Generalized geometries and N=1 vacua

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Type II sugra on $\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6$

Geometry ⟷ Fluxes

Minimal supersymmetry

⬇

$\mathcal{M}_6$ is CY —Turn on fluxes→ $\mathcal{M}_6$ ~~CY~~

Eff action: SU(3) holonomy → SU(3) x SU(3) structure

Vacua: CY → Generalized CY

Generalized complex geometry

Application on twisted tori

Supersymmetric solutions with fluxes preserving Poincare invariance

Can be obtained by variations of the superpotential (4D analysis) or directly in 10 D.

$$\delta_\epsilon \psi_M = \nabla_M \epsilon + H_{Mnp}\gamma^{np}\epsilon + e^{\phi}\sum_n \not{F}_n \gamma_M \sigma^1 \epsilon \,,$$

$$\not{F}_n = F_{a_1 \dots a_n}\gamma^{a_1 \dots a_n}$$

n=1,3,5 for IIB

- ∃ Susy requires topological condition on $\mathcal{M}_6$
- Preserved susy requires differential condition on $\mathcal{M}_6$

Only H_3: $\delta\psi_m = \nabla_m \eta + H_m \eta = 0 \quad (H_m = H_{mnp}\gamma^{np})$

$\nabla' = \nabla + H$

η is covariantly constant in a connection with torsion

torsion ↔ flux

➜ $\mathcal{M}_6$ has SU(3) structure

or SU(2) or... SU(3) x SU(3)

$\mathcal{N}$= 1

$$\varepsilon^1 = a\,\theta_+ \otimes \eta_+^1 + c.c.$$

$$\varepsilon^2 = b\,\theta_+ \otimes \eta_+^2 + c.c.$$

F ≠ 0 ⇒ relation between a and b

Orientifolds or D-brane susy: $\varepsilon^2 = \gamma^{\perp}\,\varepsilon^1$

$|a|=|b|= e^{A/2}$

$$ds_{10}^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + ds_6^2(y)$$

Rel phase of a and b depends on the D-brane (D3: a=ib / D5: a=b)

We were given enough motivation to use framework of GCG, but let's give some more...

$$\delta\psi_m = \nabla_m \eta + H_m \eta + \not{F} \gamma_m \eta = 0$$

$$(\; W \; + \; H \; + \; F \;) \; \eta = 0$$

W and F are H decomposed in representations of the structure group

Torsion ~ flux : representation by representation

Let's look at simplest case: SU(3)

torsions

↓

$$dJ = \mathrm{Im}\,(\not{W}_1\,\Omega) + \not{W}_3 + \not{W}_4 \wedge J$$

scalar: $1 \oplus 1$

primitive (2,1) : $6 \oplus \bar{6}$

$$d\,\Omega = \not{W}_1\, J^2 + \not{W}_2 \wedge J + \not{W}_5 \wedge \Omega$$

W1=W2=0	↔	complex (complex structure integrable)
W1=W3=W4=0	↔	symplectic
W1=W2=W3=W4=0	↔	Kähler
W1=W2=W3=W4=W5=0	↔	CY

In flux vacua, W ~ F

W_2 is even form, W_3 is odd form → $W_2 \sim F_2$ IIB vacua are complex

$W_3 \sim F_3$ IIA vacua are symplectic

To describe vacua of type II, we need

a mathematical construction that extends contains complex and symplectic geometry

Generalized complex geometry

Hitchin
Gualtieri

• Differential geometry on $T \oplus T^*$ sections are $v + \zeta$

Spinors Φ of O(6,6) : = p-forms

Weyl: positive chirality $S^+ \sim \Lambda^{even}$

negative chirality $S^- \sim \Lambda^{odd}$

Clifford action: $\underbrace{(v+\zeta) \bullet \Phi}_{\uparrow\,\Phi^+} = \underbrace{v^m \iota_m \Phi + \zeta_m dx^m \wedge \Phi}_{\Phi^-}$

Pure spinor: annihilator space is maximal (6-dimensional)

$$(v+\zeta) \in T \oplus T^* \text{ s.t. } (v+\zeta) \bullet \Phi = 0$$

On a manifold of SU(3) structure

$$\Phi^- = \Omega_3 = dz^1 \wedge dz^2 \wedge dz^3 \rightarrow \underbrace{\xi_i dz^i}_{E_\Omega} \wedge \Omega_3 = 0 \text{ and } v^{\bar{I}} \underbrace{\iota_{\bar{I}}}_{E_\Omega} \Omega_3 = 0 \rightarrow \Omega_3 \text{ is pure}$$

$$\Phi^+ = e^{iJ} = 1 + iJ - J^2 + \ldots \rightarrow \underbrace{v^m (\iota_m + i J_{mn} dx^n \wedge)}_{E_J} e^{iJ} = 0 \rightarrow e^{iJ} \text{ is pure}$$

1-1 correspondance between pure spinors and generalized almost complex structures $\mathcal{J}$

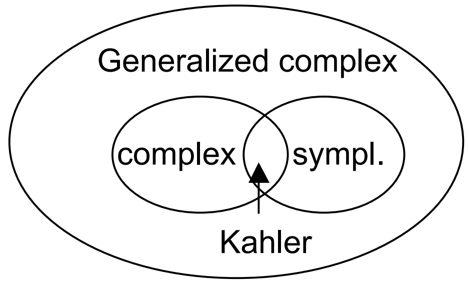
$d\Phi = (v+\xi)\,\Phi$ for some v, ξ $\longleftrightarrow$ $\mathcal{J}_\phi$ integrable generalized complex manifold

$d\Phi = 0$ $\longrightarrow$ $\mathcal{J}_\phi$ integrable generalized Calabi-Yau manifold

$d\Omega_3 = W_5 \wedge \Omega_3$ Complex manifolds → have integrable GCS

$dJ = 0$ Symplectic manifolds → have integrable GCS

- But GACS have more…



Complex: locally equivalent to $\mathbb{C}^{d/2}$

Symplectic: locally equivalent to $(\mathbb{R}^d, J)$; $J = dx^1 \wedge dx^2 + \ldots + dx^{d-1} \wedge dx^d$

Generalized complex: locally equivalent to $\mathbb{C}^k \otimes (\mathbb{R}^{d-2k}, J)$ k: rank. k=0 for symplectic
k=d/2 for complex

- How do we see the rank? Any pure spinor can be written

$$\Phi = e^A \wedge \Omega_k$$

(A: complex 2-form; Ω_k: holomorphic k-form)

such that $A^{6-k} \wedge \Omega_k \wedge \bar{\Omega}_k \neq 0$ k : rank

$$\langle \bar{\Phi}, \Phi \rangle \neq 0$$

$\Phi_- = \Omega_3$ has rank 3 $d\Omega_3 = 0$ → manifold is GCY (complex)

$\Phi_+ = e^{iJ}$ has rank 0 $de^{iJ} = 0$ → manifold is GCY (symplectic)

O(6,6) spinors are tensor products of O(6) spinors

$$\Phi_\pm = \eta^1{}_+ \otimes \eta^2{}_\pm{}^\dagger = \sum_{k=0}^{6} \frac{1}{k!} \eta^{2\,\dagger}_\pm \gamma_{i_1\ldots i_k} \eta^1_+ \gamma^{i_1\ldots i_k}$$

sum of forms

O(6)	O(6,6)
η	$\Phi^- = \eta_+ \otimes \eta^\dagger_- = \Omega_3$ (rank 3) ; $\Phi^+ = \eta_+ \otimes \eta_+^\dagger = e^{-iJ}$ (rank 0)
η^1, η^2	static SU(2)
$\eta^2_+ = c_\parallel \eta^1_+ + c_\perp (v + iv')_m \gamma^m \eta^1_-$	$\Phi_- = (c_\perp e^{-ij} + i c_\parallel \Omega_2) \wedge (v + iv')$ — rank 1 (1+3+5-form)

J, Ω in 4d plane $\perp$ v , v'

orientifolds: $\eta^2 = \gamma^\perp \eta^1$

- $c_\perp = 0 \rightarrow$ SU(3) structure
- $c_\parallel = 0 \rightarrow$ static SU(2) structure

$$\Phi_+ = (\bar{c}_\parallel e^{-ij} - i c_\perp \Omega_2) \wedge e^{-iv \wedge v'}$$

rank 2 (2+4+6-form)

$\exists\ \Phi_+, \Phi_-$ reduces structure SU(3,3) $\rightarrow$ SU(3) x SU(3)

some orientifolds only one choice
O3: SU(3) only
O4: static SU(2) only

GCY $\quad \nabla \eta_+ = 0$

$d\Phi = 0$ $\quad$ $d\Phi_+ = 0$ or $d\Phi_- = 0$

(CY: $d\Phi_+ = 0$ and $d\Phi_- = 0$)

locally $\mathbb{C}^k \otimes (\mathbb{R}^{d-2k}, J)$

SU(3) and SU(2) structures on T are particular cases of SU(3) x SU(3) on $T \oplus T^*$

Pure spinors and orientifold projection

$\Omega_{WS}\,\sigma$ → holomorphic in IIB

$\Omega_{WS}\,\sigma$ → antiholomorphic in IIA

different O-planes (O3 vs O5) ↓

CY

IIB: $\sigma\Omega_3 = \pm\Omega_3 \quad \sigma e^{-iJ} = e^{-iJ}$

IIA: $\sigma\Omega_3 = \pm\bar{\Omega}_3 \quad \sigma e^{-iJ} = e^{iJ}$

Can write this in general for SU(3) x SU(3) as an action on the pure spinors

IIB: $\sigma(\Phi^-) = \pm\lambda(\Phi^-) \qquad \sigma(\Phi^+) = \pm\lambda(\bar{\Phi}^+)$

IIA: $\sigma(\Phi^-) = \pm\lambda(\bar{\Phi}^-) \qquad \sigma(\Phi^+) = \pm\lambda(\Phi^+)$

B-field

A 2-form B acts on Φ by $e^B \Phi = (1+B+\dots) \wedge \Phi$

Φ^- and Φ^- compatible, determine metric and B field on the manifold

$$G = -\mathcal{J}_{\Phi_+} \mathcal{J}_{\Phi_-} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} \qquad \text{ex:} \; -\mathcal{J}_J \mathcal{J}_\Omega = \begin{pmatrix} 0 & IJ^{-1} \\ I^t J & 0 \end{pmatrix}$$

Twisting by H = dB

- If Φ is closed, $d(e^B \Phi) = H \wedge e^B \Phi$ ➡ $(d - H \wedge)(e^B \Phi) = 0$

  $d - H\wedge$: twisted exterior derivative

- Courant bracket can be modified to include H

Twisted Integrability $[d - H\wedge\ ]\varphi = 0$ ➡ Associated GACS integrable with $[\ ,\ ]_H$ / Associated GACS integrable: $[\Pi_{\pm}, \Pi_{\pm}]_C = 0$

SU(3) x SU(3) structure and $\mathcal{N}$=1 vacua

$$\varepsilon^1 = \theta_+ \otimes \eta^1_+ + c.c.$$
$$\varepsilon^2 = \theta_+ \otimes \eta^2_+ + c.c.$$

What does SUSY tell us about integrability of the pure spinors?

$$\Phi_\pm = \eta^1_+ \otimes \eta^{2\,\dagger}_\pm$$

SUSY: $$\delta_\epsilon \psi_m = \nabla_m \begin{pmatrix} \eta^1_+ \\ \eta^2_+ \end{pmatrix} + H_{mnp}\gamma^{np} \begin{pmatrix} \eta^1_+ \\ -\eta^2_+ \end{pmatrix} + e^\phi \sum_n \not{F}_n \gamma_m \begin{pmatrix} \eta^2_+ \\ (-1)^{Int(n/2)}\eta^1_+ \end{pmatrix} = 0$$

$$d(\eta^1 \otimes \eta^{2\,\dagger}_\pm) = dx^m \wedge \nabla_m(\eta^1_+ \otimes \eta^{2\,\dagger}_\pm)$$
$$= dx^m \wedge \left(-H_{mnp}\gamma^{np}\eta^1_+ - e^\phi \sum_n \not{F}_n \gamma_m \eta^2_+ \right) \otimes (\eta^{2\,\dagger}_\pm) + dx^m \wedge \eta^1_+ \otimes \nabla_m (\eta^{2\,\dagger}_\pm)$$

Use also dilatino and space-time gravitino equations to simplify

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IIA	IIB
$(d - H\wedge)\Phi^+ = 0$	$(d - H\wedge)\Phi^+ = i * F_B + dA \wedge \bar{\Phi}^+$
$(d - H\wedge)\Phi^- = i * F_A + dA \wedge \bar{\Phi}^-$	$(d - H\wedge)\Phi^- = 0$
↓	↓
Φ^+ is twisted closed	Φ^- is twisted closed

Φ^+ is twisted closed ⟵ generalized mirror symmetry ⟶ Φ^- is twisted closed

Susy vacua are all twisted generalized Calabi-Yau's !

IIA	IIB
$\mathcal{M}$ is symplectic ($\mathbb{R}^{3\text{x}2}$, J) in SU(3)	$\mathcal{M}$ is complex ($\mathbb{C}^3$) in SU(3)
is hybrid $\mathbb{C}^2 \otimes (\mathbb{R}^{1\text{x}2}$, J) in static SU(2)	is hybrid $\mathbb{C}^1 \otimes (\mathbb{R}^{2\text{x}2}$, J) in static SU(2)

Example: twisted torus

structure constants of Lie algebra

$(e^1, e^2, e^3, e^4, e^5, e^6)$ $$de^a = \frac{1}{2} f^a_{bc}\, e^b \wedge e^c$$

algebra nilpotent: G/Γ compact
34 classes of 6D nilmanifolds

$\Phi^- = (e^1+ie^2) \wedge (e^3+ie^4) \wedge (e^5+ie^6)$ rank 3 $\Phi^+ = \exp[i(e^1\wedge e^2 + e^3\wedge e^4 + e^5\wedge e^6)]$ rank 0 compatible

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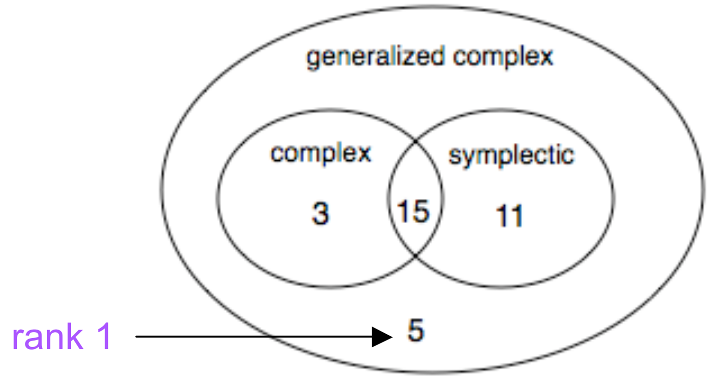
$\Phi^- = (e^1+ie^2) \exp[i(e^3\wedge ie^4 + e^5 \wedge e^6)])$ rank 1 $\Phi^+ = (e^3+ie^4) \wedge (e^5+ie^6) \exp[i\, e^1\wedge e^2]]$ rank 2 compatible

On a torus, all Φ 's are closed

On a twisted torus, not all closed

Calvacanti and Gualtieri 04

6D nilmanifolds



All generalized Calabi-Yau's

But very few N = 1 solutions...!

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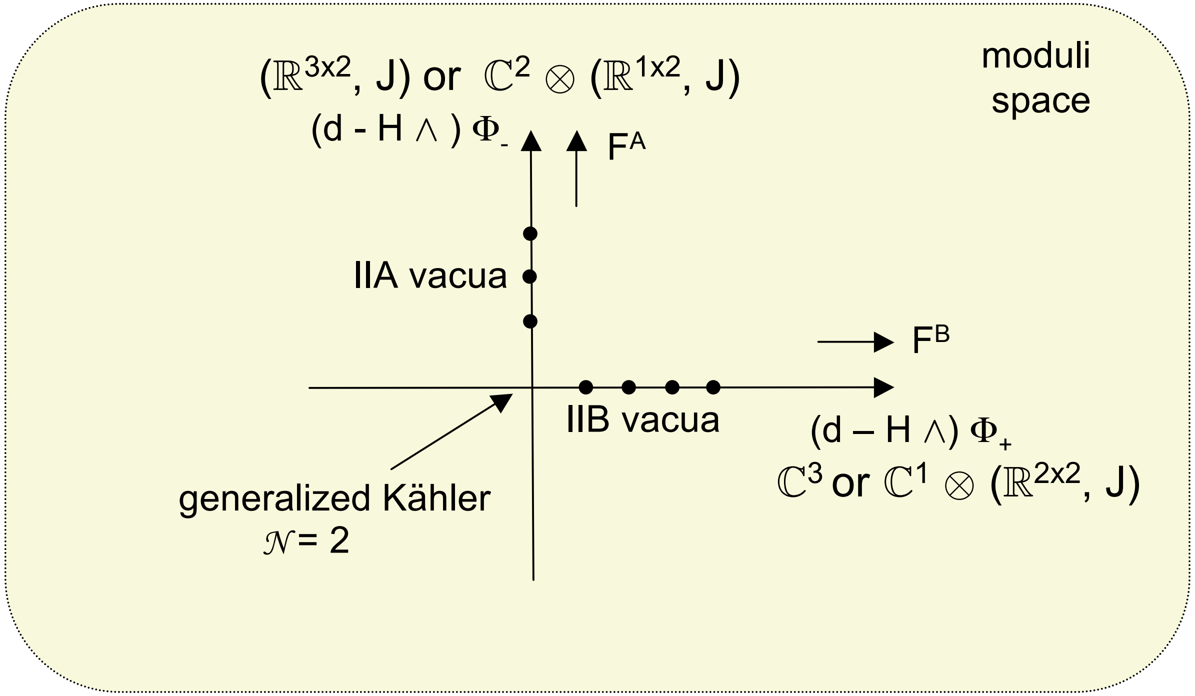
Minasian's talk

Type II and SU(3) x SU(3) structure



All $\mathcal{N} = 1$ vacua are generalized Calabi-Yau's !

Summary

- Type II on SU(3) x SU(3): pure spinors define geometry and B-field
- $\mathcal{N} = 1$ supersymmetric vacua are generalized Calabi-Yau's
- NS fluxes twist the pure spinor bundle
- RR fluxes act as obstruction for integrability of one algebraic structure

Explicit compact examples?

SU(3) x SU(3) structure is the most natural for type II on 6D manifolds.

Generalized complex geometry is tailor-made for a systematic description

of flux backgrounds.