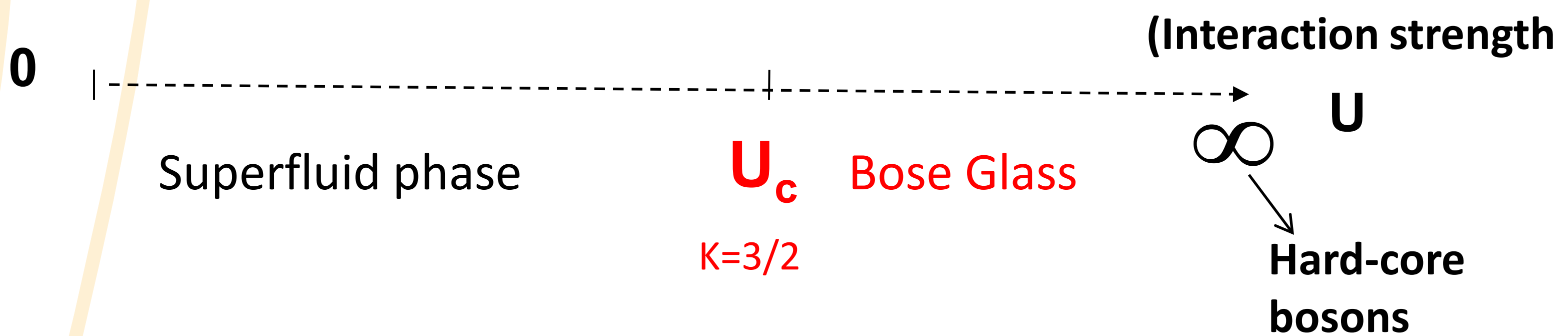
Quantum gases in a 1D disordered potential

What happens if we load **interacting** particles in a 1D disordered potential ?



- Random external potential

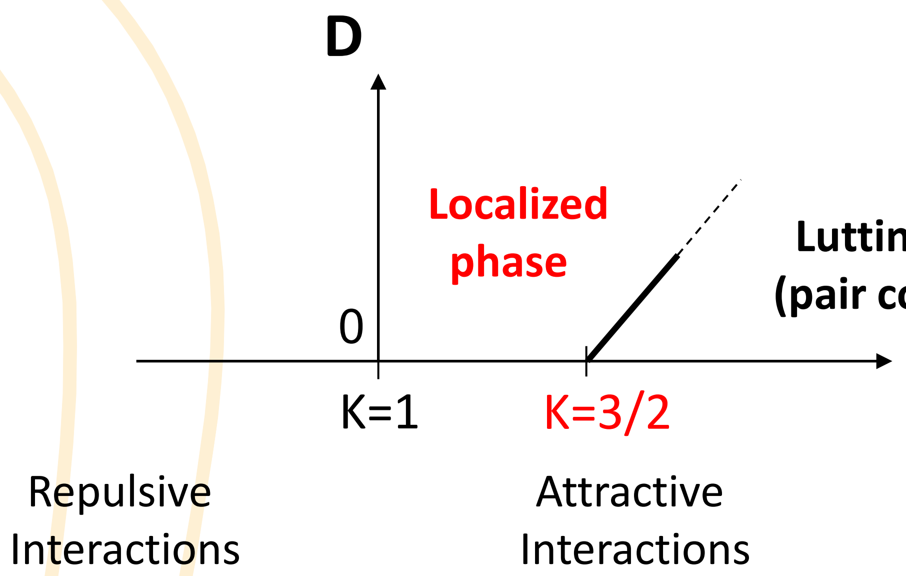
- **Bosons** in 1D with repulsive interactions



T. Giamarchi, H. Schulz, Phys. Rev. B, **37**, 325 (1988)

Quantum gases in a 1D disordered potential

- **Interacting fermions in 1D**



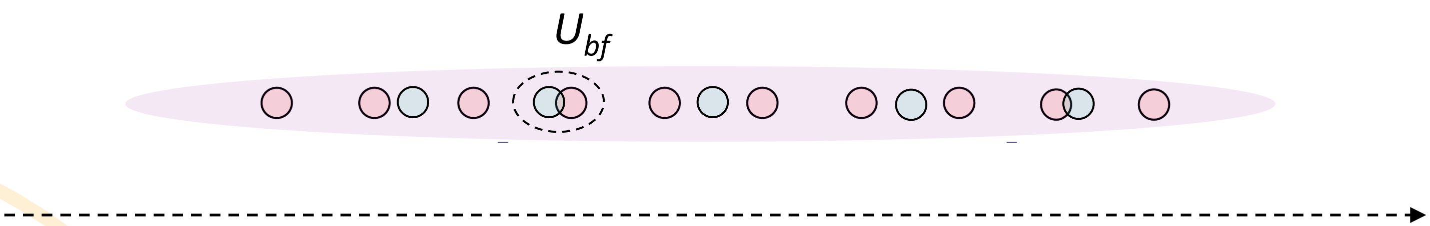
K = Luttinger parameter

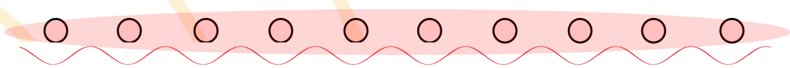
K characterizes the interactions in the system

Insulator to **superfluid** transition, for attractive interactions.

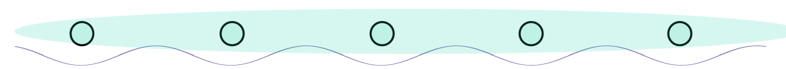
T. Giamarchi, H. Schulz, Phys. Rev. B, **37**, 325 (1988)

Bose-Fermi mixture in a 1D disordered potential

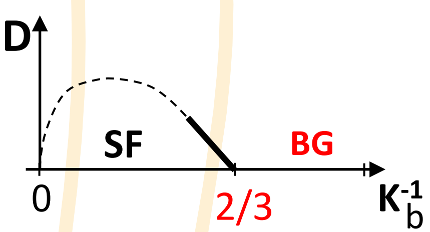


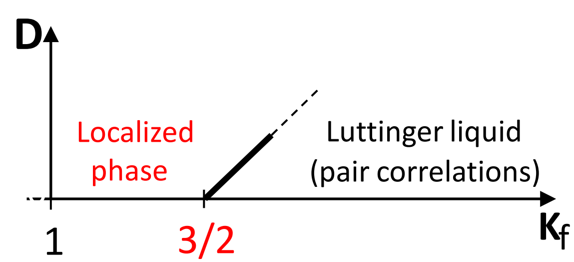


Bosons



Fermions





Questions :

- Does weak disorder decouple the species ?
- What is the mechanism of localization ? Does the localization of one species trigger the localization of the other ?
- Phase diagram ? New phases ?

Outline

1. **Cold atoms: highly tuneable correlated systems**

2. **Disordered 1D interacting systems**

3. **Bose-Fermi mixture in a disordered potential:**
   - Does fermion localization induce boson localization?
   - Phase diagram
   - Observables

4. **Conclusions and perspectives**

I-Cold atoms:

highly tuneable correlated systems

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Cold atoms: Control of dimensionality

Optical lattices

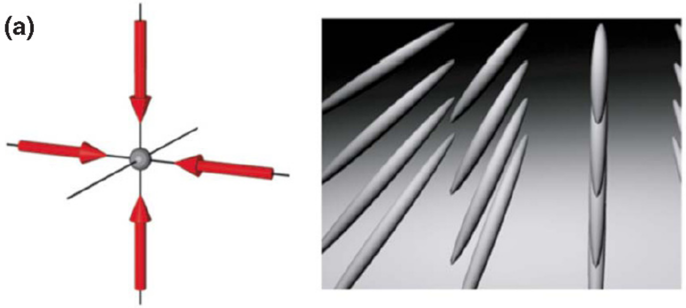
- pairs of counter-propagating beams ⟶ standing waves

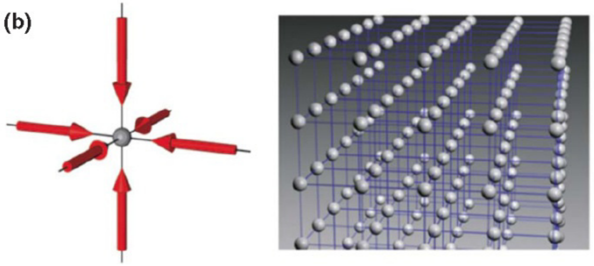
1 pair ➡ **Pancakes (quasi 2D)**

2 pairs ➡ **tubes (quasi 1D)**

3 pairs ➡ **3D cubic lattice**

1-3 dimensions



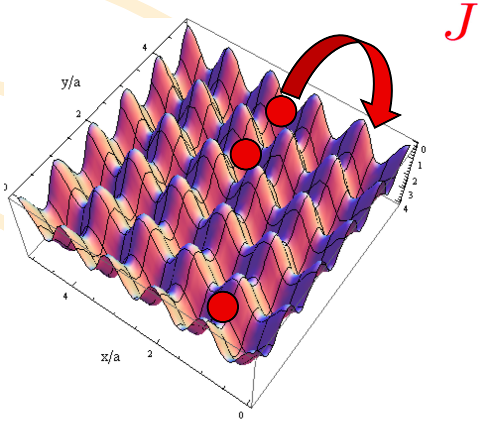


Bloch, Dalibard, and Zwerger, RMP 2008

Cold atoms: control of parameters

Hopping from site to site …

For a deep lattice: $J \sim e^{-2\sqrt{V_0/E_R}}$



V_0 *: height of the lattice*

$E_R = \frac{h^2\lambda^{-2}}{2m}$ *: recoil energy*

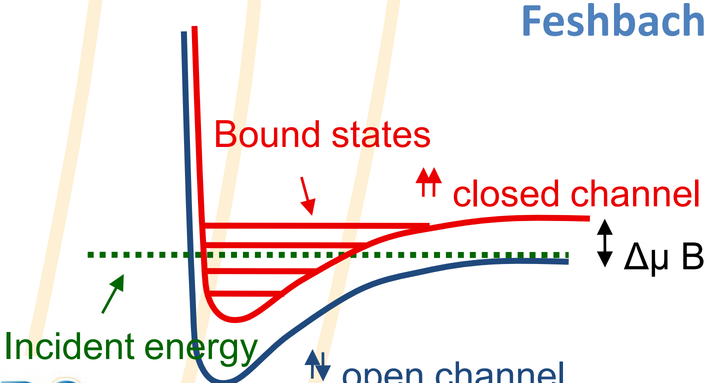
Kinetic Hamiltonian: $H = -\sum_{i,j} J_{ij} \left(a_i^\dagger a_j + a_j^\dagger a_i \right)$

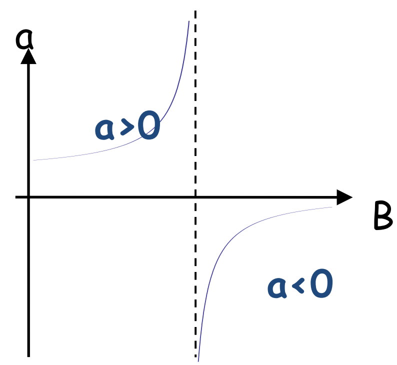
→ **tight binding approximation** (single band)

Controlling interactions

Van der Waals interactions with an *effective potential* : $V(\mathbf{r}) = \frac{4\pi\hbar^2}{2\mu} a_s \, \delta(\mathbf{r})$

Feshbach resonances





Fantastic control over sign and amplitude of interactions

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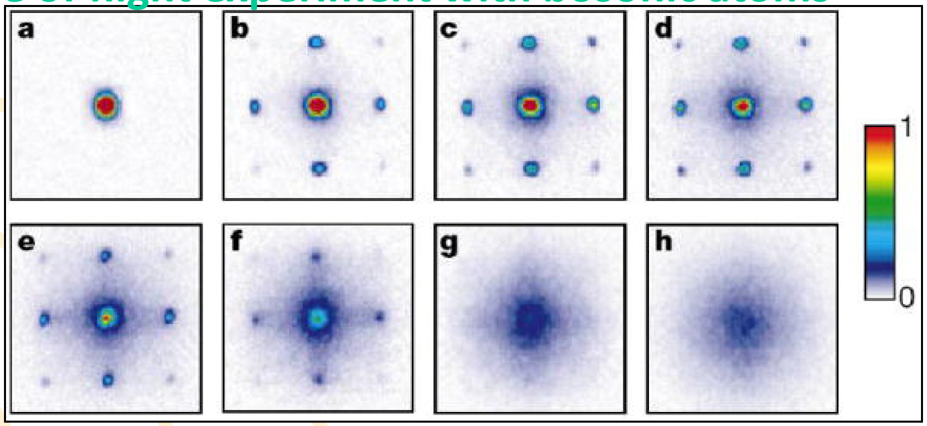
Superfluid to Mott insulator

Time of flight experiment with bosonic atoms



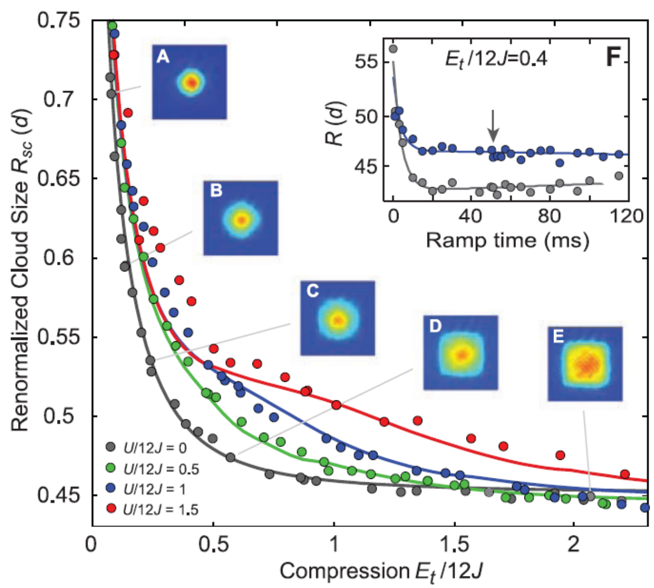
Absorption images of multiple matter wave interference patterns. These were obtained after suddenly releasing the atoms from an optical lattice potential with different potential depths V0 after a time of flight of 15 ms. Values of V0 were: a, 0 Er; b, 3 Er; c, 7 Er; d, 10 Er; e, 13 Er; f, 14 Er; g, 16 Er; and h, 20 Er.

Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms.

Greiner, Markus; Mandel, Olaf; Esslinger, Tilman; Hansch, Theodor; Bloch, Immanuel

Nature. 415, 39-44, 2002.

See also



Metallic and Insulating Phases of Repulsively Interacting Fermions in a 3D Optical Lattice

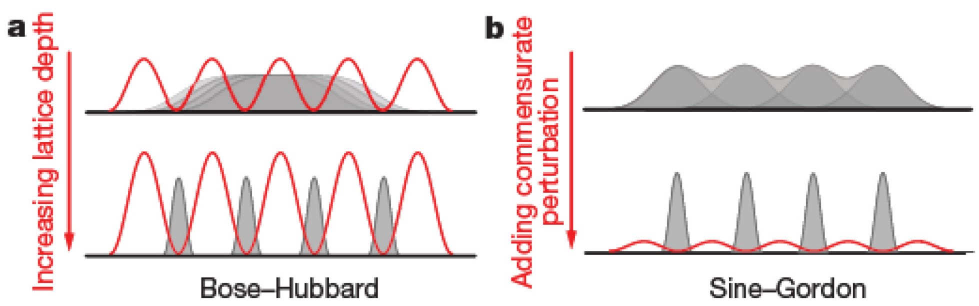
U. Schneider,[1] L. Hackermüller,[1] S. Will,[1] Th. Best,[1] I. Bloch,[1,2*] T. A. Costi,[3] R. W. Helmes,[4] D. Rasch,[4] A. Rosch[4]

Science, **322**, 1522 (2008)

Pinning transition in strongly interacting 1D systems

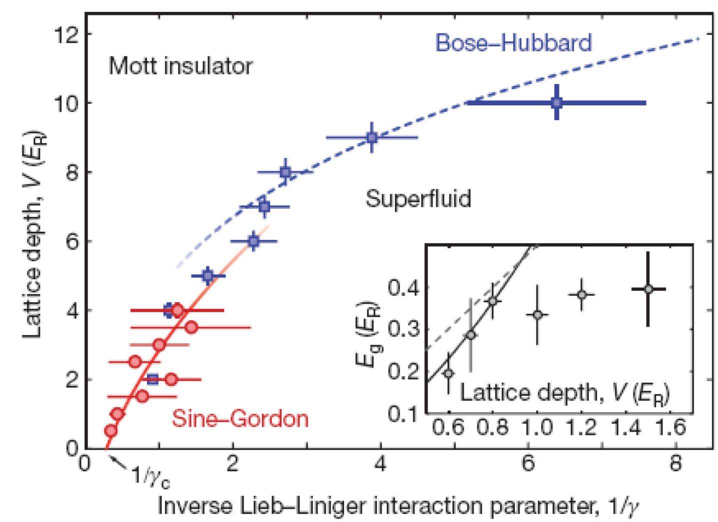
- *Pinning transition* for a Luttinger liquid of strongly interacting bosons (*Cs atoms*)

E. Haller *et al.*, Nature **466**, 597 (2010)



In 1D, provided interactions are strong enough, a periodic potential ***commensurate with the atomic density*** will pin the gas and open a gap, ***no matter how weak the potential.***

→ ***Crossover*** from the **Bose-Hubbard** regime for weak interactions to the **sine-Gordon** model for large interactions



Interaction parameter tuned by a Feshbach resonance

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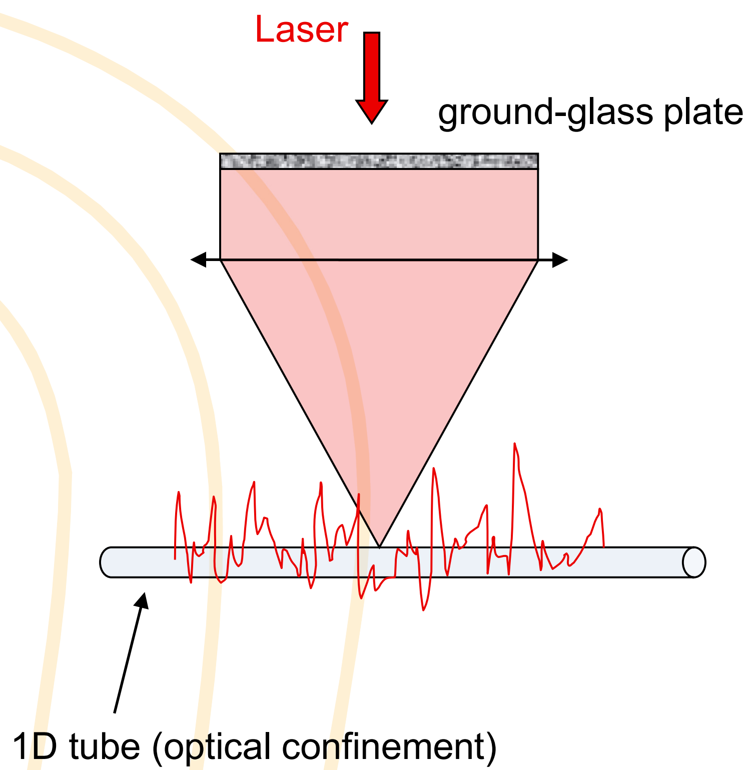
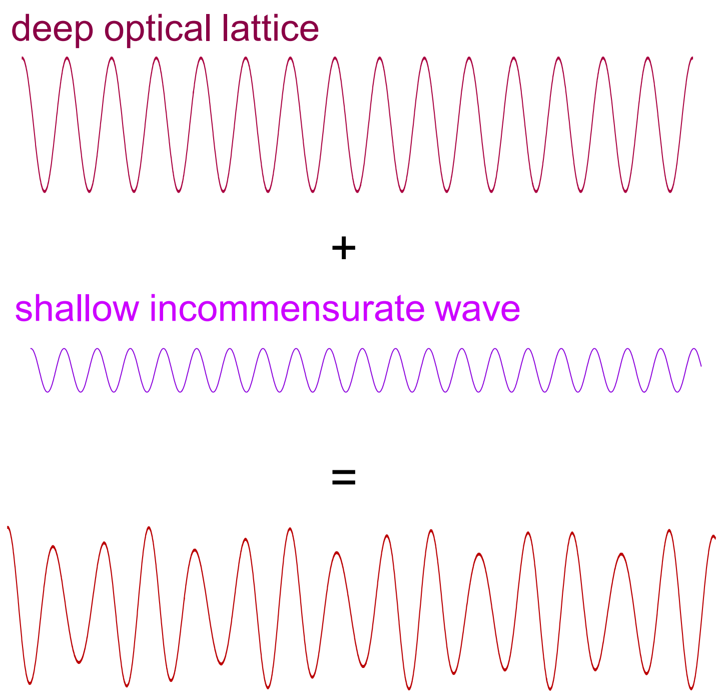
Control of the disorder potential

- 2 main classes of experimental systems



Speckle

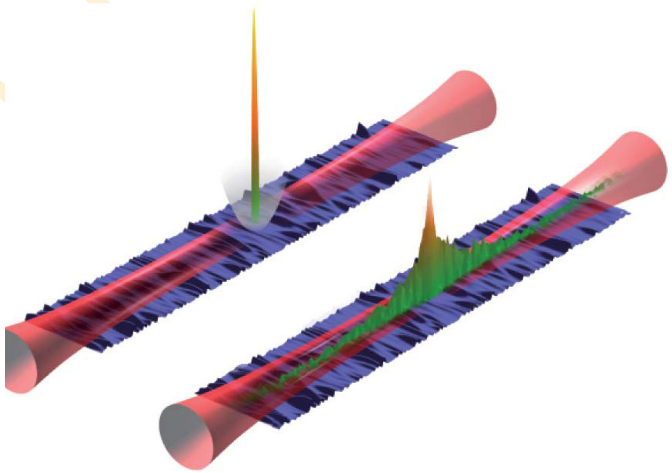


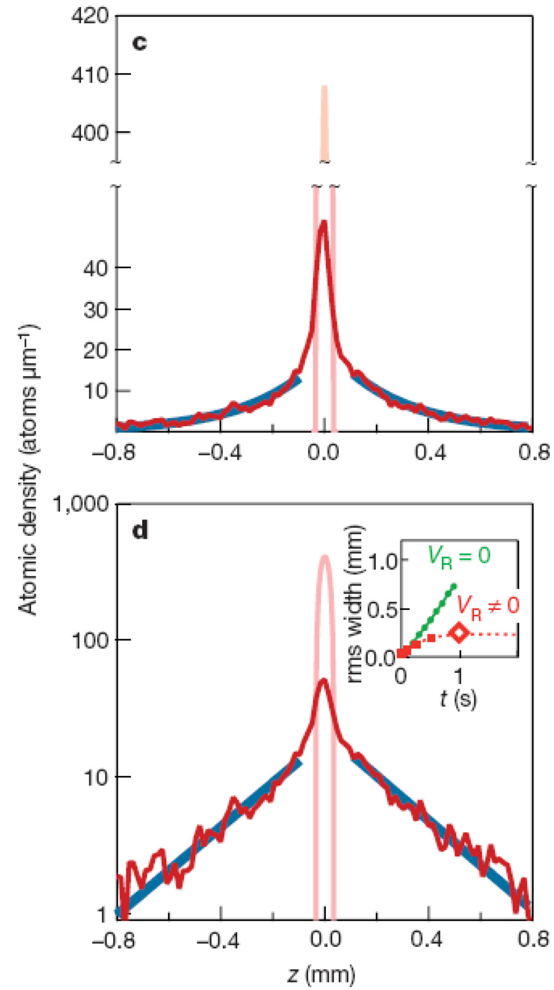
Incommensurate potentials

Direct observation of Anderson localization

- weakly interacting ***BEC*** of ^{87}Rb in a speckle potential

→ ***Anderson localization***





J. Billy, *et al*. Direct observation of Anderson Localization of matter-waves in a controlled disorder. Nature, **453**, 891 (2008)

See also G. Roati et al., Nature, **453**, 895 (2008)

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II-Disorder 1D interacting systems

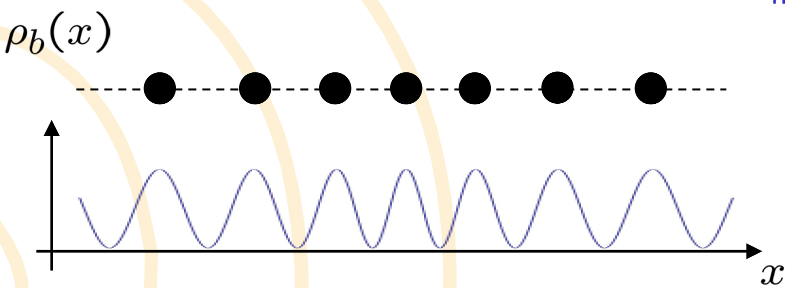
Treating interacting particles in 1D: bosonization

- Hydrodynamic approach

  - Rewrite the dynamics of the system in terms of *density fluctuations* :

$$\rho_b(x) = \sum_i \delta(x - x_i) \longleftrightarrow \rho_b(x) = \rho_b - \frac{1}{\pi}\nabla\phi_b(x) + \rho_b \cos[2\pi\rho_b x - 2\phi_b(x)] + \dots$$

mean density — long wavelength **density** fluctuations — Higher harmonics



- Creation operator:

$$\psi_b^\dagger(x) = \sqrt{\rho_b(x)}\, e^{-i\theta_b(x)}$$

phase field

$$\left[\frac{1}{\pi}\partial_x\theta_b(x), \phi_b(y)\right] = i\delta(x - y)$$

↳ Bosonic fields

$$H_b = \frac{\hbar v_b}{2\pi}\int dx \left[K_b\,[\nabla\theta_b(x)]^2 + \frac{1}{K_b}\,[\nabla\phi_b(x)]^2\right]$$

« kinetic energy » — « Interaction energy »

v_b is the velocity of phonons in the quasi-condensate.

K_b is the so-called Luttinger parameter and is related to the interaction strength

Haldane, PRL (1981)

Remarks on Luttinger parameters for bosons and fermions ($U_{bf} = 0$)

Fermions			Bosons	
$K_f = 1$	no interaction	⟺	$K_b = \infty$	non-interacting
$K_f < 1$	repulsive		$K_b = 1$	Infinite repulsion
$K_f > 1$	attractive			

Fermion field ⟺ Bose field

$$\langle \psi_f(x)\psi_f^+(0) \rangle \sim \frac{1}{|x|^{(K_f + K_f^{-1})/2}}$$

$$\langle \psi_b(x)\psi_b^+(0) \rangle \sim \frac{1}{|x|^{1/2K_b}}$$

POWER LAWS

Adding a commensurate potential

- External lattice potential:

$$H_{\text{ext}} = \int dx \; V(x)\rho(x) \qquad \text{with} \qquad V(x) = V_0 \cos(2\pi\rho_b x + \varphi)$$

- Full low-energy Hamiltonian

$$H_b = \frac{\hbar v_b}{2\pi}\int dx \left[K_b\,[\nabla\theta_b(x)]^2 + \frac{1}{K_b}\,[\nabla\phi_b(x)]^2\right] - V_0 \int dx \; \cos[2\phi_b(x)]$$

Quantum fluctuations

Minimize fluctuations in ϕ_b

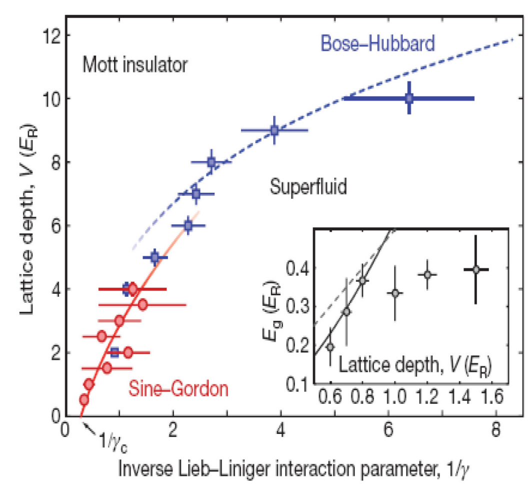
Lock ϕ_b to optimal value

1) Variationnal calculation: *look for best quadratic action.*

→ Gap in the excitation spectrum $\Delta \sim (V_0)^{\frac{1}{2-K}}$ for $K < 2$

2) Renormalization group calculation, perturbative in V_0

→ BKT transition at $K = 2$



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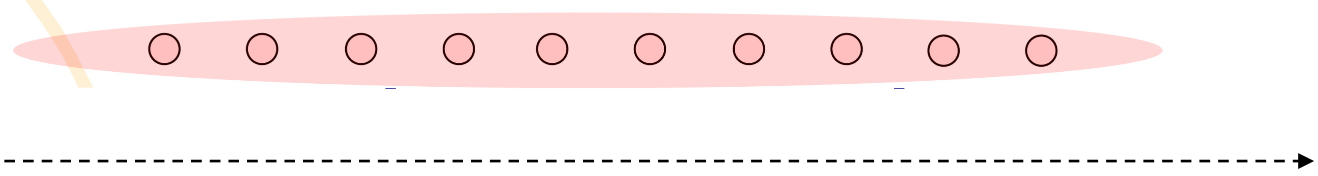
Adding a disorder potential

➡ 1D disordered quantum gases … another kind of pinning transition

- **Random external potential**



→ White noise, the $q = 2\pi\rho_b$ Fourier component tries to pin the density wave.

$$H = \frac{\hbar v}{2\pi} \int dx \left[K \left[\nabla\theta(x)\right]^2 + \frac{1}{K} \left[\nabla\phi(x)\right]^2 - 2\rho|\xi(x)| \cos\left[2\phi(x) - 2\lambda(x)\right] \right]$$

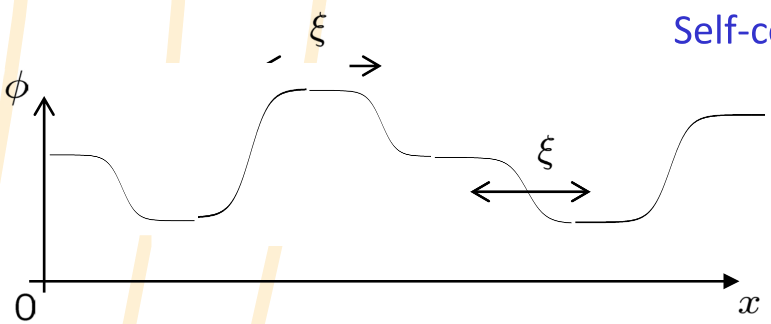
random phase

A variational argument

$$H = \frac{\hbar v}{2\pi}\int dx \left[K\left[\partial_x\theta(x)\right]^2 + \frac{1}{K}\left[\partial_x\phi(x)\right]^2 - 2\rho|\xi(x)|\cos\left[2\phi(x) - 2\lambda(x)\right]\right]$$

random phase

- The density wave tries to adjust its phase ϕ to the random phase, in order to minimize its potential energy.
- any adjustment costs « elastic » energy through $[\partial_x\phi(x)]^2$
- quantum fluctuations work against the pinning of the density wave through $[\partial_x\theta(x)]^2$



Self-consistent calculation including quantum fluctuations:

$$\xi \sim \left(\frac{1}{D}\right)^{\frac{1}{3-2K}}$$ ***Localization length***

Y. Suzumura, H. Fukuyama
J. Phys. Soc. Jap., **8**, 2870 (1983)

This simple picture is in agreement with a RG analysis in replica space

T. Giamarchi, H. Schulz, Phys. Rev. B, **37**, 325 (1988)

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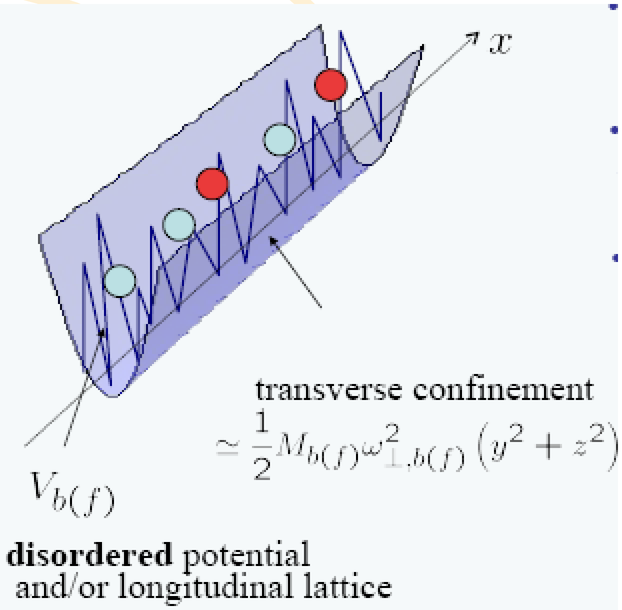
III- Bose-Fermi Mixture in a 1D disordered potential

Experimental considerations

- Array of tubes, with a **tight transverse confinement**.

Mixtures of **bosons** and **spinless fermions** e.g. ^{87}Rb ^{40}K, ^{7}Li ^{6}Li



- Transverse motion in the **ground state** of the harmonic oscillator.
- VdW interactions are **effectively point like.**

→ $U_b = 2\hbar\omega_{\perp,b}\, a_{bb}$

→ $U_{bf} = 2\hbar \frac{1}{M_r} \frac{M_f\omega_{\perp,f} M_b\omega_{\perp,b}}{M_f\omega_{\perp,f} + M_b\omega_{\perp,b}}\, a_{bf}$

a_{bb} and a_{bf} are two scattering lengths

- 2 knobs to control $\mathbf{U_b}$ and $\mathbf{U_{bf}}$ independantly.

$\omega_{\perp,b}$ a_{bf} **Feshbach resonance**

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Disordered Bose-Fermi mixture

Clean case

^{87}Rb – ^{40}K ; ^{7}Li – ^{6}Li ; ^{23}Na – ^{6}Li

- Bosons interact repulsively. Spinless fermions are non interacting.

- Bosons and fermions interact, repulsively of attractively.

→ *Effective (attractive) intra-species interactions*

Disordered Bose-Fermi mixture

- **Is the usual mechanism of localization affected ?**

→ Spinless fermions *should be localized*

→ Bosons should undergo a *transition*

- **Does the disorder decouple the species or does the localization of one species trigger the localization of the other because of interactions ?**

- **What is the phase diagram ?**

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Disordered 1D Bose-Fermi mixtures

Model

- In the continuum

Kinetic energy — Random potential

$$H = \int dx\, \psi_f^\dagger(x)\left[-\frac{\hbar^2}{2M_f}\partial_x^2 + V_f(x)\right]\psi_f(x) + \int dx\, \psi_b^\dagger(x)\left[-\frac{\hbar^2}{2M_b}\partial_x^2 + V_b(x)\right]\psi_b(x)$$

$$+\frac{U_b}{2}\int dx\;\; \rho_b(x)\left(\rho_b(x)-1\right) \; + U_{bf}\int dx\;\; \rho_b(x)\rho_f(x)$$

Bose-Bose interaction — Bose-Fermi interaction

- Interaction parameters

- Integrating out transverse degrees of freedom:

$$U_b = 2\hbar\omega_{\perp,b}\; a_{bb}$$

$$U_{bf} = 2\hbar\frac{1}{M_r}\frac{M_f\omega_{\perp,f}M_b\omega_{\perp,b}}{M_f\omega_{\perp,f}+M_b\omega_{\perp,b}}\; a_{bf}$$

- Disorder

$$\overline{V_f(x)V_f(x')} = D_f\,\delta(x-x') \qquad \overline{V_f(x)} = 0$$

$$\overline{V_b(x)V_b(x')} = D_b\,\delta(x-x') \qquad \overline{V_b(x)} = 0$$

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1D Bose-Fermi mixtures: clean case

➡ Model (continued)

- $$H_b = \frac{\hbar v_b}{2\pi}\int dx \left[K_b\,[\nabla\theta_b(x)]^2 + \frac{1}{K_b}\,[\nabla\phi_b(x)]^2\right]$$

is the *universal* low-energy Hamiltonian of a 1D Bose gas – *Luttinger liquid* theory.

K_b and v_b are model-dependant, $K_b \geq 1$ for repulsive interactions

- $$H_f = \frac{\hbar v_f}{2\pi}\int dx \left[K_f\,[\nabla\theta_f(x)]^2 + \frac{1}{K_f}\,[\nabla\phi_f(x)]^2\right]$$

$K_f = 1$ for non interacting fermions
v_f is the Fermi velocity

- $$H_{bf} = \frac{U_{bf}}{\pi^2}\int dx\, \nabla\phi_b(x)\nabla\phi_f(x) \quad + g\int dx\, \cos[2\pi(\rho_f-\rho_b)x+2(\phi_f(x)-\phi_b(x))]$$

Bose-Fermi interactions *couple* two Luttinger liquids

1D Bose-Fermi mixtures: clean case

Model (continued)

- $$H_b = \frac{\hbar v_b}{2\pi} \int dx \left[K_b \left[\nabla\theta_b(x)\right]^2 + \frac{1}{K_b} \left[\nabla\phi_b(x)\right]^2 \right]$$

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(second term crossed out:) $\rho_f \neq \rho_b$

Bose-Fermi interactions *couple* two Luttinger liquids ***through density fluctuations.***

1D Bose-Fermi mixtures : clean case

Linear transformation $\phi_b = b_+\phi_+ + b_-\phi_- \quad \phi_f = f_+\phi_+ + f_-\phi_-$

$$H^0 = \sum_{\alpha=\pm} \frac{v_\alpha}{2\pi} \int dx \left[(\partial_x \theta_\alpha(x))^2 + (\partial_x \phi_\alpha(x))^2 \right]$$

- sound velocity

$$v_\pm^2 = 1/2(v_f^2 + v_b^2) \pm 1/2\sqrt{(v_f^2 - v_b^2)^2 + 4g^2 v_f^2 v_b^2}$$

- dimensionless coupling $g = \frac{U_{bf}}{\pi}\sqrt{\frac{K_f K_b}{v_f v_b}}$

- instability: $|g| \to 1$ (Wentzel-Bardeen instability)

Cazalilla and Ho, PRL 2003; Mathey et al., PRL 2004

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Disordered 1D Bose-Fermi mixtures

Model (continued)

- Perturb the 2-component Luttinger liquid fixed point with disorder

$$H^{\alpha}_{dis} = \int dx \left[-\frac{1}{\pi}\gamma_\alpha(x)\partial_x\phi_\alpha(x) + \rho_\alpha\xi_\alpha(x)e^{-i2\phi_\alpha(x)} + h.c. \right]$$

$\alpha = f, b$

Random chemical potential
(forward scattering on disorder)

Pinning of the density wave
(backscattering on disorder)

$q = 2\pi\rho_b$ ou $2k_F$

$$\gamma_\alpha(x) = \frac{1}{L}\sum_{q\sim 0} e^{iqx}V_{\alpha,q}$$ can be gauged away in the quadratic Hamiltonian.

$$\xi_\alpha(x) = \frac{1}{L}\sum_{q\sim 0} e^{iqx}V_{\alpha,q-2\pi\rho_\alpha}$$ is responsible for the localization. $\overline{\xi_\alpha(x)\xi^*_\alpha(x')} = D_\alpha\delta(x-x')$

- Competition between :
  - the random potential that tries to pin the phase
  - the « elastic » energy (coming from the interactions)
  - the quantum fluctuations

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Replica trick

Replica Trick:

$$\overline{\ln Z} = \lim_{n \to 0} (\overline{Z^n} - 1)/n$$

$$\overline{Z^n} = \int \prod_{a=1}^{n} D\Phi_b^a D\Phi_f^a \; e^{-S_{rep}}$$

Impurity average can be performed

Replicated action:

$$S_{rep} = S_0 + S_{dis}^{rep}$$

$$S_0 = \sum_a \sum_{j=f,b} \frac{1}{2\pi K_j} \int dx d\tau \left[\frac{1}{v_j} \left(\partial_\tau \phi_j^a \right)^2 + v_j \left(\partial_x \phi_j^a \right)^2 \right] + \frac{U_{bf}}{\pi^2} \int dx d\tau \partial_x \phi_f^a \partial_x \phi_b^a ,$$

$$S_{dis}^{rep} = -D_f \rho_f^2 \sum_{a,b} \int dx \int d\tau d\tau' \cos(2\phi_f^a(x,\tau) - 2\phi_f^b(x,\tau'))$$

$$-D_b \rho_b^2 \sum_{a,b} \int dx \int d\tau d\tau' \; \cos(2\phi_b^a(x,\tau) - 2\phi_b^b(x,\tau'))$$

replica-interaction generated

Renormalization group equations

Relevance of disorder:

$$\frac{d\tilde{D}_f}{dl} = (3 - X_f)\tilde{D}_f(l)$$
$$\frac{d\tilde{D}_b}{dl} = (3 - X_b)\tilde{D}_b(l)$$

$$\tilde{D}_f = 4\pi D_f \rho_f^2 / (\Lambda^3 v_f^2)$$

$$X_f = \frac{2K_f(1 + t\sqrt{1-g^2})}{\sqrt{1-g^2}\sqrt{1+2t\sqrt{1-g^2}+t^2}}$$

$$X_b = \frac{2K_b(t + \sqrt{1-g^2})}{\sqrt{1-g^2}\sqrt{1+2t\sqrt{1-g^2}+t^2}}$$

$$t = v_f / v_b$$

Localization length:

$$\xi_{loc}^f \sim a\left(\frac{1}{\tilde{D}_f}\right)^{\frac{1}{3-X_f}}$$

$$\xi_{loc}^b \sim a\left(\frac{1}{\tilde{D}_b}\right)^{\frac{1}{3-X_b}}$$

Feed-back of disorder:

$$\frac{dK_f}{dl} = -K_f^3 \tilde{D}_f C$$
$$\frac{dv_f}{dl} = -v_f K_f^2 \tilde{D}_f C$$

$$\frac{dK_b}{dl} = -K_b^3 \tilde{D}_b C$$
$$\frac{dv_b}{dl} = -v_b K_b^2 \tilde{D}_b C$$

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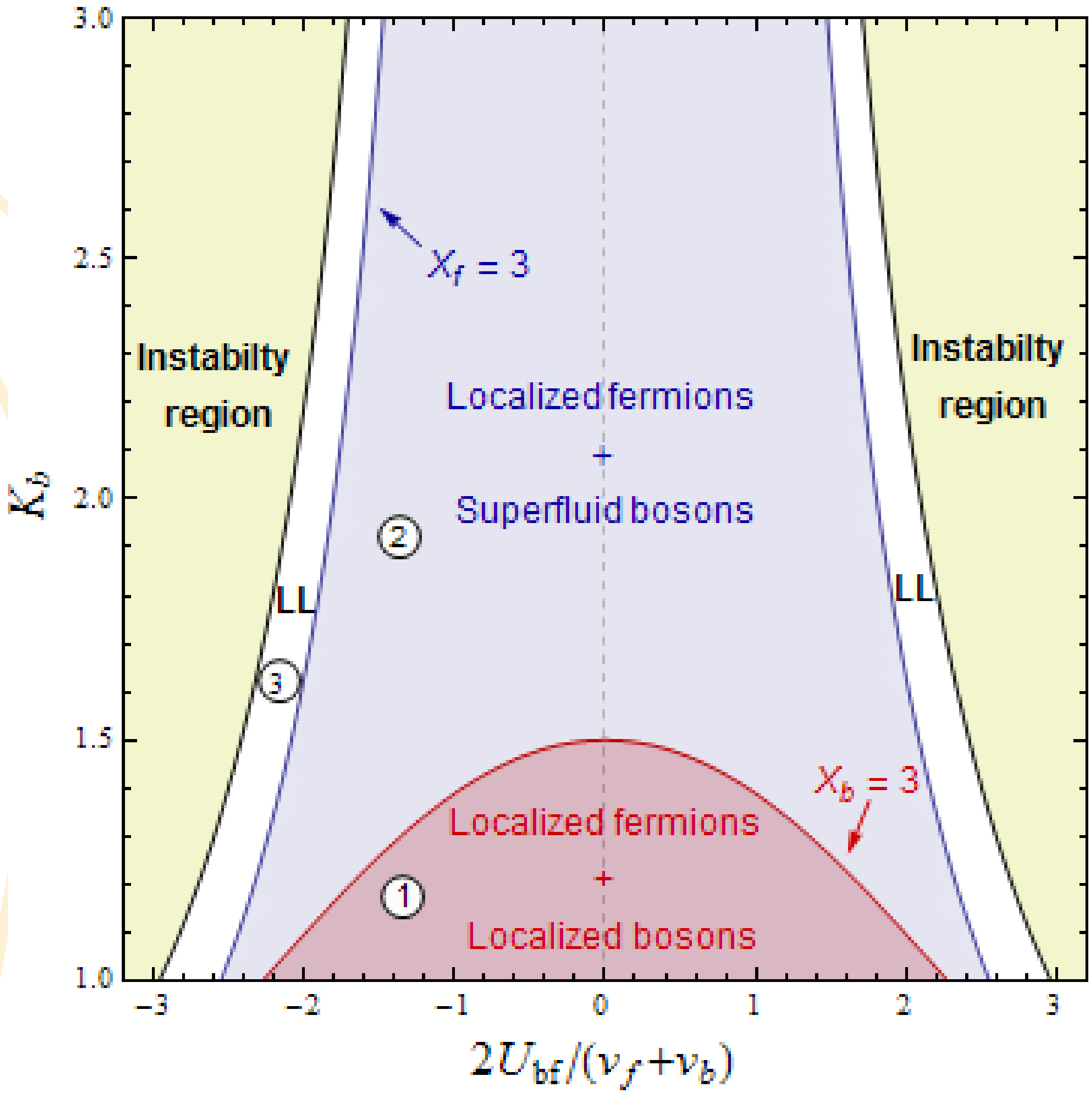
Phase diagram obtained from simple RG



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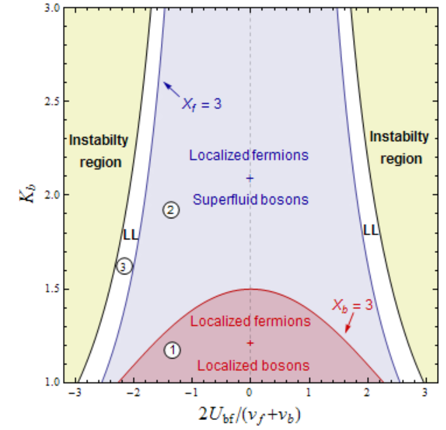
Beyond simple RG



Are bosons really superfluid when $K_b < 3/2$?

In the regime $X_b > 3$ and $X_f < 3$ the RG predicts that fermions are localized while bosons are superfluid.

<u>However</u>, fermionic density fluctuations become `` gapped'' beyond $\Lambda \ll 1/\xi_f$

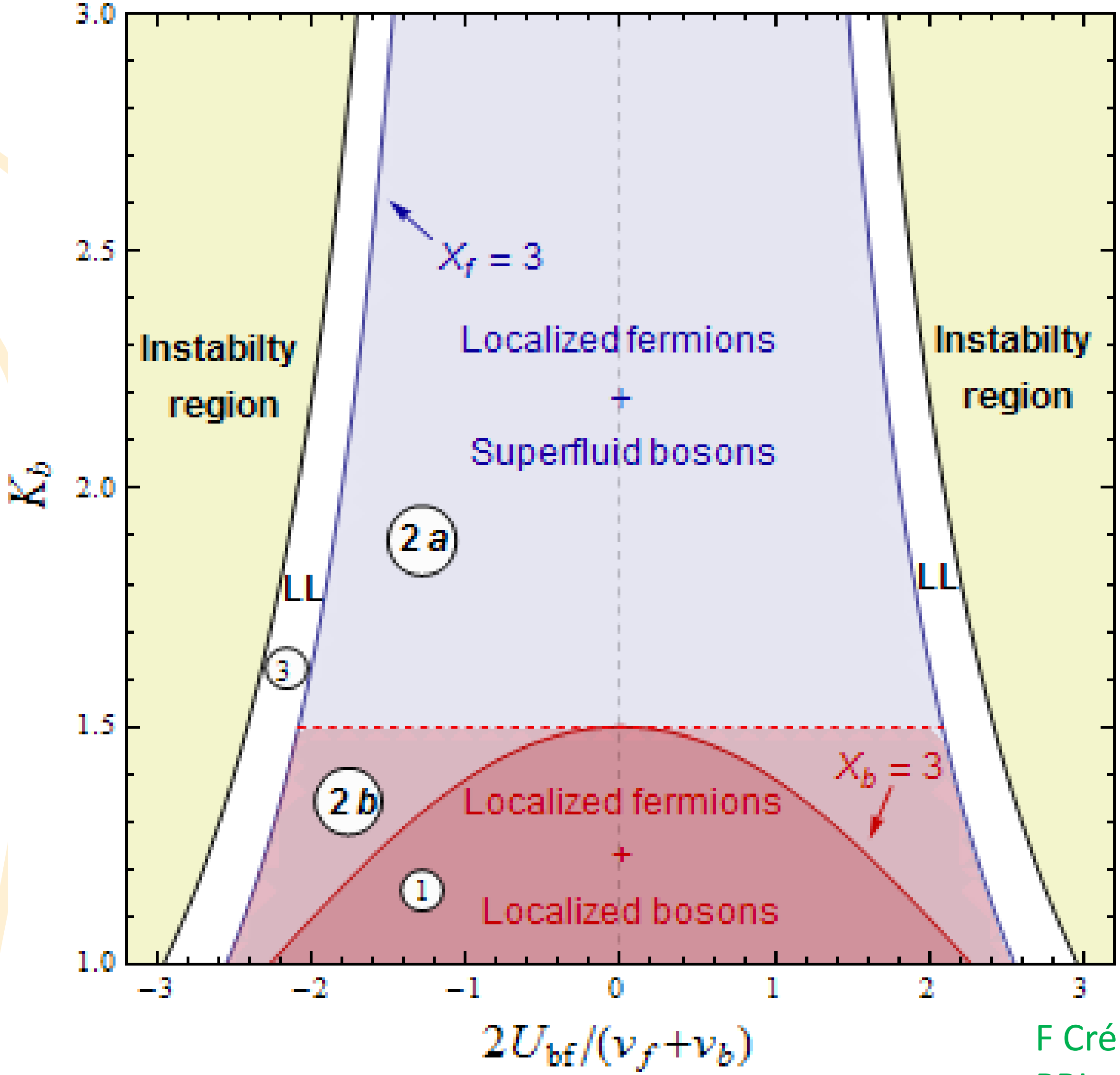
RG must be done in two steps:

$$\frac{d\log \tilde{D}_b}{dl} = \begin{cases} 3 - X_b & \text{if } \Lambda \gg 1/\xi_f \\ 3 - 2K_b & \text{if } \Lambda \ll 1/\xi_f \end{cases}$$

This extends the phase where bosons do localize

This argument is substantiated by a more elaborated approach: the Gaussian variationnal method in replica space

Phase diagram obtained from 2-step RG



F Crépin, G. Zaránd, PS
PRL., **105**, 115301 (2010)

A word on the Gaussian variational method

- Replicated action:

$$S_0 = \sum_{a=1}^{n} \sum_{\alpha=f,b} \frac{\hbar}{2\pi K_\alpha} \int dx d\tau \left[\frac{1}{v_\alpha} (\partial_\tau \varphi_\alpha^a)^2 + v_\alpha (\partial_x \varphi_\alpha^a)^2\right] + \frac{U_{bf}}{\pi^2} \int dx d\tau \; \partial_x \varphi_f^a \partial_x \varphi_b^a$$

$$S_{\text{dis}} = -\frac{D_f \rho_f^2}{\hbar} \sum_{a,b} \int dx \int d\tau d\tau' \cos\left[2\varphi_f^a(x,\tau) - 2\varphi_f^b(x,\tau')\right]$$

$$-\frac{D_b \rho_b^2}{\hbar} \sum_{a,b} \int dx \int d\tau d\tau' \; \cos\left[2\varphi_b^a(x,\tau) - 2\varphi_b^b(x,\tau')\right]$$

- One looks for the best quadratic action

$$S = \frac{1}{2}\frac{1}{\beta L} \sum_{q, i\omega_n} \varphi_\alpha^a(q, i\omega_n) (G^{-1})_{\alpha\beta}^{ab}(q, i\omega_n) \varphi_\beta^b(-q, -i\omega_n)$$

with: $(G^{-1})_{\alpha\beta}^{ab} = (G_0^{-1})_{\alpha\beta}^{ab} - \sigma_{\alpha\beta}^{ab}$

self-energy

by minimizing the variational free energy:

$$F_{\text{var}} = -1/\beta \ln \text{Tr}\, e^{-S_G} + \langle S - S_G \rangle_G / \beta$$

Mézard, Parisi, 91'

Giamarchi, Le Doussal, 96'

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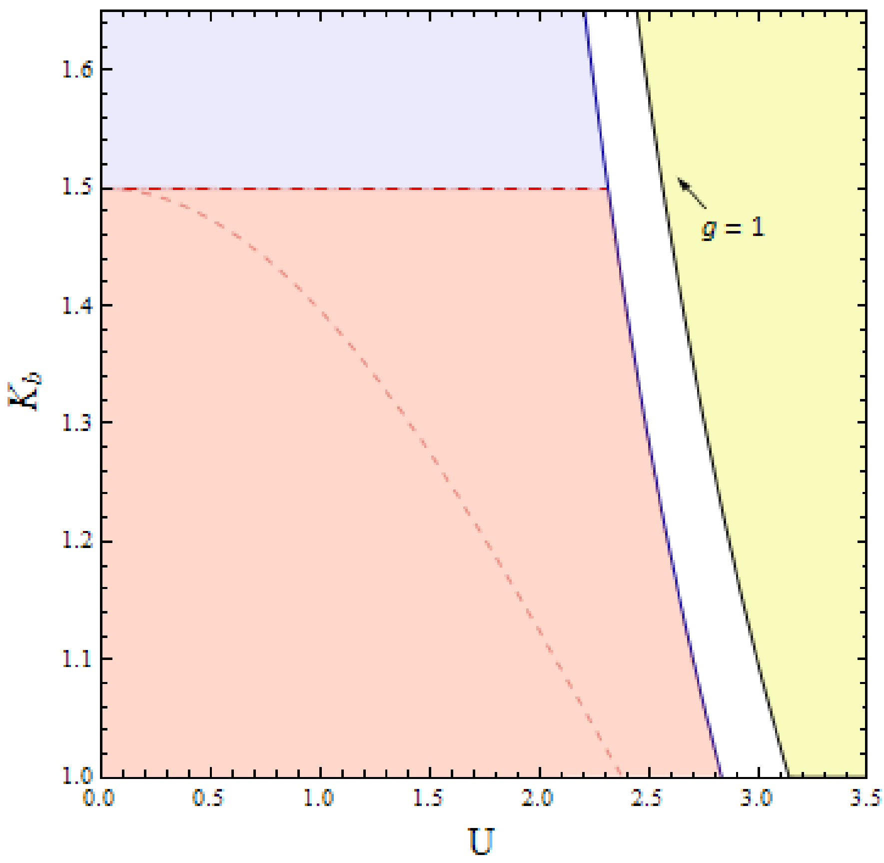
Disordered 1D Bose-Fermi mixtures

➡ Gaussian variationnal method, in replica space



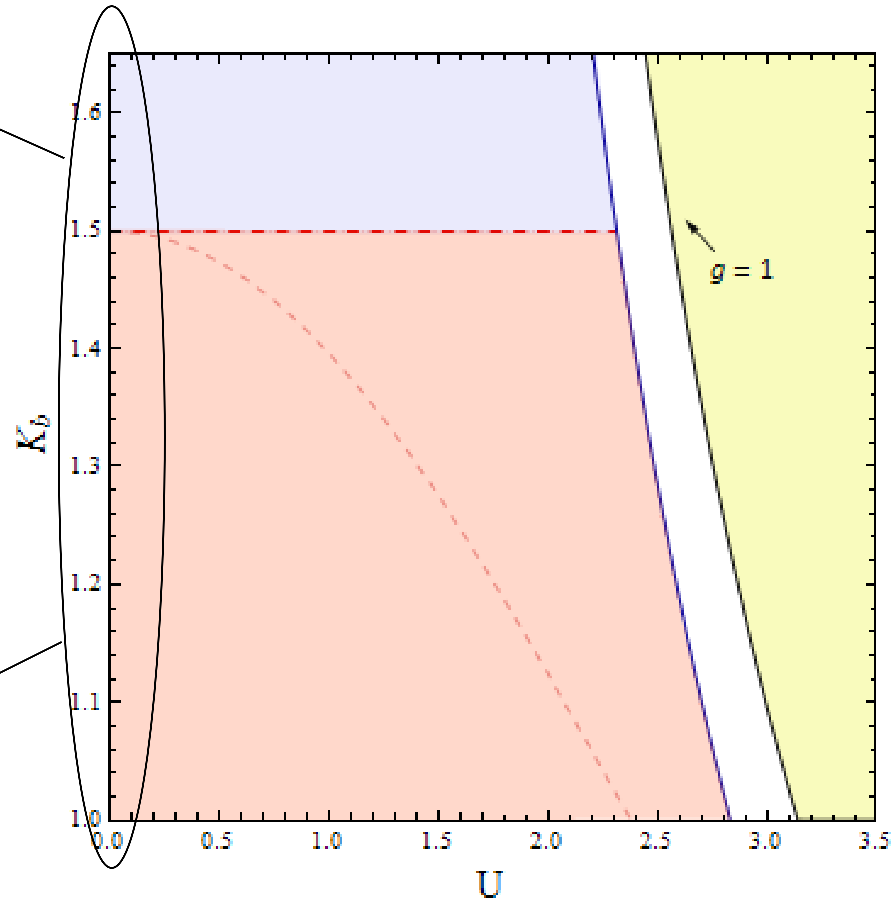
Disordered 1D Bose-Fermi mixtures

➡ Gaussian variationnal method, in replica space

Fermions are localized
(1RSB)

Boson superfluid
(RS)

Fermions are localized
(1RSB)
+
Boson are localized
(1RSB)



Disordered 1D Bose-Fermi mixtures

- Gaussian variationnal method, in replica space

Fermions are localized
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Fermions are localized
(1RSB)
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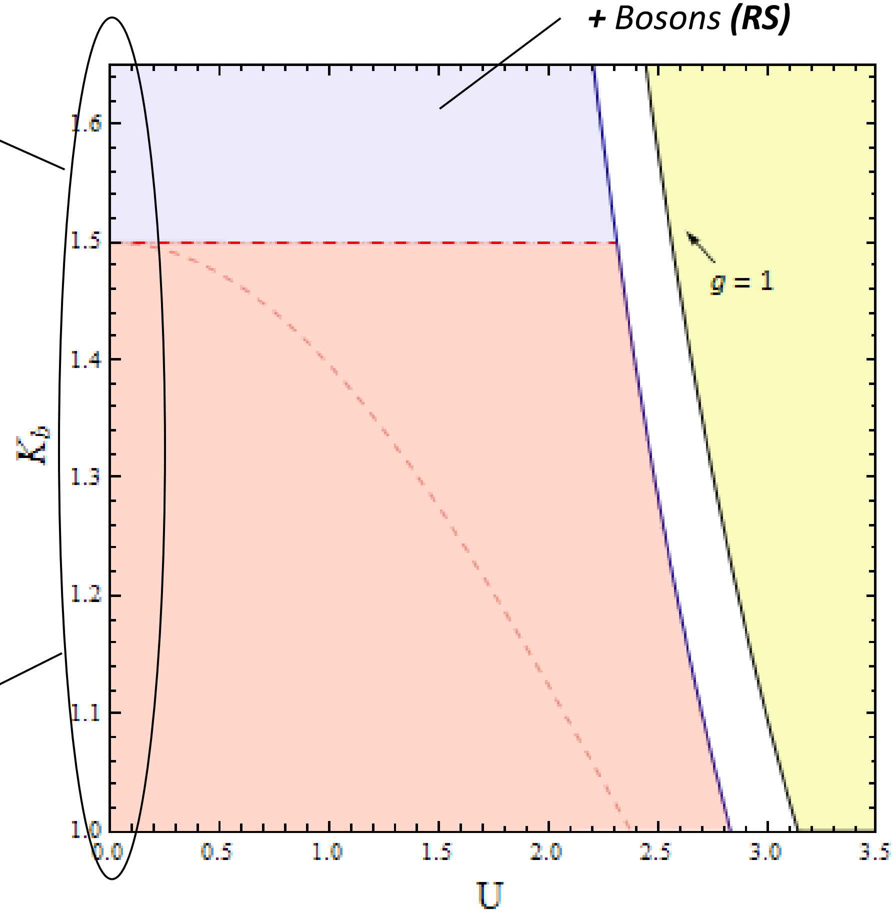
(1RSB)

Fermions ***(1RSB)***
+ *Bosons* ***(RS)***

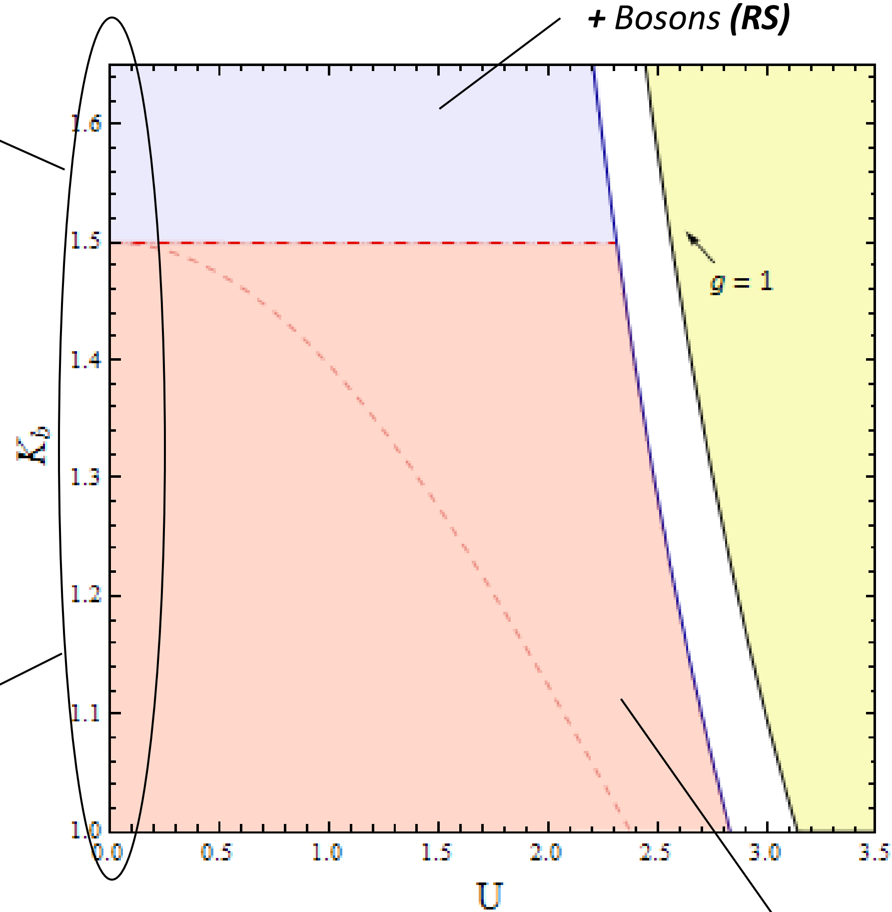


Disordered 1D Bose-Fermi mixtures

- Gaussian variationnal method, in replica space

Fermions ***(1RSB)***
+ *Bosons* ***(RS)***

Fermions are localized
(1RSB)
+
Boson superfluid
(RS)



Fermions are localized
(1RSB)
+
Boson are localized
(1RSB)

Fully localized
Fermions ***(2RSB)***
+ *Bosons* ***(1RSB)***

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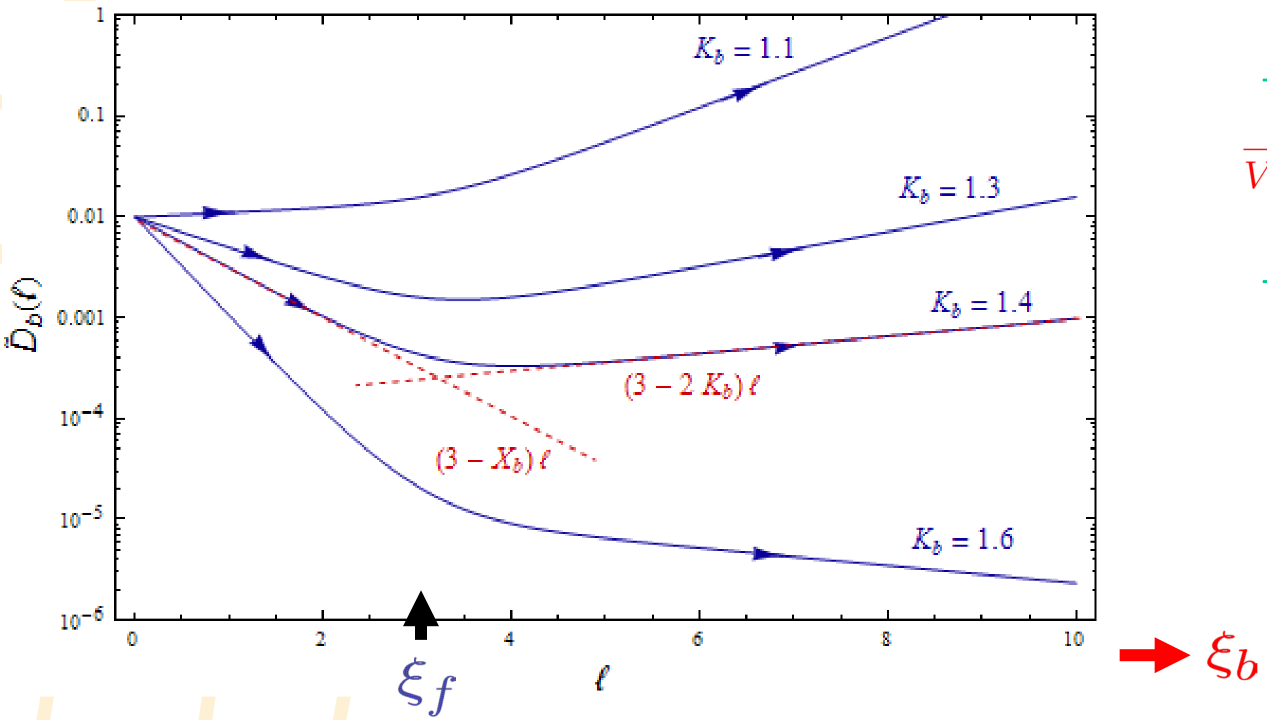
Example of typical flow trajectories

- Modified RG flow from the variational solution (1RSB + RS)



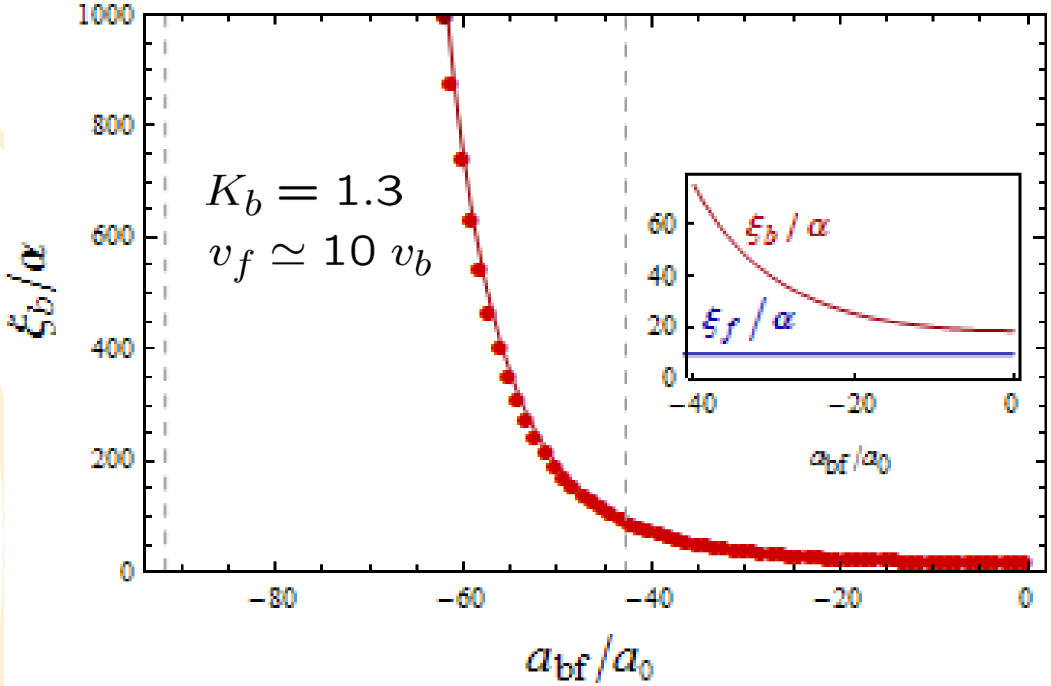
- Disorder strength $D_b(l)$:

$$\overline{V_b(x)V_b(x')} = D_b\,\delta(x - x')$$

- Short-distance cutoff:

$$\alpha(\ell) = \alpha_0 e^{\ell}$$

Boson localization length

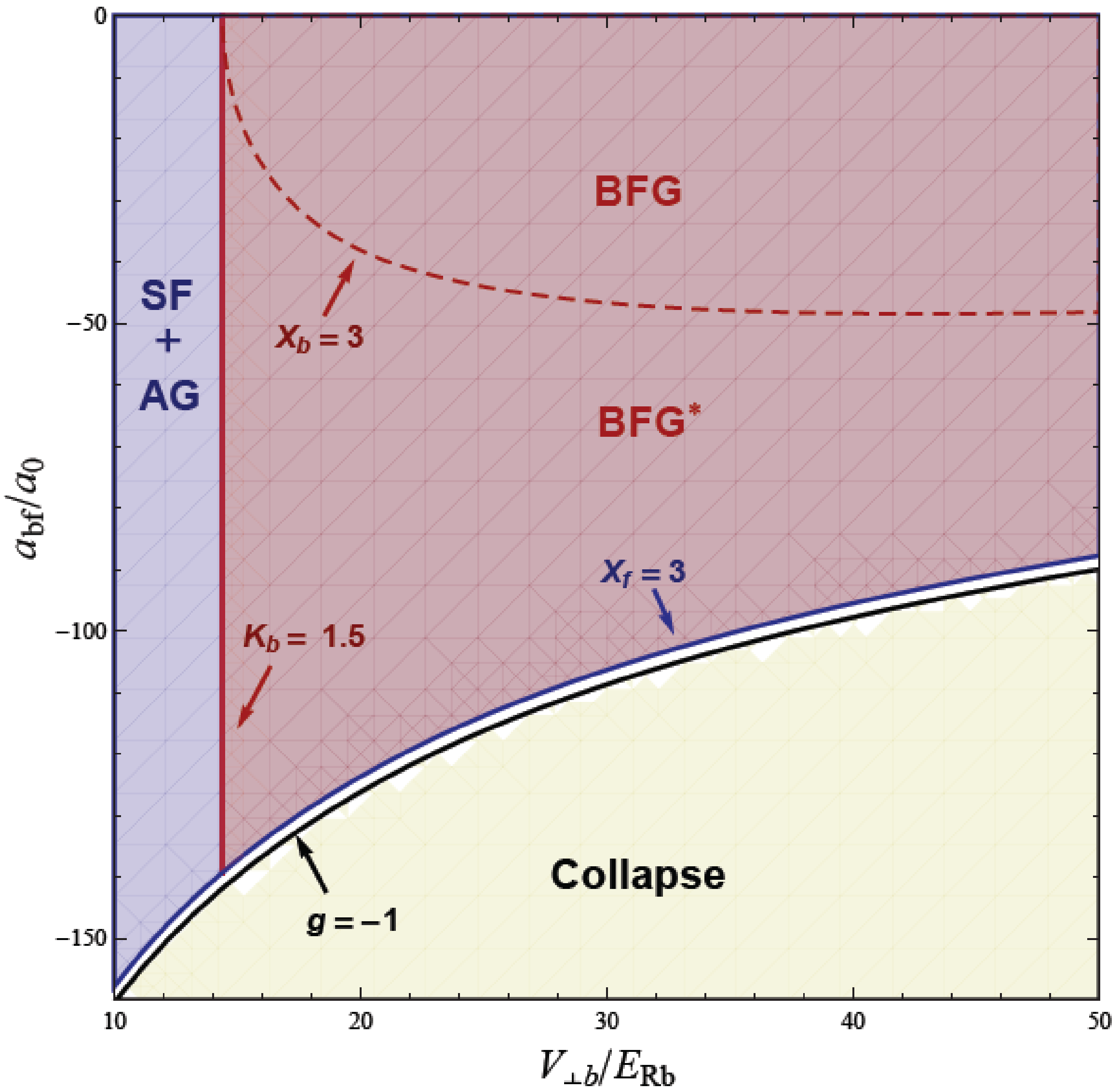


Results for a ^{87}Rb – ^{40}K mixture.

- In the fully localized phase, *both species are localized*, but interactions are still very important, since *the bosonic localization length is controlled by Bose-Fermi interactions*.

F Crépin, G. Zaránd, PS
PRL **105**, 115301 (2010)

Phase diagram for ^{87}Rb - ^{40}K mixture



Observables : Bragg scattering

Bragg scattering experiments give access to the dynamic structure factor

See D. Clément et al., PRL 09'

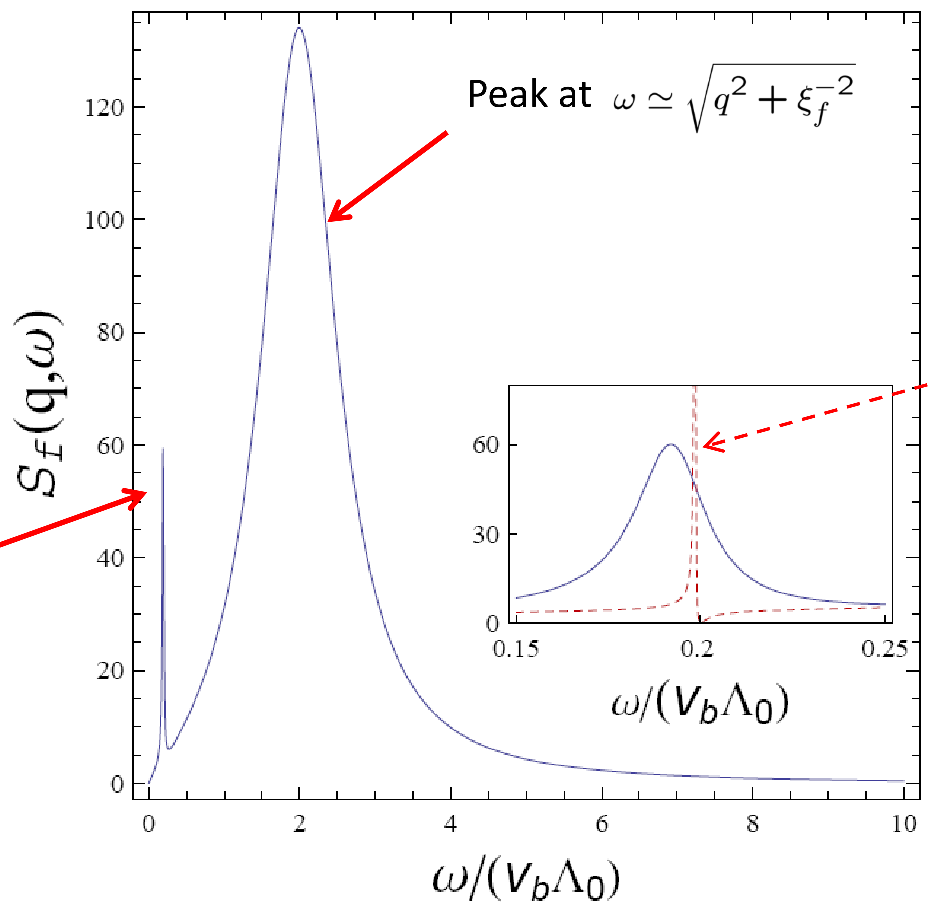
$$S_{b/f}(q,\omega) = \int dtdx \; e^{iqx - i\omega t} \overline{\langle \rho_{b/f}(x,t)\rho_{b/f}(0,0)\rangle}$$

Fermionic structure factor in the Bose-Fermi glass phase

Peak at $\omega \simeq \sqrt{q^2 + \xi_f^{-2}}$

In the AG+SFB phase

Peak at $\omega \simeq \sqrt{q^2 + \xi_b^{-2}}$



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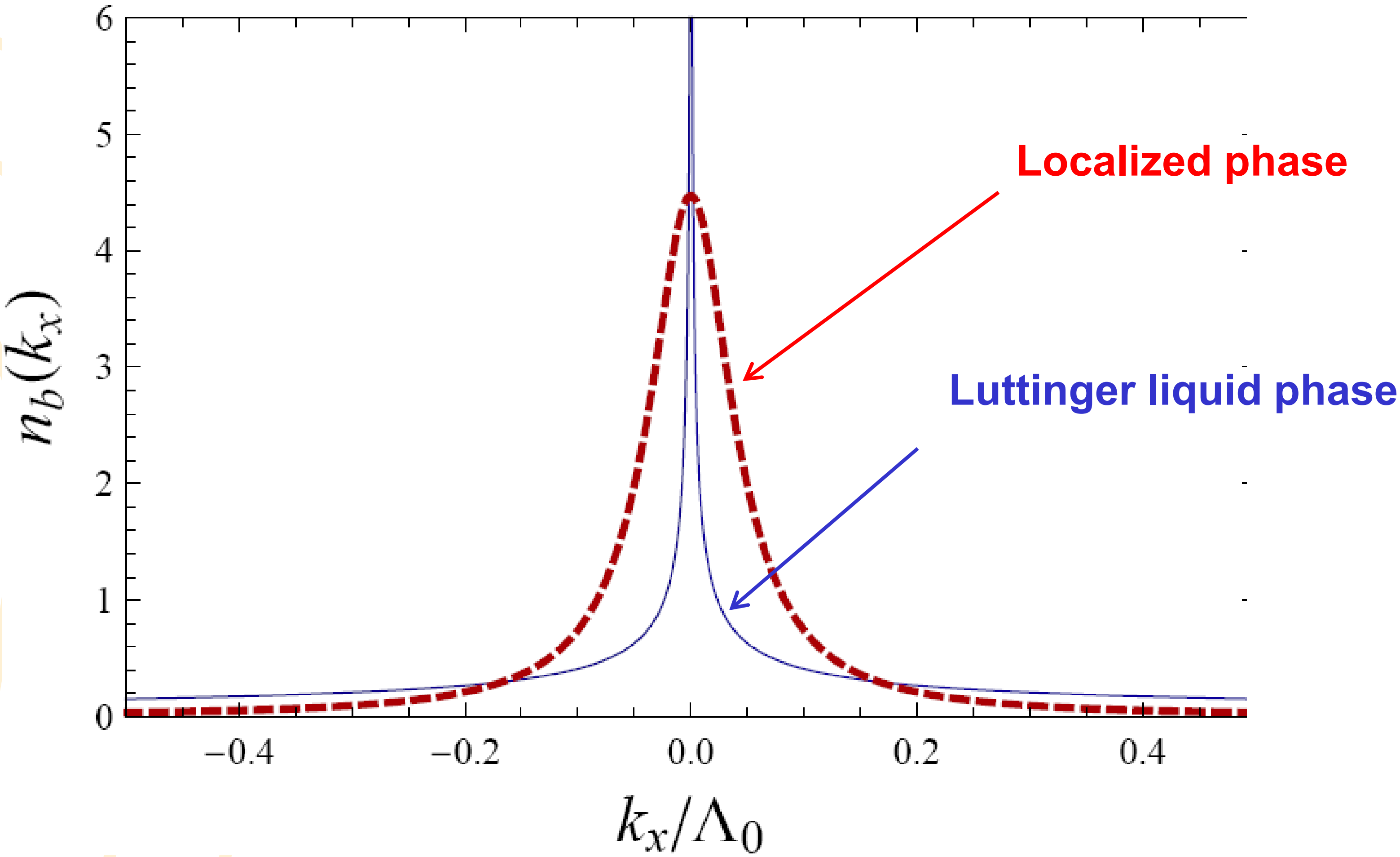
PRL., **105**, 115301 (2010)

Observables 2:time of flight



Average bosonic density after a time of flight t

Conclusions

- We have studied a Bose-Fermi mixture in a disordered potential

Several phases: **Bose-Fermi glass**,
coexisting localized/LL phases

Localization length of bosons controlled by fermions

- Phase transitions can be detected by
  - Bragg scattering experiement

- **Open question:** $\rho_f = \rho_b$

➡ Composite fermion glass ?
competition between various insulating orders

F Crépin, G. Zaránd, PS, PRL, **105**, 115301 (2010)
F Crépin, G. Zaránd, PS, arXiv 1109.xxxx (2011)

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A zest of integral equations ...

$$(G^{-1})^c_{\alpha\beta} = \sum_b (G^{-1})^{ab}_{\alpha\beta} \qquad B^{ab}_{bb}(0,\tau) = \langle[\phi^a_b(0,\tau) - \phi^b_b(0,0)]^2\rangle_G$$

$$B^{ab}_{ff}(0,\tau) = \langle[\phi^a_f(0,\tau) - \phi^b_f(0,0)]^2\rangle_G$$

$$\begin{aligned}(G^{-1})^c_{ff} &= (G_0^{-1})^{ab}_{ff} + 2\int_0^\beta d\tau(1-\cos(\omega_n\tau))U'_f(\widetilde{B_{ff}}(\tau)) \\ &+ 2\beta(\delta_{n0}-1)\int_0^1 du\ U'_f(B_{ff}(u)) \qquad (10)\end{aligned}$$

$$\begin{aligned}(G^{-1})^c_{bb} &= (G_0^{-1})^{ab}_{bb} + 2\int_0^\beta d\tau(1-\cos(\omega_n\tau))U'_b(\widetilde{B_{bb}}(\tau)) \\ &+ 2\beta(\delta_{n0}-1)\int_0^1 du\ U'_b(B_{bb}(u)) \qquad (11)\end{aligned}$$

$$(G^{-1})^c_{fb} = (G^{-1})^c_{bf} = \widetilde{(G_0^{-1})}_{bf} \qquad (12)$$

with $U_\alpha(z) = 2\rho_\alpha^2 D_\alpha e^{-2z}$

$$\widetilde{B_{ff}}(\tau) = \frac{2}{\beta L}\sum_{q,i\omega_n}\widetilde{G_{ff}}(q,i\omega_n)(1-\cos\omega_n\tau)$$

$$B_{ff}(u) = \frac{2}{\beta L}\sum_{q,i\omega_n}\widetilde{G_{ff}}(q,i\omega_n) - G_{ff}(q,0,u)$$

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Solution of resulting integral equations

$$[G^{(c)}]^{-1}(\omega_n, q) \equiv \begin{pmatrix} \frac{1}{\pi K_b}(\frac{1}{v_b}\omega_n^2 + v_b q^2) + I_b(\omega_n) + \Sigma_b(1-\delta_{n,0}) & \frac{1}{\pi}\sqrt{\frac{v_f v_b}{K_f K_b}} g\, q^2 \\ \frac{1}{\pi}\sqrt{\frac{v_f v_b}{K_f K_b}} g\, q^2 & \frac{1}{\pi K_f}(\frac{1}{v_f}\omega_n^2 + v_f q^2) + I_f(\omega_n) + \Sigma_f(1-\delta_{n,0)} \end{pmatrix}$$

- Replica symmetry breaking (~ ``gap”)

$\Sigma_b > 0$ if bosons localized

$\Sigma_f > 0$ if fermions localized

$$\Sigma_{b,f} \sim \xi_{b,f}^{-2}$$

- $\lim_{\omega_n \to 0} I_{b,f}(\omega_n) = 0$,
- $I_{b,f}(\omega_n)$ and $\Sigma_{b,f}$ must be determined self-consistently

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