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Rough paths for physicists

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Rough path theory emerged in the 1990's motivated by practical and conceptual questions. The general context is that of controlled differential equations, for example $dU_t = f(U_t)dX_t$ where X_t is a known property of a system, changing with time, and U_t is an unknown whose evolution is to be computed via the equation. The emphasis is on the regularity of X_t as a function of time, and its consequences on that of U_t, making the necessary assumptions on f.

If X_t is smooth, we are on familiar grounds, but what if X_t, though continuous, has little regularity? In that case, it is the rule rather than the exception that Euler's discretization $U_{t_{n+1}} = U_{t_n} + f(U_{t_n})(X_{t_{n+1}} - X_{t_n})$ does not converge when the mesh goes to 0. Approaching X_t more and more closely by smooth functions $X_t^{(n)}$, and seeing if the corresponding $U_t^{(n)}$s get close to something, has its own difficulties. What is the right notion of "close"? Does $\lim_{n\to\infty} U_t^{(n)}$, when it exists, depend on the approximation sequence $X_t^{(n)}$ and how?

Rough path theory gives an answer to these questions in a form suggestive of deep relations to physics (power counting, counterterms, etc) that I will try to explain. The short answer is that depending on the regularity of X_t, making unambiguous sense of the equation $dU_t = f(U_t)dX_t$ requires to supplement X_t with new data which roughly (!) speaking replace some undefined integrals by a formal, yet concrete, mathematical object: a rough path structure.

Plan:
1. Motivations. The sewing lemma and Young integrals.
2. Rough paths, combinatorics and regularity. A waltz with Brownian motion.
3. Controlled rough paths, rough integrals and rough differential equations.
Rough integrals versus stochastic integrals, the Itō stochastic area.
4. Thermalization of a particle in a magnetic field. Outlook.

Course website: https://courses.ipht.fr/?q=en/node/309