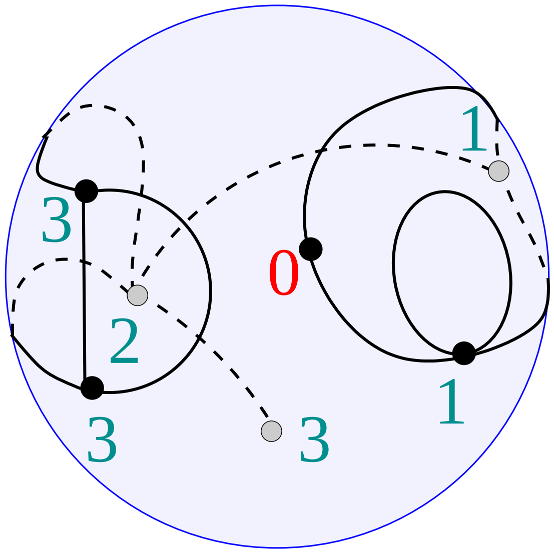
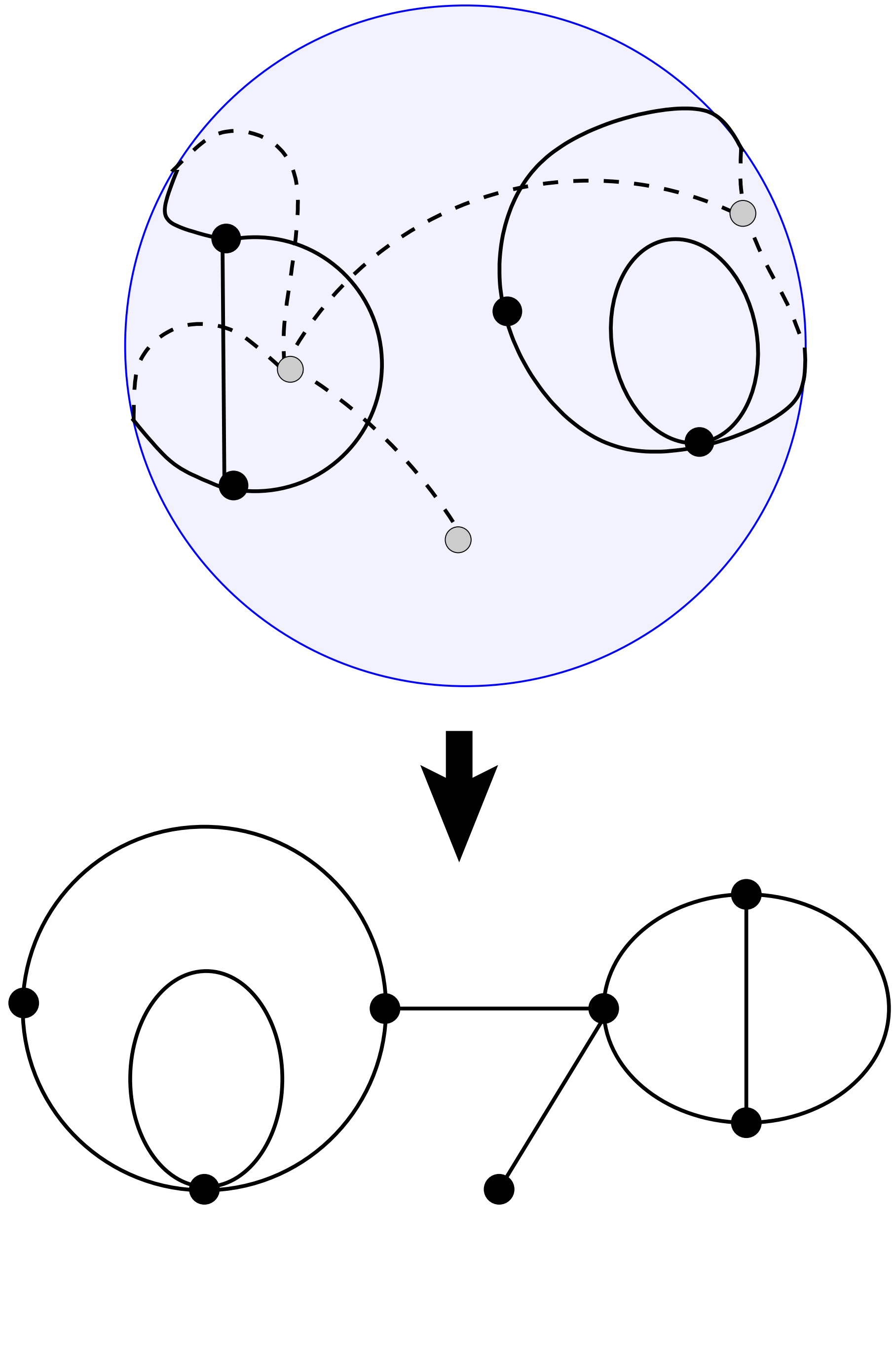
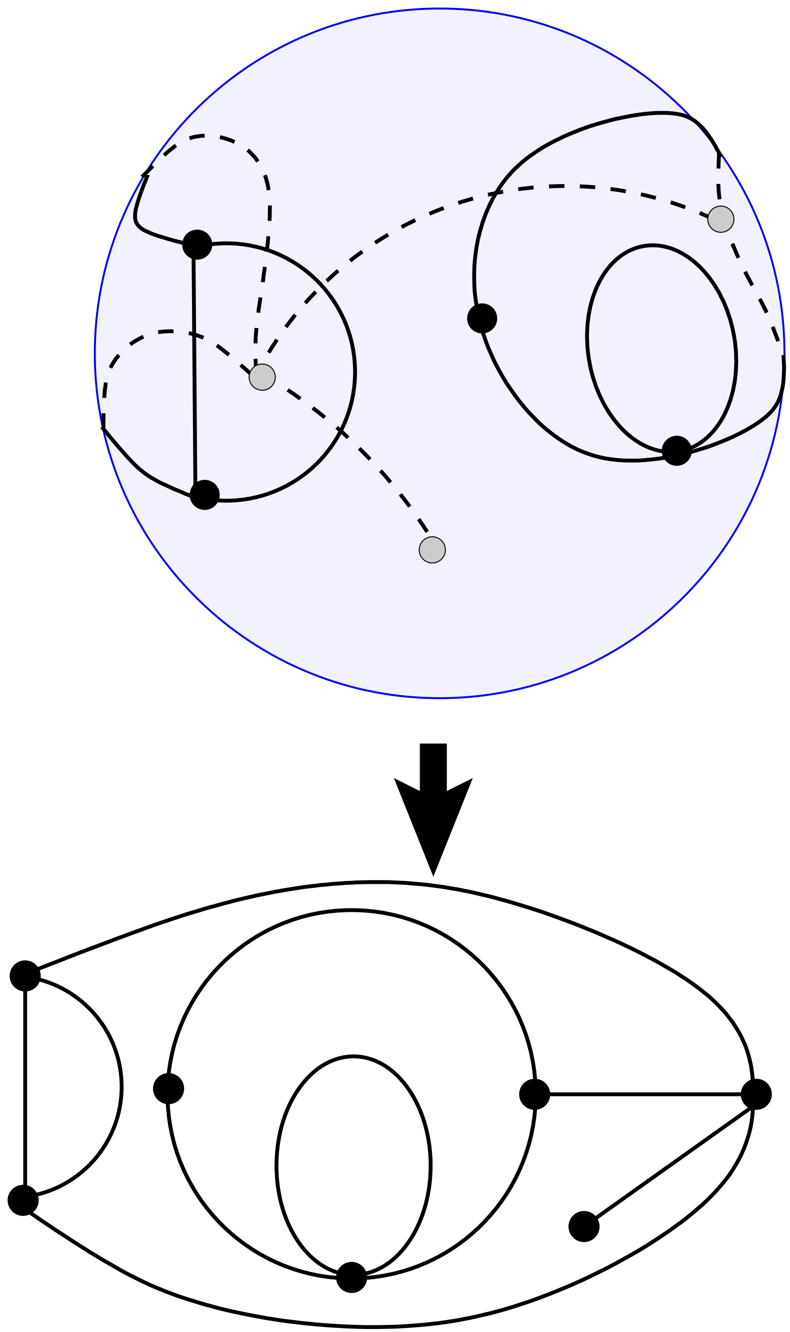
Statistics of distances in planar maps

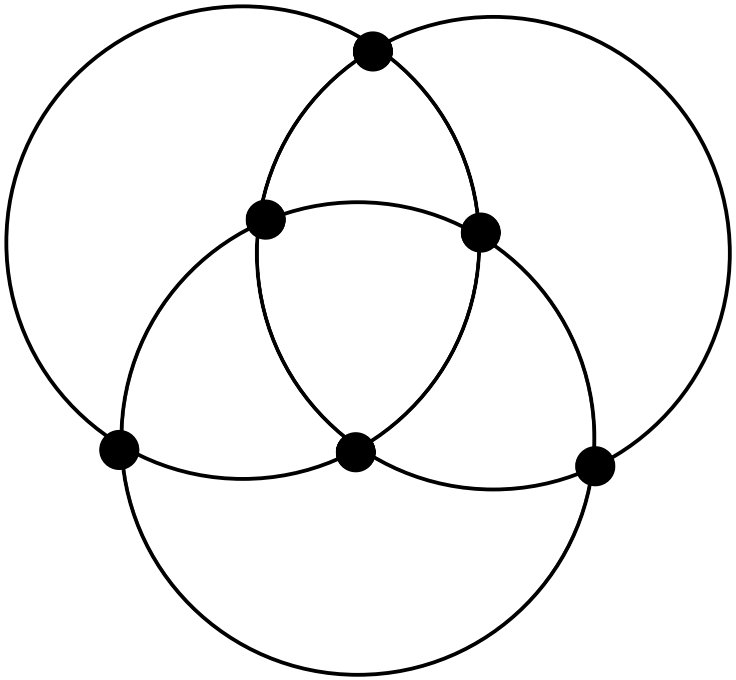
Emmanuel Guitter

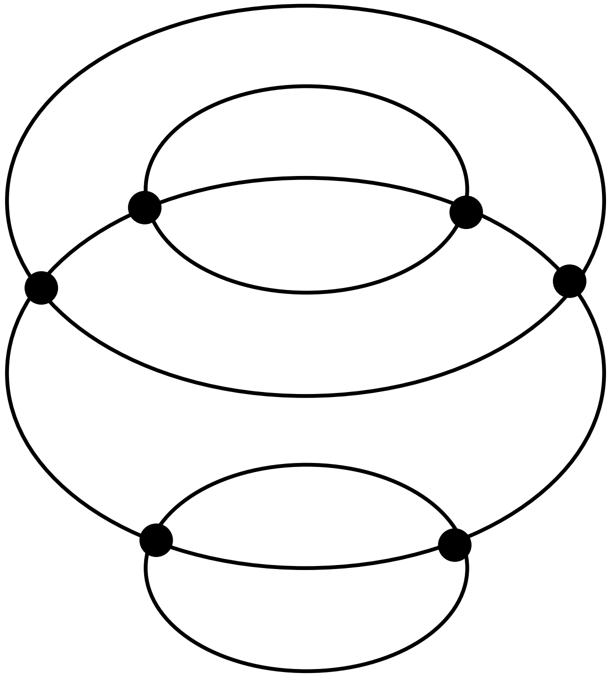


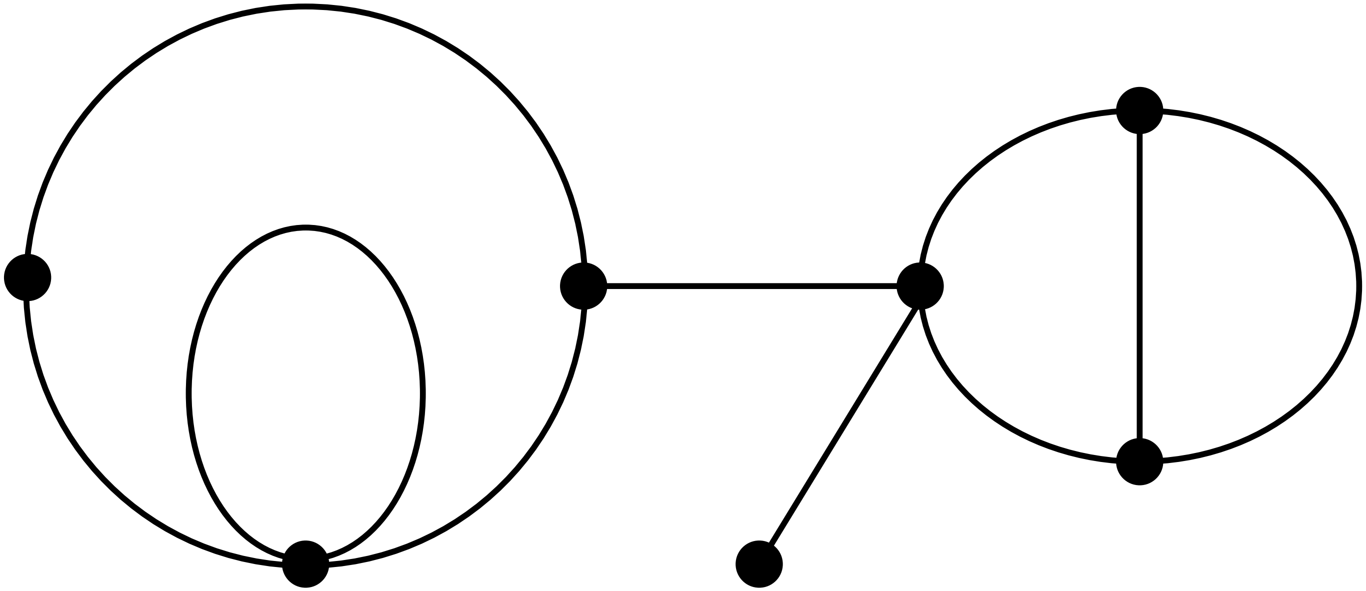
maps and distances: generalities

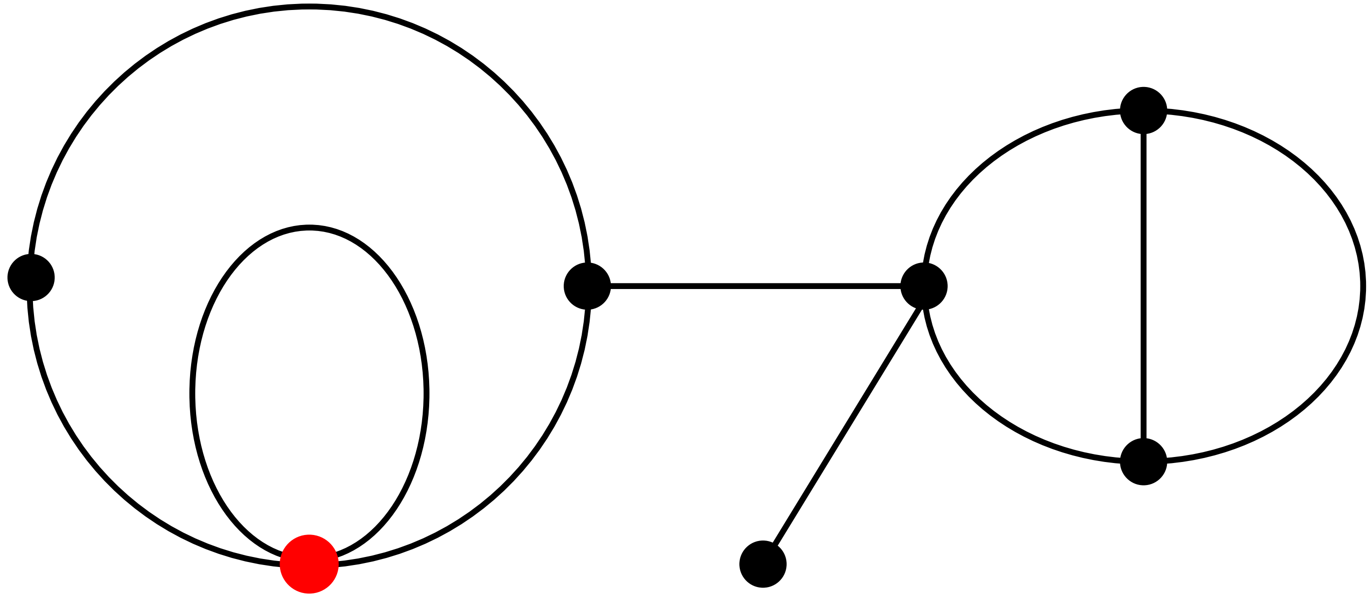




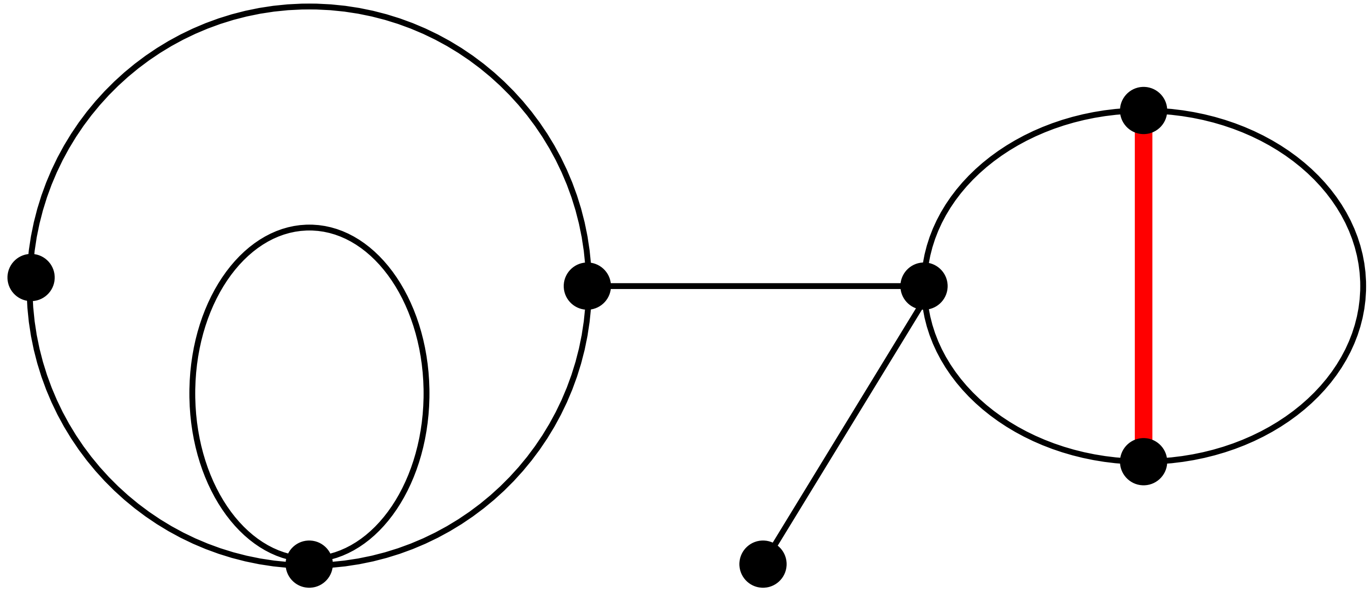




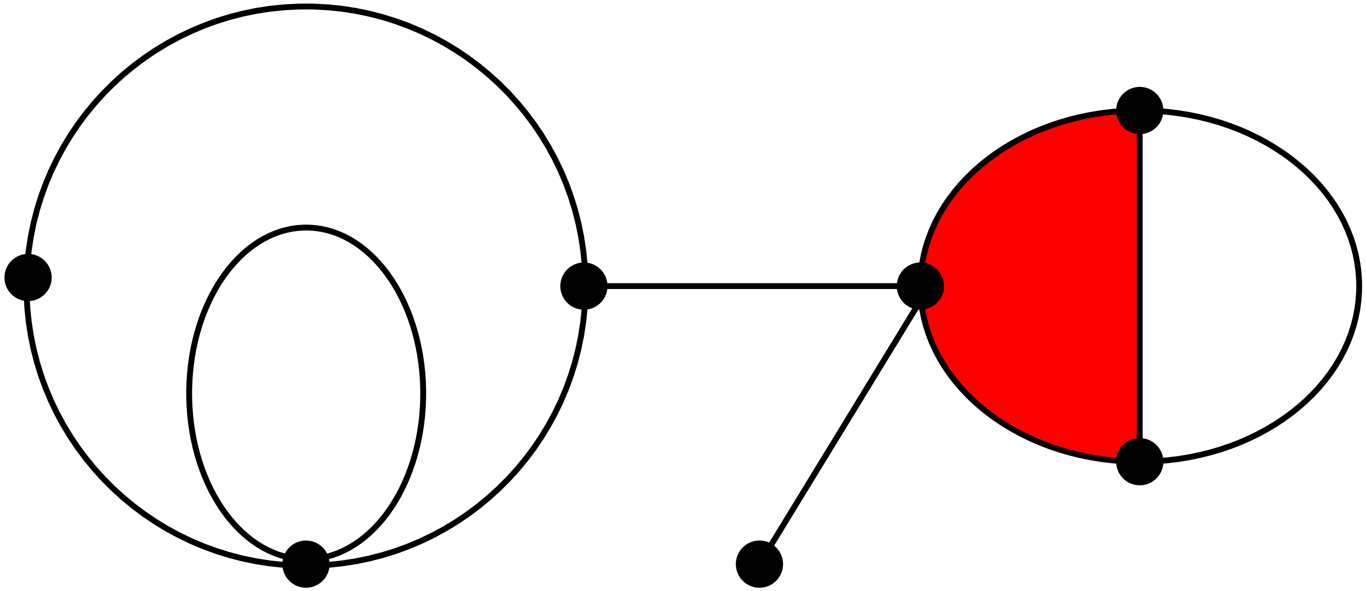




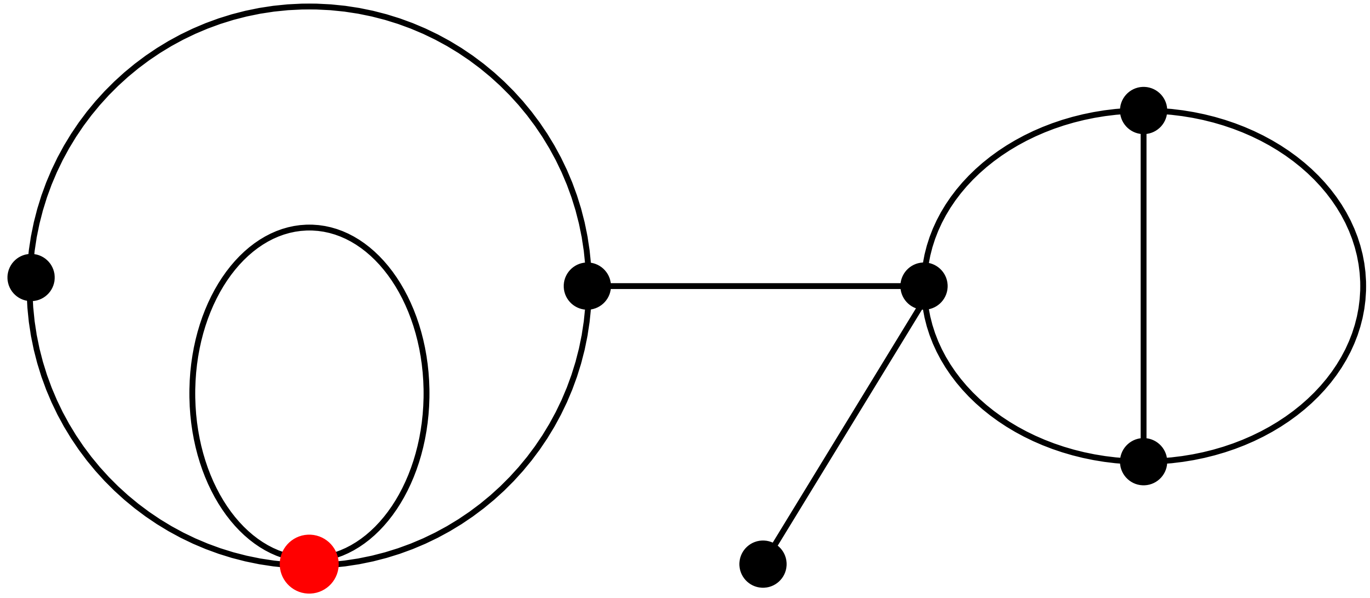
◇ vertices (here of degree 4)



- vertices
- edges (with possibly loops or multiple edges)

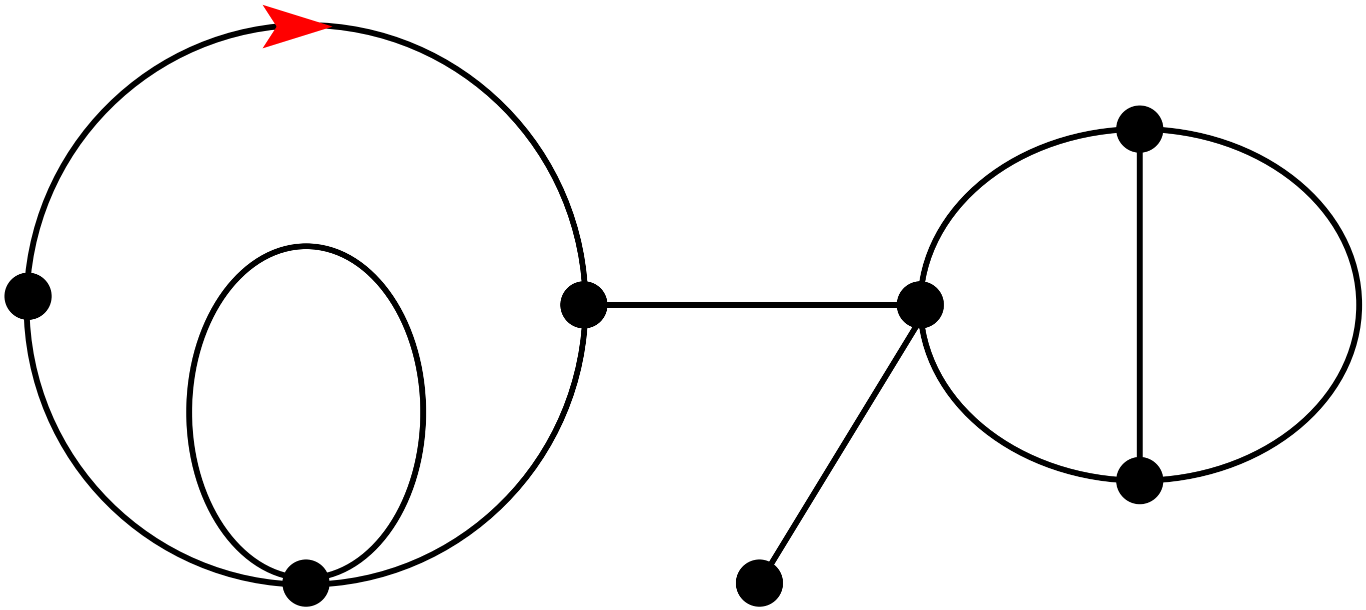


- vertices
- edges
- faces *with a single boundary* (here of degree 3)



- vertices
- edges
- faces

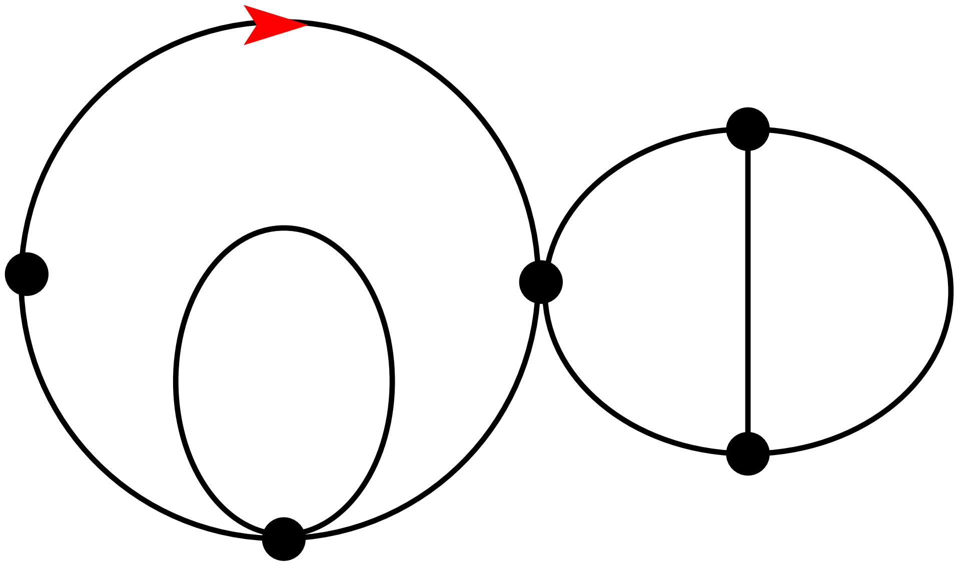
→ pointed maps



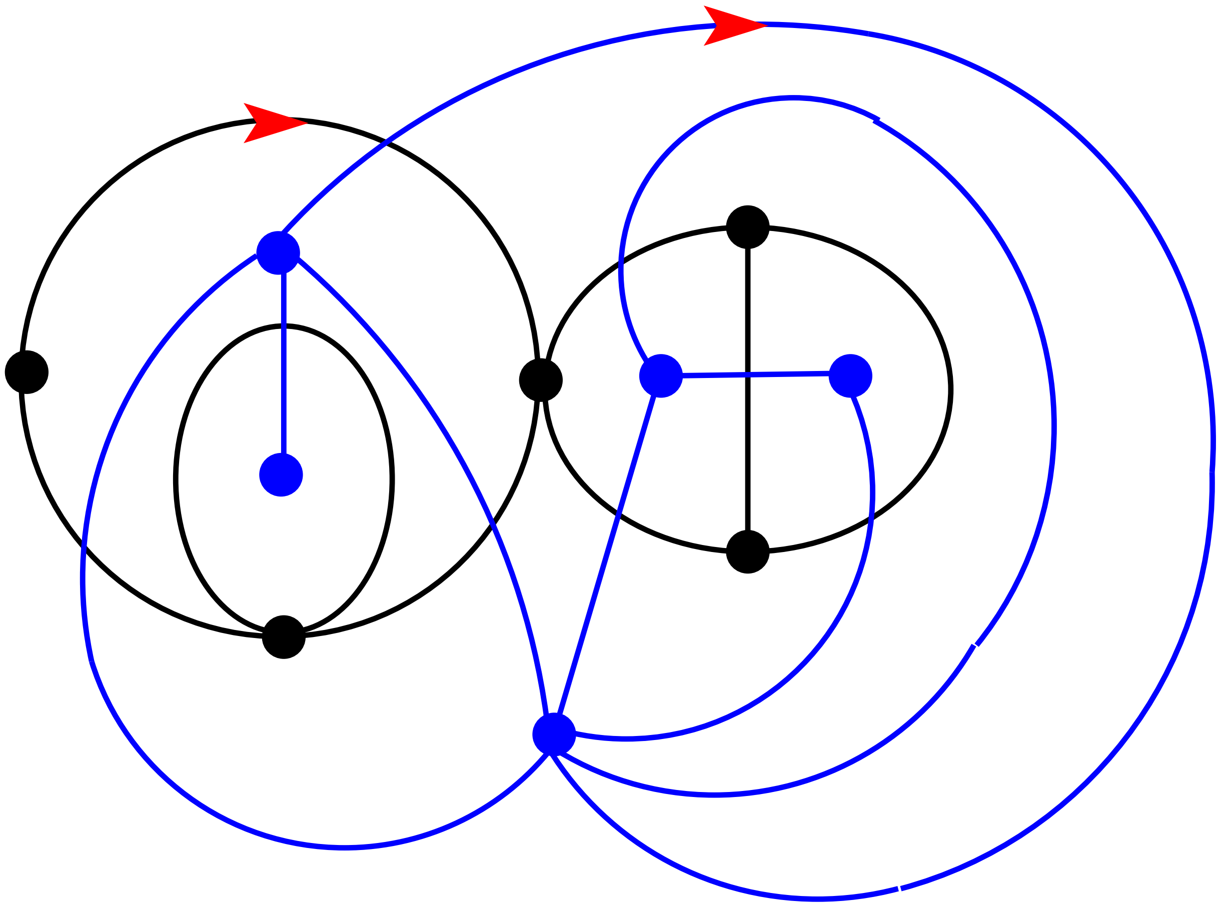
- vertices
- edges
- faces

→ rooted maps

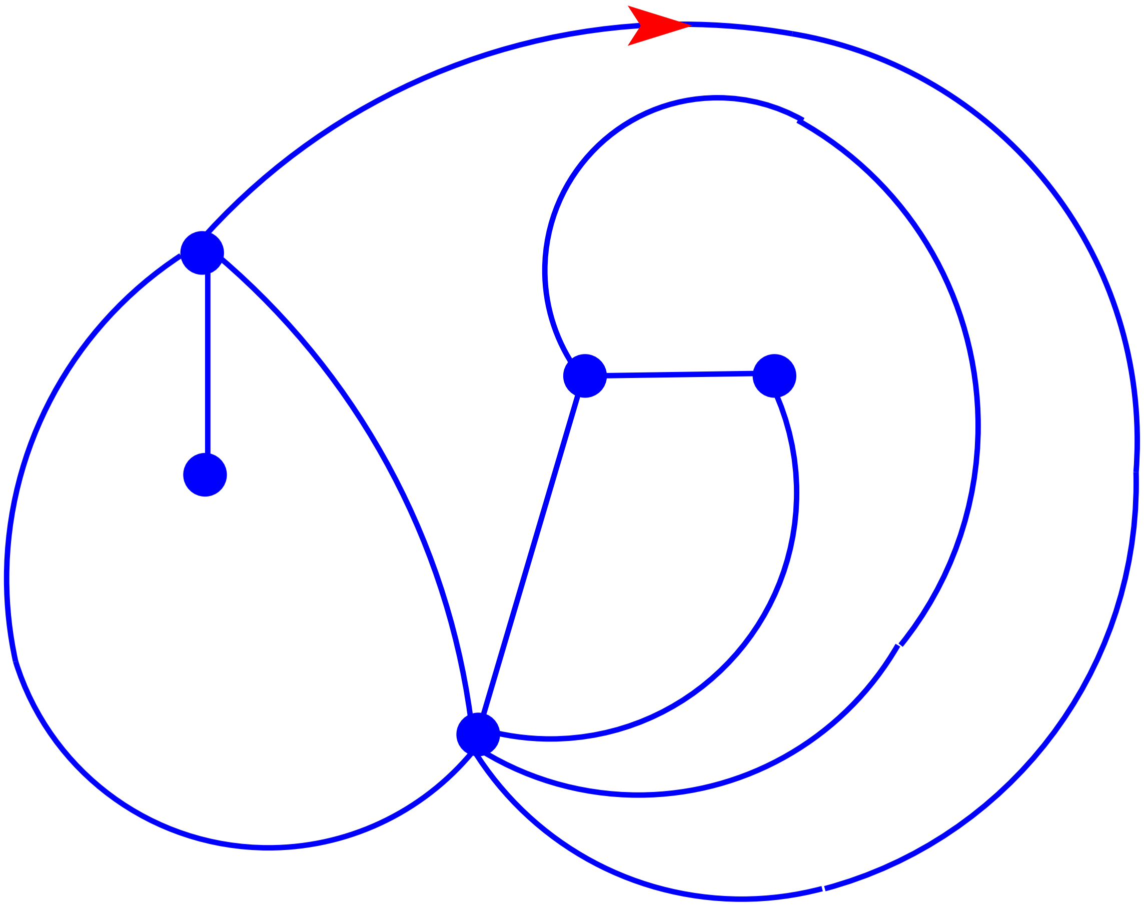
dual (rooted) map



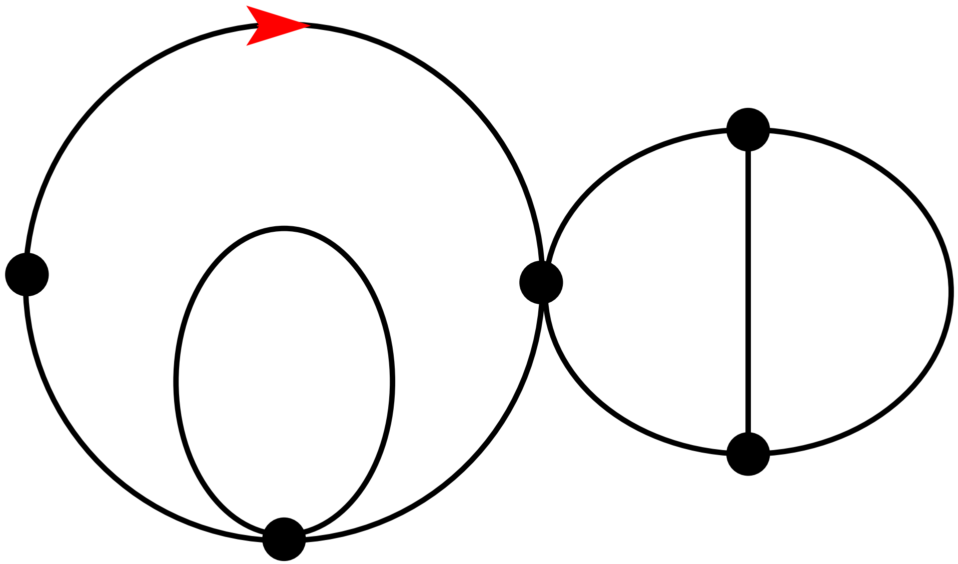
dual (rooted) map



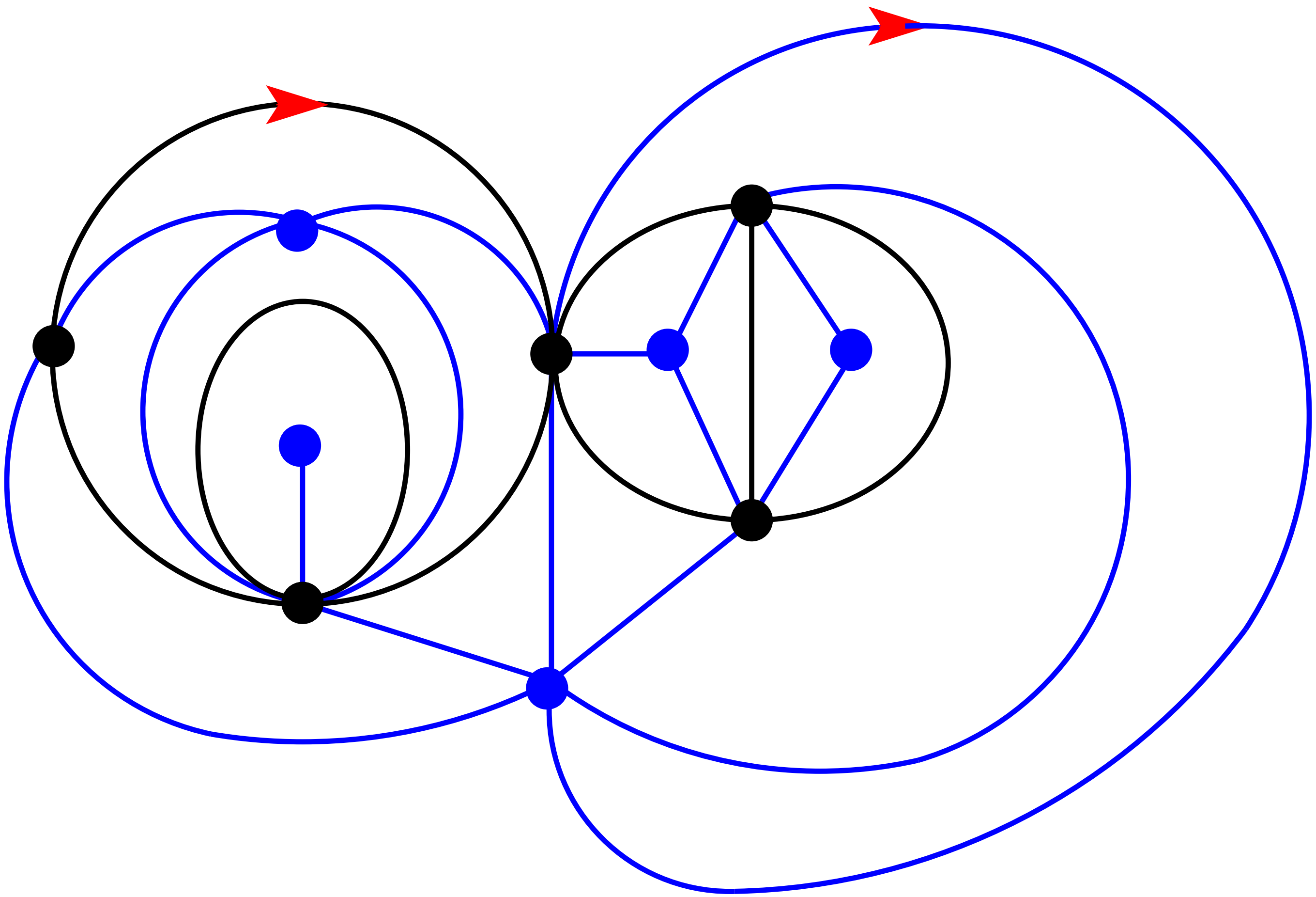
dual (rooted) map



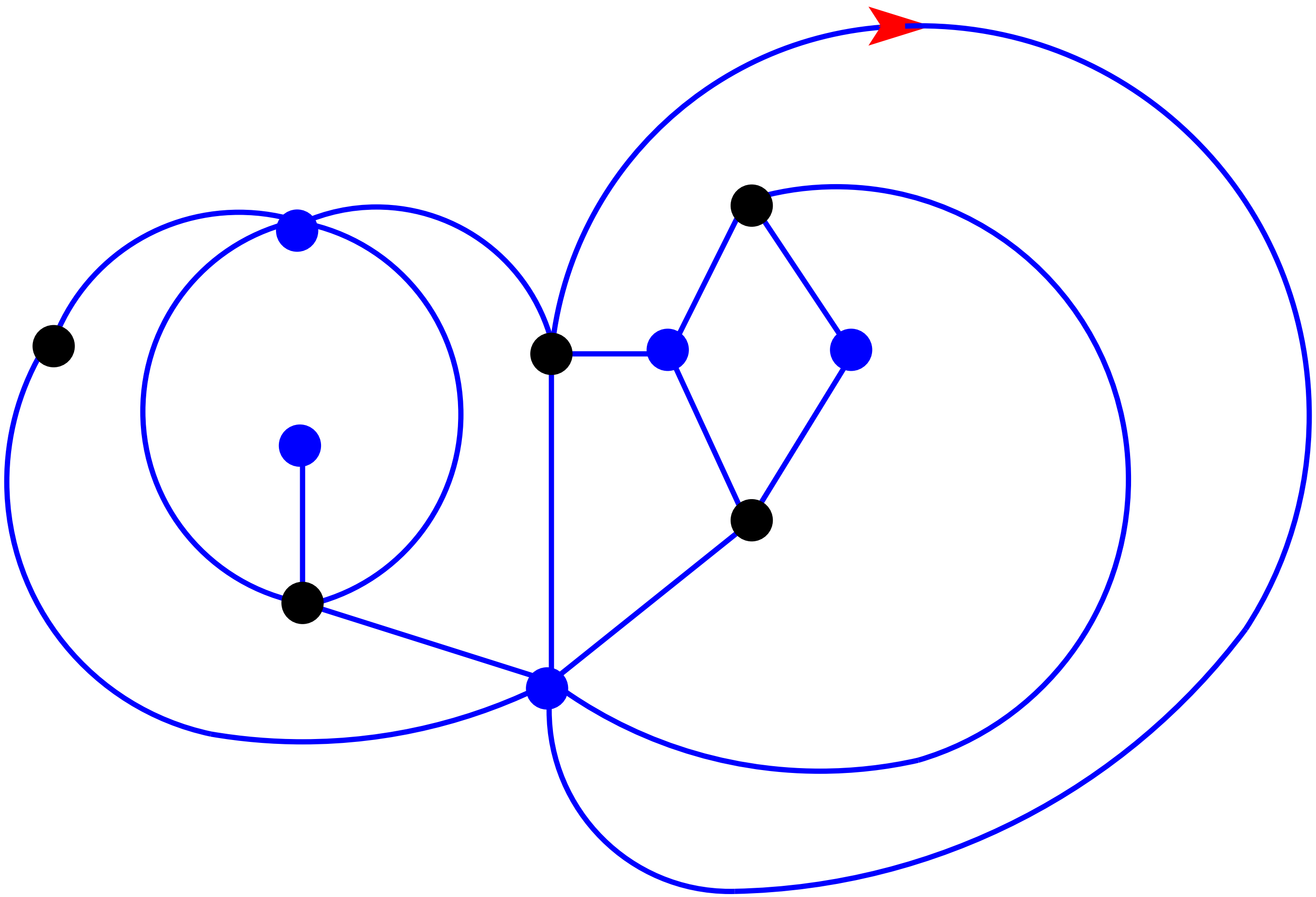
coding of a (rooted) map by a (rooted) quadrangulation

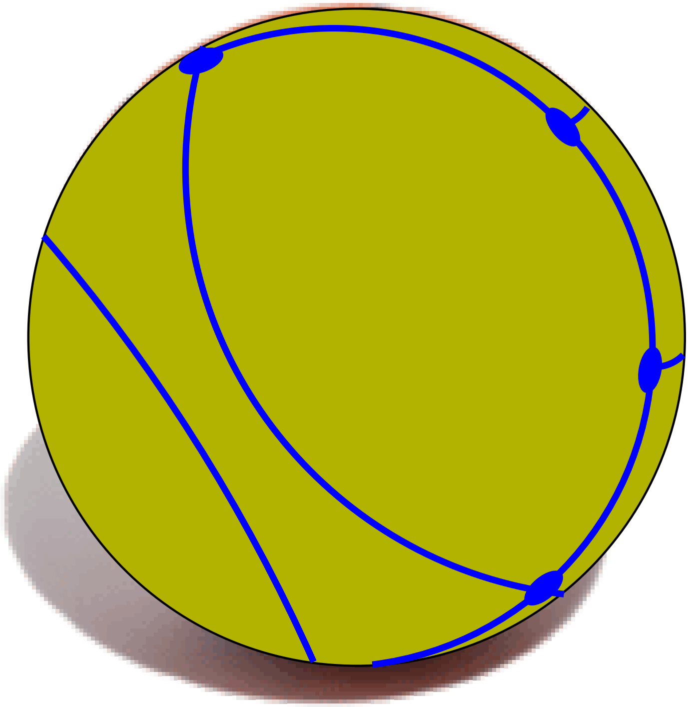


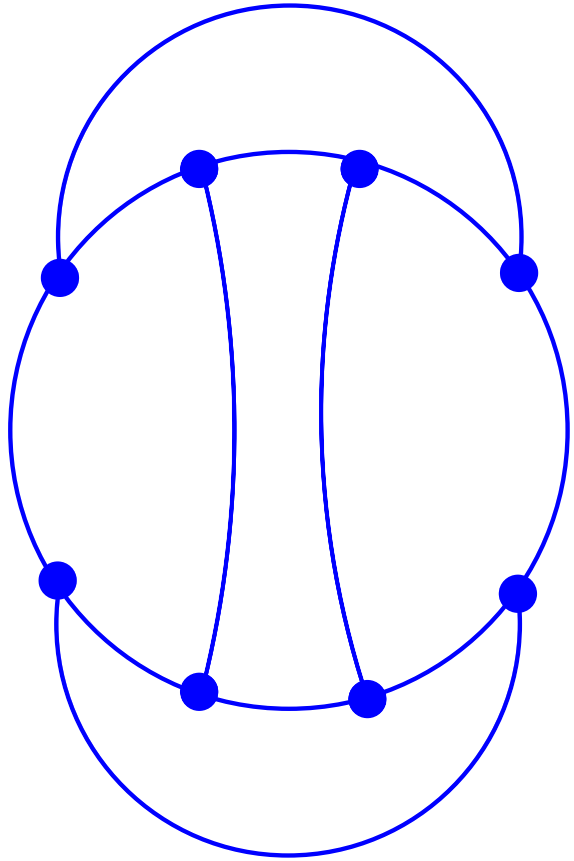
coding of a (rooted) map by a (rooted) quadrangulation

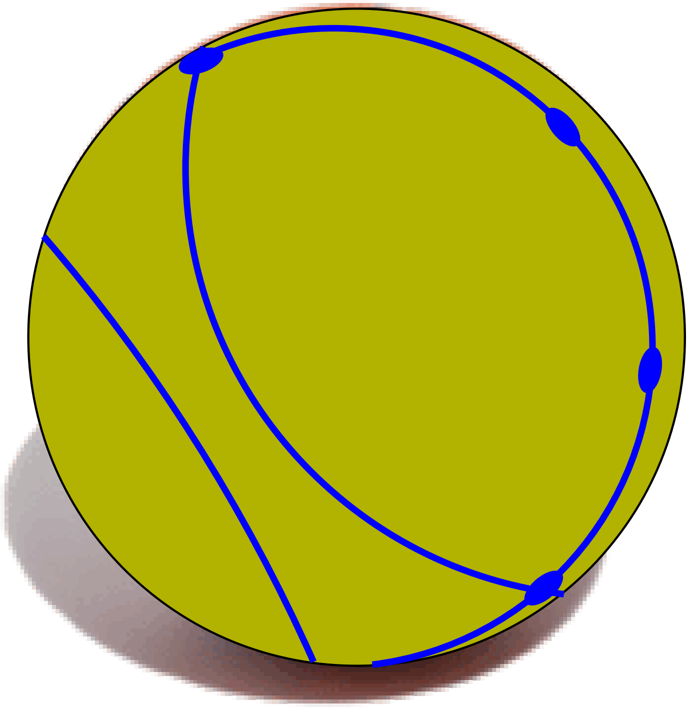


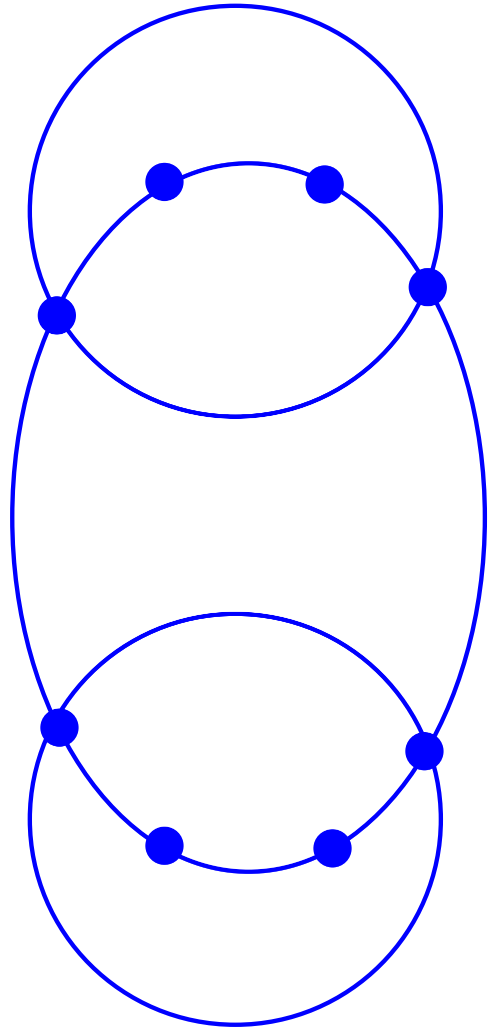
coding of a (rooted) map by a (rooted) quadrangulation

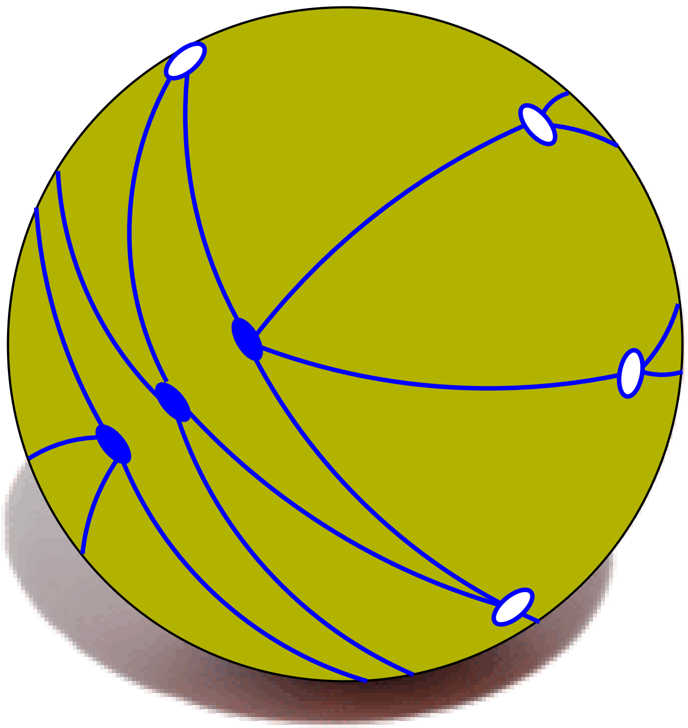


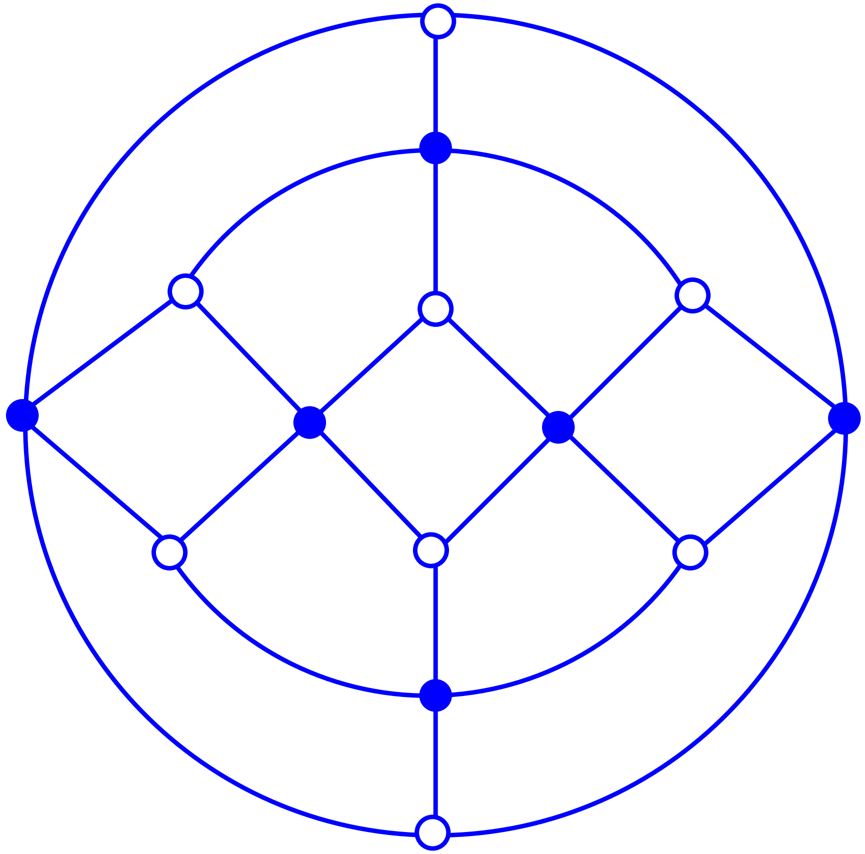


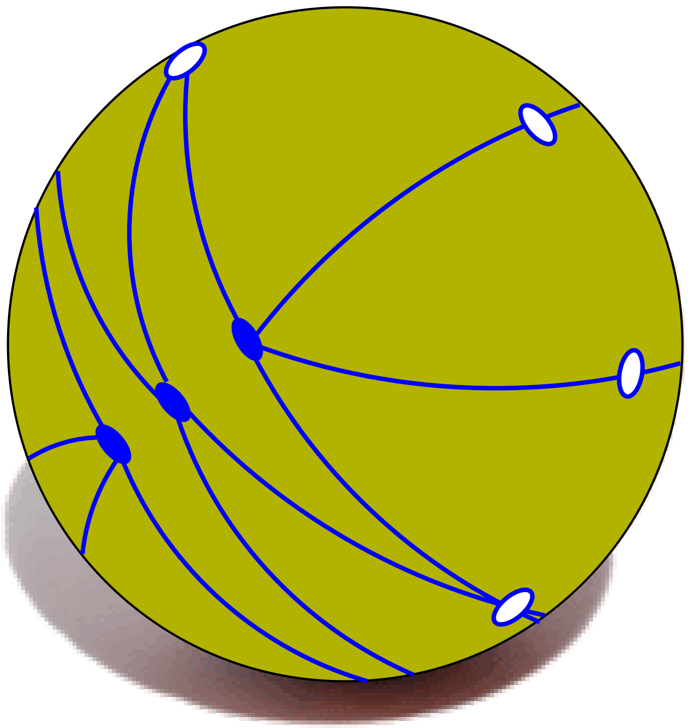


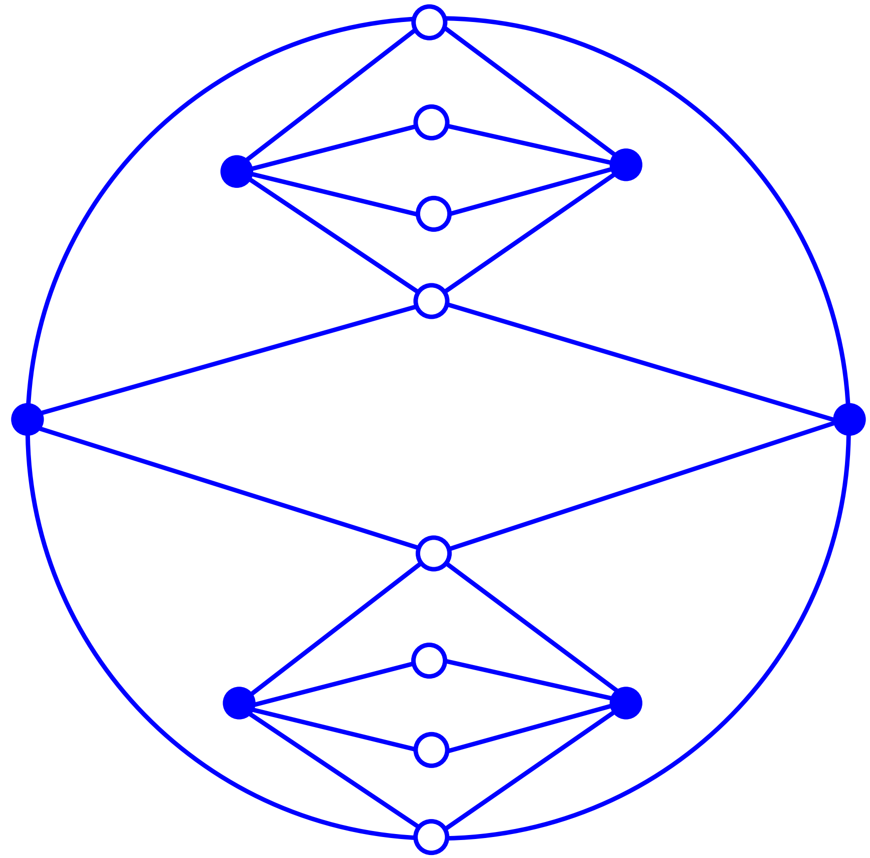






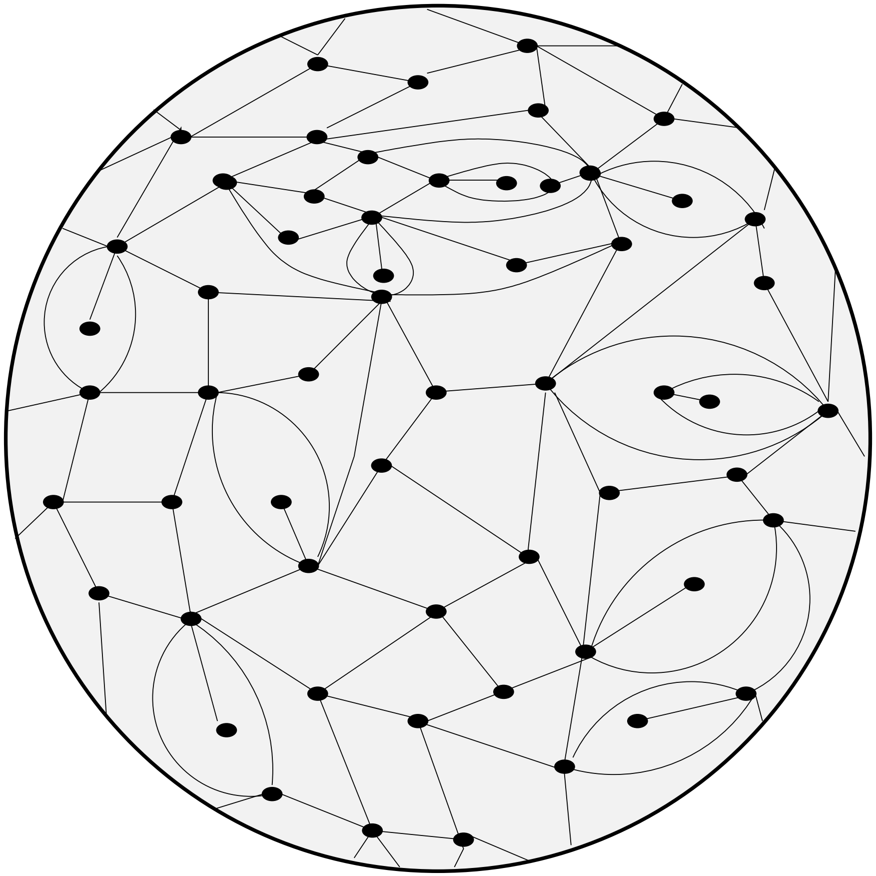






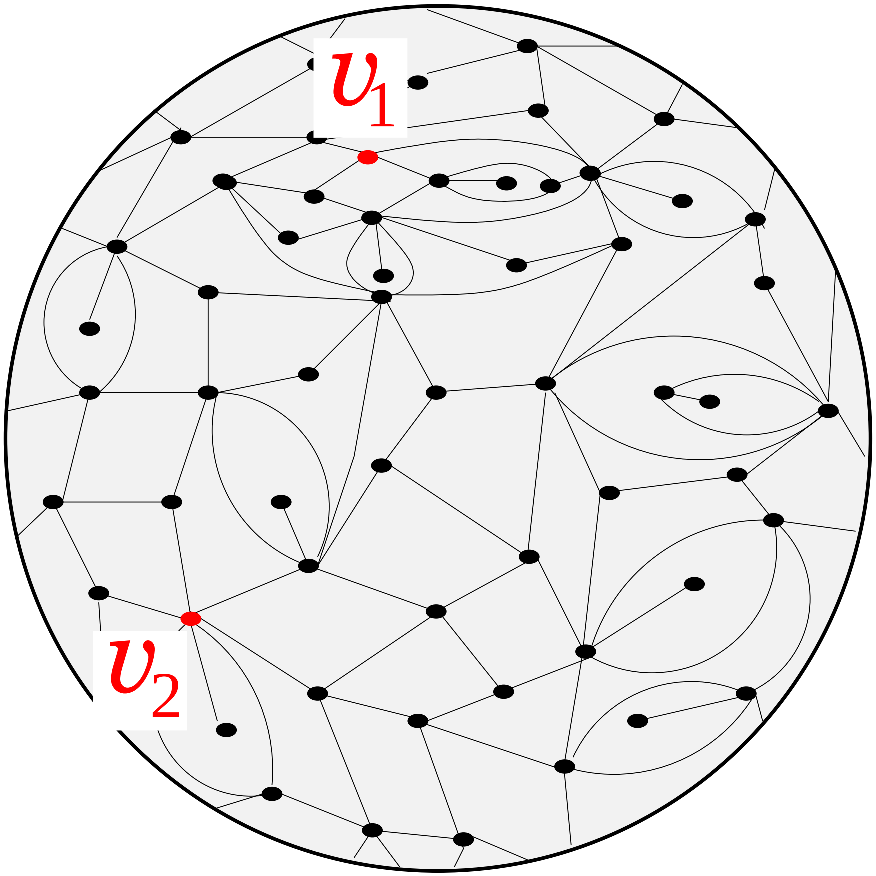
simple enumeration problems

enumerate, say planar quadrangulations with F faces



distance statistics

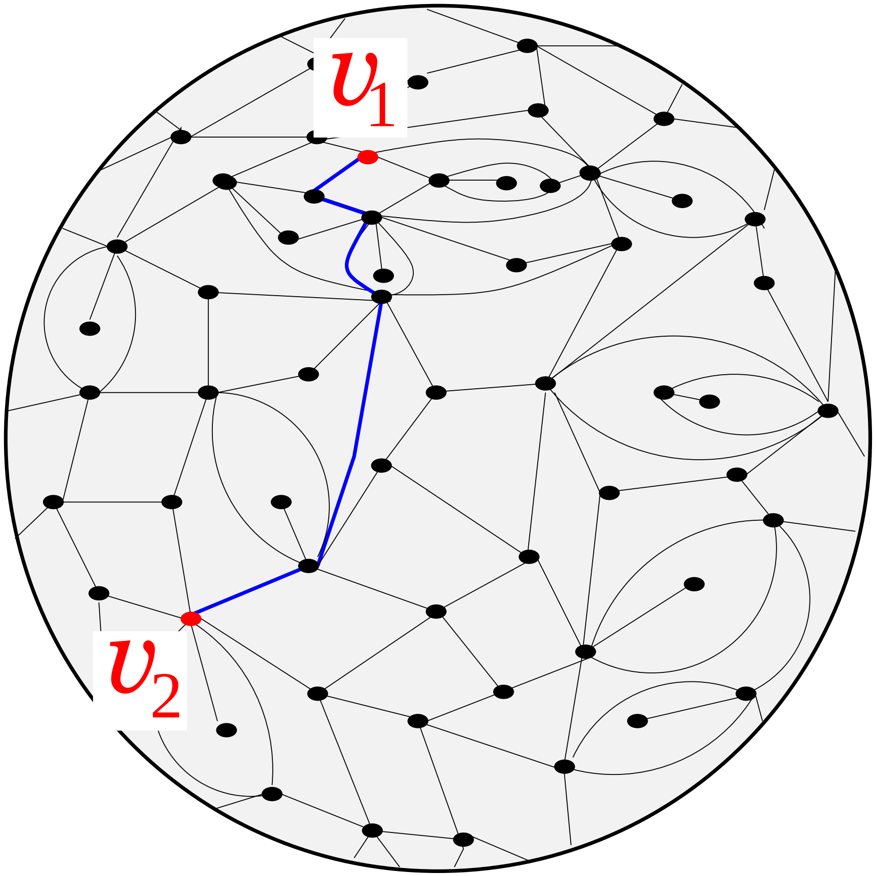
enumerate, say planar quadrangulations with F faces



and with 2 marked vertices

distance statistics

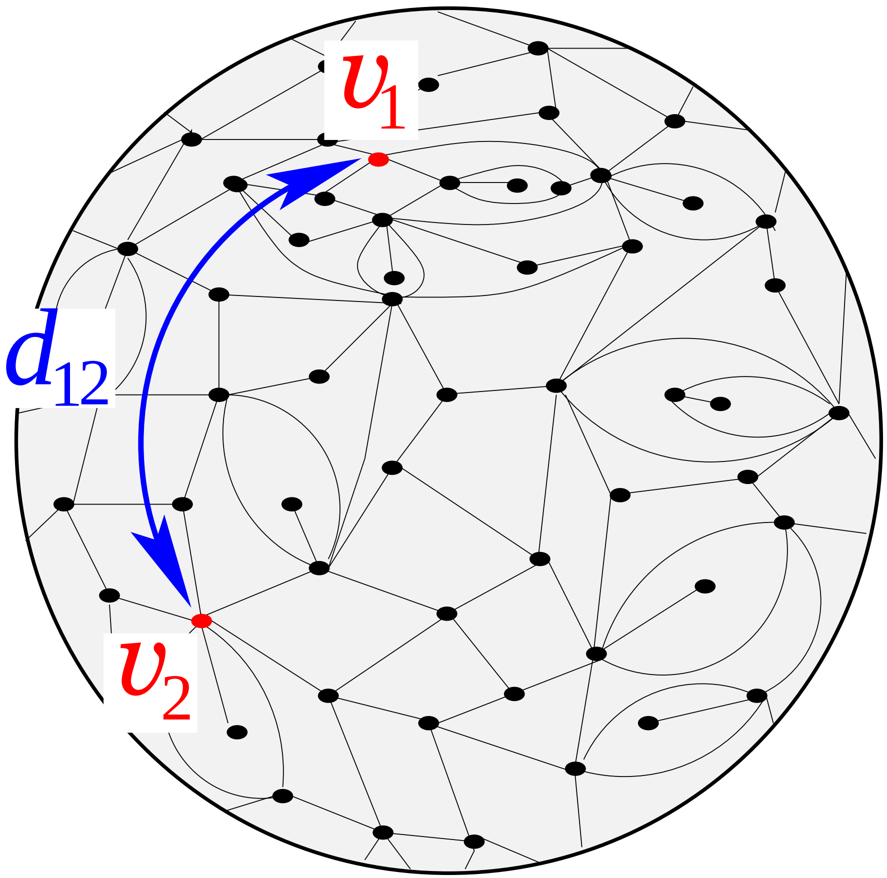
enumerate, say planar quadrangulations with F faces



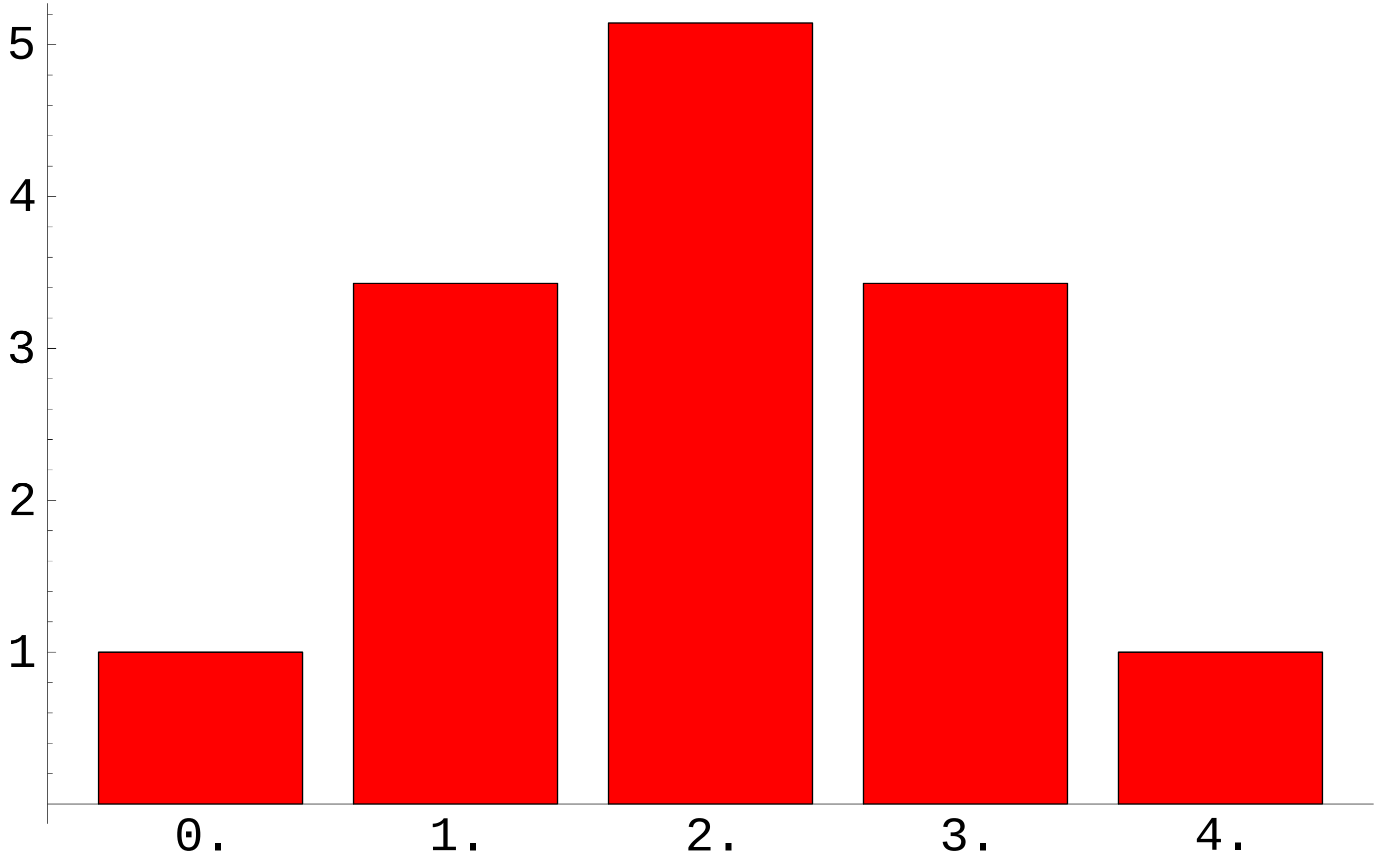
and with 2 marked vertices

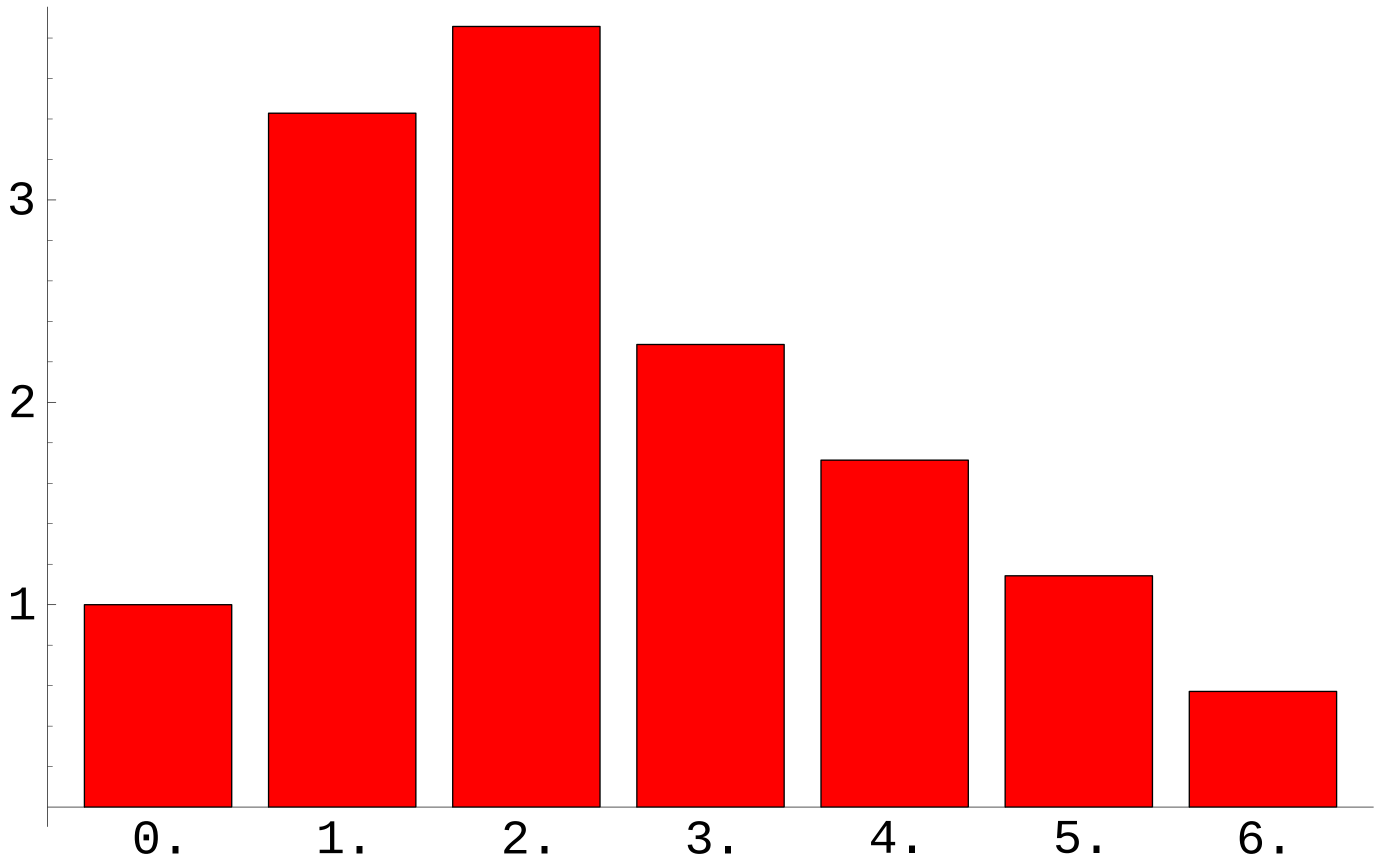
distance statistics

enumerate, say planar quadrangulations with F faces

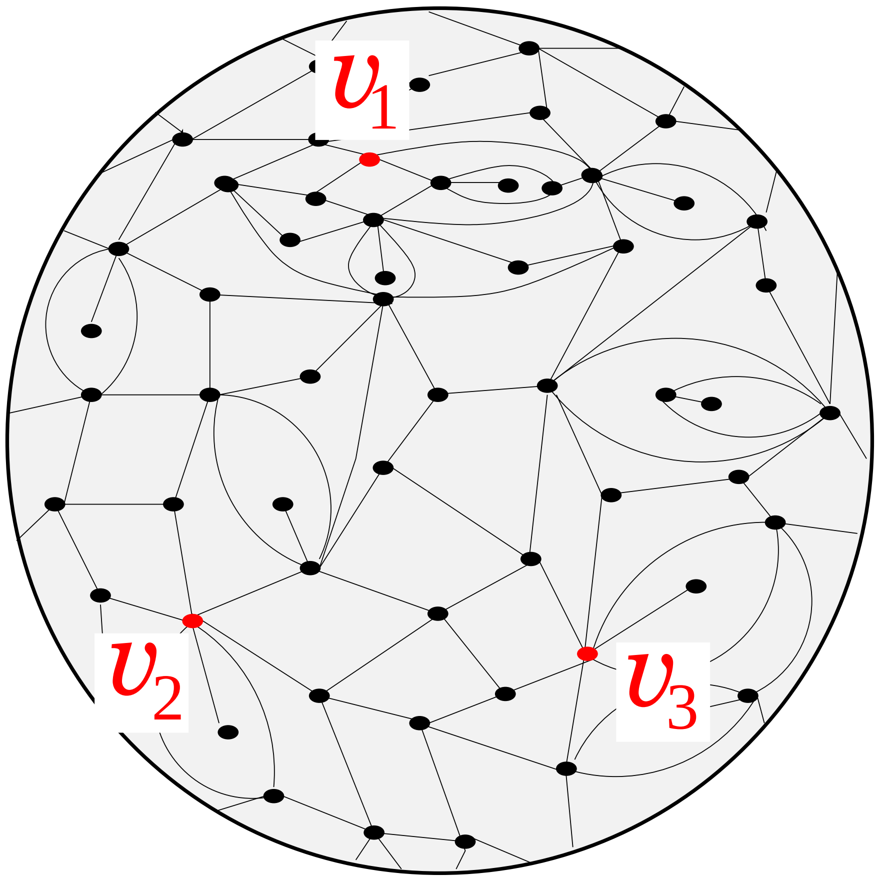


and with 2 marked vertices at prescribed distance
$\longrightarrow$ distance profile


5
4
3
2
1
0.
1.
2.
3.
4.

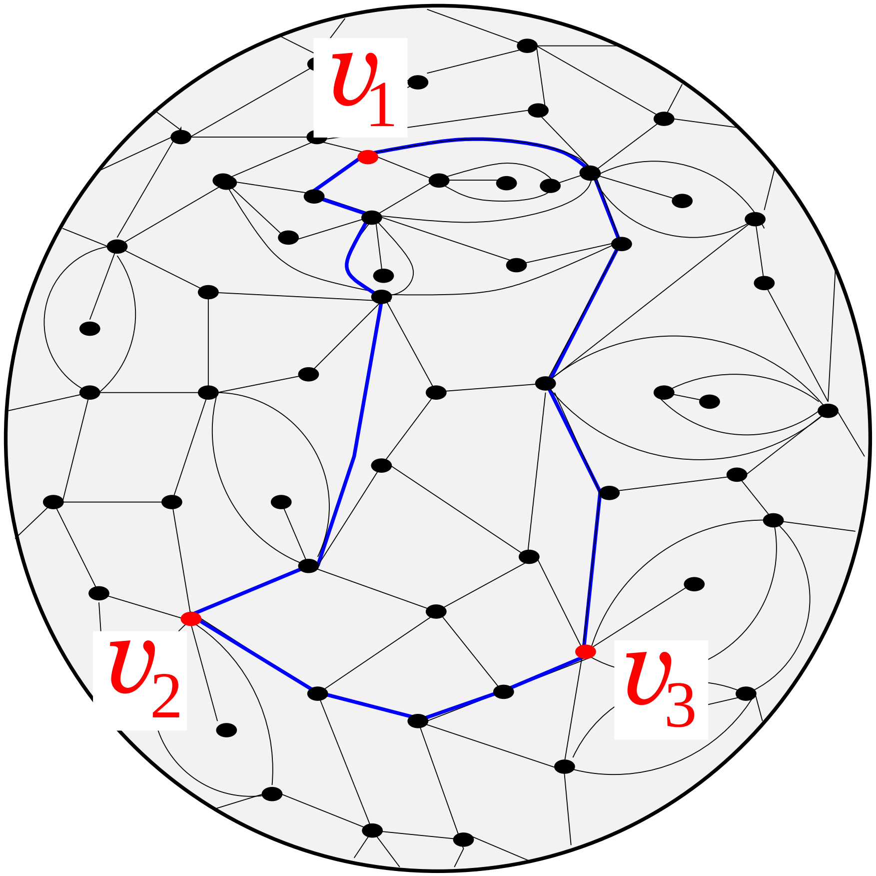

3
2
1
0.
1.
2.
3.
4.
5.
6.

enumerate, say planar quadrangulations with F faces



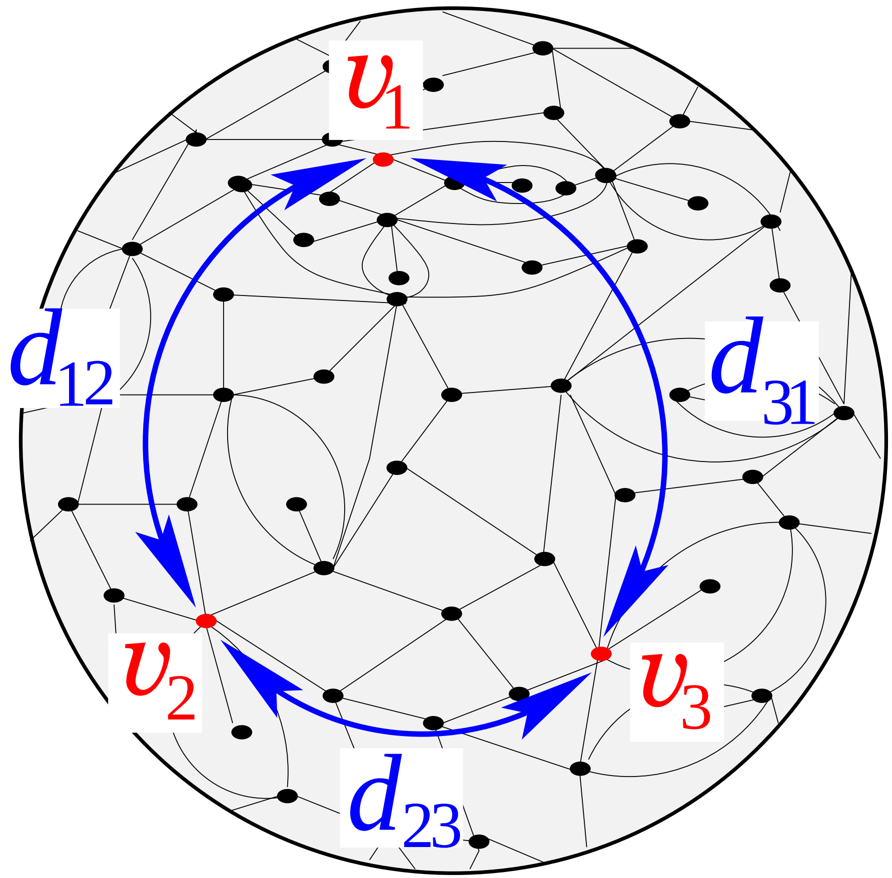
and with 3 marked vertices

enumerate, say planar quadrangulations with F faces



and with 3 marked vertices

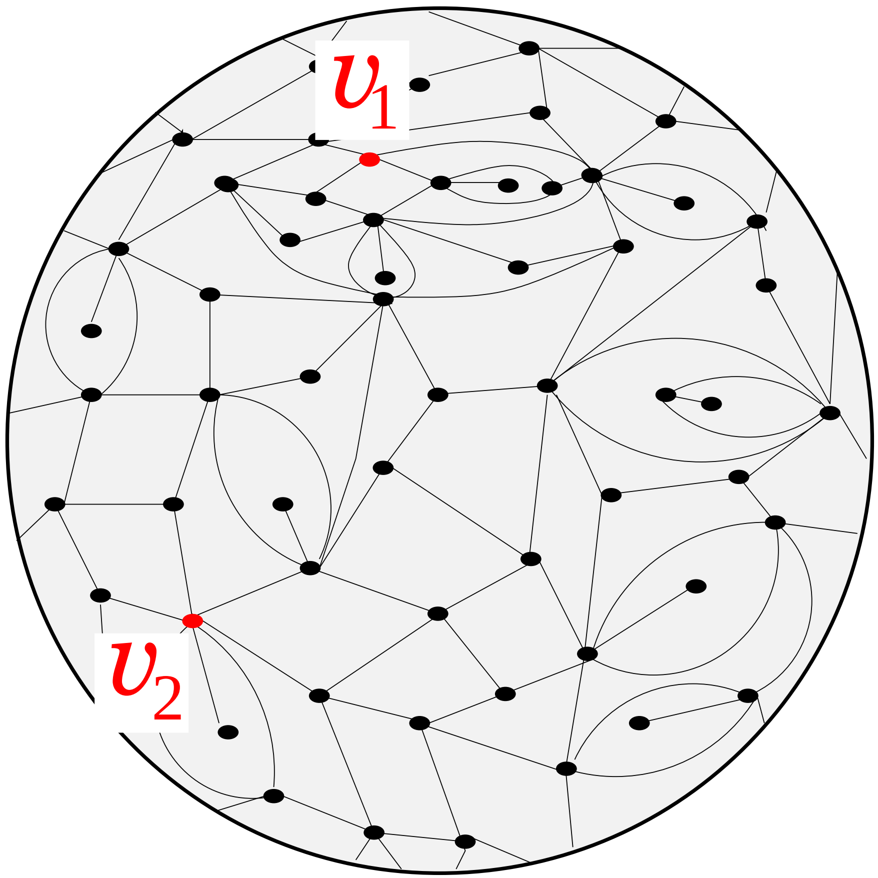
enumerate, say planar quadrangulations with F faces



and with 3 marked vertices at prescribed pairwise distances

number of geodesics

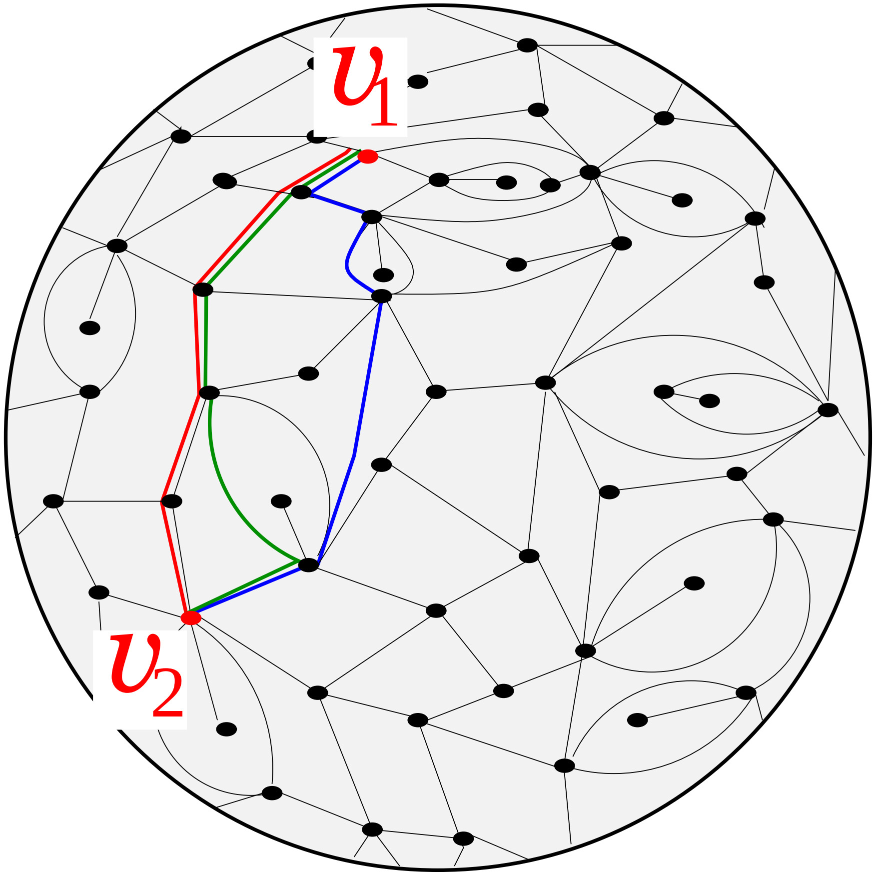
enumerate, say planar quadrangulations with F faces



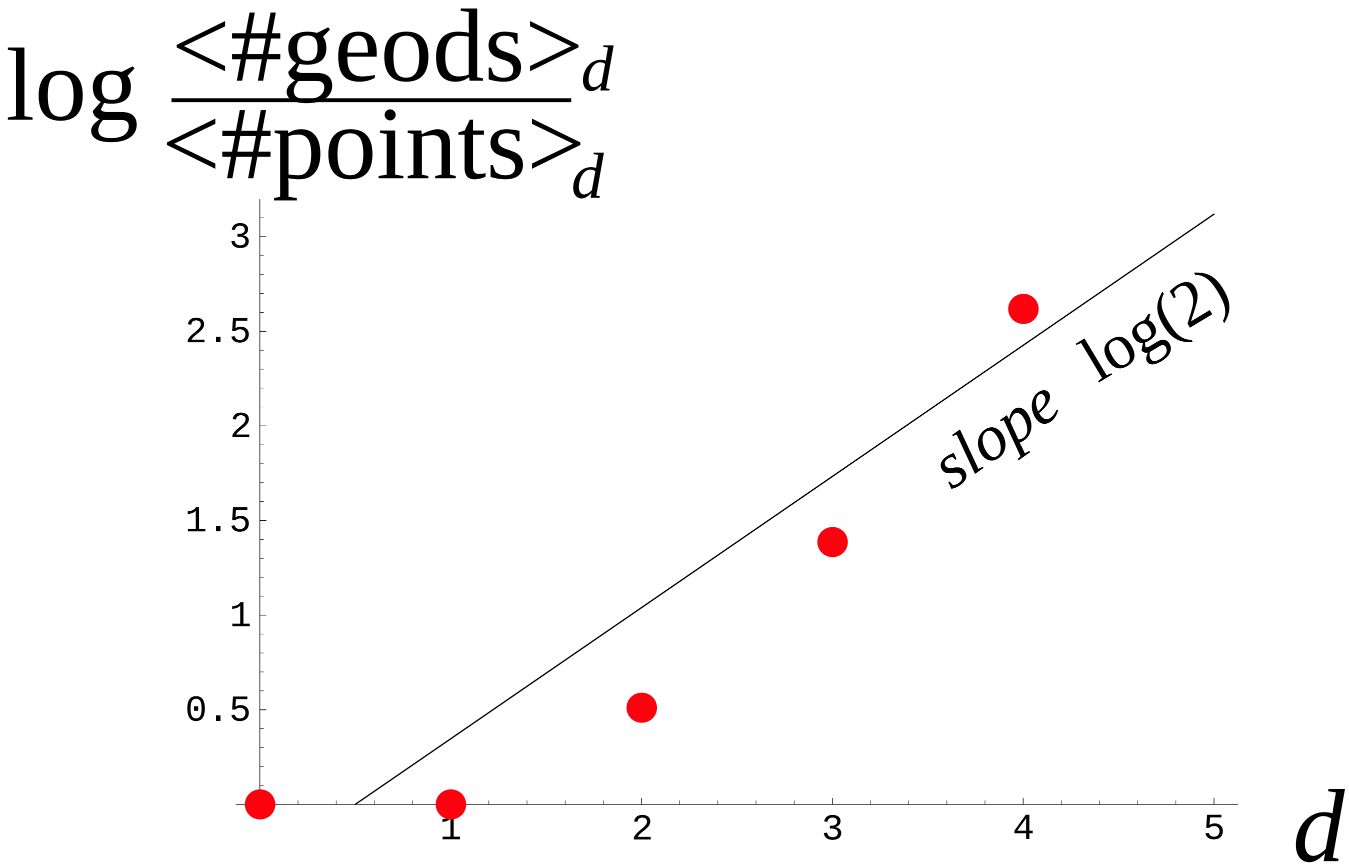
with 2 marked vertices

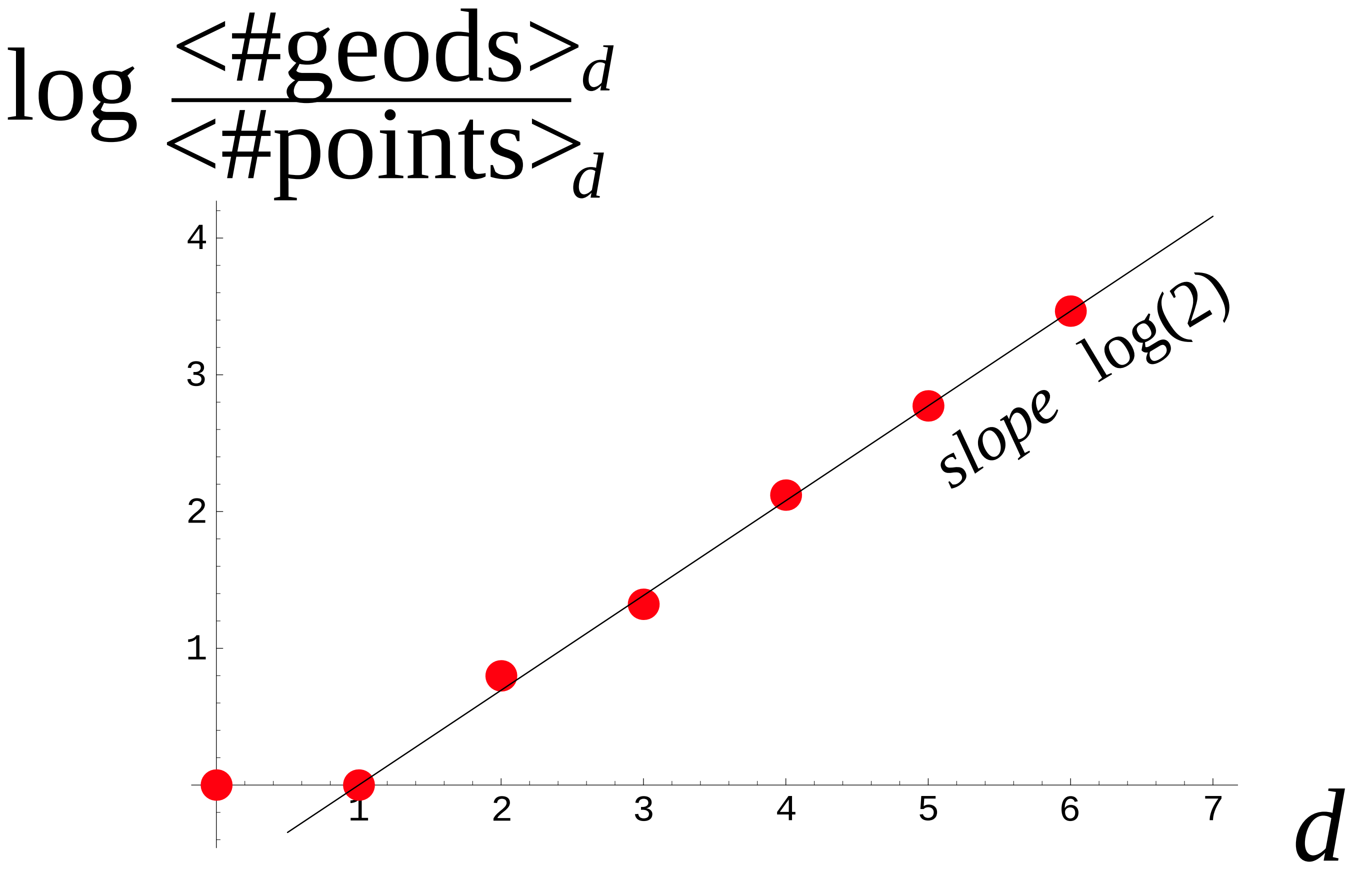
number of geodesics

enumerate, say planar quadrangulations with F faces



with 2 marked vertices
with marked geodesic paths $\rightarrow$ number of geodesic paths


log <#geods>_d / <#points>_d
3
2.5
2
1.5
1
0.5
1
2
3
4
5
d
slope log(2)

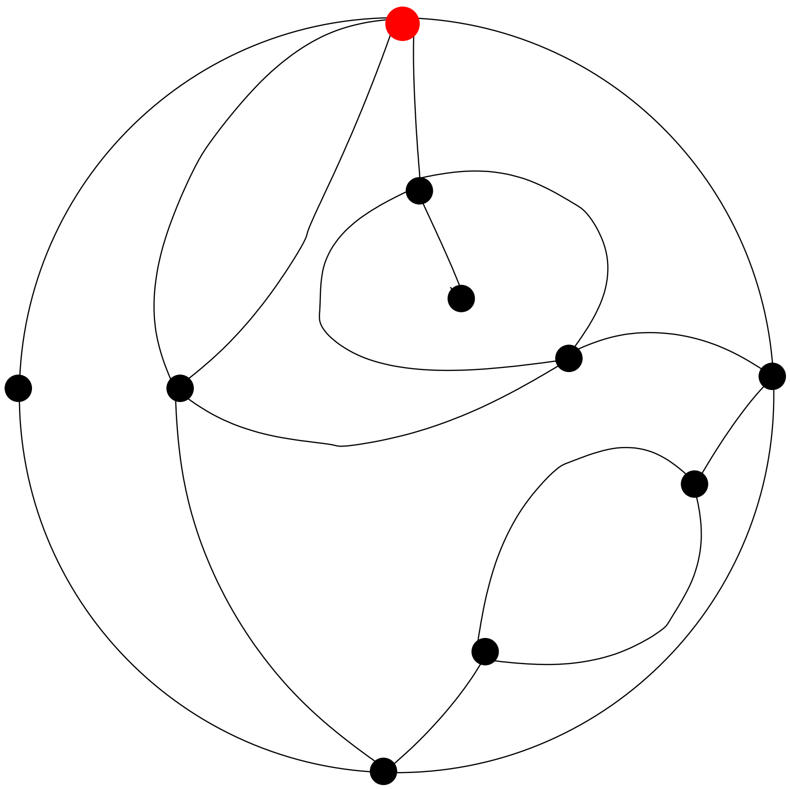


log <#geods>_d / <#points>_d
4
3
2
1
1
2
3
4
5
6
7
d
slope log(2)

the bijection with mobiles

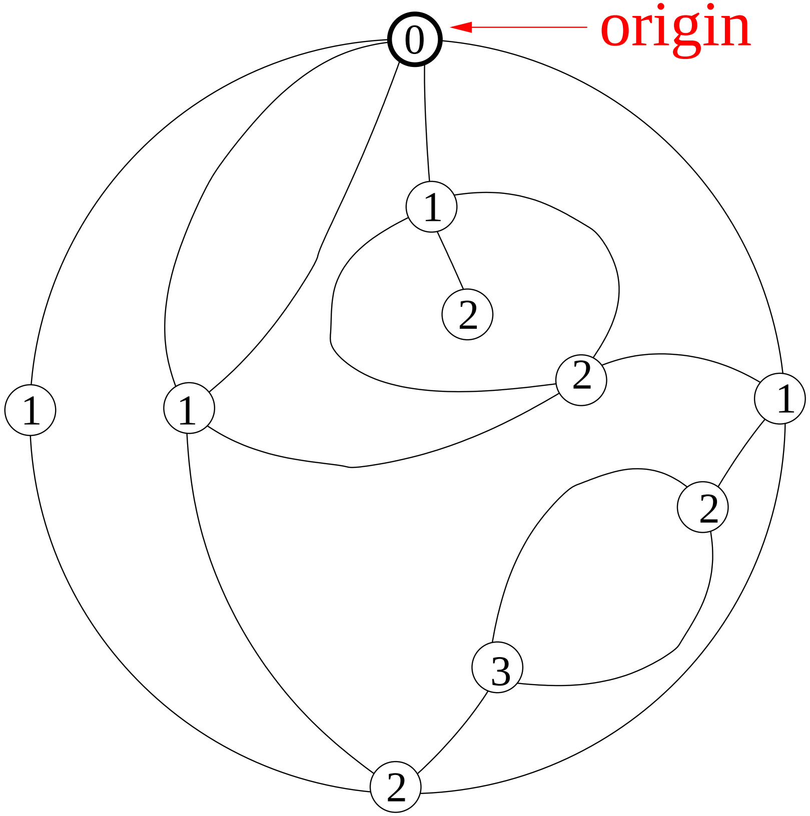
from maps to well-labeled mobiles

starting from a pointed planar map with even-valent faces



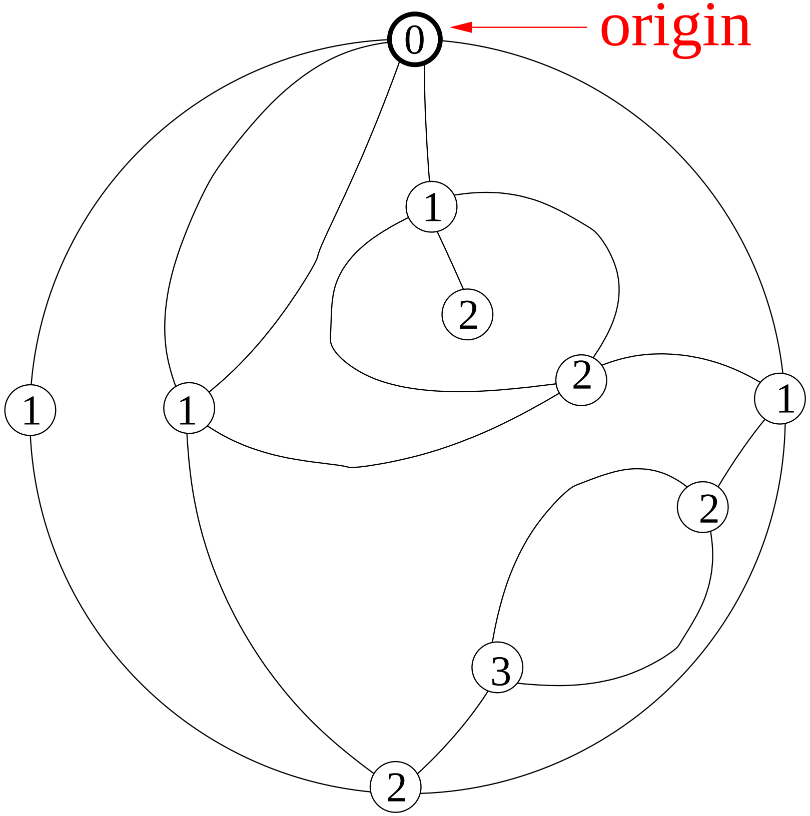
from maps to well-labeled mobiles

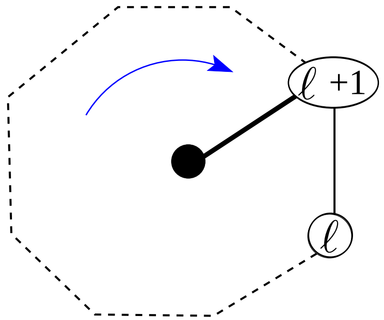
starting from a pointed planar map with even-valent faces



from maps to well-labeled mobiles

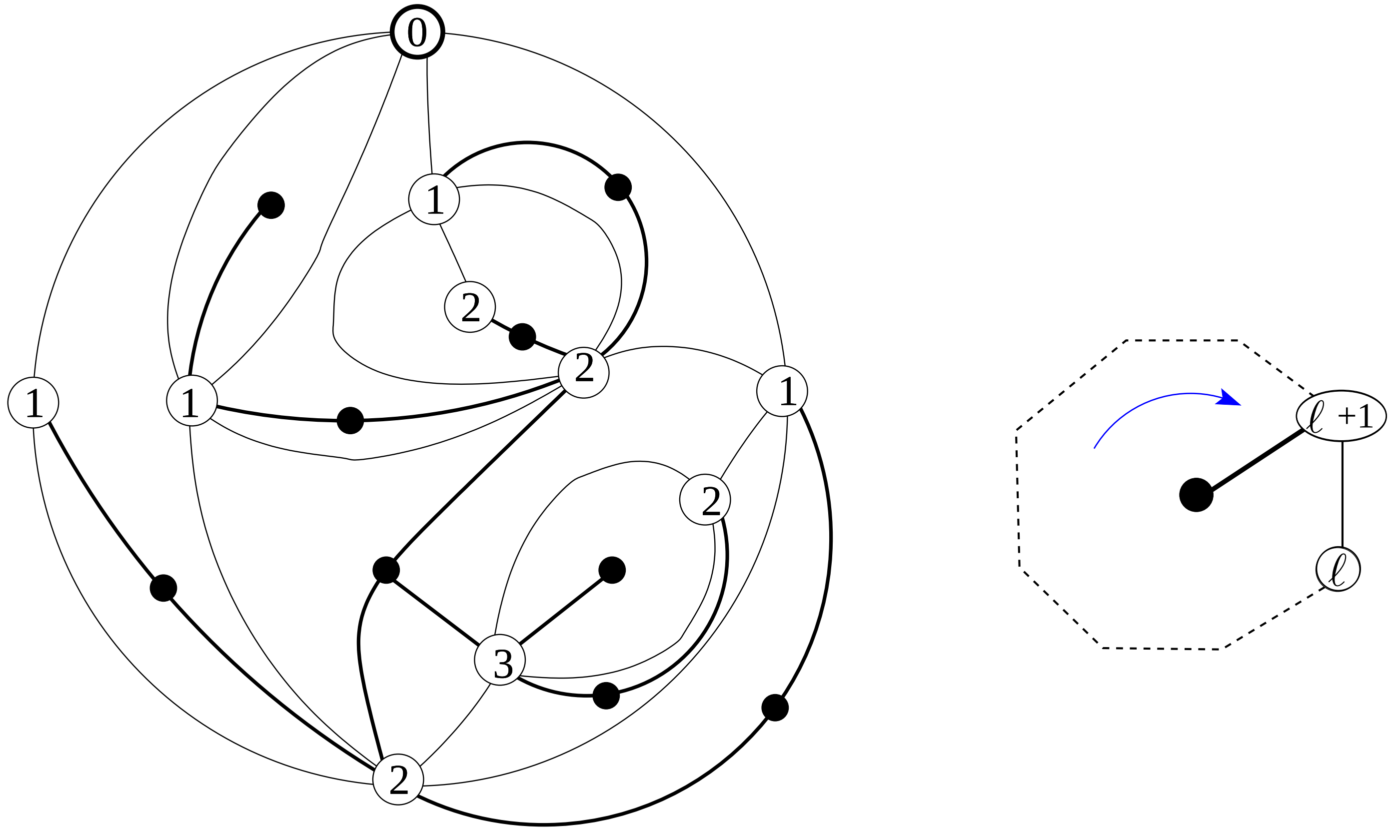
starting from a pointed planar map with even-valent faces





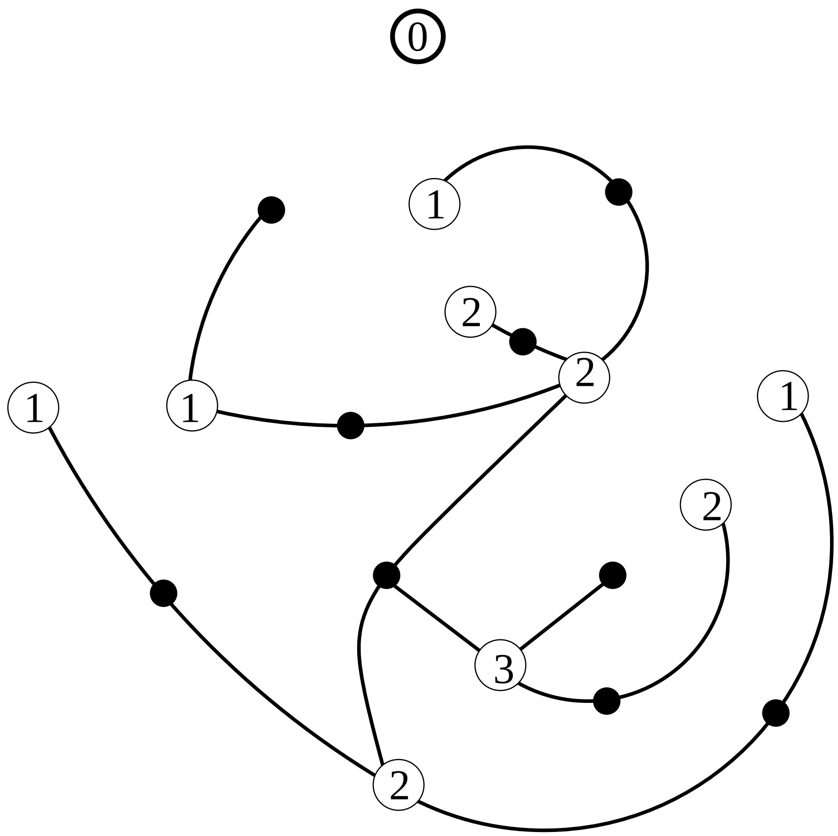
from maps to well-labeled mobiles

starting from a pointed planar map with even-valent faces



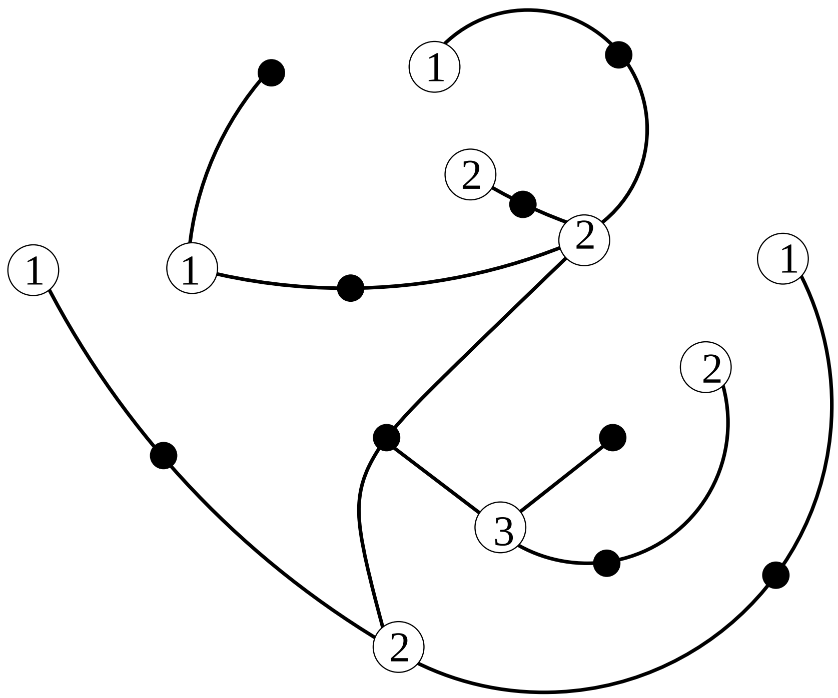
from maps to well-labeled mobiles

starting from a pointed planar map with even-valent faces



from maps to well-labeled mobiles

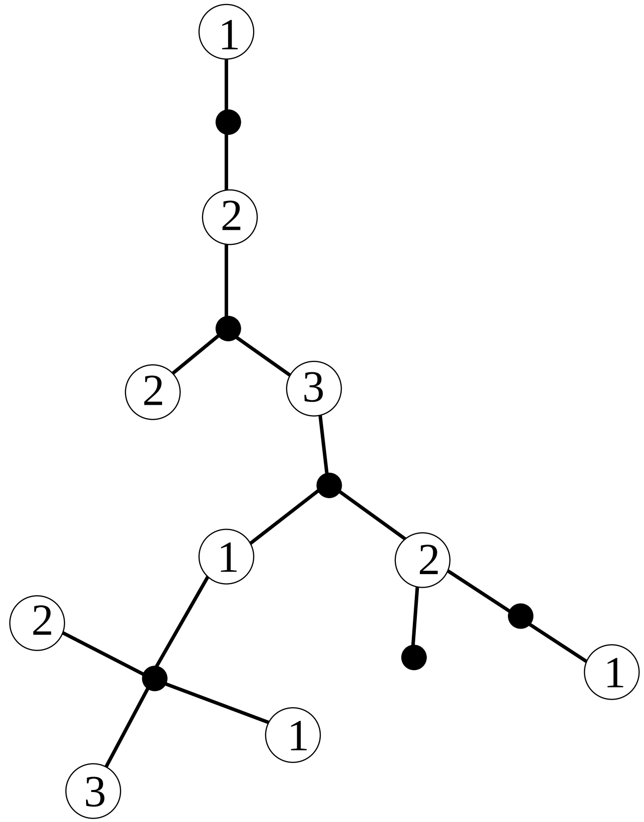
starting from a pointed planar map with even-valent faces



end up with a well-labeled mobile

well-labeled mobiles

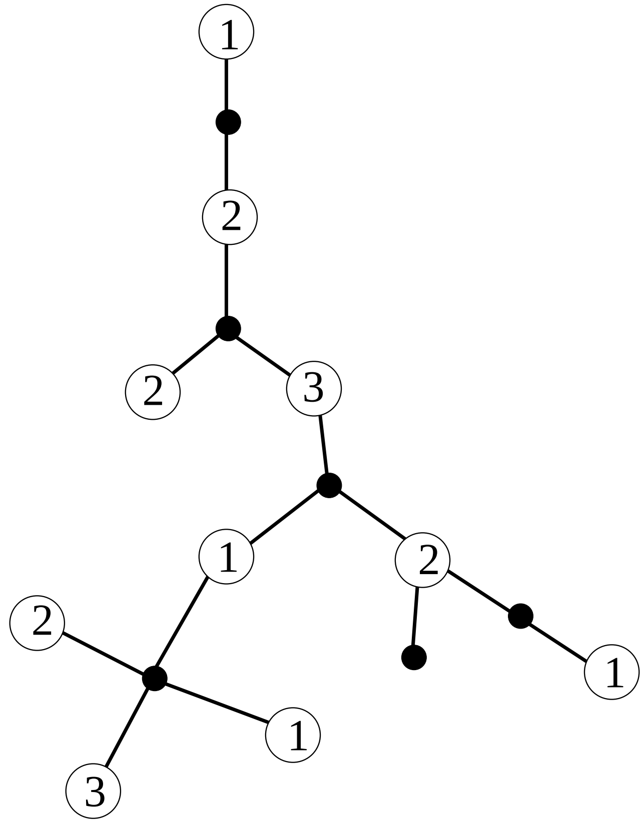
well-labeled:



well-labeled mobiles

well-labeled:

(i) positive integer labels

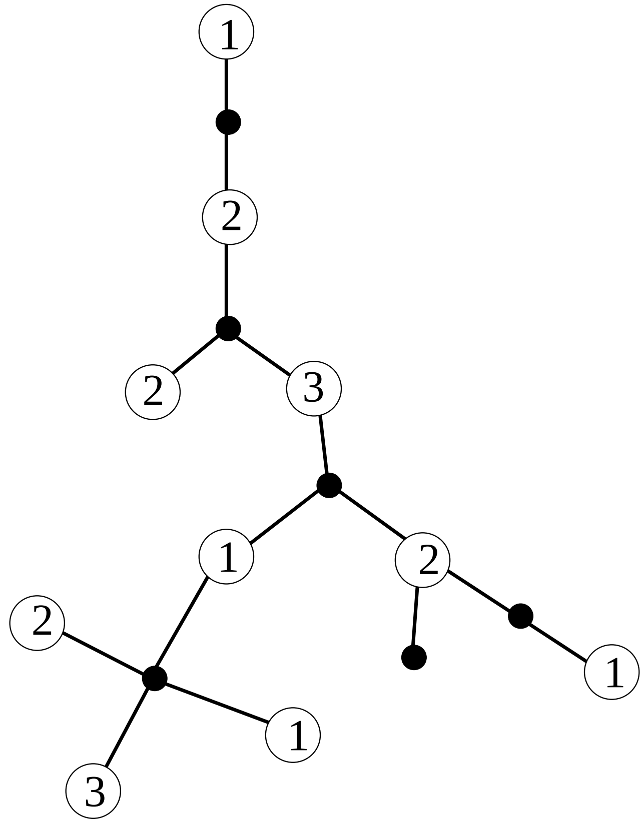


well-labeled mobiles

well-labeled:

(i) positive integer labels

(ii) at least one label 1



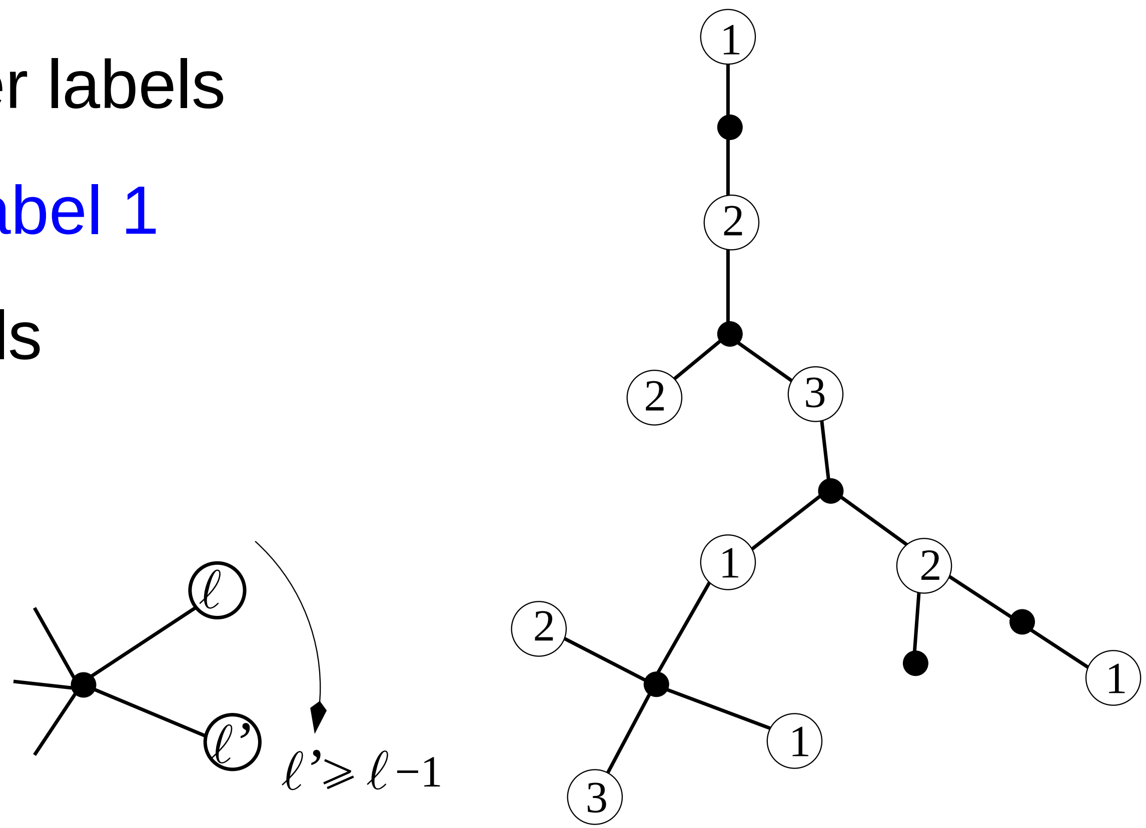
well-labeled mobiles

well-labeled:

(i) positive integer labels

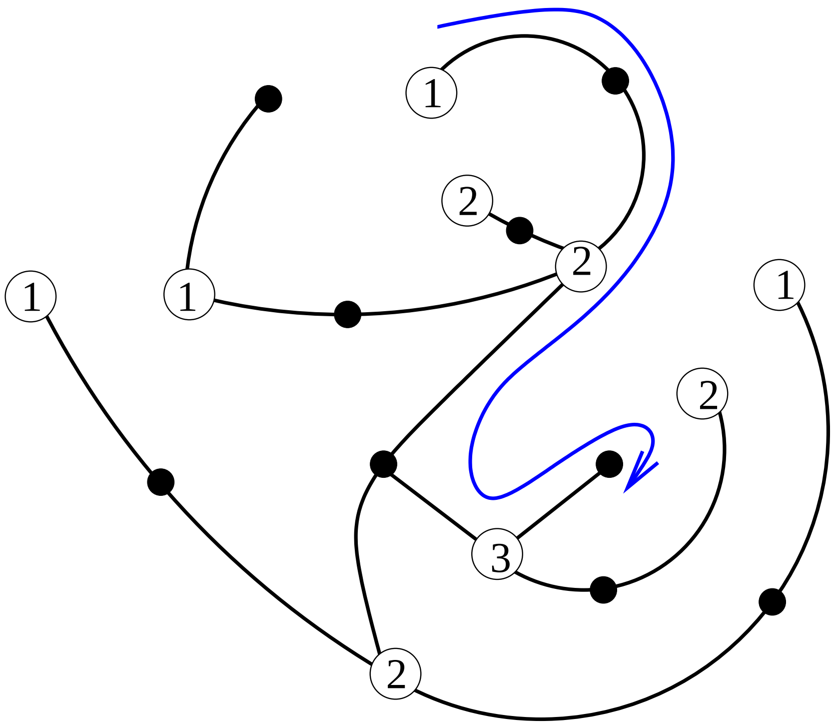
(ii) at least one label 1

(iii) rules on labels



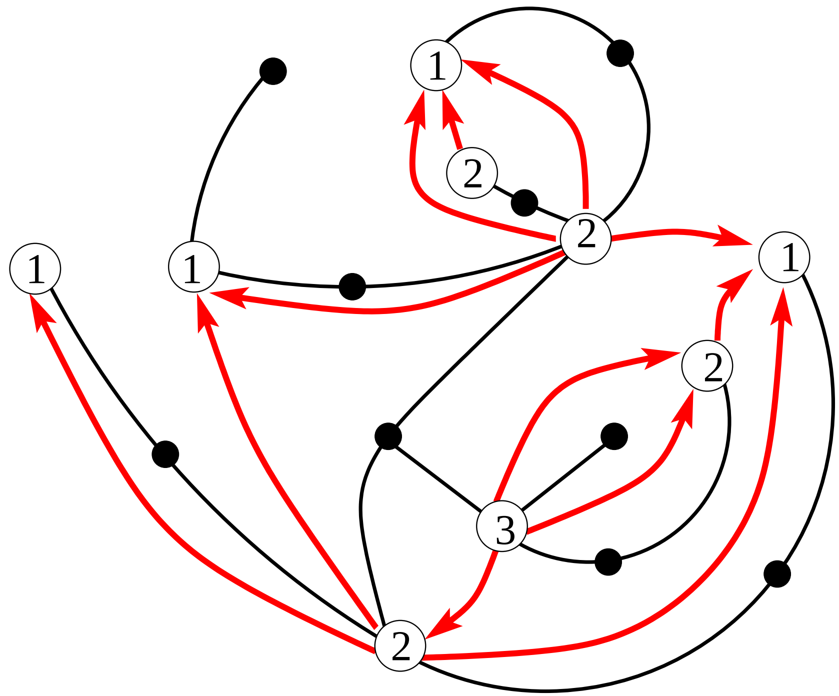
well-labeled mobiles → maps

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



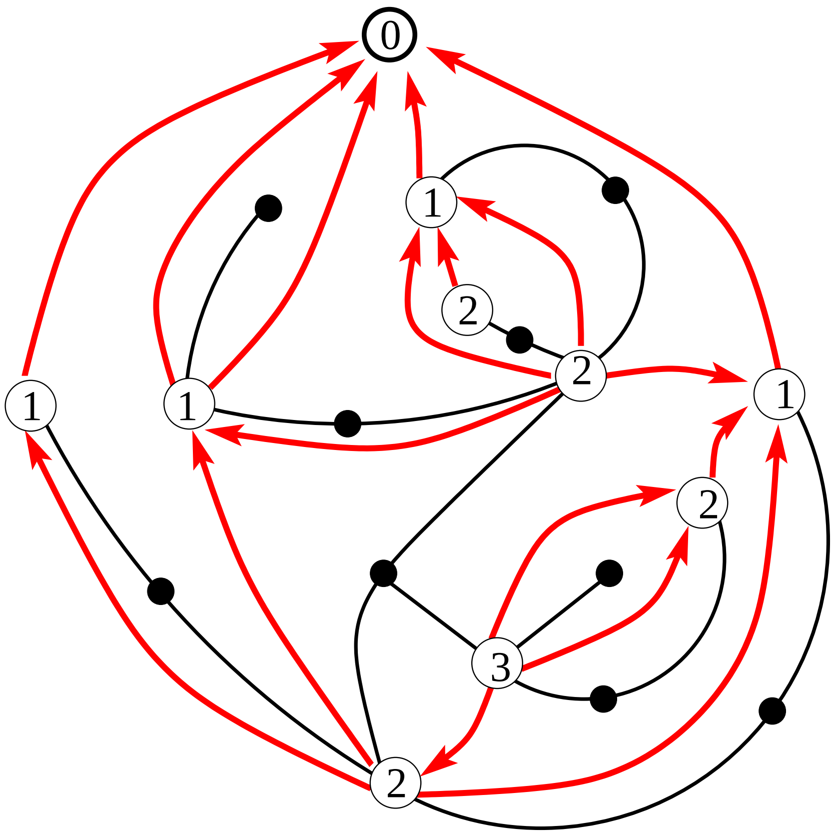
well-labeled mobiles → maps

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



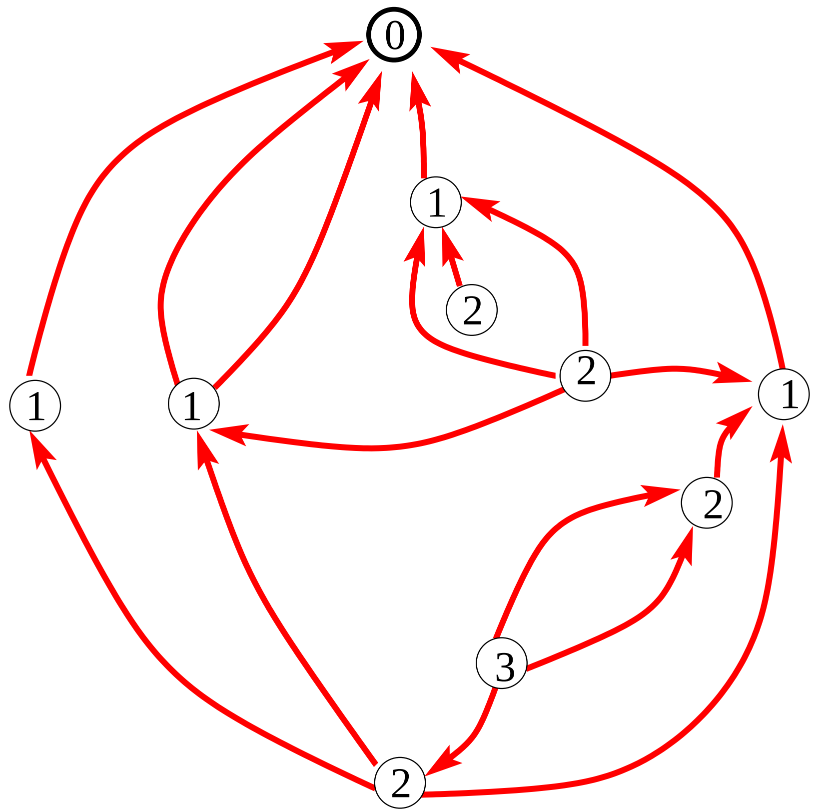
well-labeled mobiles → maps

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



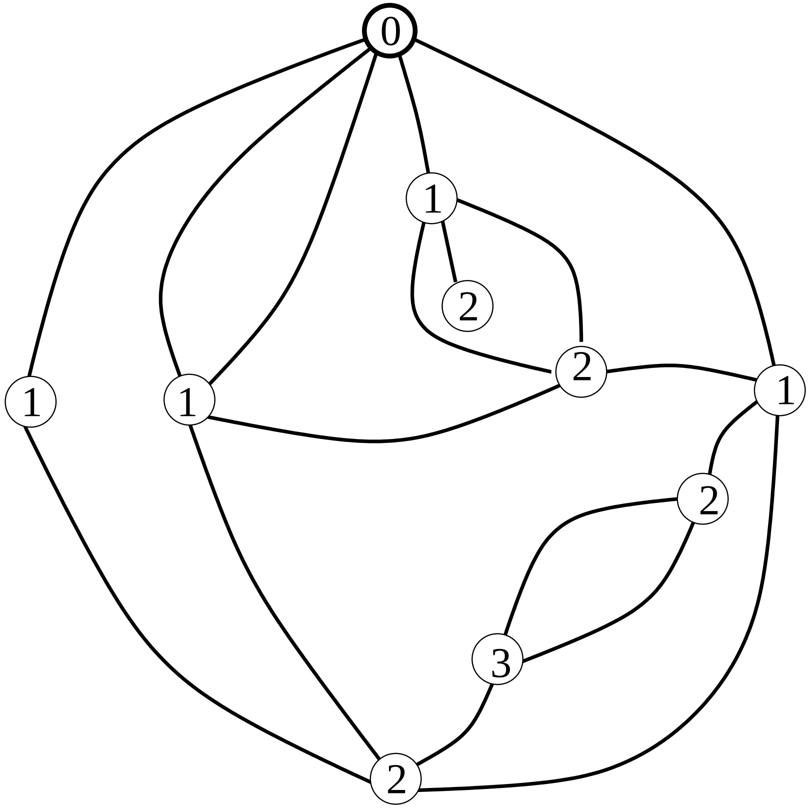
well-labeled mobiles → maps

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



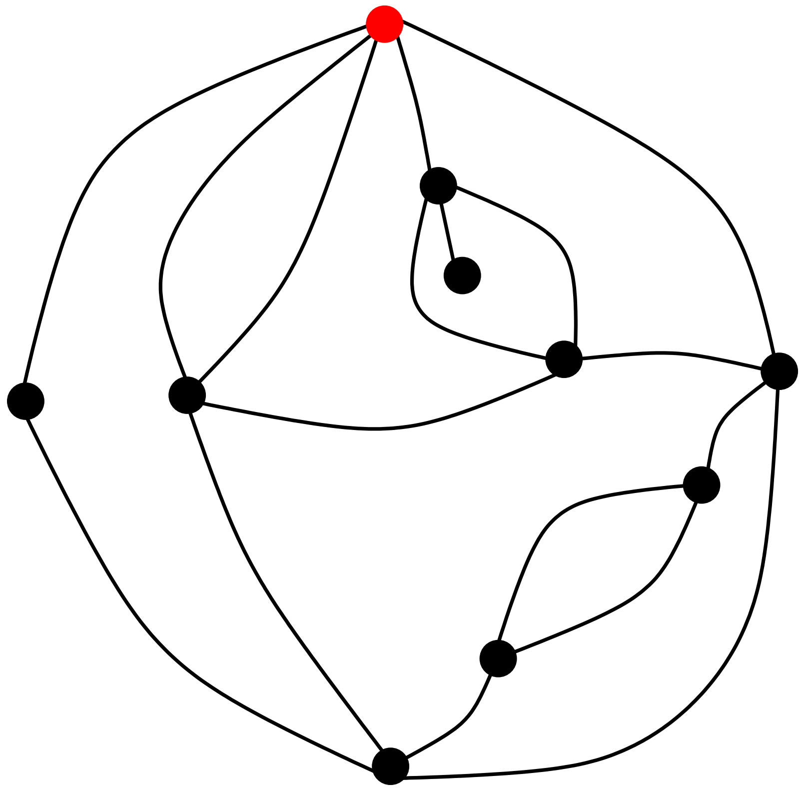
well-labeled mobiles → maps

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



well-labeled mobiles → maps

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$

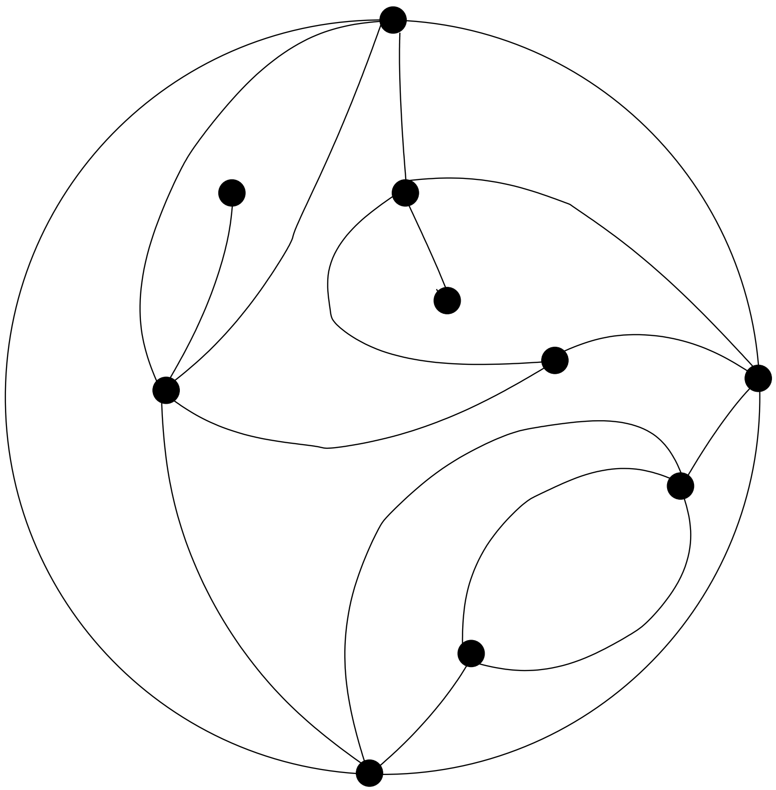


other species of trees

mobilaceae family

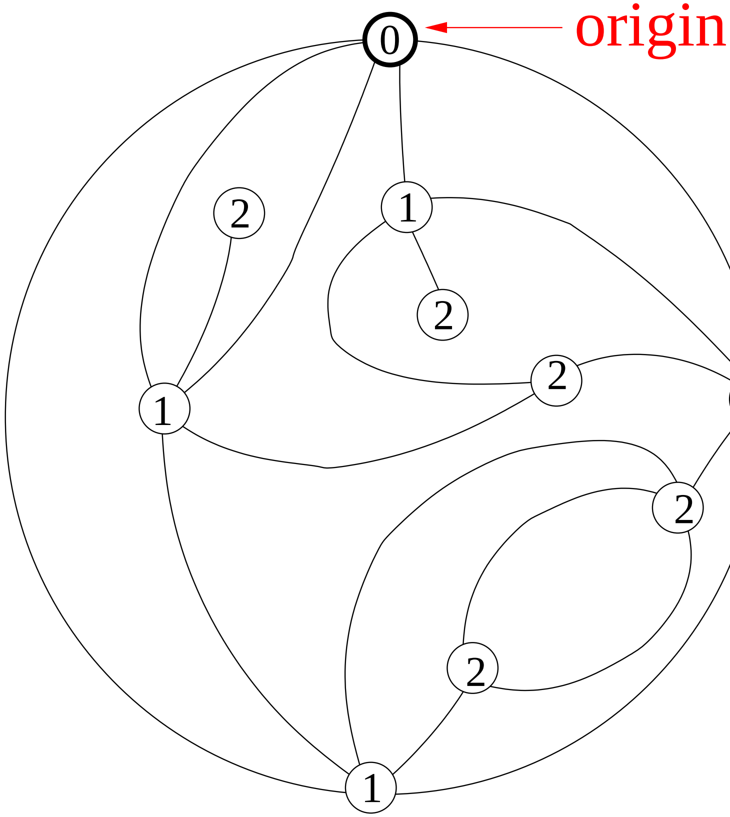
arbitrary degrees

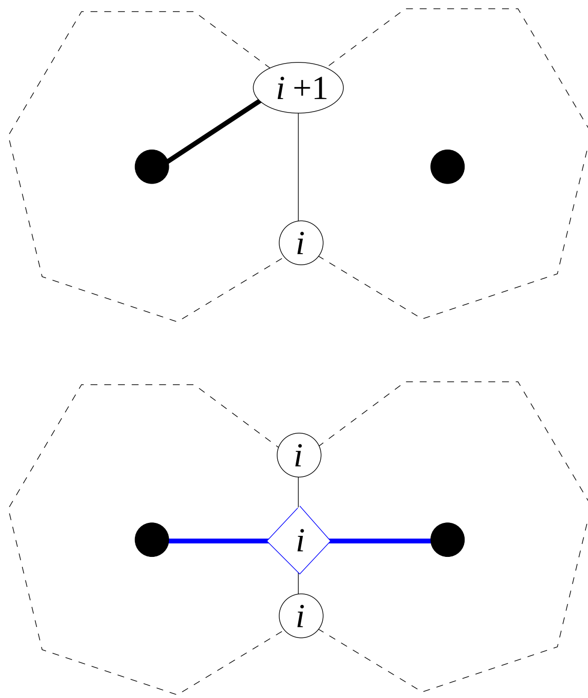
start with a pointed planar map



arbitrary degrees

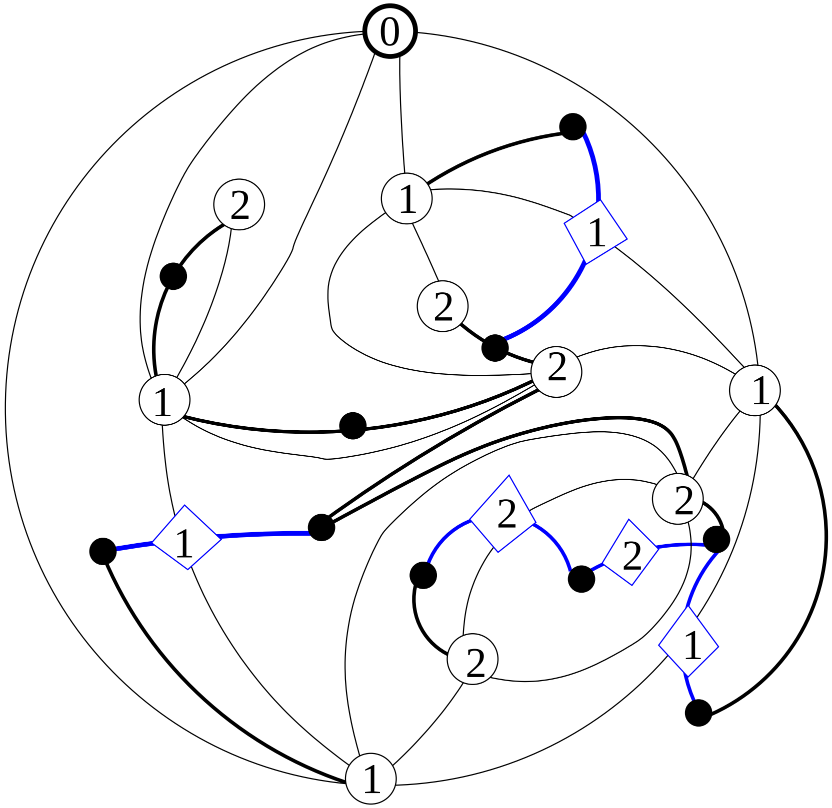
start with a pointed planar map

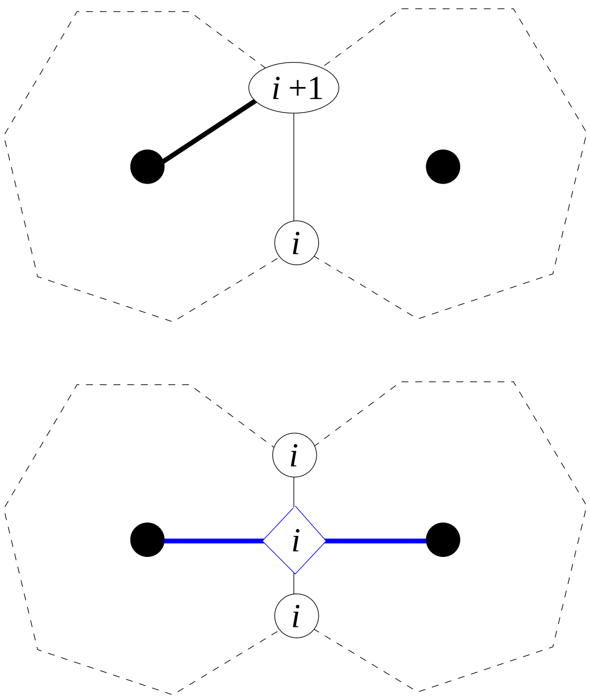




arbitrary degrees

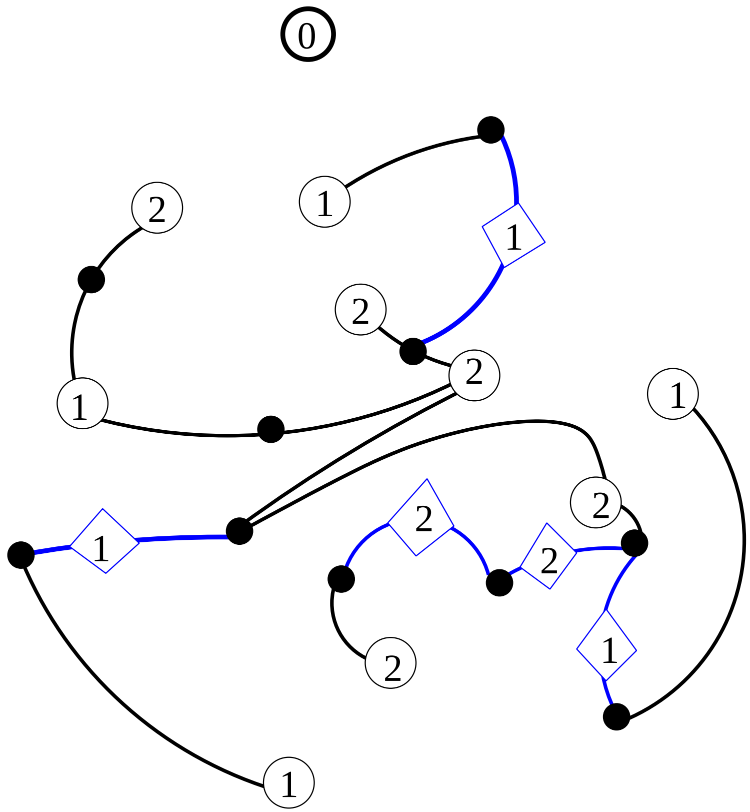
start with a pointed planar map



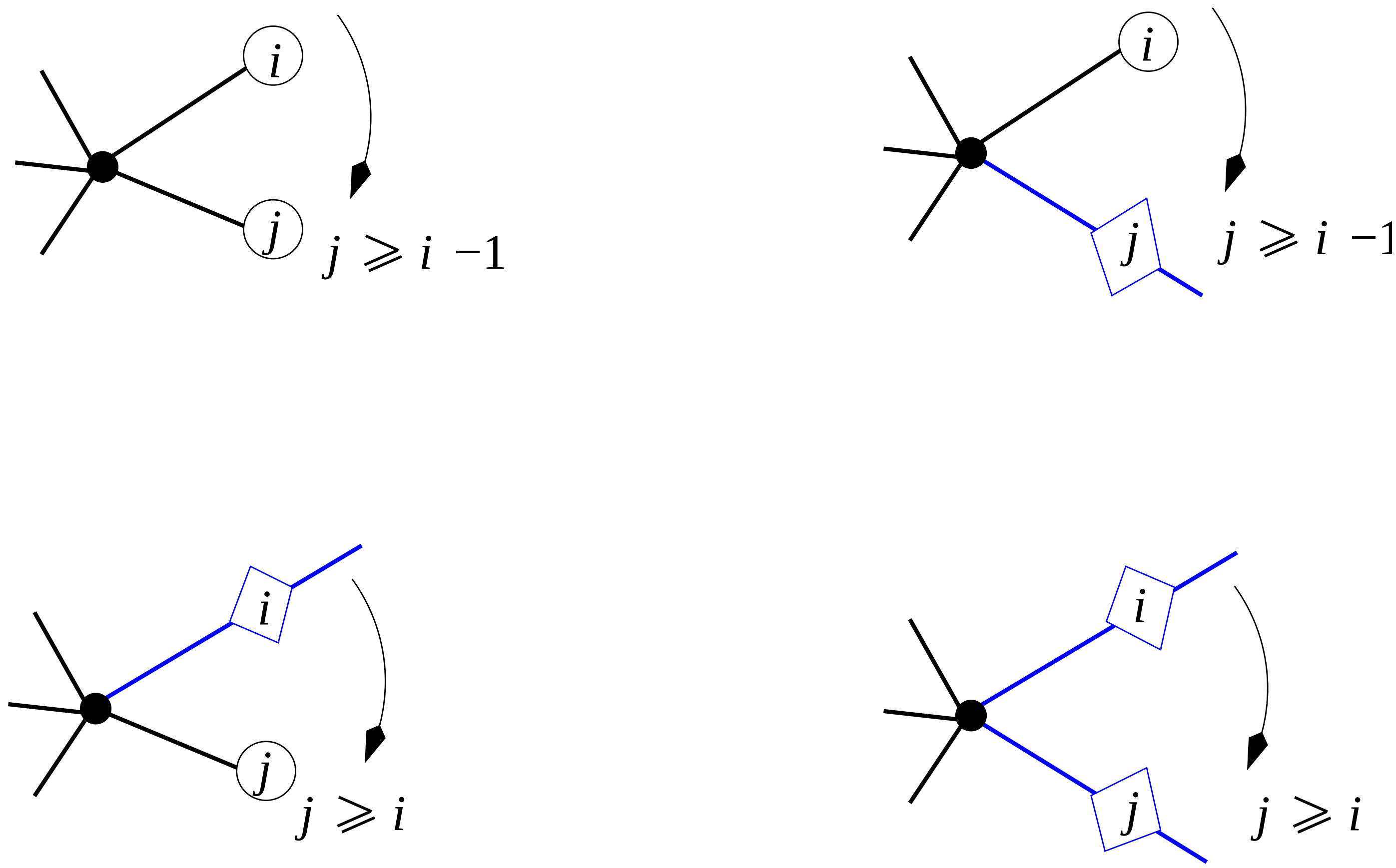


arbitrary degrees

start with a pointed planar map

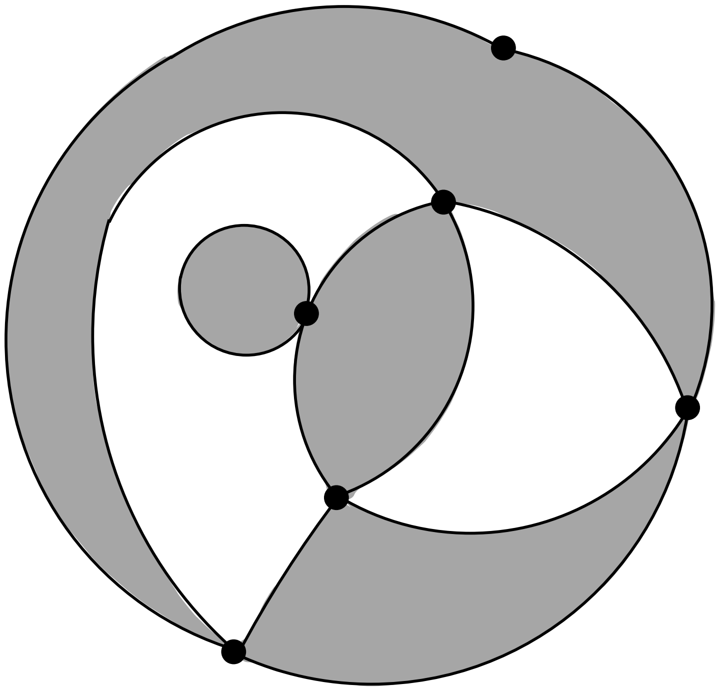


end up with a new type of mobile


i
j
j ⩾ i −1
i
j
j ⩾ i −1
i
j
j ⩾ i
i
j
j ⩾ i

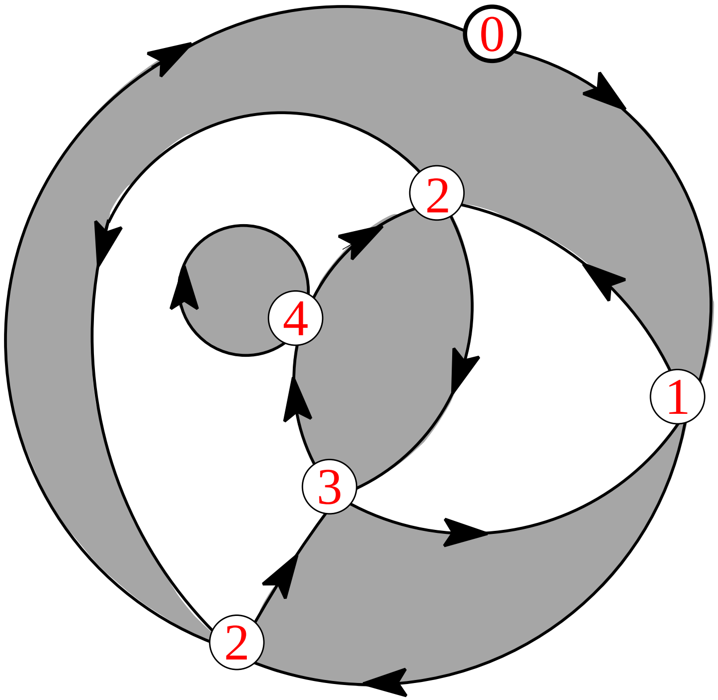
eulerian maps

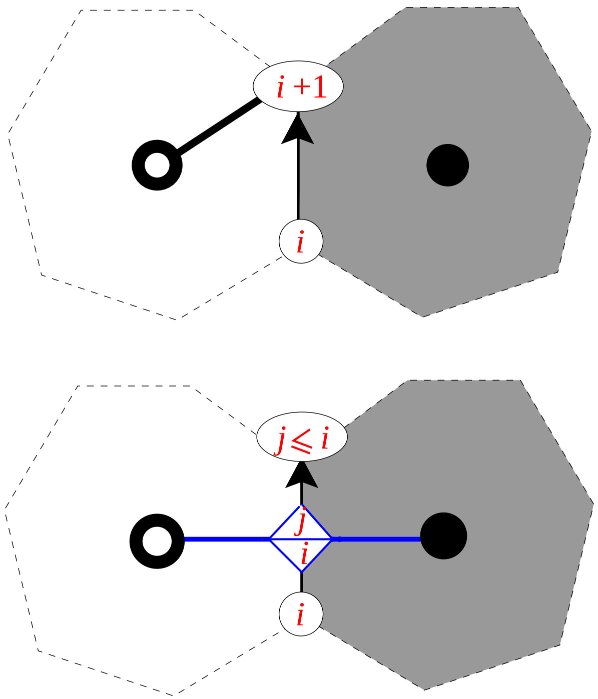
start with an eulerian (face bi-colored) planar map



eulerian maps

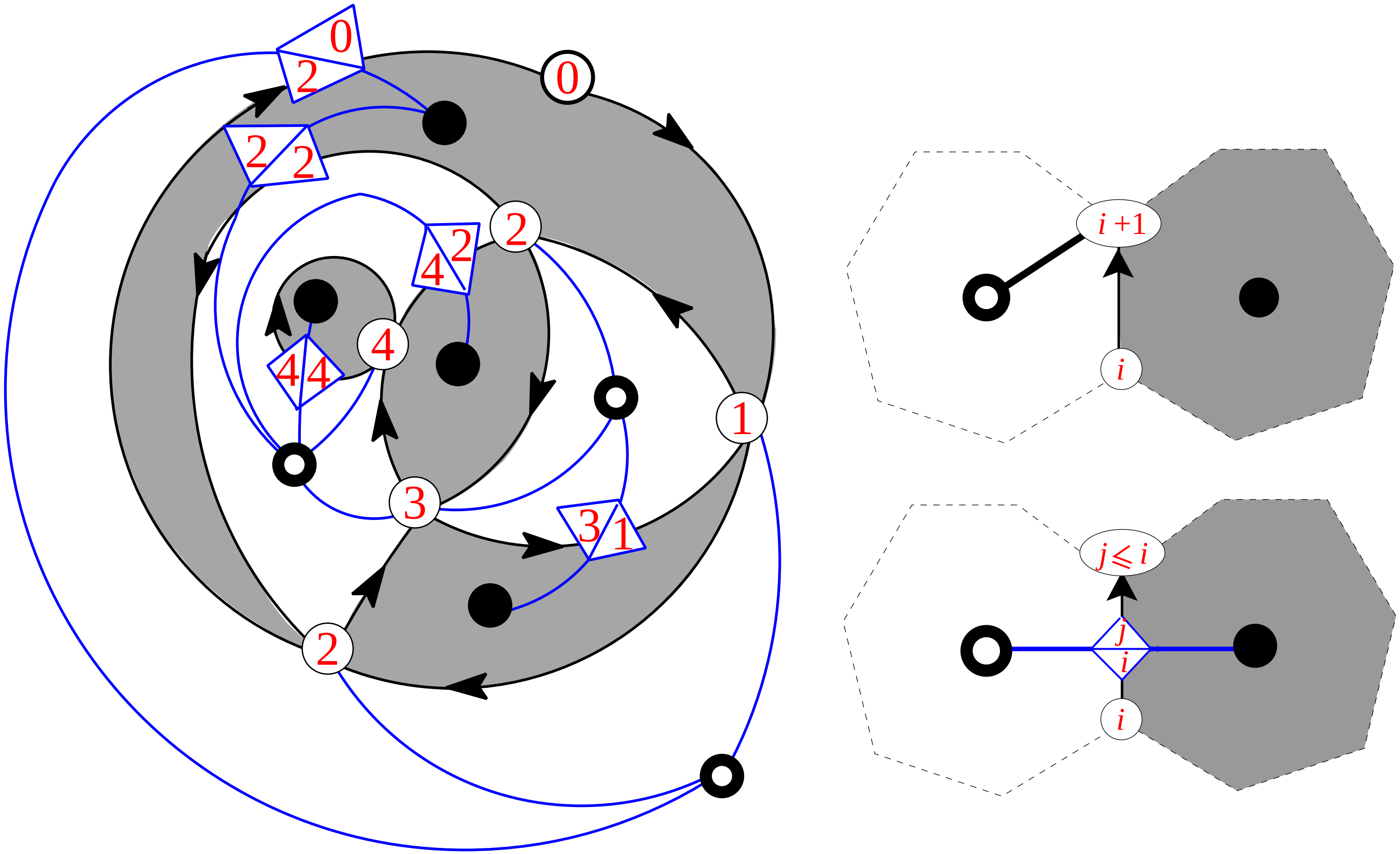
start with an eulerian (face bi-colored) planar map





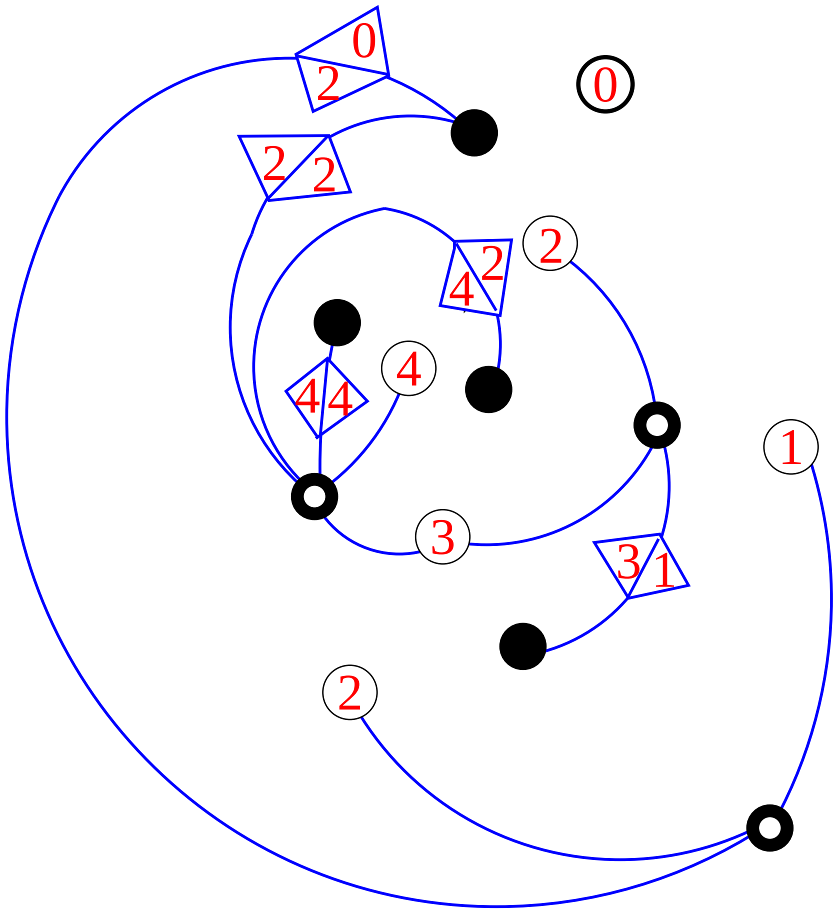
eulerian maps

start with an eulerian (face bi-colored) planar map

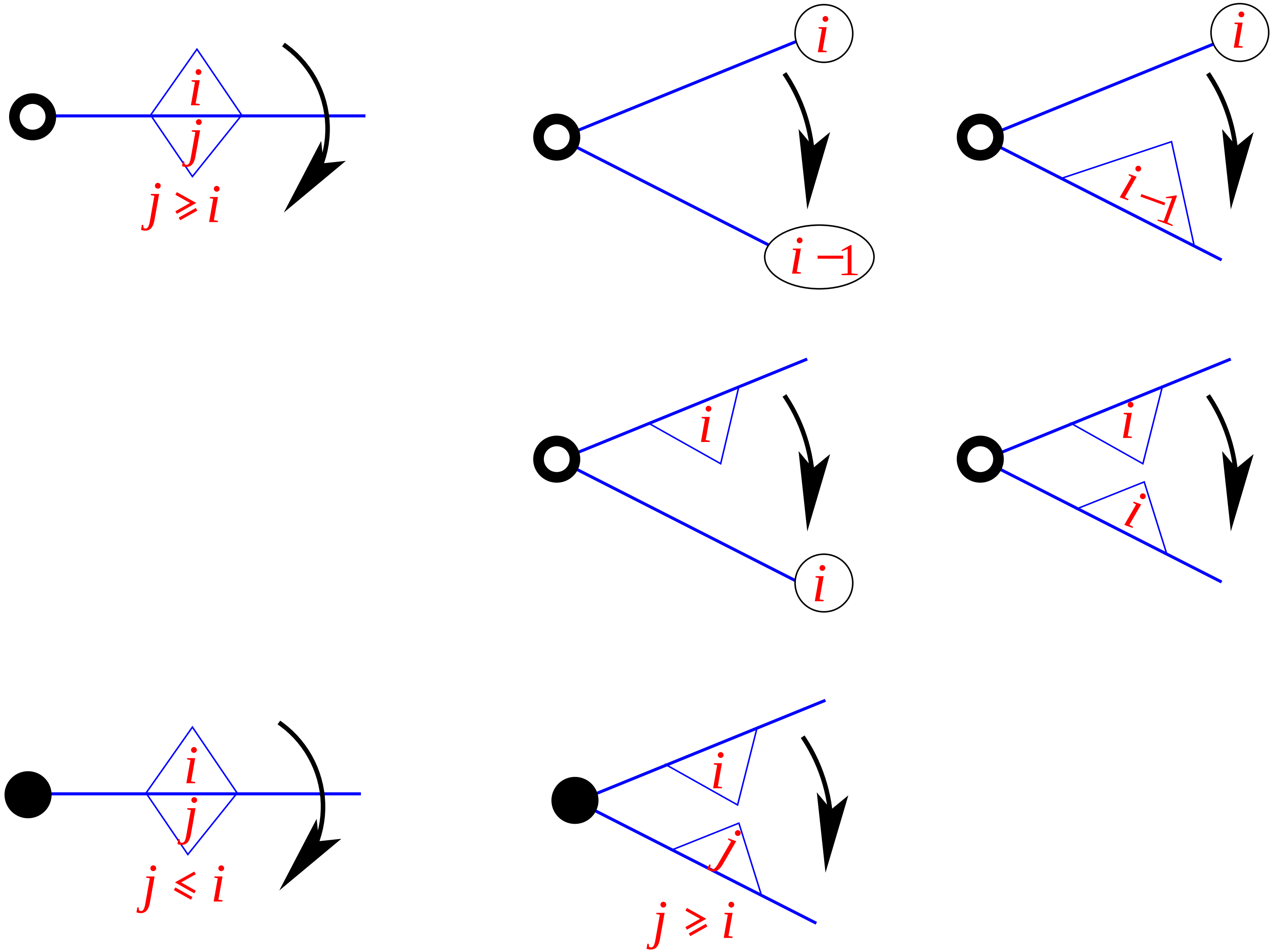


eulerian maps

start with an eulerian (face bi-colored) planar map

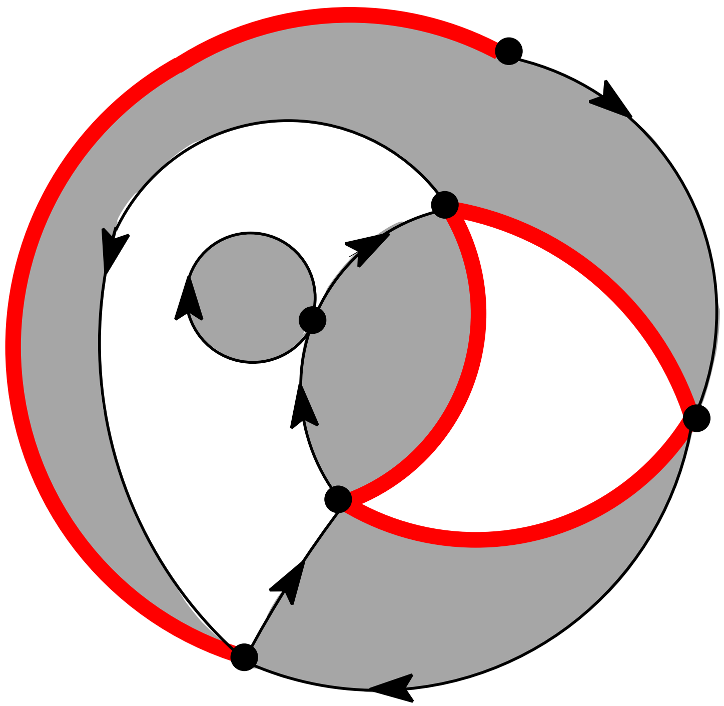


end up with a new type of mobile


i
j
j ≥ i
i
i − 1
i
i − 1
i
i
i
i
i
j
j ≤ i
i
j
j ≥ i

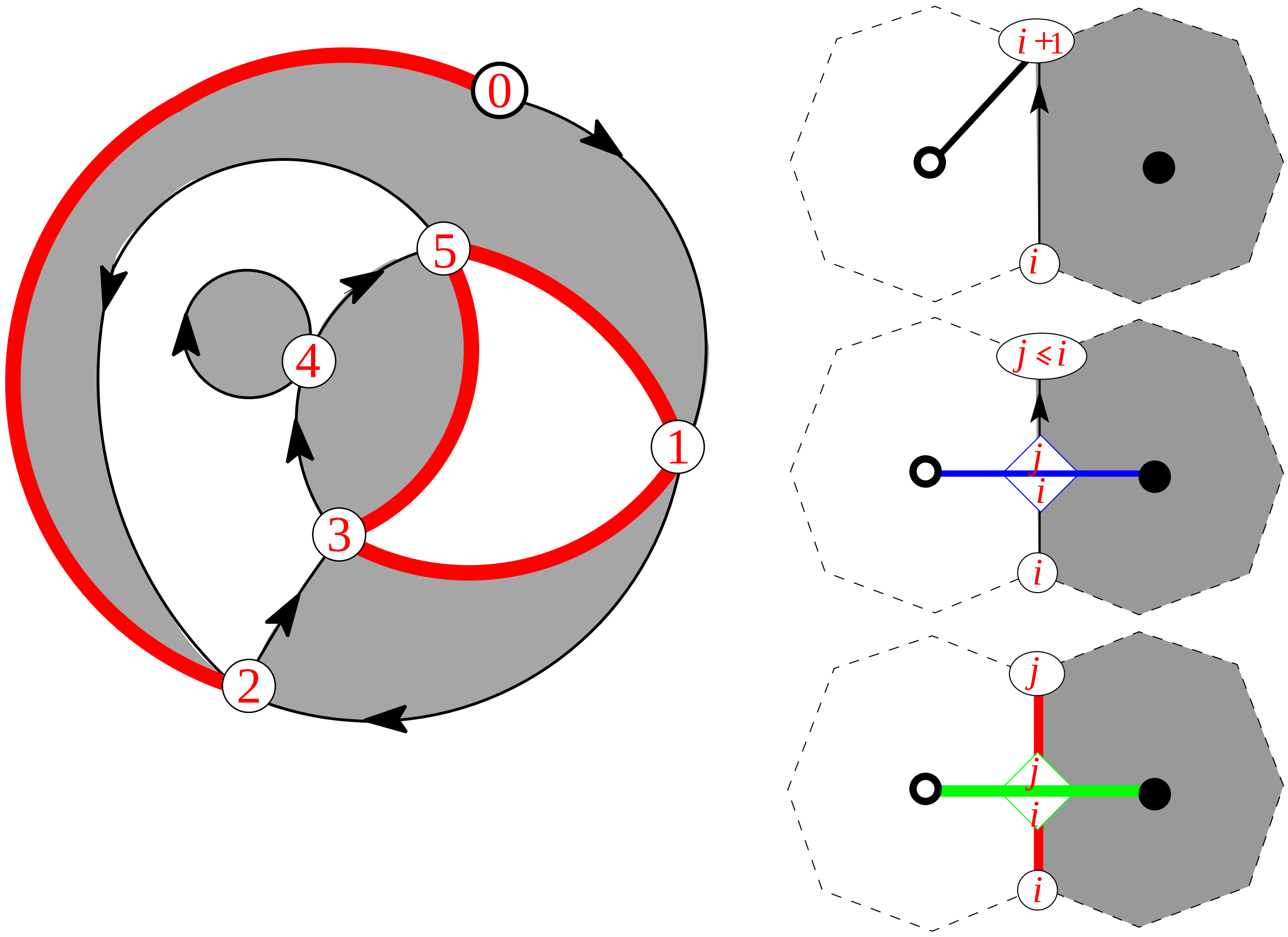
eulerian maps with blocked edges

start with an eulerian planar map with blocked edges



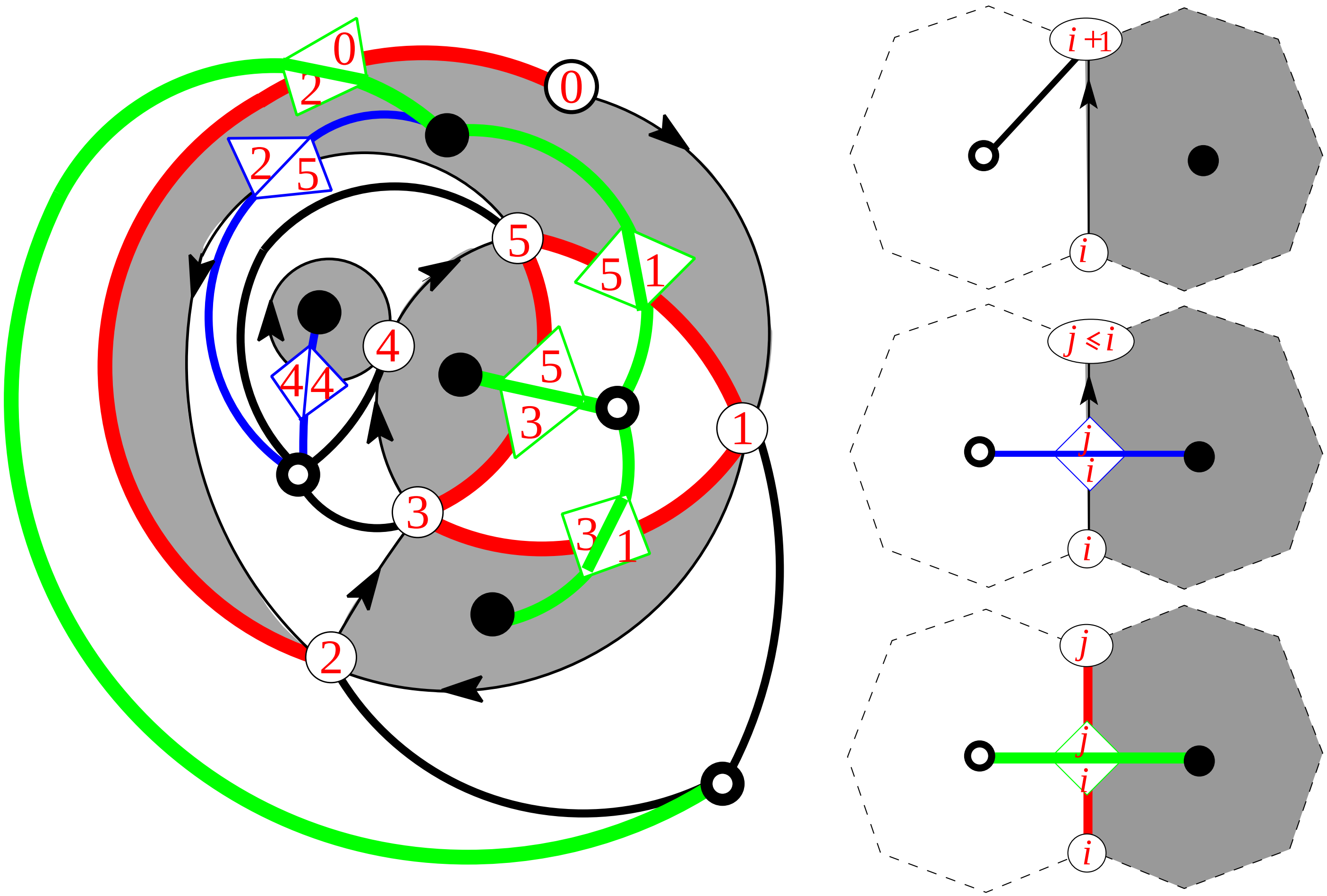
eulerian maps with blocked edges

start with an eulerian planar map with blocked edges



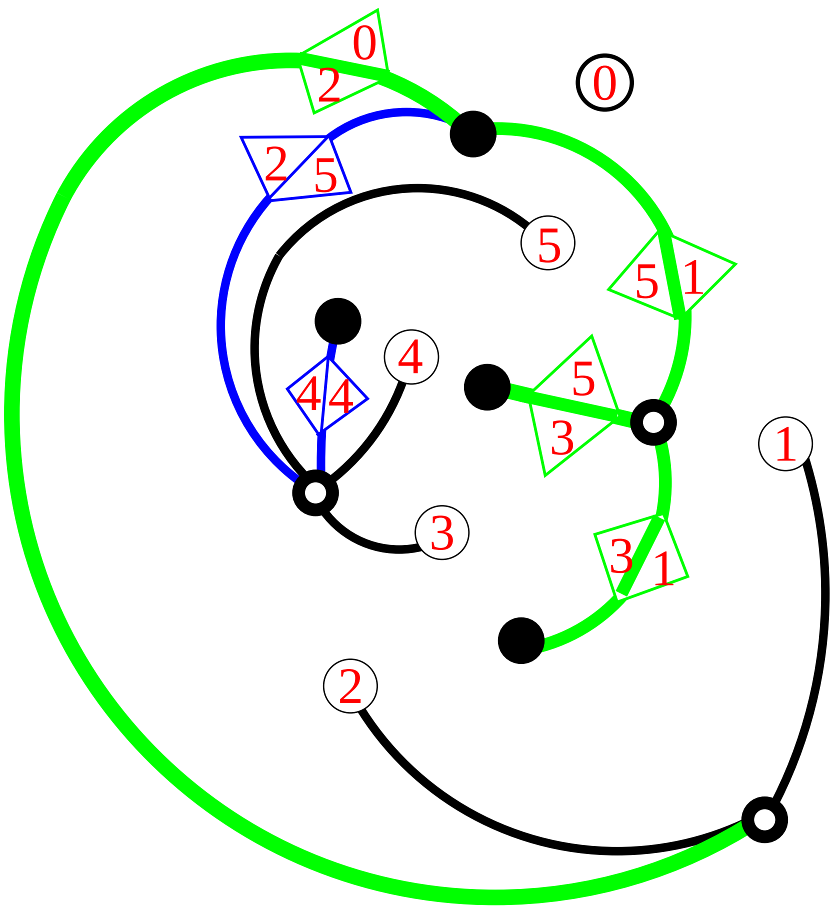
eulerian maps with blocked edges

start with an eulerian planar map with blocked edges



eulerian maps with blocked edges

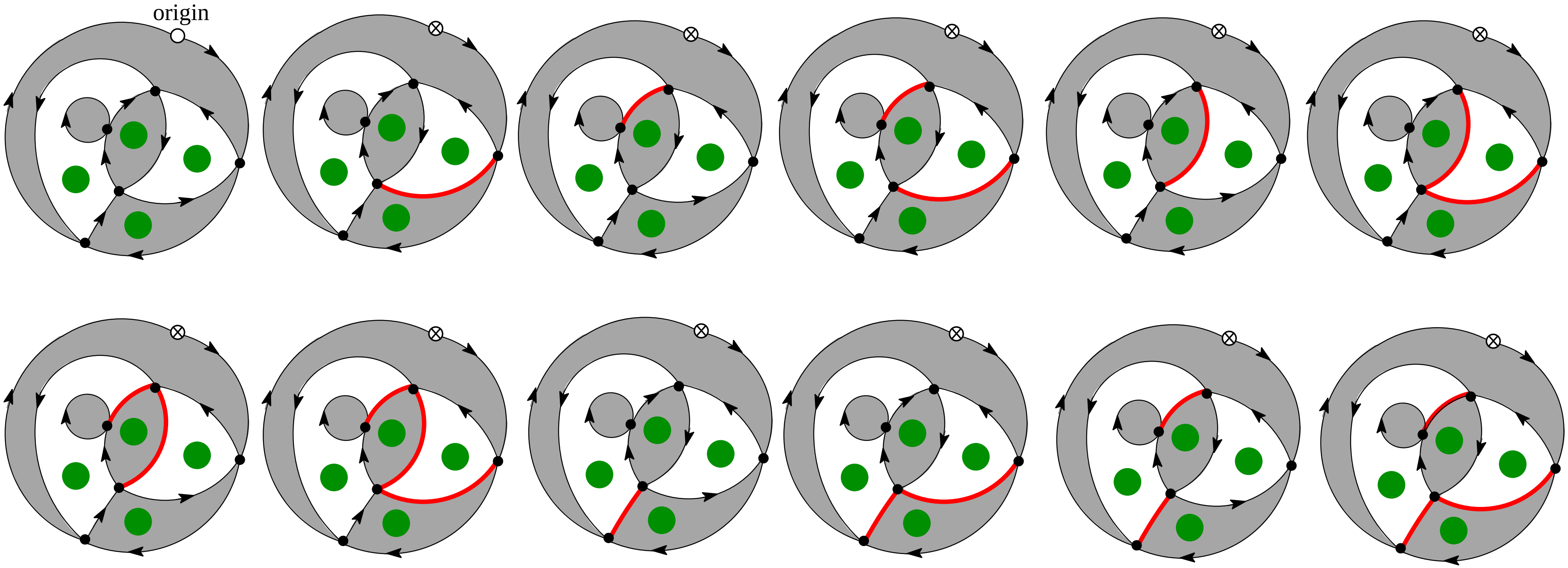
start with an eulerian planar map with blocked edges



end up with a new type of mobile

eulerian maps with hard particles

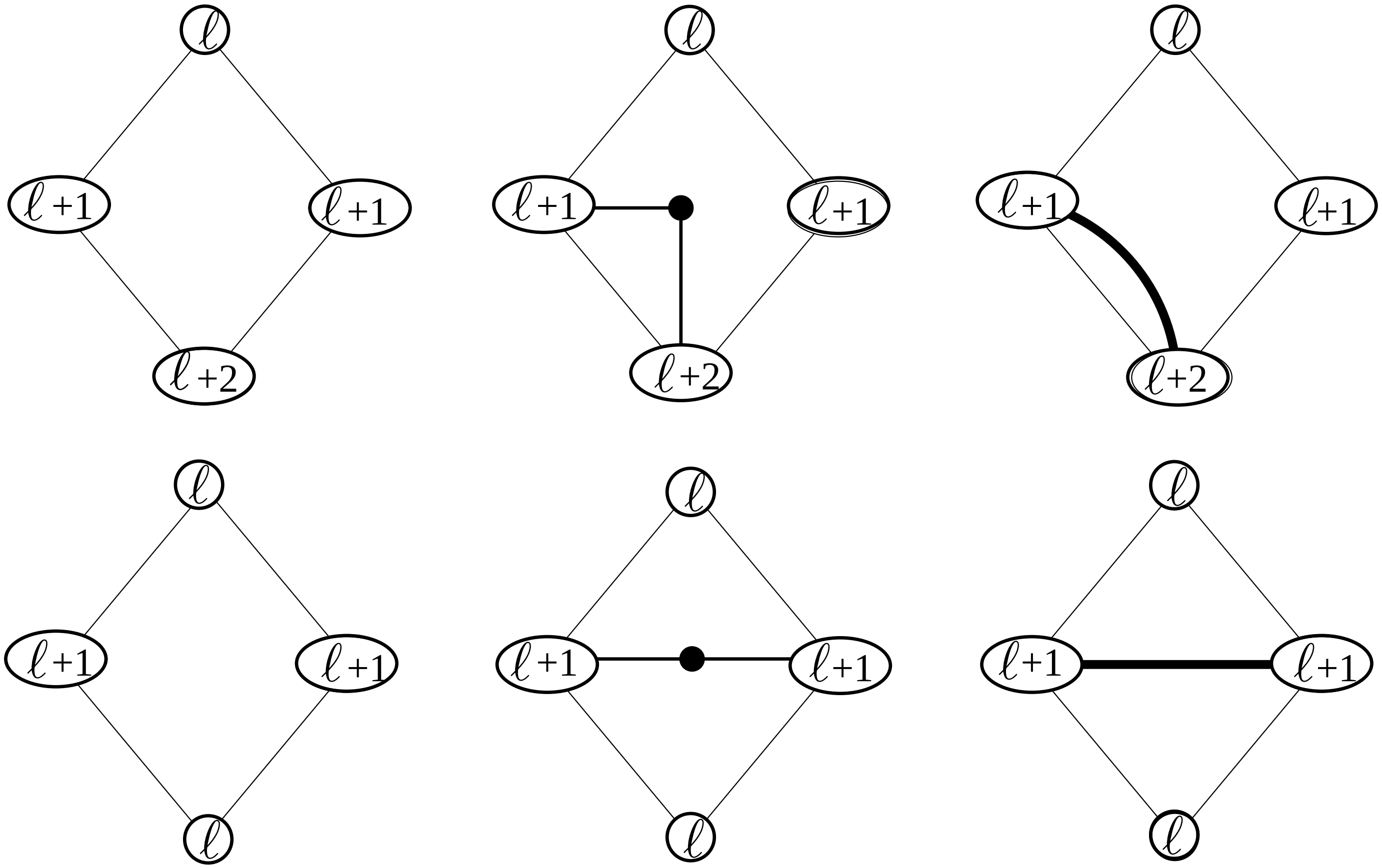
◇ Consider eulerian maps with at most 1 particle per face
◇ Decide to block or not edges between two occupied faces



◇ Weight -1 per blocked edge $\rightarrow$ selects hard-particle configurations

generating functions for quadrangulations

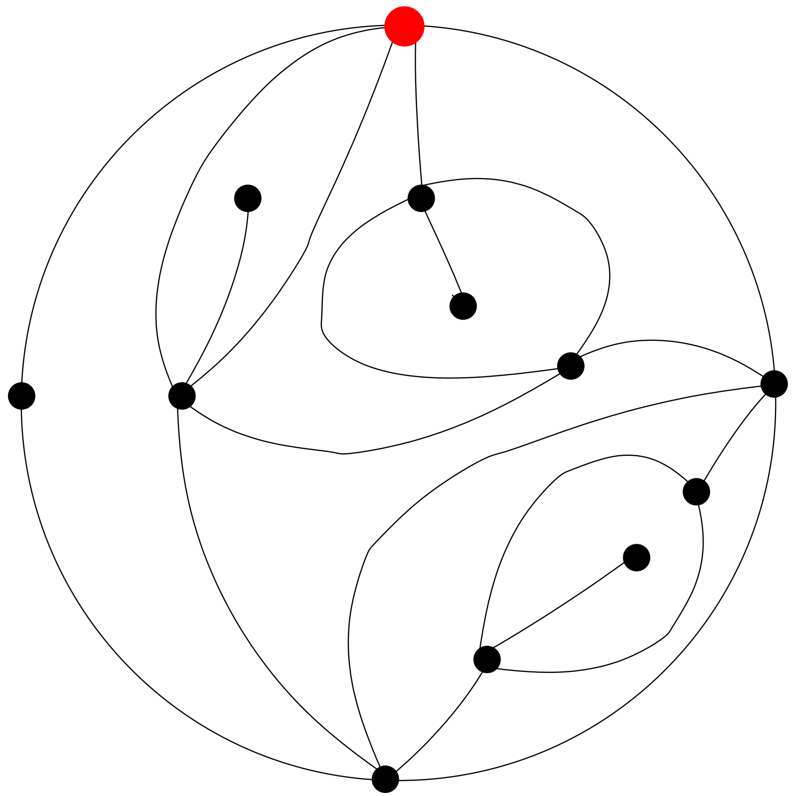
case of quadrangulations



Schaeffer's bijection

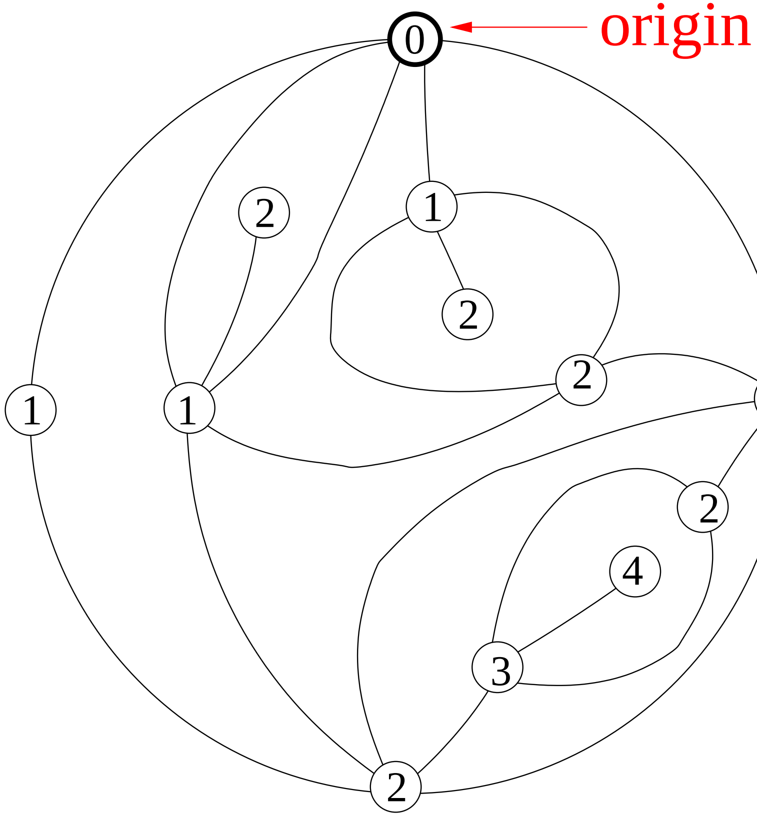
quadrangulations → well-labeled trees

start with a pointed planar quadrangulation



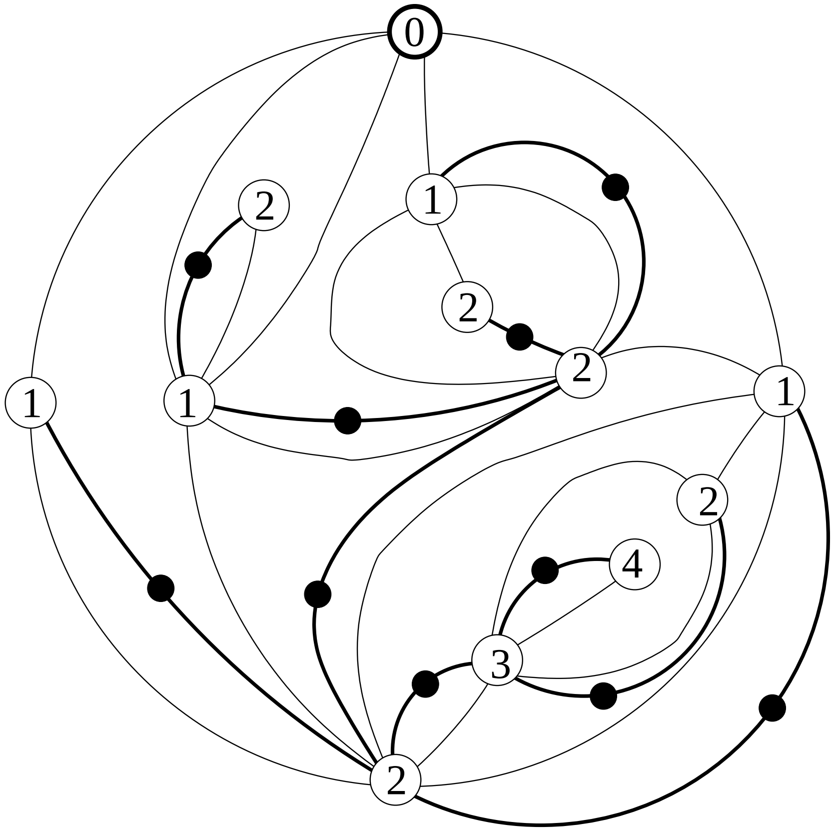
quadrangulations → well-labeled trees

start with a pointed planar quadrangulation



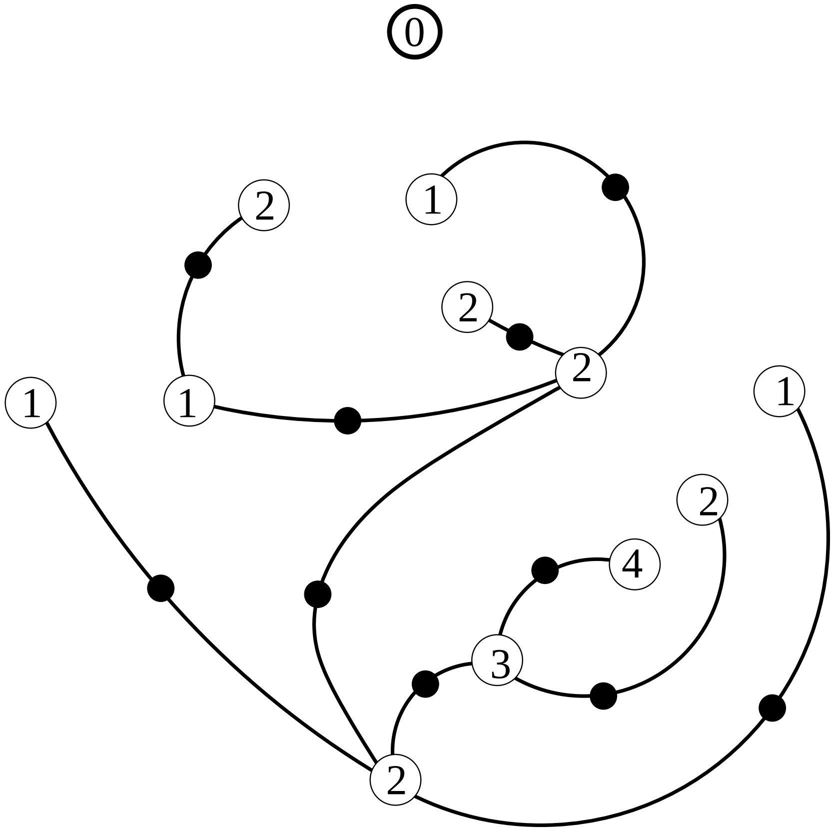
quadrangulations → well-labeled trees

start with a pointed planar quadrangulation



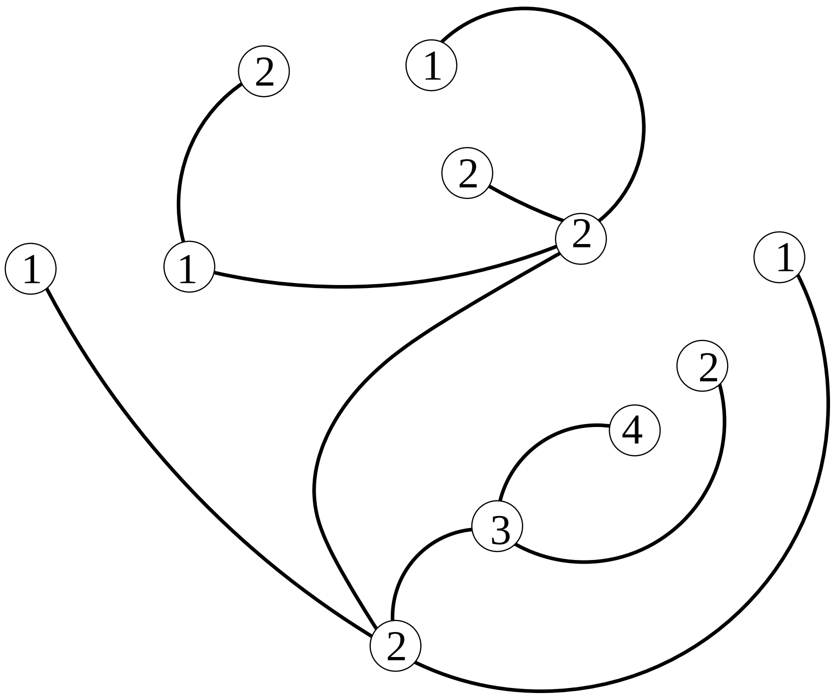
quadrangulations → well-labeled trees

start with a pointed planar quadrangulation



quadrangulations → well-labeled trees

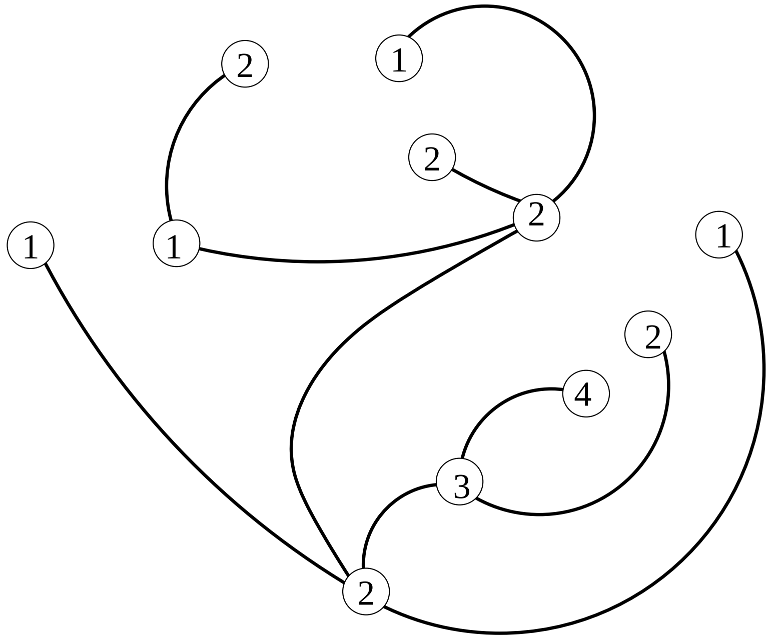
start with a pointed planar quadrangulation



end up with a planar well-labeled tree

well-labeled trees

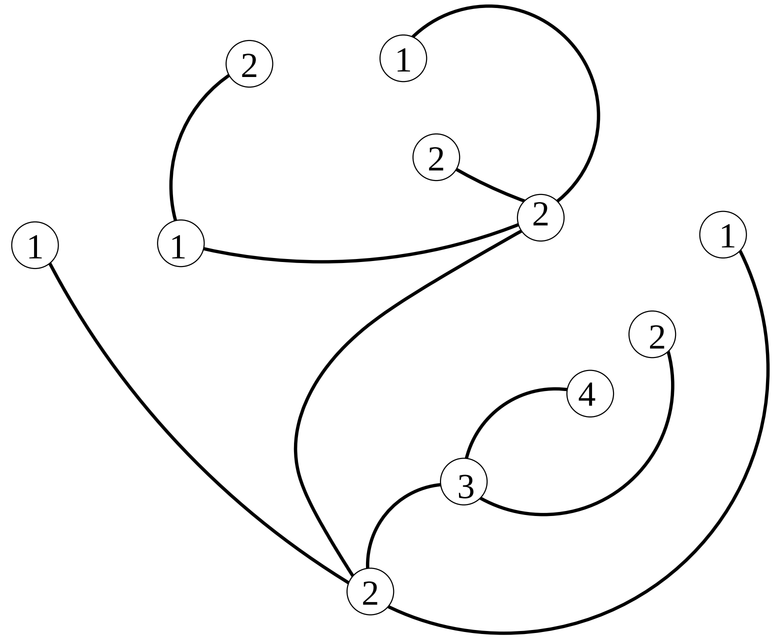
well-labeled:



well-labeled trees

well-labeled:

(i) positive integer labels

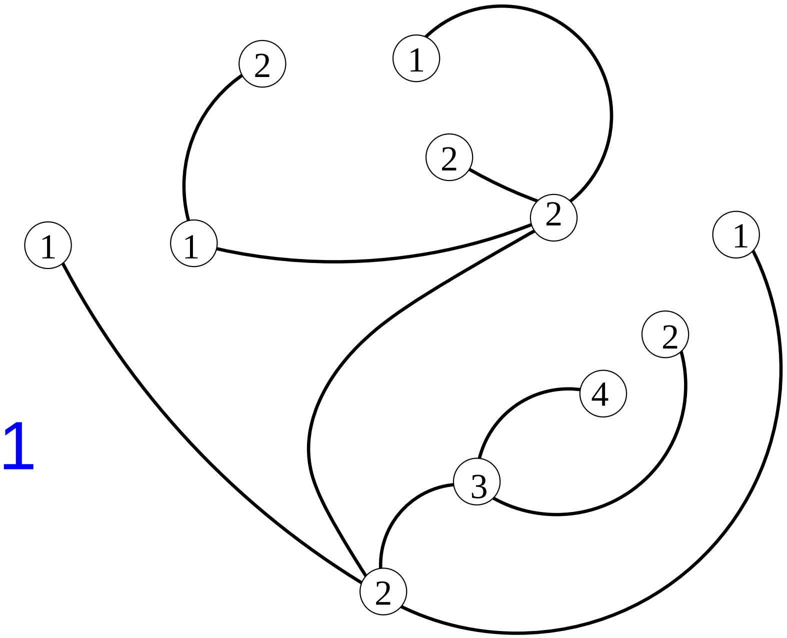


well-labeled trees

well-labeled:

(i) positive integer labels

(ii) there is at least one label 1



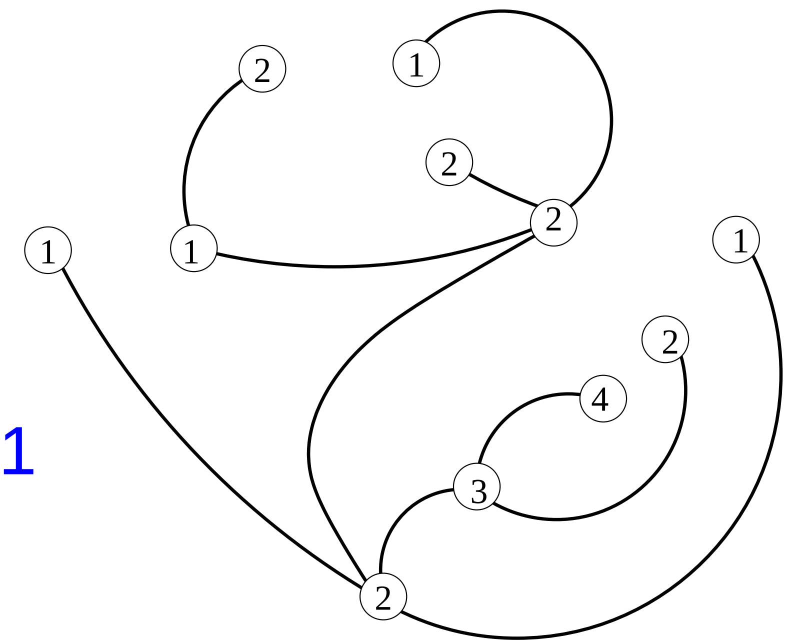
well-labeled trees

well-labeled:

(i) positive integer labels

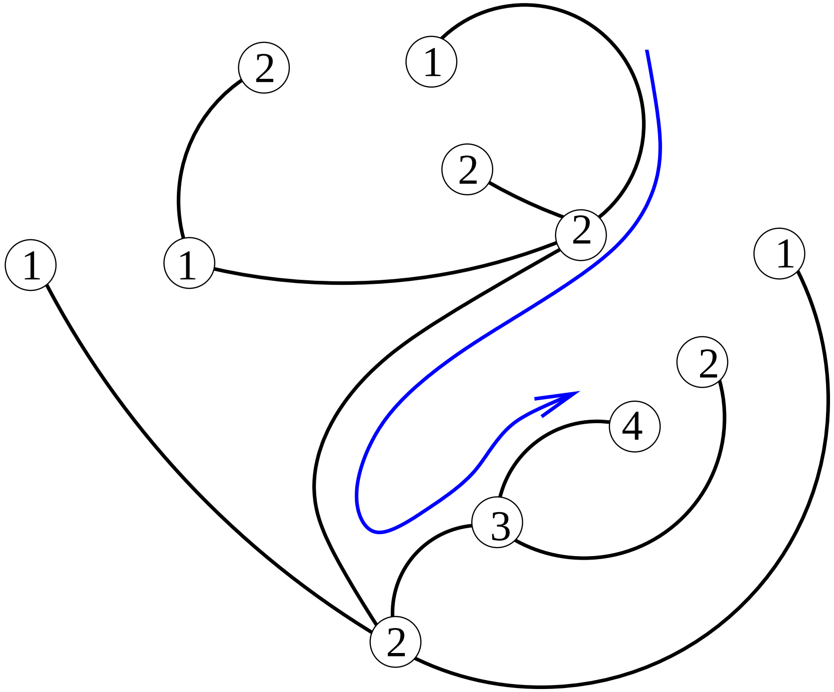
(ii) there is at least one label 1

(iii) labels vary by at most 1
between neighbors



well-labeled trees → quadrangulations

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



well-labeled trees → quadrangulations

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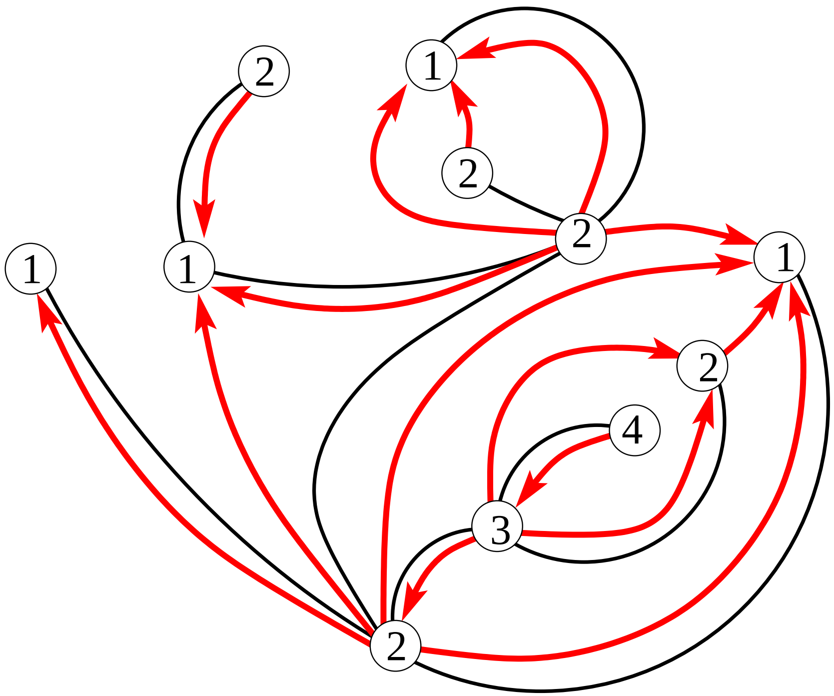
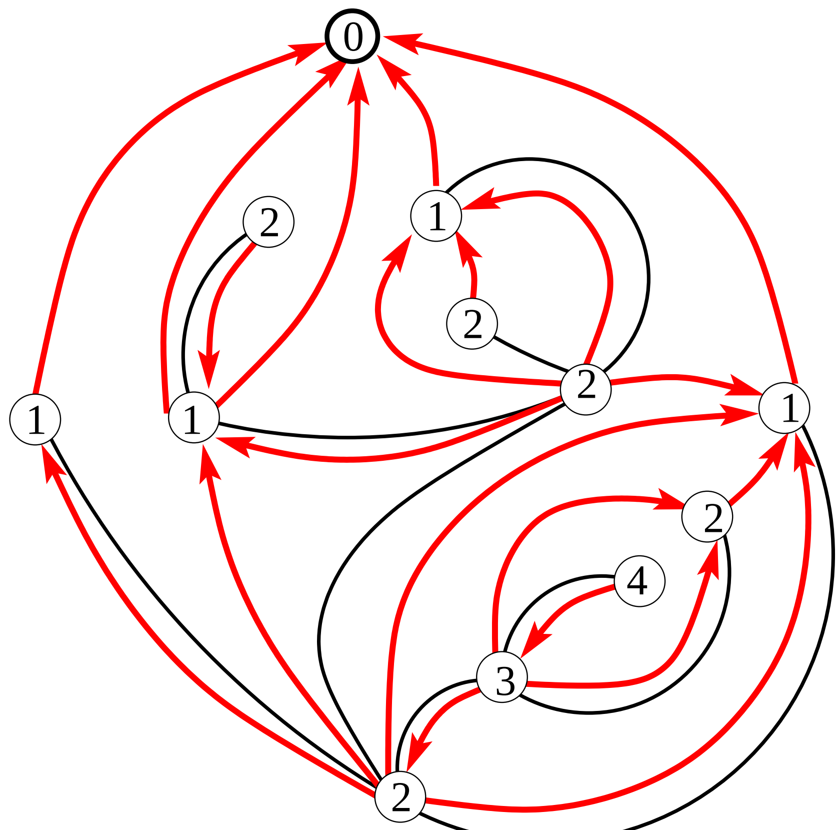


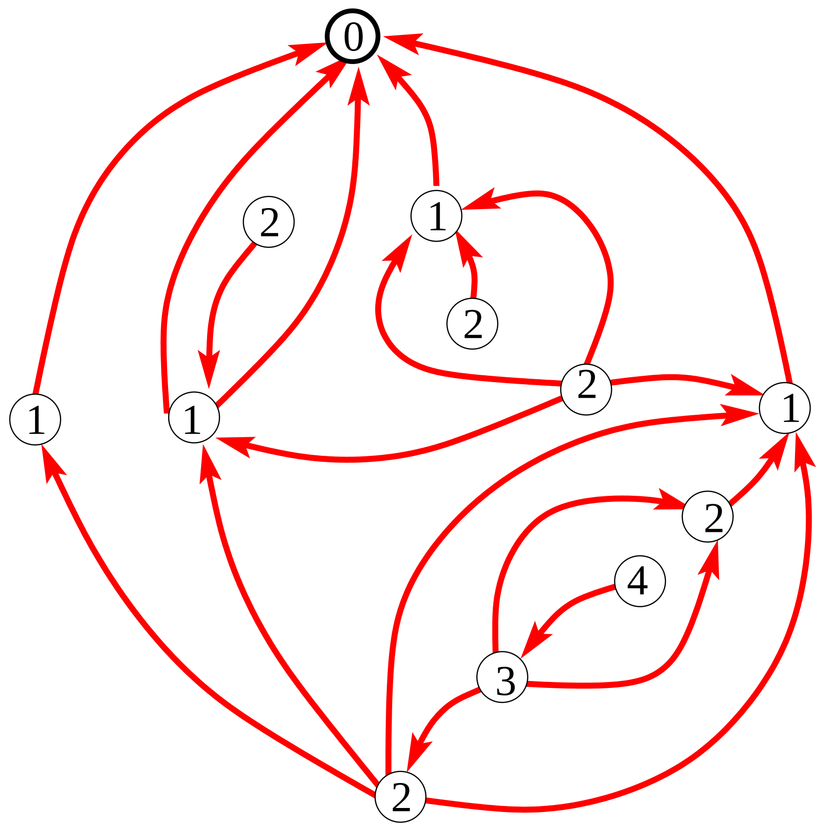
well-labeled trees → quadrangulations

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



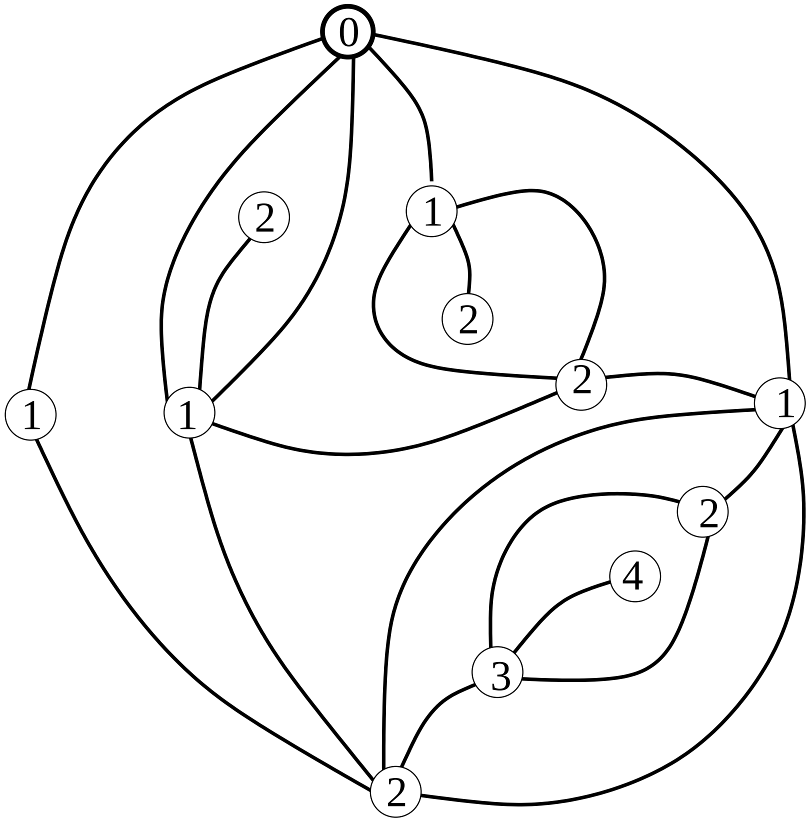
well-labeled trees → quadrangulations

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



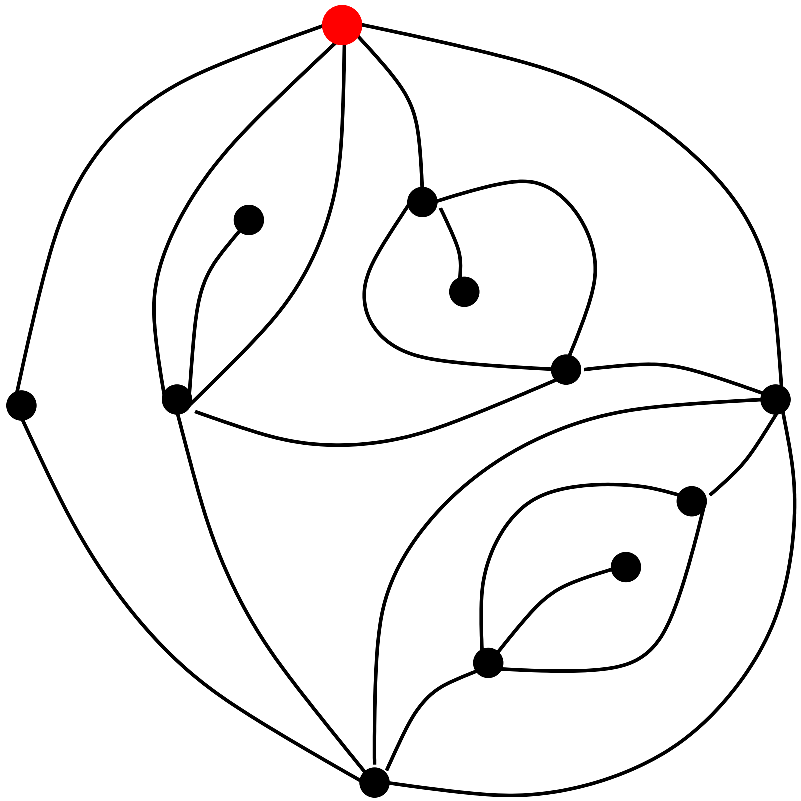
well-labeled trees → quadrangulations

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



well-labeled trees → quadrangulations

going clockwise around the tree, each corner ℓ has a successor $\ell - 1$



map-tree correspondence

pointed planar quadrangulation

(with an origin vertex)

well-labeled tree

map-tree correspondence

pointed planar quadrangulation *(with an origin vertex)* — well-labeled tree

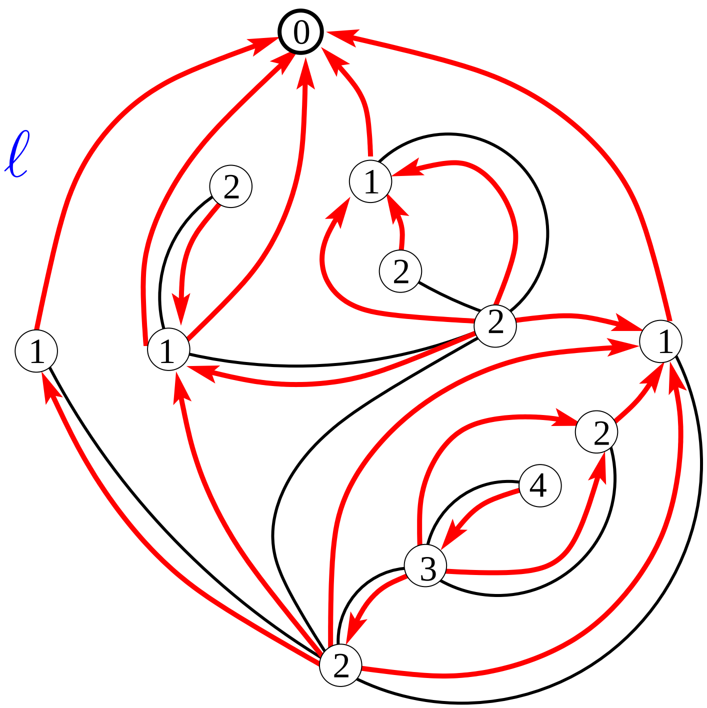
vertices at distance ℓ from the origin — vertices labeled ℓ

map-tree correspondence

pointed planar quadrangulation *(with an origin vertex)* | well-labeled tree

vertices at distance ℓ from the origin | vertices labeled ℓ

edges $(\ell - 1) \leftrightarrow \ell$ | corner labeled ℓ



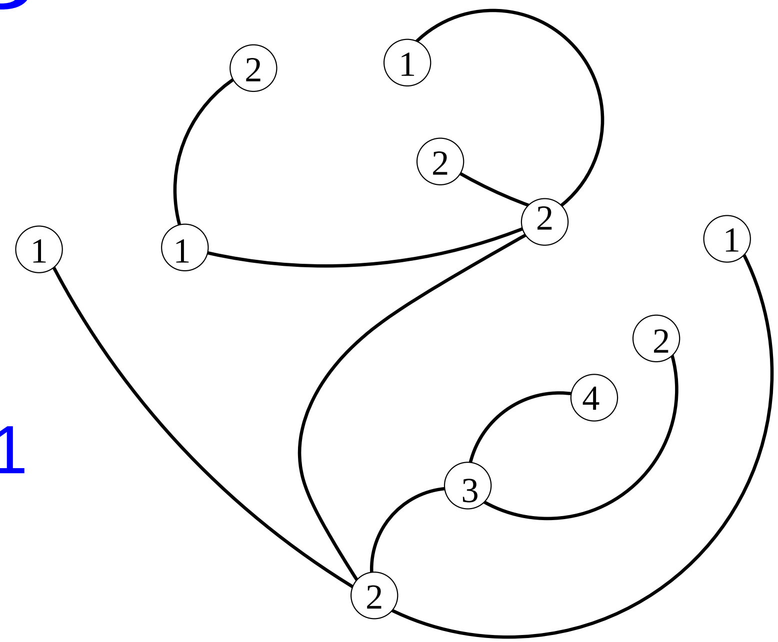
map-tree correspondence

pointed planar quadrangulation *(with an origin vertex)*	well-labeled tree
vertices at distance ℓ from the origin	vertices labeled ℓ
edges $(\ell - 1) \leftrightarrow \ell$	corner labeled ℓ
marked edge $(\ell - 1) \leftrightarrow \ell$	planted at a corner labeled ℓ

map-tree correspondence

pointed planar quadrangulation *(with an origin vertex)*	well-labeled tree
vertices at distance ℓ from the origin	vertices labeled ℓ
edges $(\ell - 1) \leftrightarrow \ell$	corner labeled ℓ
marked edge $(\ell - 1) \leftrightarrow \ell$	planted at a corner labeled ℓ
rooted planar quadrangulation *(with a root edge)*	well-labeled tree planted at a corner labeled 1

generating functions



well-labeled:

(i) positive integer labels

(ii) there is at least one label 1

(iii) labels vary by at most 1
between neighbors

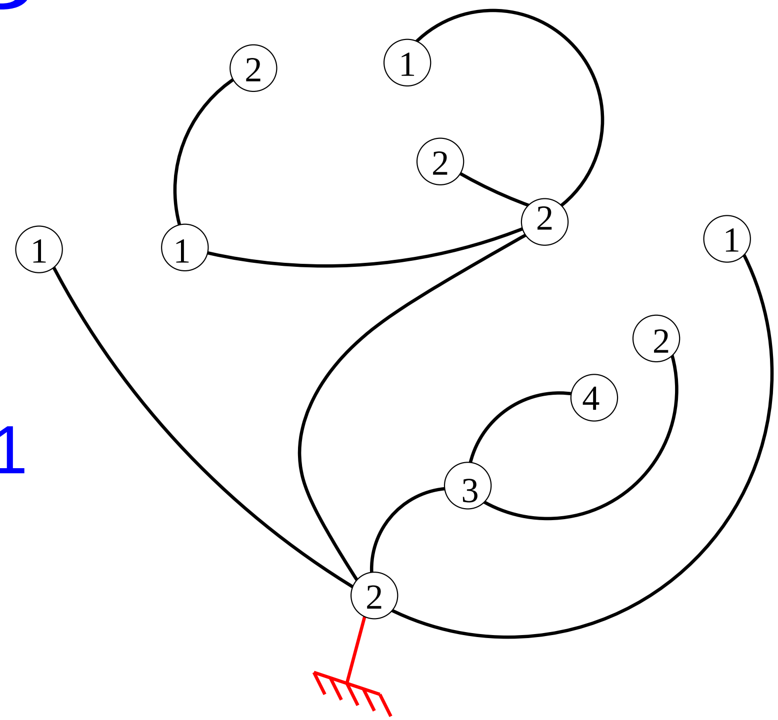
generating functions

well-labeled:

(i) positive integer labels

(ii) ~~there is at least one label 1~~

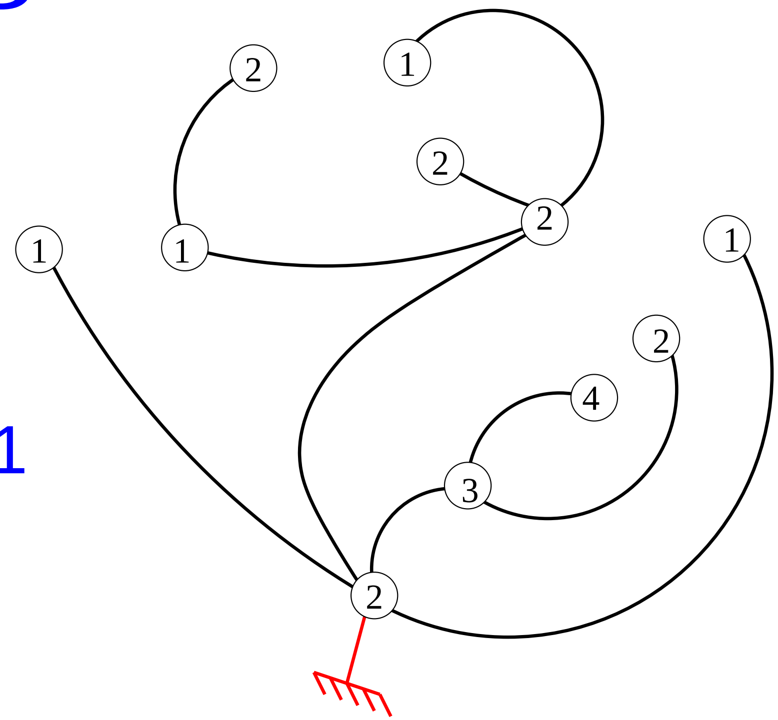
(iii) labels vary by at most 1
between neighbors



gen. func. for trees planted at a corner with label ℓ with a weight g per edge:

- without cond. (ii) $\rightarrow R_\ell(g)$

generating functions



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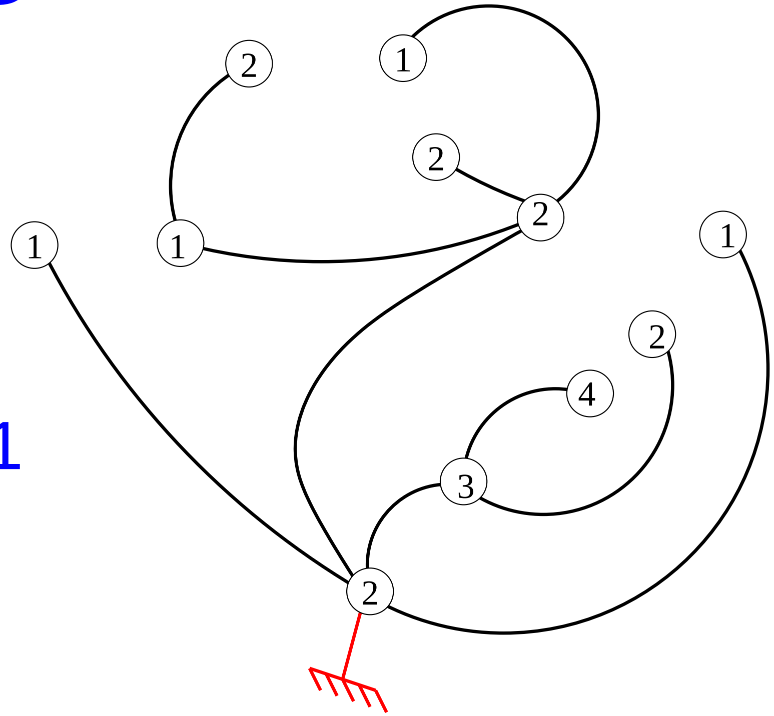
gen. func. for trees planted at a corner with label ℓ with a weight g per edge:

- without cond. (ii) $\rightarrow R_\ell(g)$
- with cond. (ii) $\rightarrow G_\ell(g) = R_\ell(g) - R_{\ell-1}(g)$, $R_0 \equiv 0$

generating functions



well-labeled:

(i) positive integer labels

(ii) there is at least one label 1

(iii) labels vary by at most 1
between neighbors

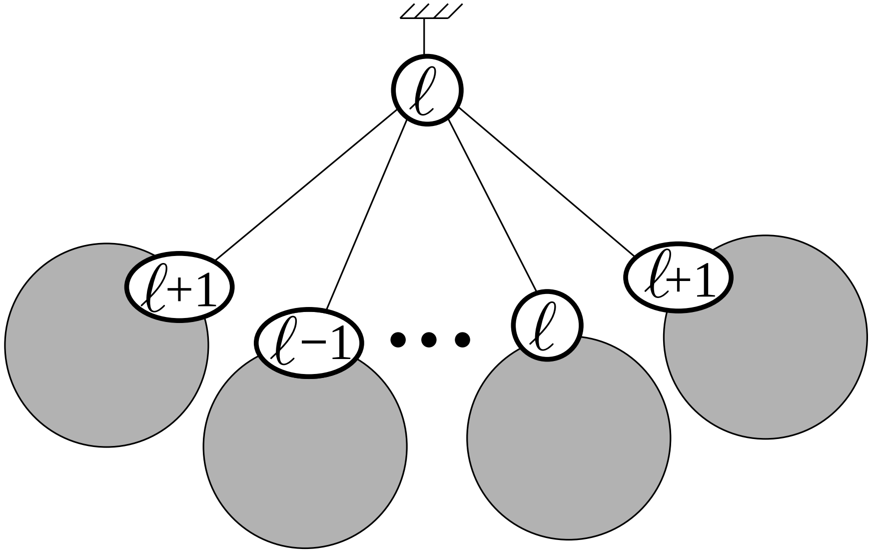
gen. func. for trees planted at a corner with label ℓ with a weight g per edge:

- without cond. (ii) $\rightarrow R_\ell(g)$
- with cond. (ii) $\rightarrow G_\ell(g) = R_\ell(g) - R_{\ell-1}(g)$, $\quad R_0 \equiv 0$

$\rightarrow G_1 = R_1$: gen. func. for rooted planar quadrangulations

recursion relations

$$R_\ell = \frac{1}{1 - g(R_{\ell+1} + R_\ell + R_{\ell-1})}$$



with $R_0 = 0$.

$R_\ell \stackrel{\ell\to\infty}{\longrightarrow} R$ with $R = 1/(1 - 3\,g\,R)$, namely

$$R = \frac{1 - \sqrt{1 - 12\,g}}{6g}$$

R is the gen. func. of quadrangulations with an origin and a marked edge

$$R|_{g^n} = 3^n \operatorname{cat}(n)$$

with

$$\operatorname{cat}(n) \equiv \frac{1}{n+1}\binom{2n}{n}$$

$$\vec{Q}(n) = \frac{2}{n+2} \times 3^n \operatorname{cat}(n)$$

$$Q^{\bullet}(n) = \frac{1}{2n} \times 3^n \operatorname{cat}(n)$$

$$Q(n) = \frac{1}{2n(n+2)} \times 3^n \operatorname{cat}(n)$$

solution

$$R_\ell = R\,\frac{(1-x^\ell)(1-x^{\ell+3})}{(1-x^{\ell+1})(1-x^{\ell+2})} = R\,\frac{[\ell][\ell+3]}{[\ell+1][\ell+2]}$$

where

$$[\ell] \equiv \frac{1-x^\ell}{1-x}$$

and where $x + x^{-1} + 1 = 1/(g\,R^2)$, namely

$$x = \frac{1-24g-\sqrt{1-12g}+\sqrt{6}\sqrt{72g^2+6g+\sqrt{1-12g}-1}}{2(6g+\sqrt{1-12g}-1)}$$

statistics of the distance between two points

two-point function

a marked origin $+$ a marked vertex at distance $m = d_{12}$
$\Leftrightarrow$ well-labeled tree with a marked vertex with label m

two-point function

a marked origin $+$ a marked vertex at distance $m = d_{12}$
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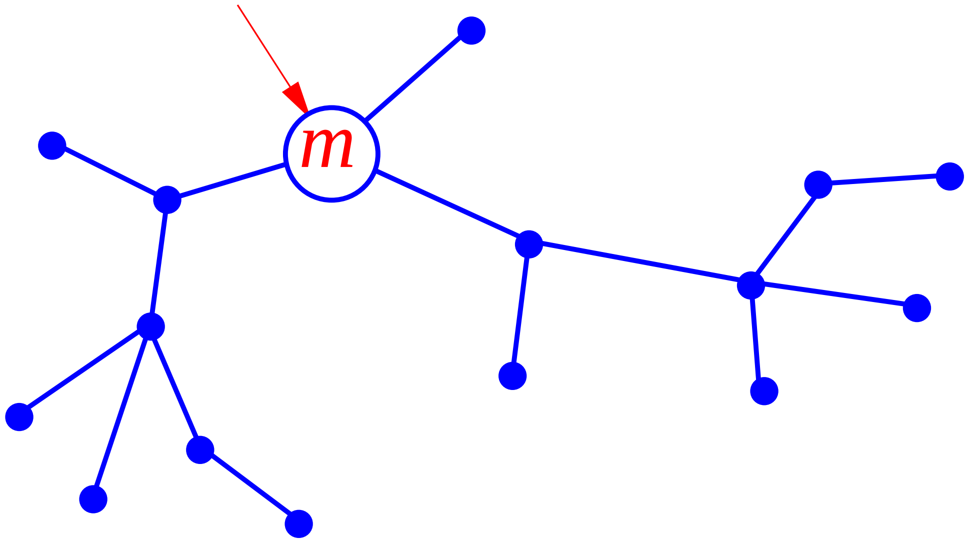
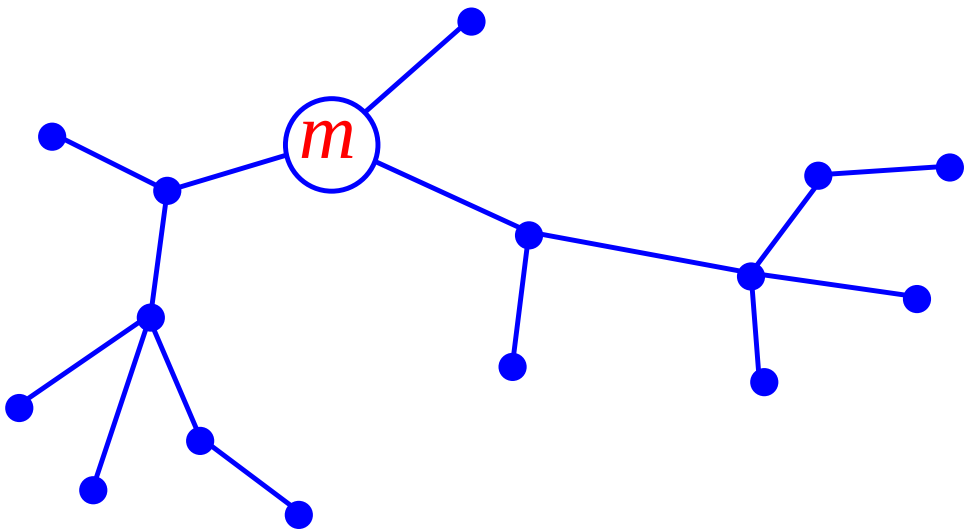


◇ marked corner with label m: R_m

two-point function

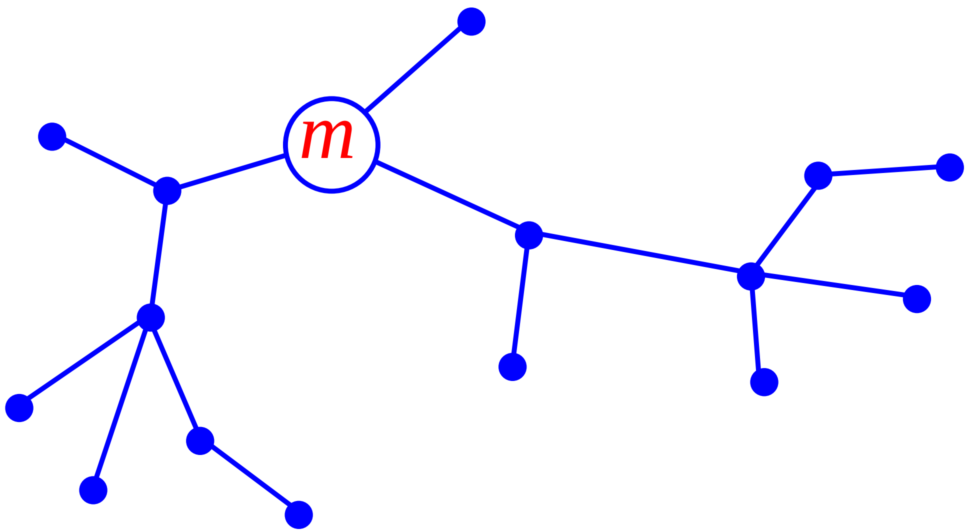
a marked origin $+$ a marked vertex at distance $m = d_{12}$
$\Leftrightarrow$ well-labeled tree with a marked vertex with label m



◇ marked vertex with label m: $L_m = \log R_m$

two-point function

a marked origin $+$ a marked vertex at distance $m = d_{12}$
$\Leftrightarrow$ well-labeled tree with a marked vertex with label m

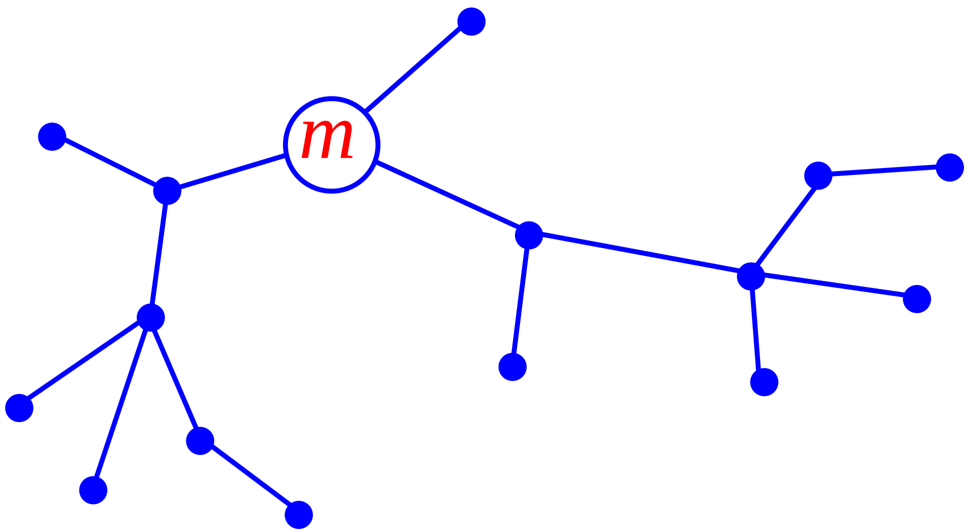


◇ marked vertex with label m: $L_m = \log R_m$
◇ impose $\min_{v \in \text{ tree}} \ell(v) \geq 1$

two-point function

a marked origin $+$ a marked vertex at distance $m = d_{12}$

$\Leftrightarrow$ well-labeled tree with a marked vertex with label m



- ◇ marked vertex with label m: $L_m - L_{m-1} = \log(R_m/R_{m-1})$
- ◇ impose $\min_{v\in \text{ tree}} \ell(v) = 1$

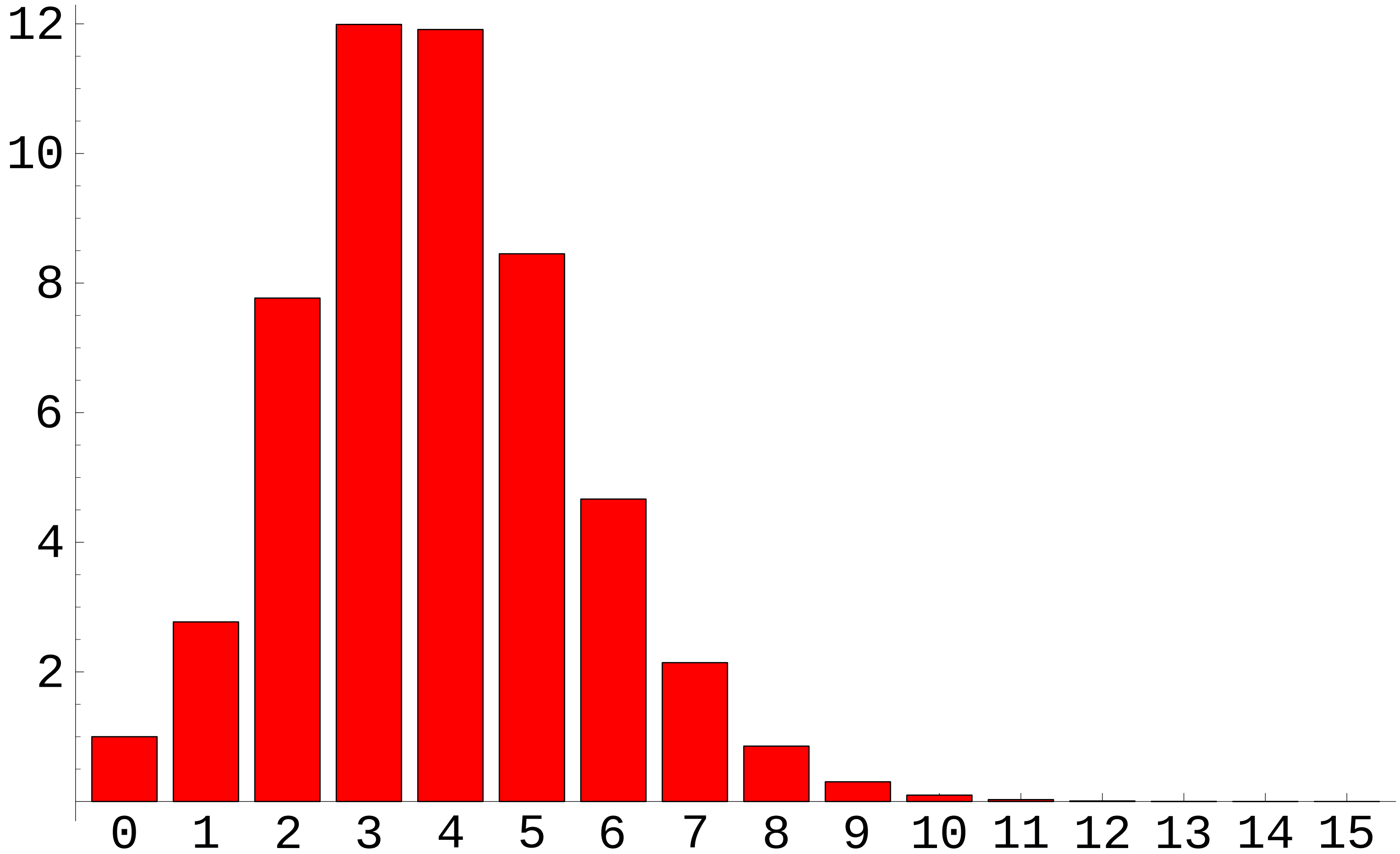
two-point function

a marked origin $+$ a marked vertex at distance $m = d_{12}$
$\Leftrightarrow$ well-labeled tree with a marked vertex with label m

$$Q_{d_{12}}(g) = \begin{cases} \log\left(\frac{([d_{12}])^2[d_{12}+3]}{[d_{12}-1]([d_{12}+2])^2}\right) & \text{for } d_{12} \geq 2 \\ \log\left(R\frac{[1][4]}{[2][3]}\right) & \text{for } d_{12} = 1 \end{cases}$$

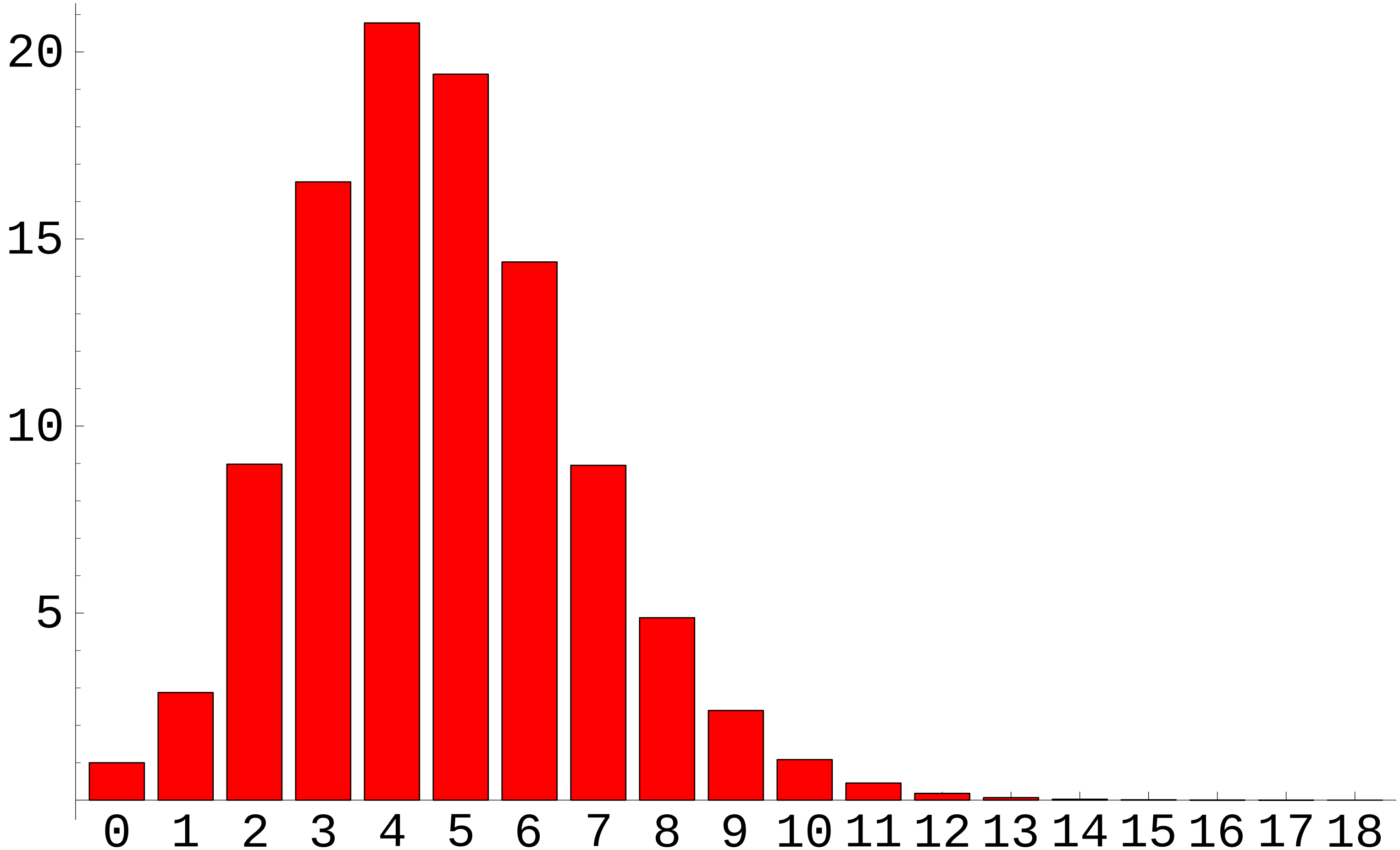
$\equiv$ generating function for doubly-pointed quadrangulations whose two marked (and distinguished) vertices are at distance d_{12} from each other

distance profile



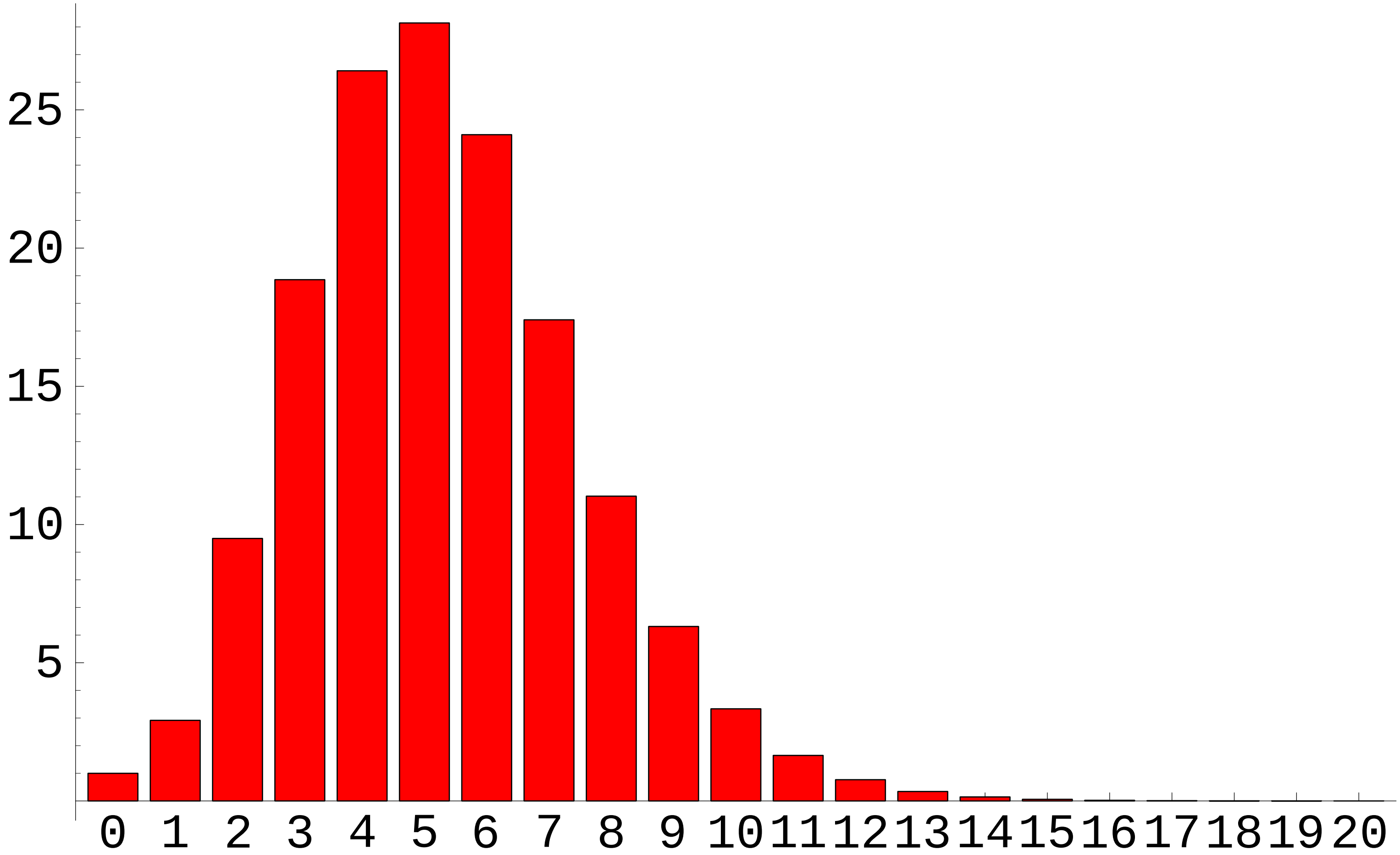
$n = 50$

distance profile



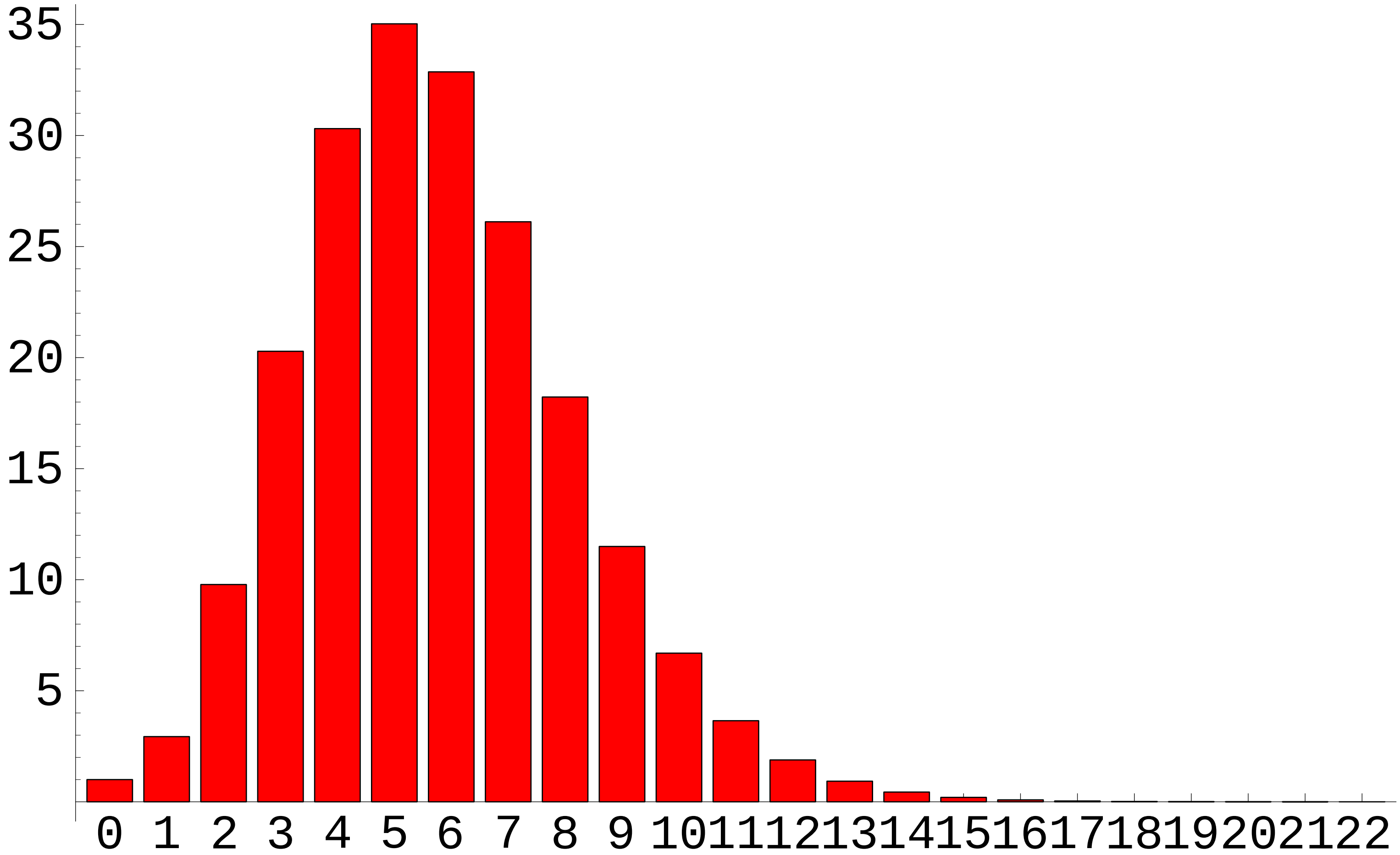
$n = 100$

distance profile



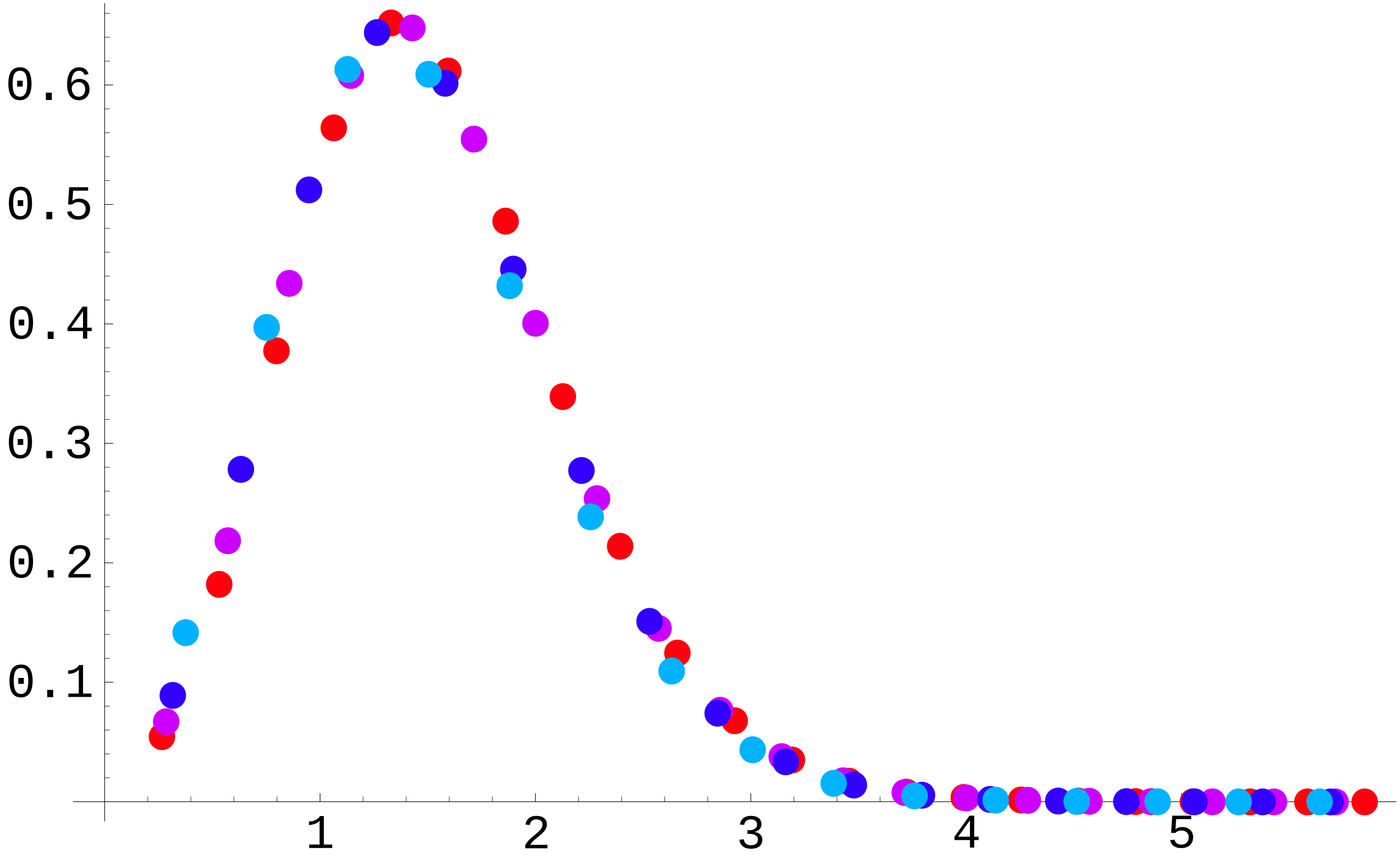
$n = 150$

distance profile



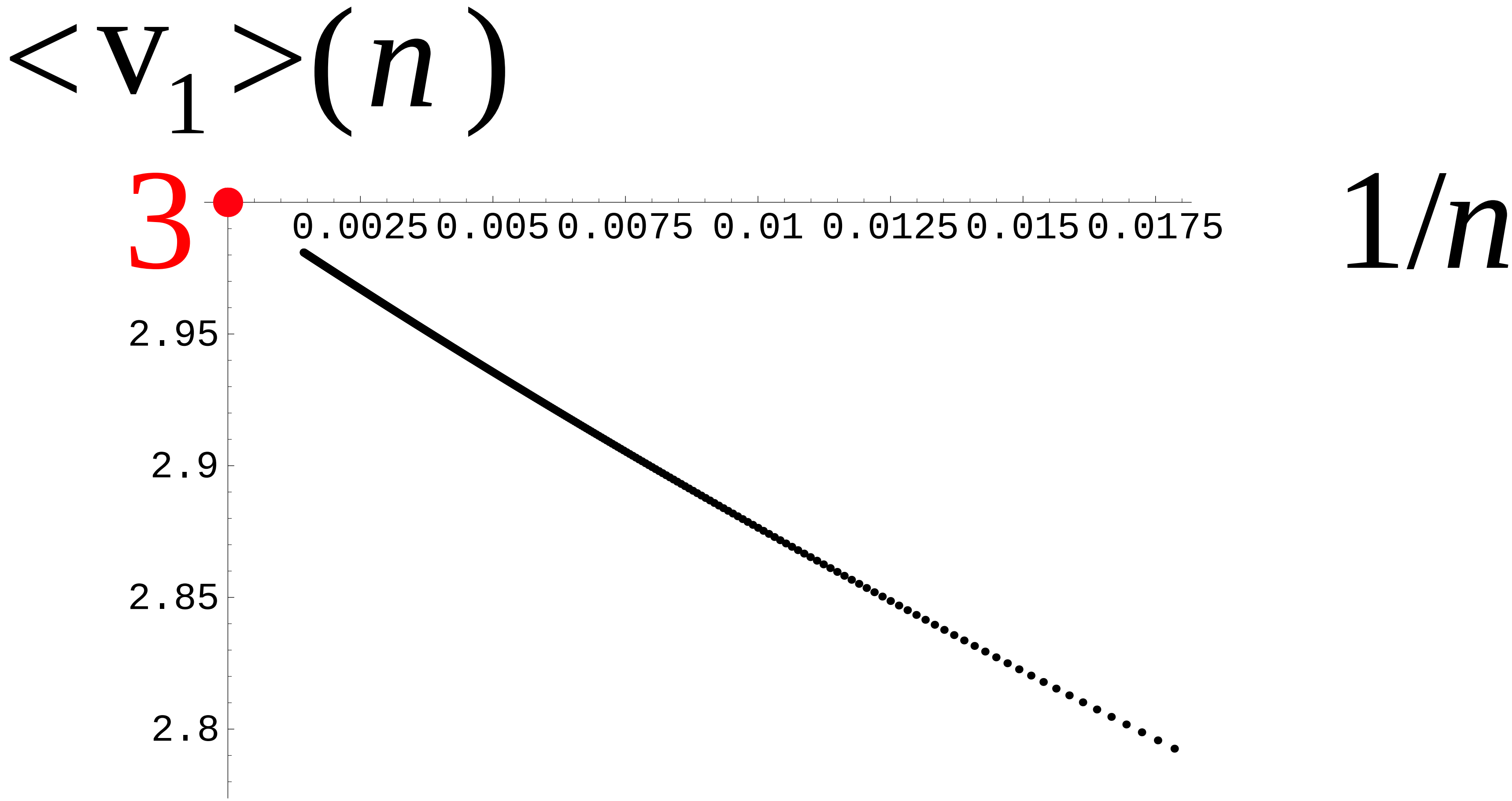
$n = 200$

distance profile



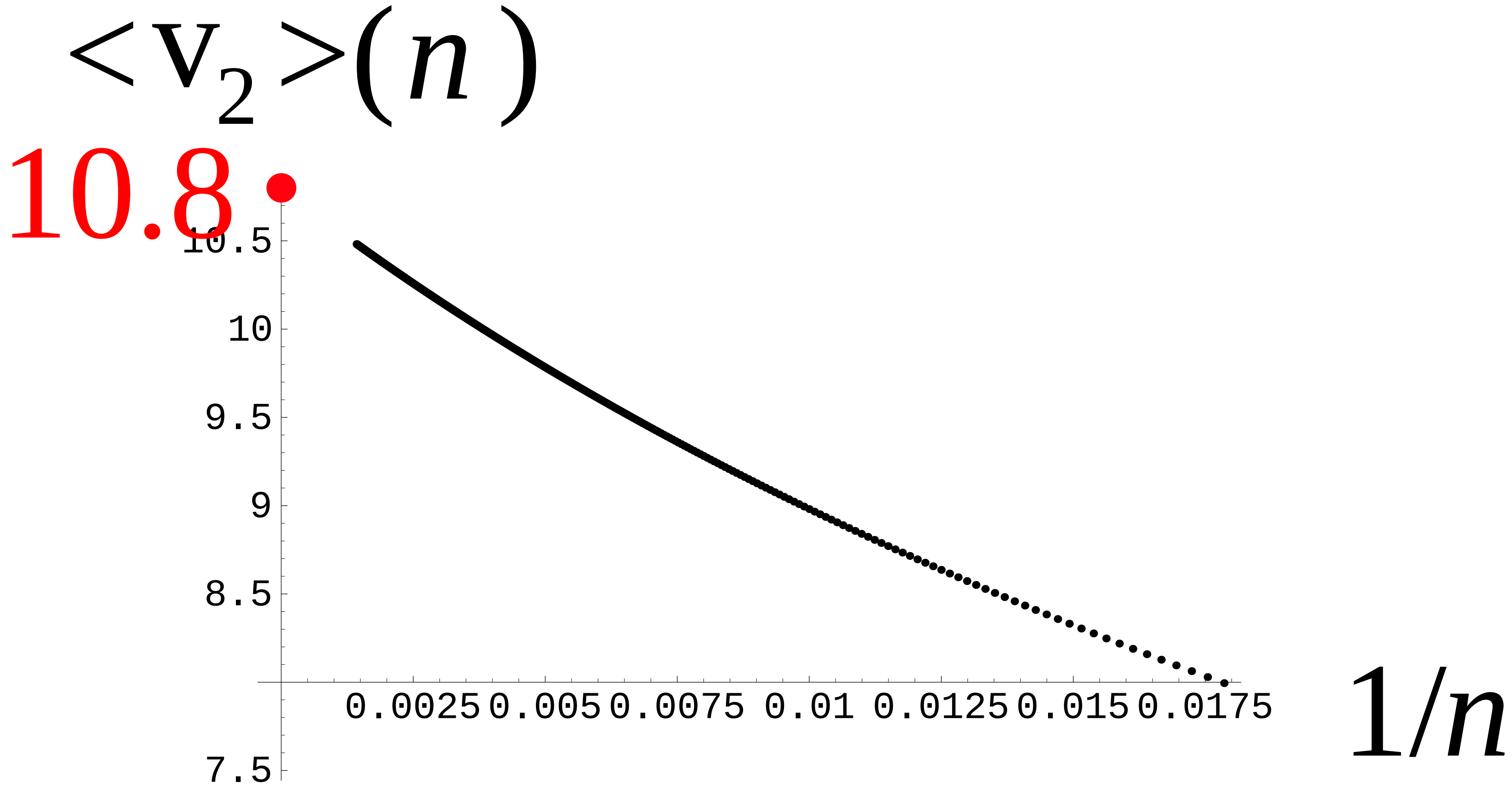
rescaled profiles

local limit



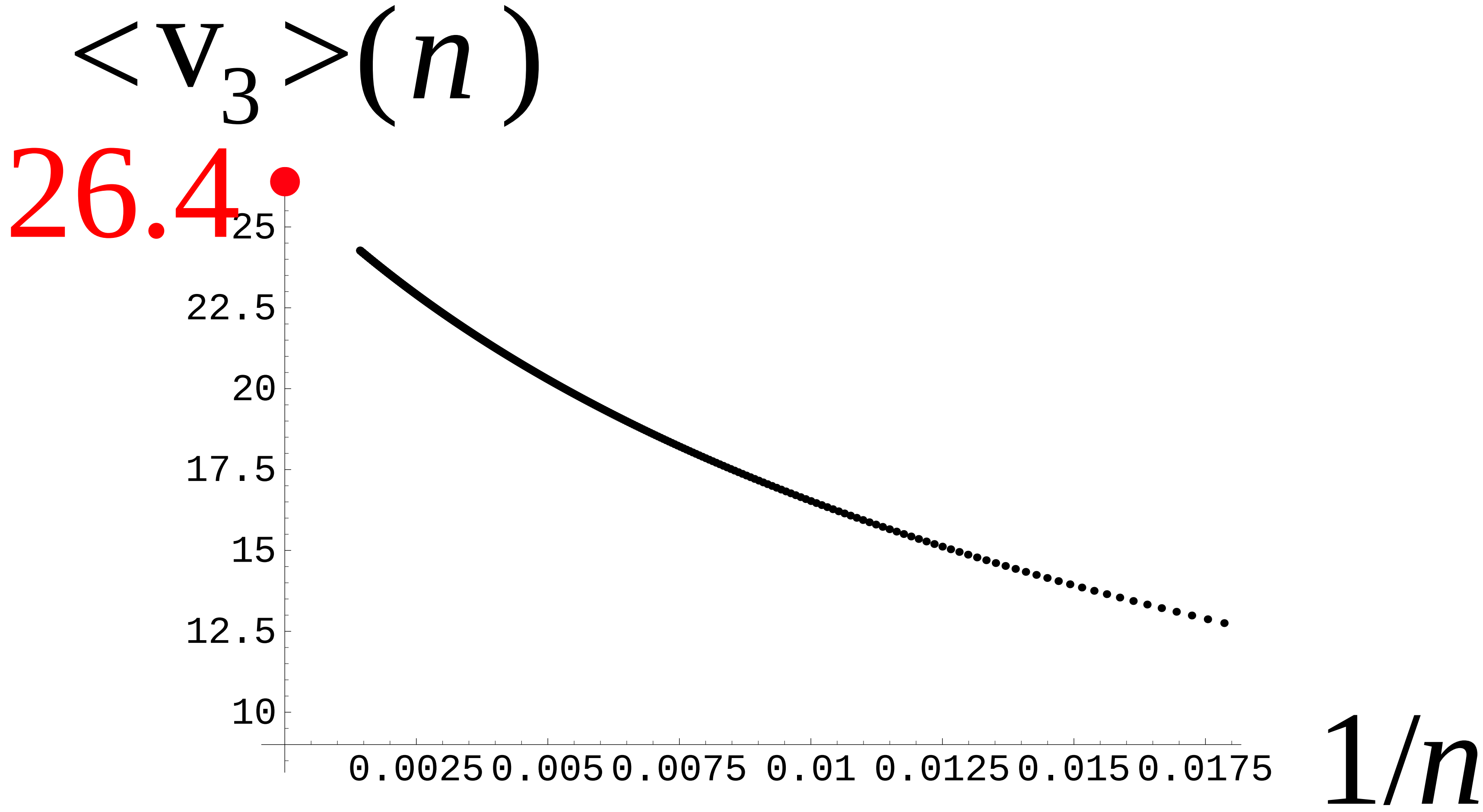
immediate neighbors

local limit



next-nearest neighbors

local limit



next-next-nearest neighbors

limit laws

for large maps

local limit

write

$$g = \frac{1}{12}(1 - \epsilon^2)$$

$$R_\ell = \alpha_\ell + \beta_\ell\, \epsilon + \gamma_\ell\, \epsilon^2 + \delta_\ell \epsilon^3 + \cdots$$

$$\alpha_\ell = \frac{2\ell(\ell+3)}{(\ell+1)(\ell+2)} \quad \beta_\ell = 0 \quad \gamma_\ell = -\frac{\ell(\ell+3)(3\ell^2+9\ell-2)}{5(\ell+1)(\ell+2)}$$

$$\delta_\ell = \frac{\ell(\ell+3)(5\ell^4+30\ell^3+59\ell^2+42\ell+4)}{35(\ell+1)(\ell+2)}$$

and the leading singularity (odd power in ϵ) gives

$$R_\ell|_{g^n} \sim \frac{12^n}{\sqrt{\pi} n^{5/2}} \frac{3}{4}\, \delta_\ell$$

In[1]:= $g = \frac{1}{12}\left(1-\epsilon^2\right);\ R = \frac{2}{1+\epsilon};\ x = X\ /.\ \text{Solve}\left[X + \frac{1}{X} + 1 == \frac{1}{g R^2}, X\right][[1]];$

$Rl := R\,\frac{(1-x^l)\,(1-x^{l+3})}{(1-x^{l+1})\,(1-x^{l+2})};$

```
Simplify[Series[Rl, {ε, 0, 3}]]
```

Out[2]= $\frac{2\,l\,(3+l)}{2+3\,l+l^2} - \frac{\left(l\,(-6+25\,l+18\,l^2+3\,l^3)\right)\epsilon^2}{5\,(2+3\,l+l^2)} + \frac{l\,(12+130\,l+219\,l^2+149\,l^3+45\,l^4+5\,l^5)\,\epsilon^3}{35\,(2+3\,l+l^2)} + O[\epsilon]^4$

```
In[3]:= Factor[CoefficientList[Normal[%], ε]]
```

Out[3]= $\left\{\frac{2\,l\,(3+l)}{(1+l)\,(2+l)},\ 0,\ -\frac{l\,(3+l)\,(-2+9\,l+3\,l^2)}{5\,(1+l)\,(2+l)},\ \frac{l\,(3+l)\,(4+42\,l+59\,l^2+30\,l^3+5\,l^4)}{35\,(1+l)\,(2+l)}\right\}$

In[4]:= $\delta[l\_] := \frac{l\,(3+l)\,(4+42\,l+59\,l^2+30\,l^3+5\,l^4)}{35\,(1+l)\,(2+l)};\ \text{Factor}\left[\frac{3}{2}\,(\delta[l]-\delta[l-1])\right]$

Out[4]= $\frac{6\,(-1+2\,l+l^2)\,(4+14\,l+27\,l^2+20\,l^3+5\,l^4)}{35\,l\,(1+l)\,(2+l)}$

distance statistics

the average number $\langle e_\ell \rangle$ of edges at distance ℓ *(i.e. $\ell - 1 \leftrightarrow \ell$)* in infinite quadrangulations is

$$\langle e_\ell \rangle = \lim_{n\to\infty} \frac{(R_\ell - R_{\ell-1})|_{g^n}}{R|_{g^n}/(2n)} = \frac{3}{2}(\delta_\ell - \delta_{\ell-1})$$

one gets

$$\langle e_\ell \rangle = \frac{6}{35} \frac{(\ell^2 + 2\ell - 1)(5\ell^4 + 20\ell^3 + 27\ell^2 + 14\ell + 4)}{\ell(\ell+1)(\ell+2)}$$

$$\overset{\ell\to\infty}{\sim} \frac{6}{7}\ell^3$$

$\rightarrow$ fractal dimension $d_F = 4$

NB: $\langle e_1 \rangle = 4$ *obvious from Euler's relation*

$$\log(R_\ell) = \tilde{\alpha}_\ell + \tilde{\beta}_\ell\,\epsilon + \tilde{\gamma}_\ell\,\epsilon^2 + \tilde{\delta}_\ell\epsilon^3 + \cdots$$

$$\tilde{\beta}_\ell = 0 \quad \tilde{\delta}_\ell = \frac{5\ell^4 + 30\ell^3 + 59\ell^2 + 42\ell + 4}{70}$$

and the leading singularity gives

$$\log(R_\ell)|_{g^n} \sim \frac{12^n}{\sqrt{\pi}n^{5/2}}\,\frac{3}{4}\,\tilde{\delta}_\ell$$

the average number $\langle v_\ell \rangle$ of vertices at distance ℓ in infinite quadrangulations is given by

$$\langle v_\ell \rangle = \frac{3}{35} \Big((\ell+1)(5\ell^2 + 10\ell + 2) + \delta_{\ell,1} \Big)$$

$$\overset{i\to\infty}{\sim} \frac{3}{7}\ell^3$$

first values:

$$\langle e_1 \rangle = 4 \qquad \langle e_2 \rangle = 19 \qquad \langle e_3 \rangle = \frac{1234}{25}$$

$$\langle v_1 \rangle = 3 \qquad \langle v_2 \rangle = \frac{54}{5} \qquad \langle v_3 \rangle = \frac{132}{5}$$

scaling limit

take ℓ large as $\ell = u\,\epsilon^{-1/2}$ with u finite $\rightarrow$ scaling function $\mathcal{F}$:

$$R_\ell = 2(1 - \epsilon\,\mathcal{F}(u)) + \mathcal{O}(\epsilon^{3/2})$$

whose small u behavior can be read off the local limit

$$\alpha_\ell = 2 - \frac{4}{u^2}\epsilon + \mathcal{O}(\epsilon^{3/2}),\ \gamma_\ell \epsilon^2 = -\frac{3u^2}{5}\epsilon + \mathcal{O}(\epsilon^{3/2}),\ \delta_\ell \epsilon^3 = \frac{u^4}{7}\epsilon + \mathcal{O}(\epsilon^{3/2})$$

$$R_\ell = 2 - \epsilon\left(\frac{4}{u^2} + \frac{3u^2}{5} - \frac{u^4}{7} + \mathcal{O}(u^5)\right) + \mathcal{O}(\epsilon^{3/2})$$

from the exact solution, one finds

$$\mathcal{F}(u) = 1 + \frac{3}{\sinh^2\left(\sqrt{3/2}\,u\right)}$$

scaling limit (fixed n)

by a change of variables $g \to V \equiv gR$, we have

$$R_\ell|_{g^n} = \oint \frac{dg}{2i\pi g^{n+1}} R_\ell(g) = \oint \frac{dV(1-6V)}{2i\pi(V(1-3V))^{n+1}} R_\ell(g)$$

for large n and in the scaling limit

$$\ell = r n^{1/4}$$

do a saddle point calculation

$$V = \frac{1}{6}\left(1 + \mathrm{i}\frac{\xi}{\sqrt{n}}\right), \quad g = \frac{1}{12}\left(1 + \frac{\xi^2}{n}\right)$$

use previous formulas with

$$\epsilon = \frac{-\mathrm{i}\,\xi}{\sqrt{n}} \qquad u = r\,\sqrt{-\mathrm{i}\,\xi}$$

$$R_\ell = 2\left(1 + \frac{\mathrm{i}\xi}{\sqrt{n}}\mathcal{F}(r\sqrt{-\mathrm{i}\xi})\right) \text{ where } \mathcal{F}(u) = 1 + \frac{3}{\sinh^2(\sqrt{3/2}\,u)}$$

hence the large n limit

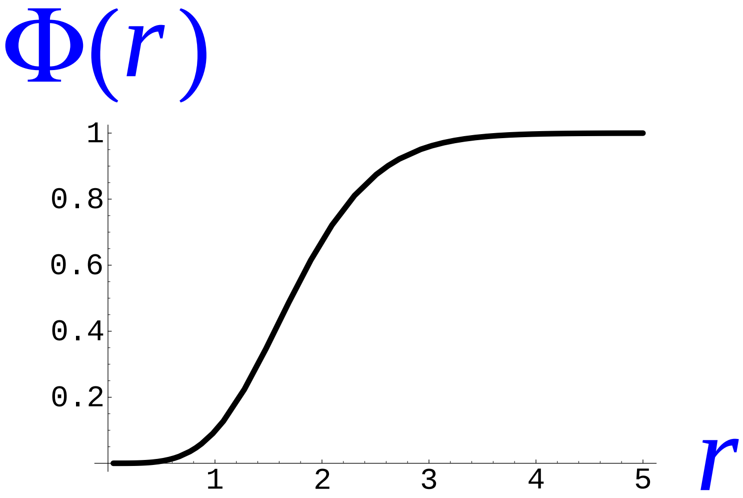
$$R_\ell|_{g^n} \sim 2\frac{12^n}{\pi n^{3/2}} \int_{-\infty}^{\infty} d\xi \xi^2 e^{-\xi^2} \left(\mathcal{F}\left(r\sqrt{-\mathrm{i}\xi}\right)\right)$$

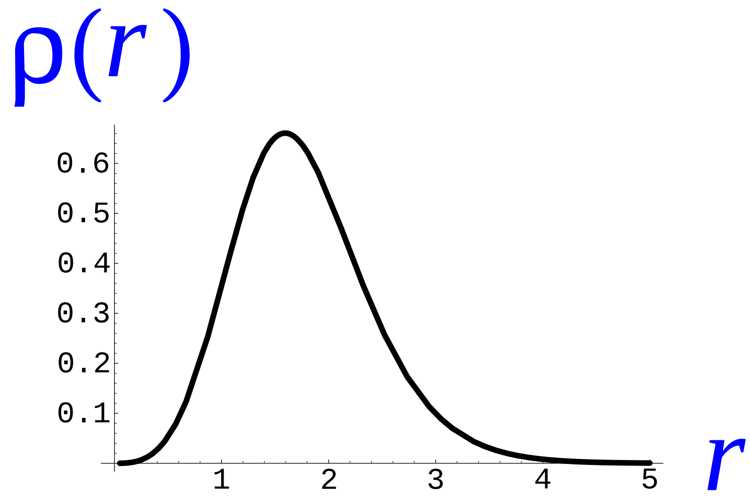
probability $\Phi(r)$ for a point (vertex or edge) to be at geodesic distance less than r:

$$\Phi(r) = \frac{4}{\sqrt{\pi}} \int_0^\infty d\xi \xi^2 e^{-\xi^2} \left\{ 1 - 6 \frac{1 - \cosh(r\sqrt{3\xi}) \cos(r\sqrt{3\xi})}{\left(\cosh(r\sqrt{3\xi}) - \cos(r\sqrt{3\xi})\right)^2} \right\}$$

probability density $\rho(r)$ for a point (vertex or edge) to be at geodesic distance r

$$\rho(r) = \frac{d\Phi(r)}{d\,r}$$



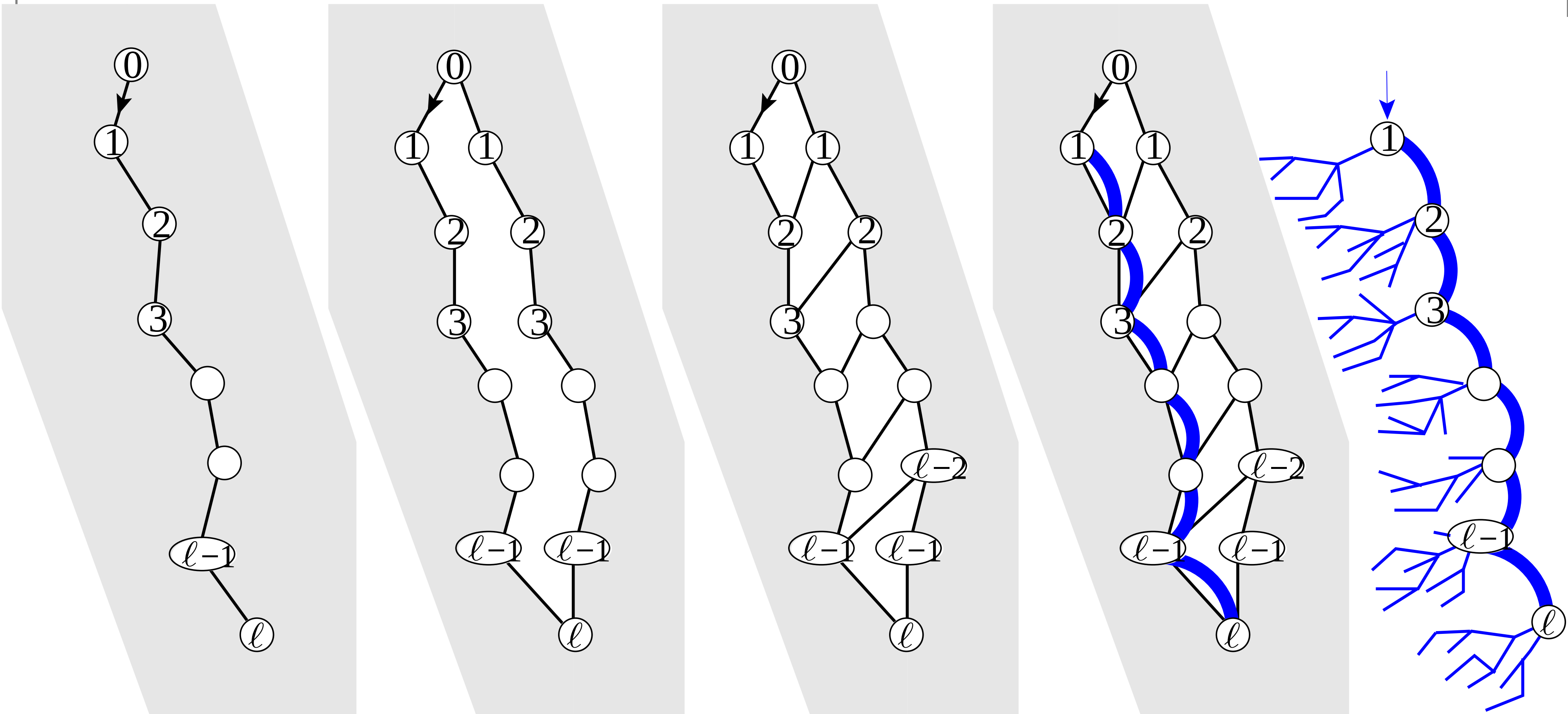


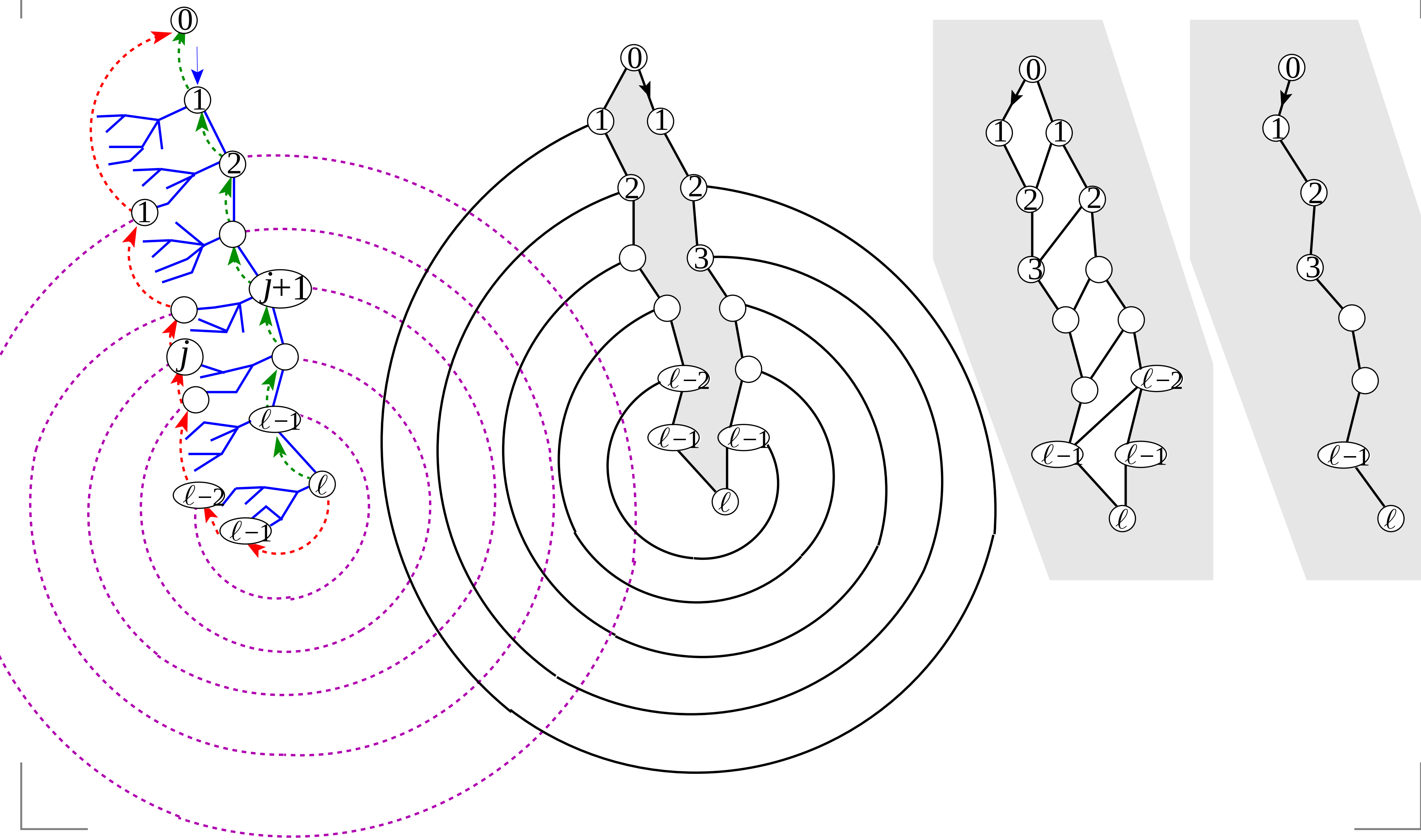
$$\rho(r) \overset{r\to 0}{\sim} \frac{3}{7} r^3, \quad \rho(r) \overset{r\to\infty}{\sim} \exp\left(-\frac{3}{4}\, 3^{2/3} r^{4/3}\right)$$

in agreement with $\langle v_\ell \rangle$ and Fisher's law $\delta = \frac{4}{3} = \frac{1}{1-\nu}$ with $\nu = \frac{1}{4} = \frac{1}{d_F}$

statistics of geodesics

quadrang. with a marked geodesic

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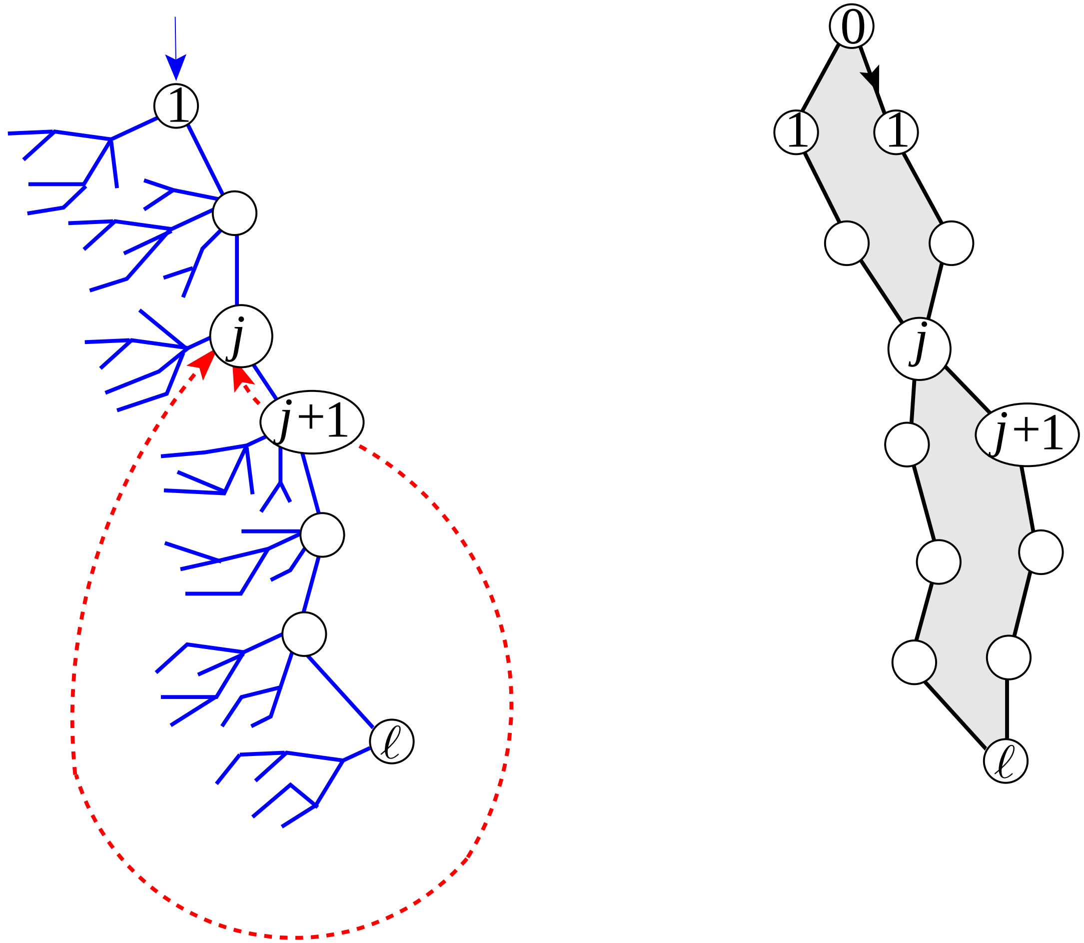
3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j
j+1
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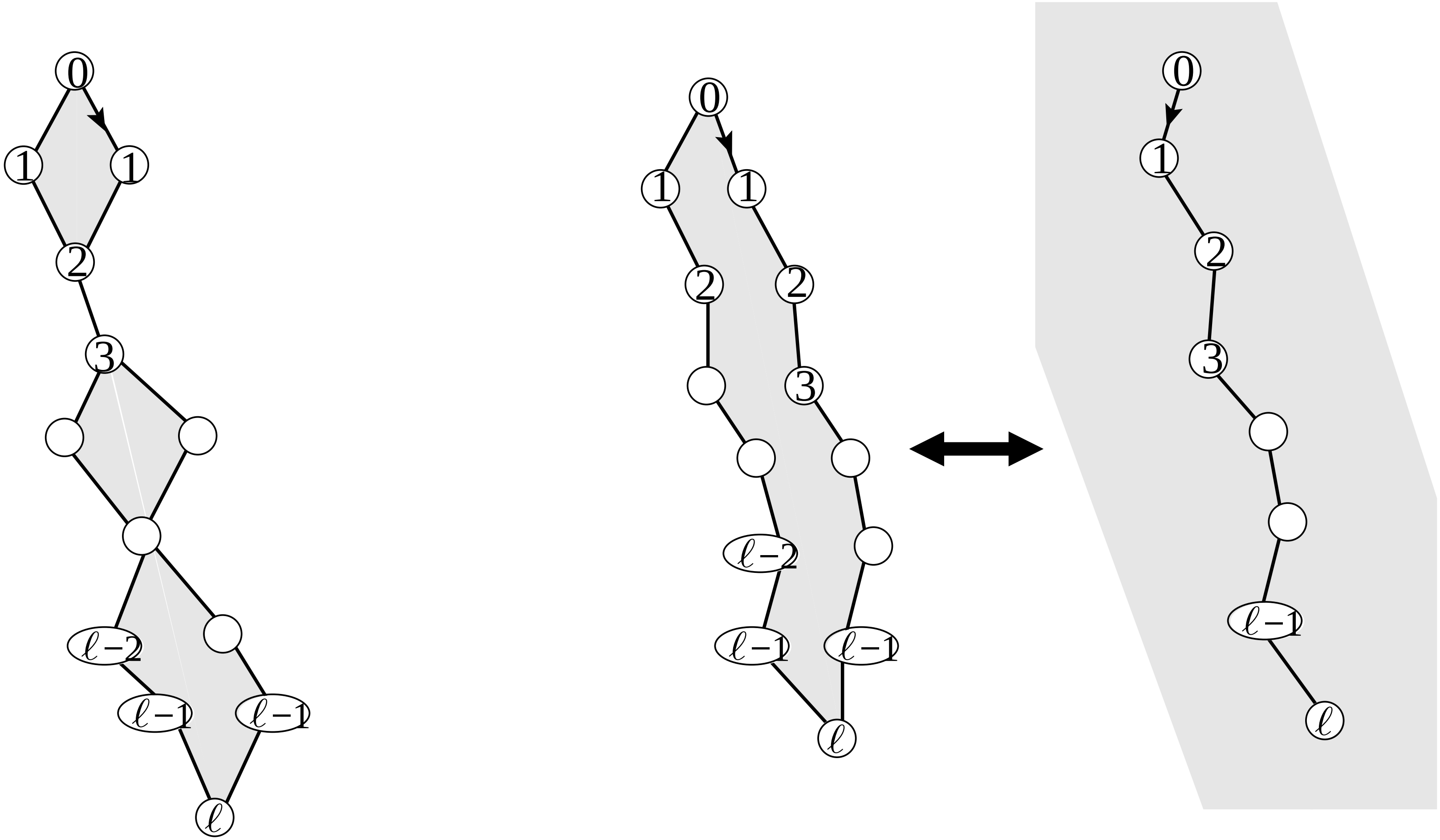
j
j+1
ℓ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0
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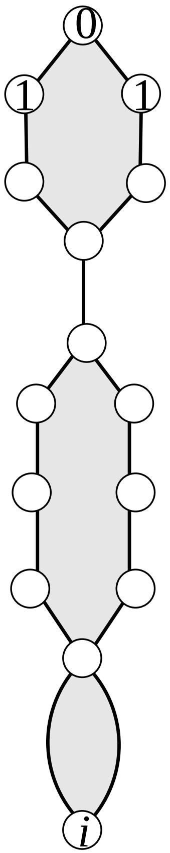
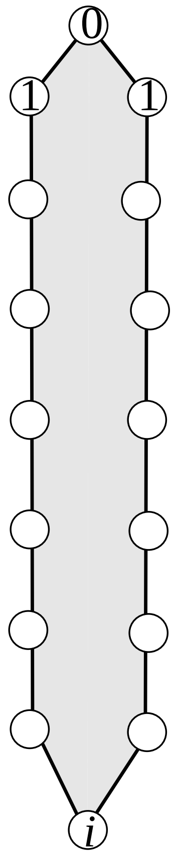
ℓ

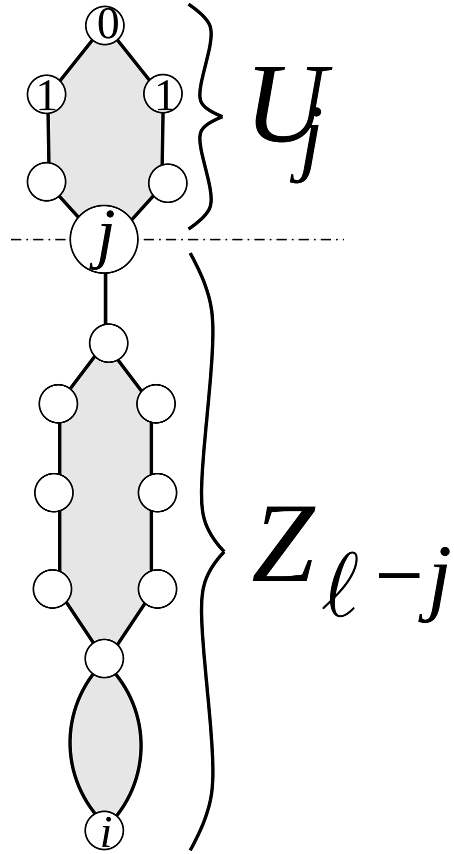
$Z_\ell =$



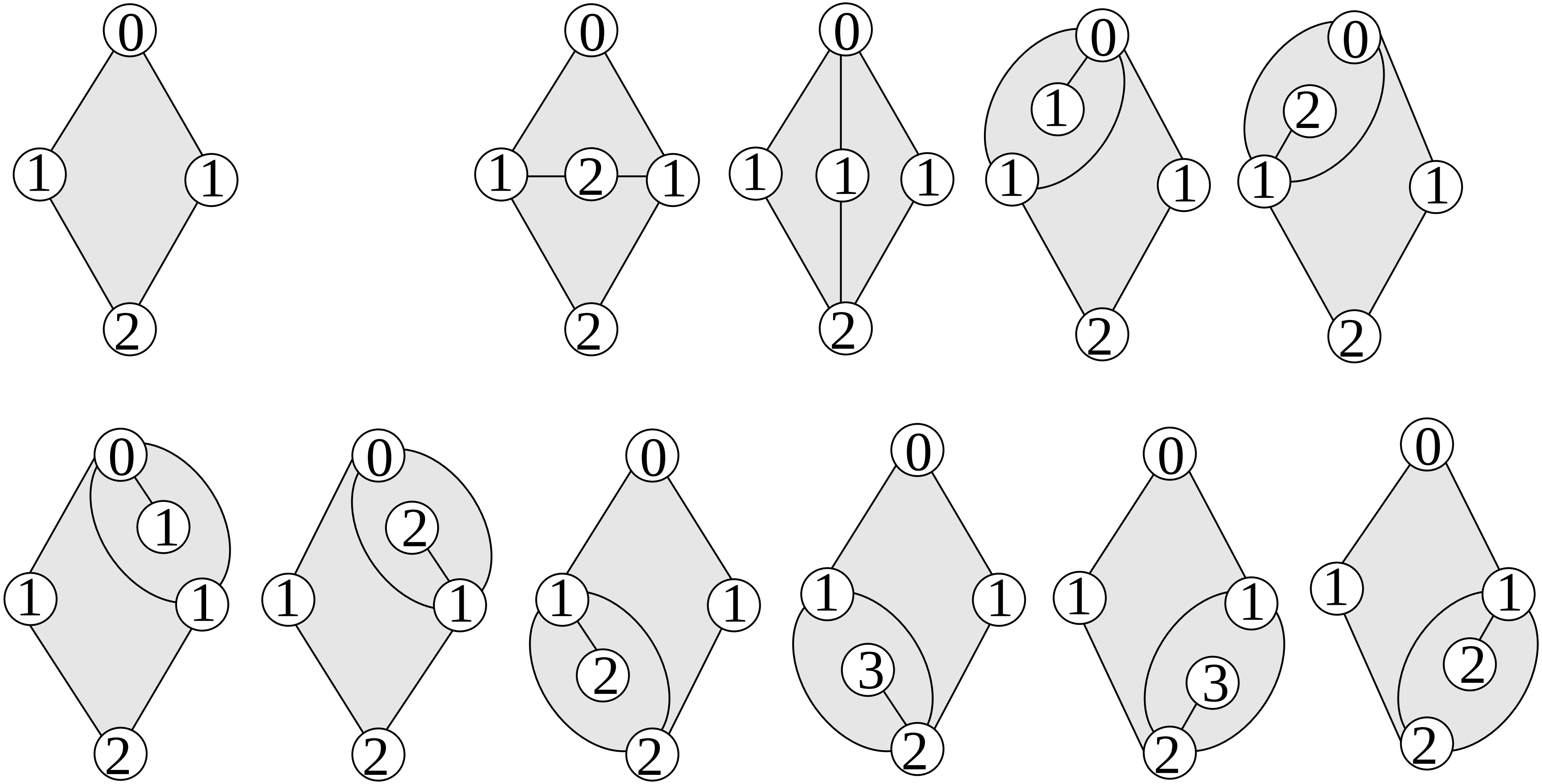
$U_\ell =$

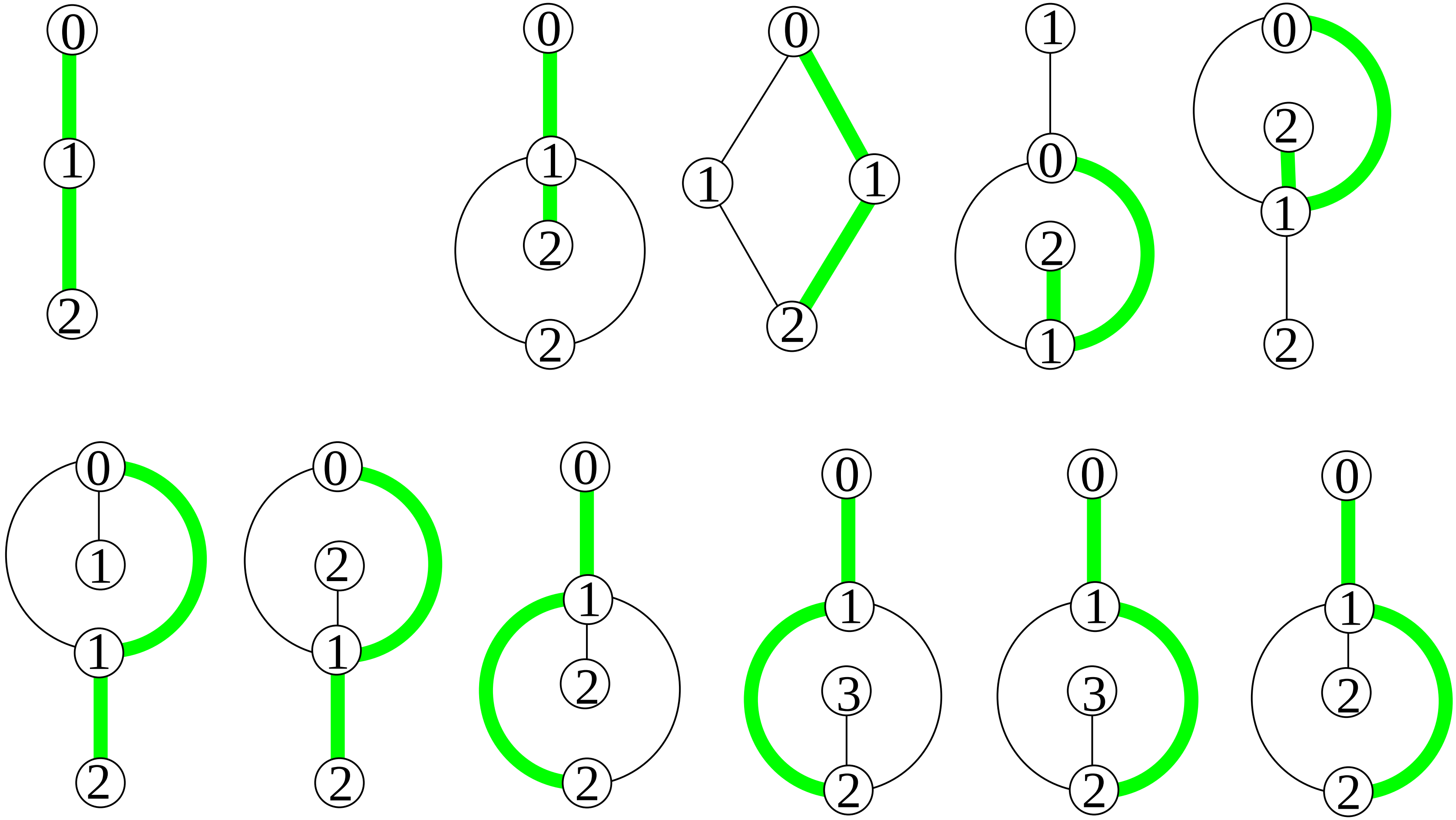


$= Z_\ell - \sum_{j=1}^{\ell-1}$

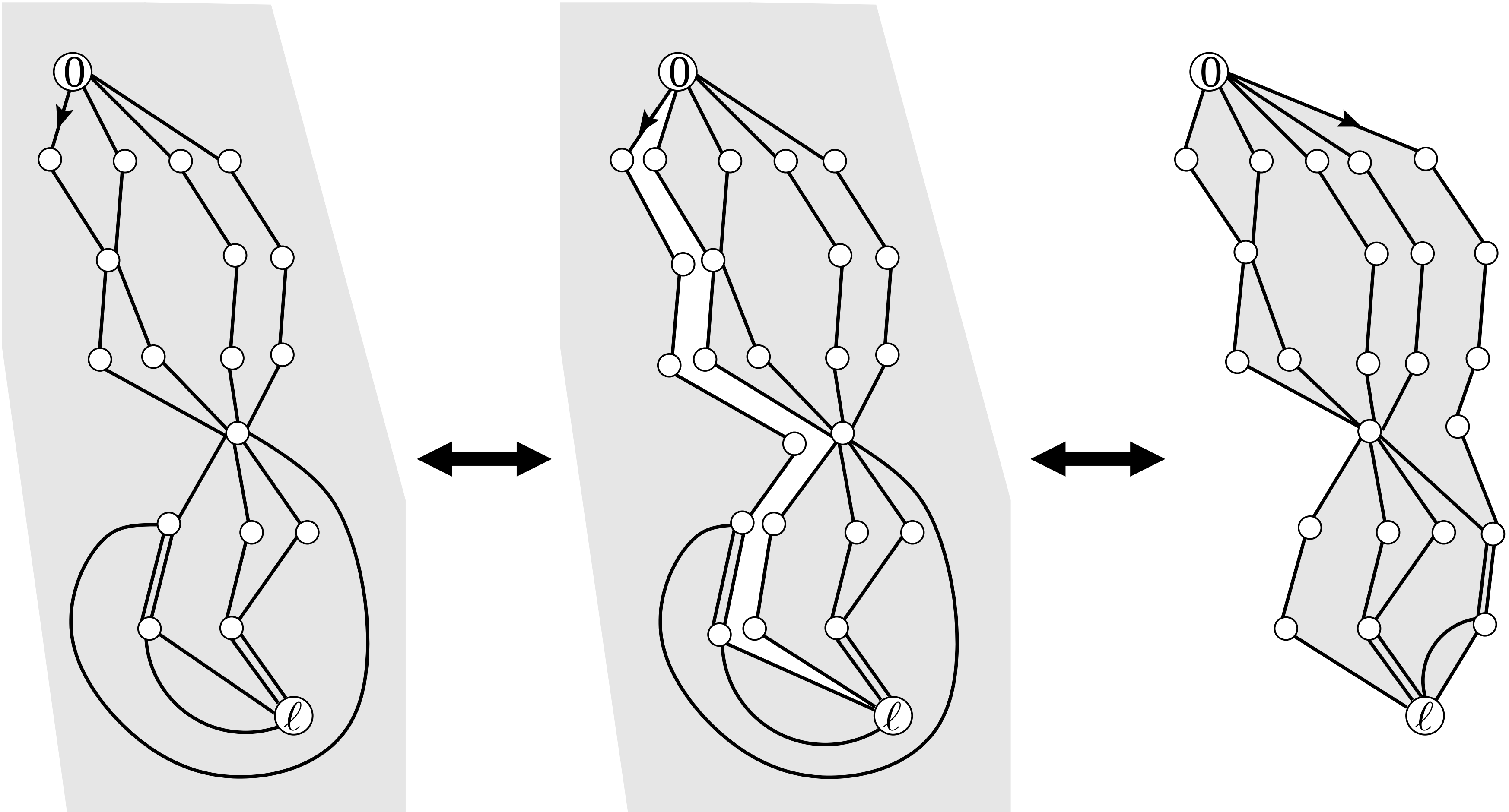


$$U_2(g) = g + 10g^2 + \ldots$$




0 1 2
0 1 2 2
0 1 1 2
1 0 2 1
0 2 1 2
0 1 1 2
0 2 1 2
0 1 2 2
0 1 3 2
0 1 3 2
0 1 2 2

collection of geodesics



local limit

$$Z_\ell = A_\ell - C_\ell\, \epsilon^2 + \frac{2}{3} D_\ell \epsilon^3 + \cdots$$

$$A_\ell = 2^\ell \frac{l+3}{3(\ell+1)}, \quad D_\ell = 2^\ell \frac{\ell(\ell+2)(\ell+3)(\ell+4)(3\ell^2+12\ell+13)}{420(\ell+1)}$$

$$U_\ell = a_\ell - c_\ell\, \epsilon^2 + \frac{2}{3} d_\ell \epsilon^3 + \cdots$$

$$a_\ell = A_\ell - \sum_{j=1}^{\ell-1} a_j A_{\ell-j}, \quad d_\ell = D_\ell - \sum_{j=1}^{\ell-1} (a_j D_{\ell-j} + d_j A_{\ell-j})$$

introduce $\hat{A}(t) = \sum_{\ell \geq 1} A_\ell t^\ell$, $\cdots$

$$\hat{a}(t) = \frac{\hat{A}(t)}{1+\hat{A}(t)}, \quad \hat{d}(t) = \frac{\hat{D}(t)}{(1+\hat{A}(t))^2}$$

$\longrightarrow$ exact expression for $\hat{d}(t)$

$$\hat{d}(t) = 4t + \frac{80}{3}t^2 + 132t^3 + \cdots$$

$$\langle \text{geods} \rangle_1 = d_1 = 4$$

$$\langle \text{geods} \rangle_2 = d_2 = \frac{80}{3}$$

$$\langle \text{geods} \rangle_3 = d_3 = 132$$

large ℓ behavior of d_ℓ ?

$$A_\ell \sim \frac{2^\ell}{3}(1+\frac{2}{\ell}) \to \hat{A}(t) \sim \frac{1}{3(1-2t)} - \frac{2}{3}\log(1-2t)$$

$$D_\ell \sim \frac{2^\ell \ell^5}{140} \to \hat{D}(t) \sim \frac{6}{7(1-2t)^6}$$

$$\hat{a}(t) \sim 1-3(1-2t)-6(1-2t)^2\log(1-2t) \to a_\ell \sim \frac{2^\ell\,12}{\ell^3}$$

$$\hat{d}(t) \sim \frac{54}{7(1-2t)^4} \to d_\ell \sim \frac{2^\ell\,9\ell^3}{7}$$

$$d_\ell \sim (3\times 2^\ell)\times\frac{3}{7}\ell^3$$

collection of geodesics

$$U_\ell^{(k)} = a_\ell^{(k)} - c_\ell^{(k)}\,\epsilon^2 + \frac{2}{3} d_\ell^{(k)} \epsilon^3 + \cdots$$

$$d_\ell^{(k)} = k \times (3 \times 2^\ell)^k \times \frac{3}{7}\ell^3$$

$$\tilde{U}_\ell^{(k)} = \tilde{a}_\ell^{(k)} - \tilde{c}_\ell^{(k)}\,\epsilon^2 + \frac{2}{3} \tilde{d}_\ell^{(k)} \epsilon^3 + \cdots$$

$$\tilde{d}_\ell^{(k)} = k\ (a_\ell)^{k-1} d_\ell$$

$$\tilde{d}_\ell^{(k)} = k \times (3 \times 2^\ell)^k \times 4^{k-1} \frac{3}{7}\ell^{6-3k}$$

number of vertices at distance ℓ reached by k avoiding geods $= 4^{k-1}\frac{3}{7}\ell^{6-3k}$

scaling limit (exponents)

the average number of pairs of points linked by k avoiding geods and at rescaled distance in the range $[r, r+dr]$ behaves as

$$n \times (n^{1/4})^{6-3k} n^{1/4} dr \times \rho^{(k)}(r)$$

with

$$\rho^{(k)}(r) \overset{r \to 0}{\sim} r^{6-3k}$$

$$\Rightarrow n^{(11-3k)/4}$$

$k = 1 : \quad n^2$
$k = 2 : \quad n^{5/4}$
$k = 3 : \quad n^{1/2}$

scaling limit (scaling functions)

$g = \frac{1}{12}(1-\epsilon^2)$, $\ell = u\,\epsilon^{-1/2}$

$$R_\ell \sim 2(1-\epsilon\,\mathcal{F}(u)), \quad \frac{Z_\ell}{2^\ell} \sim \frac{1}{3} + \epsilon^{1/2}\,\mathcal{H}(u), \quad \frac{U_\ell}{2^\ell} \sim \epsilon^{3/2}\,\mathcal{L}(u)$$

$$\mathcal{F}(u) = -3\frac{d}{du}\mathcal{H}(u), \quad \mathcal{L}(u) = 9\frac{d^2}{du^2}\mathcal{H}(u) = -3\frac{d}{du}\mathcal{F}(u)$$

scaling limit $\ell = r\,n^{1/4}$

$$U_\ell|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}}(3\cdot 2^\ell)\;n\,\rho(r)\;\frac{1}{n^{1/4}}$$

$$U_\ell^{(k)}|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}}k(3\cdot 2^\ell)^k\;n\,\rho(r)\;\frac{1}{n^{1/4}}$$

$\rightarrow$ no new scaling function

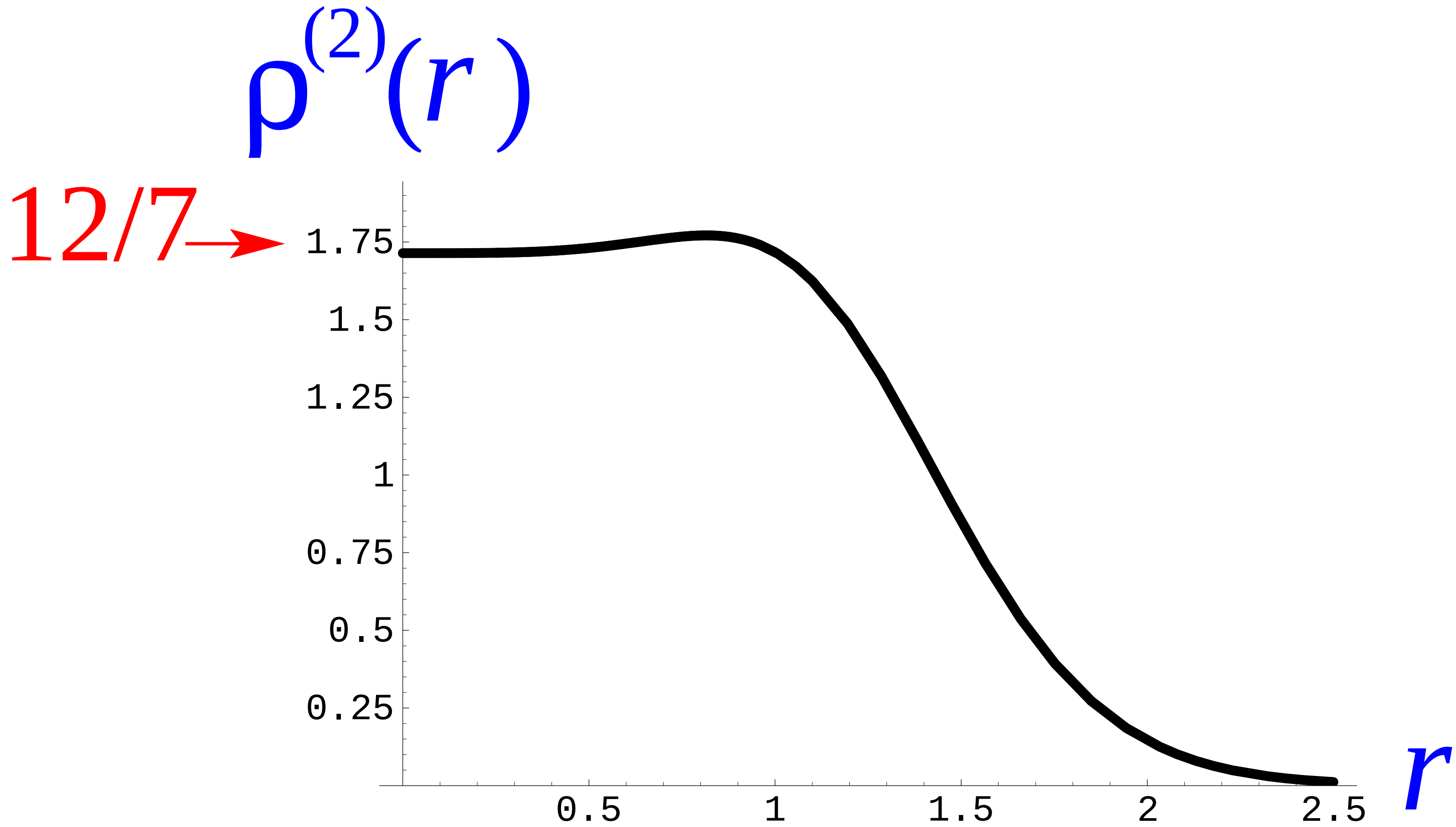
$$\frac{\tilde{U}_\ell^{(2)}}{2^{2\ell}} = \left(\frac{U_\ell}{2^\ell}\right)^2 \sim \epsilon^3 \; (\mathcal{L}(u))^2$$

$$n\tilde{U}_\ell^{(2)}|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} 2(3 \cdot 2^\ell)^2 \; n^{5/4} \, \rho^{(2)}(r) \; \frac{1}{n^{1/4}}$$

with the new scaling function

$$\rho^{(2)}(r) = \frac{1}{9\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, \xi^4 e^{-\xi^2} \big(\mathcal{L}(r\sqrt{-\mathrm{i}\xi})\big)^2$$

$\rho^{(2)}(r)$



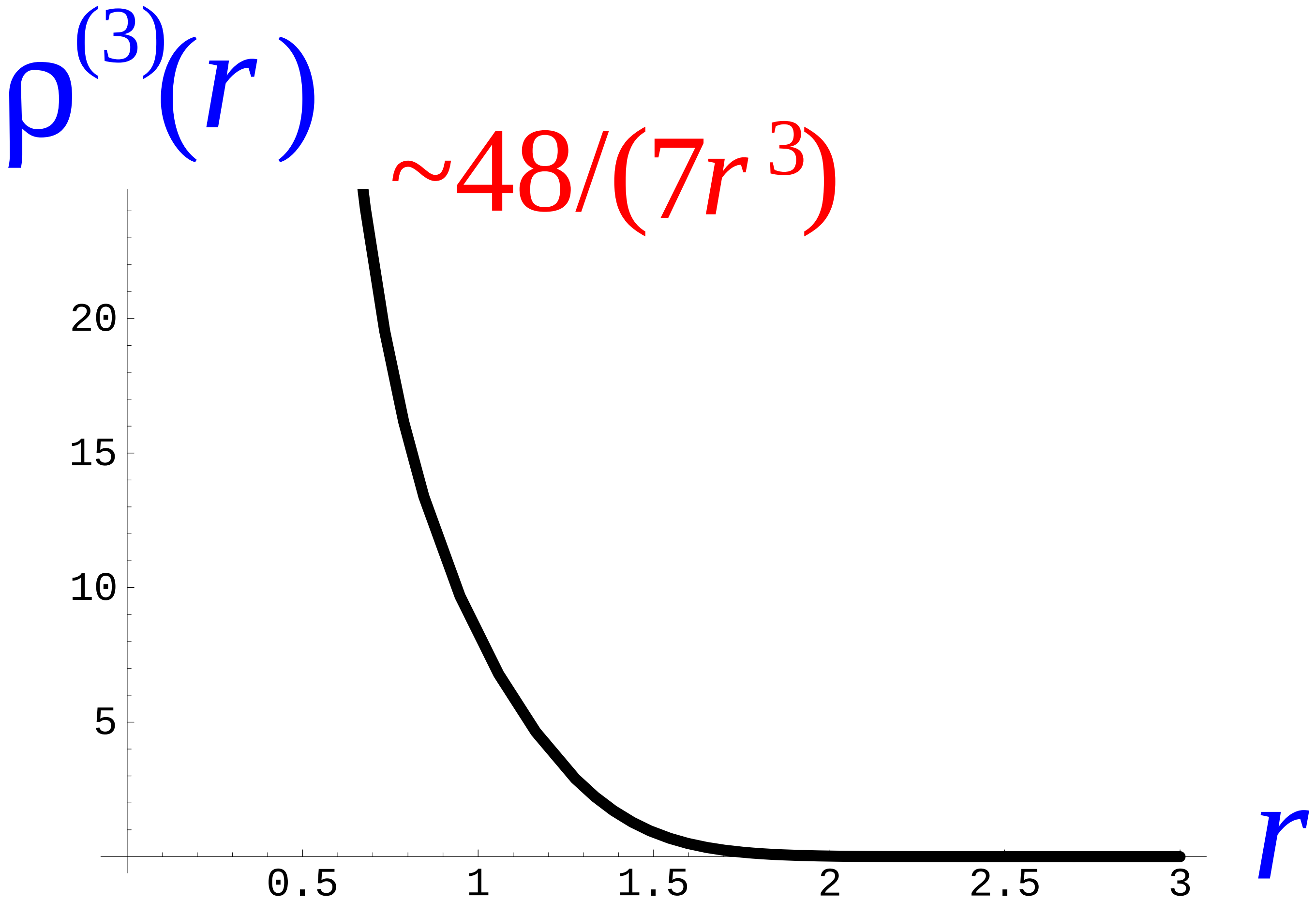
$$\frac{\tilde{U}_\ell^{(3)}}{2^{3\ell}} = \left(\frac{U_\ell}{2^\ell}\right)^3 \sim \epsilon^{9/2} \left(\mathcal{L}(u)\right)^3$$

$$n\tilde{U}_\ell^{(3)}|_{g^n} \sim \frac{12^n}{2\sqrt{\pi}n^{5/2}} 3(3\cdot 2^\ell)^3 \; n^{1/2} \, \rho^{(3)}(r) \; \frac{1}{n^{1/4}}$$

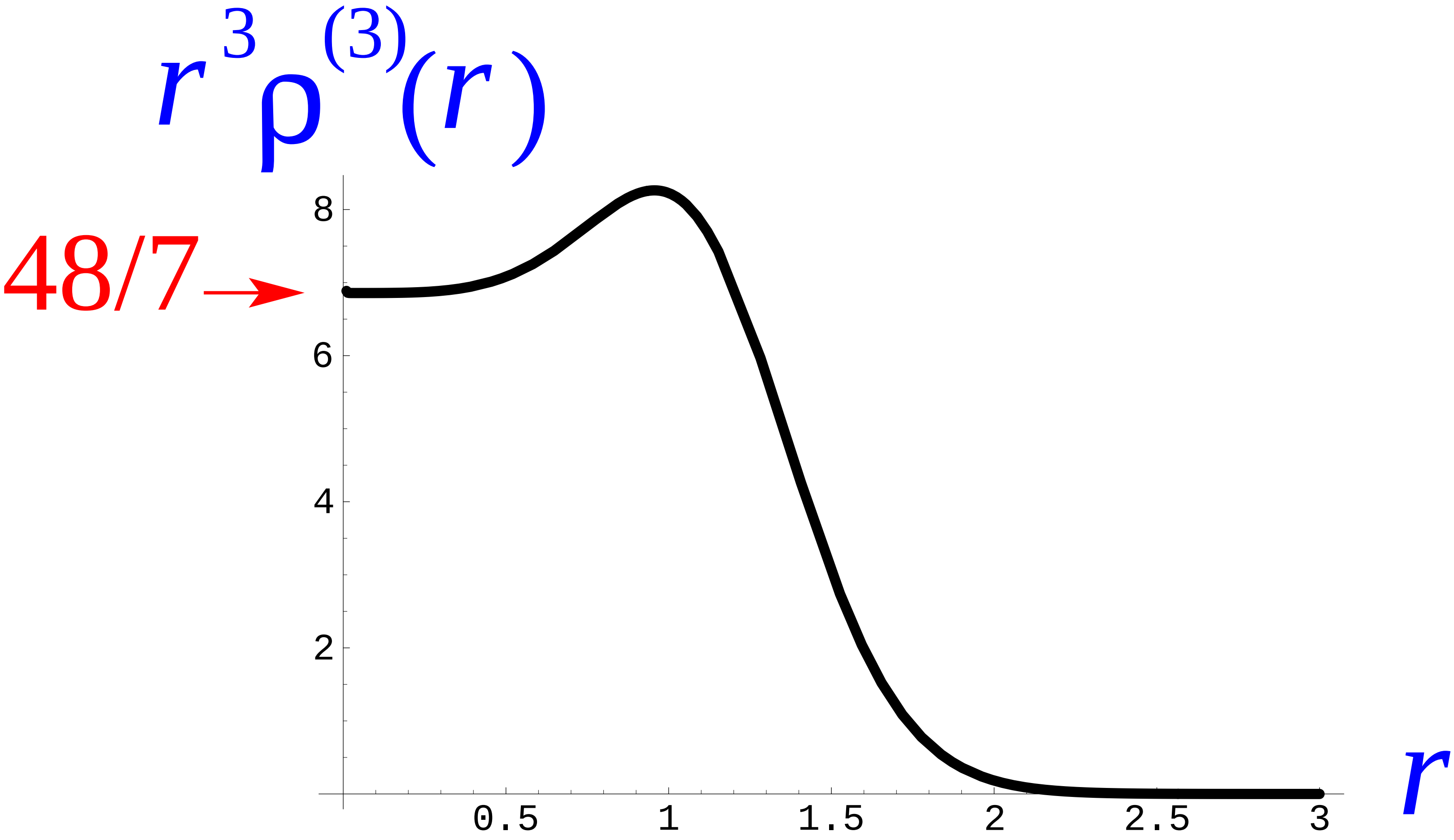
with the new scaling function

$$\rho^{(3)}(r) = \frac{2}{81\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, \frac{\xi^5}{\mathrm{i}} e^{-\xi^2} \sqrt{-\mathrm{i}\xi} \Big(\mathcal{L}(r\sqrt{-\mathrm{i}\xi})\Big)^3$$

$\rho^{(3)}(r)$



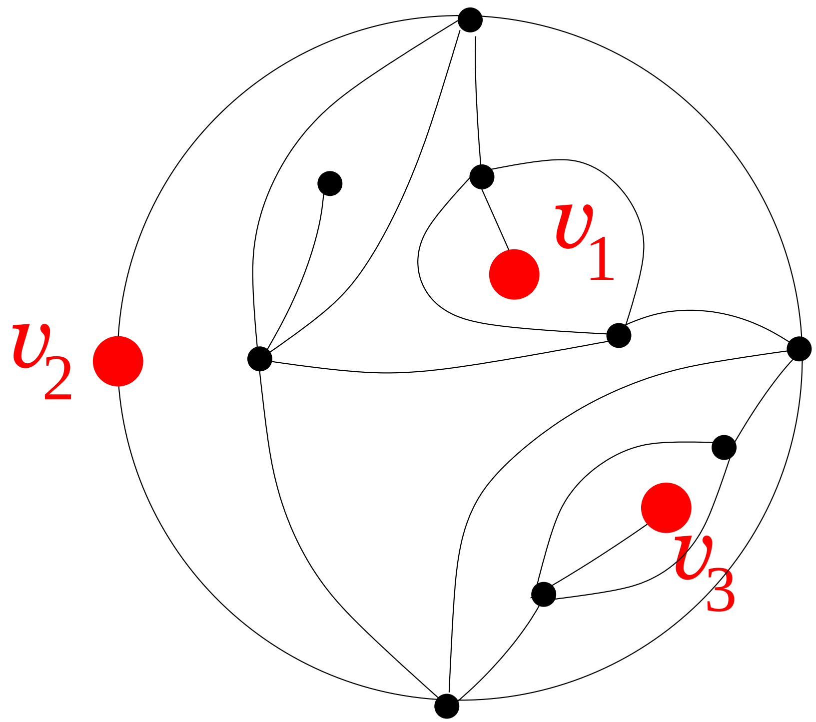
$\rho^{(3)}(r)$



three-point statistics

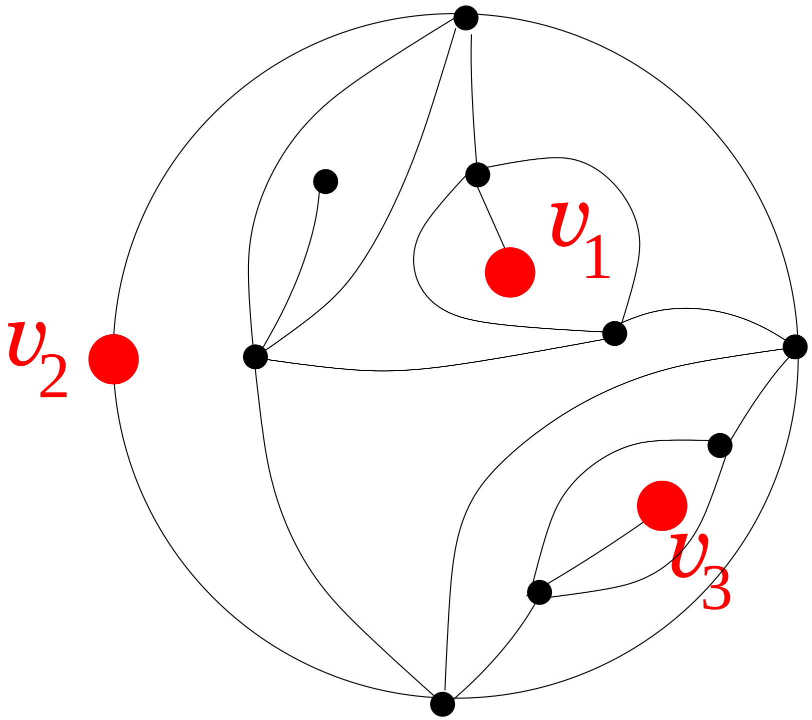
Miermont's bijection

start with a multiply-pointed planar quadrangulation with p marked vertices (=sources) distinguished as $v_1, \cdots, v_p$ and satisfying $d(v_i, v_j) \geq 2$

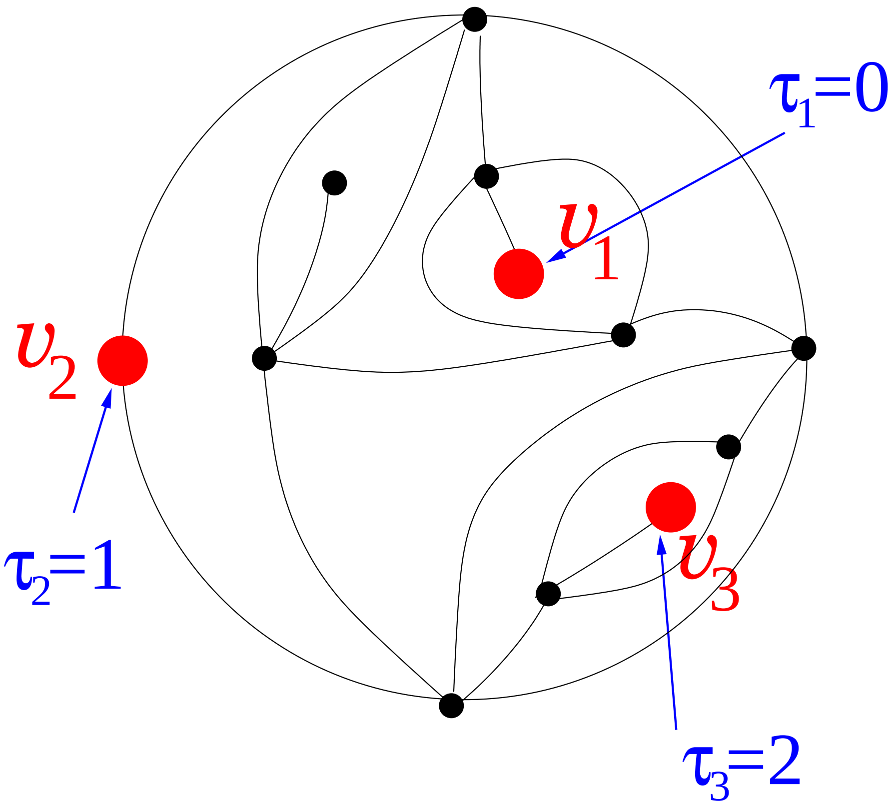


Miermont's bijection

natural labeling: $\ell(v) \equiv \min_{j=1,\ldots p} d(v, v_j)$



one can favor/penalize some of the sources by attaching to each source v_i a delay τ_i =integer



this defines a “delayed distance" from v to the source v_j:

$$\ell_j(v) \equiv d(v, v_j) + \tau_j$$

a vertex v now receives the label:

$$\ell(v) \equiv \min_{j=1,\dots p} \ell_j(v) = \min_{j=1,\dots p} (d(v, v_j) + \tau_j)$$

which is the “distance" to the closest source, where the distance from v_j incorporates a penalty τ_j

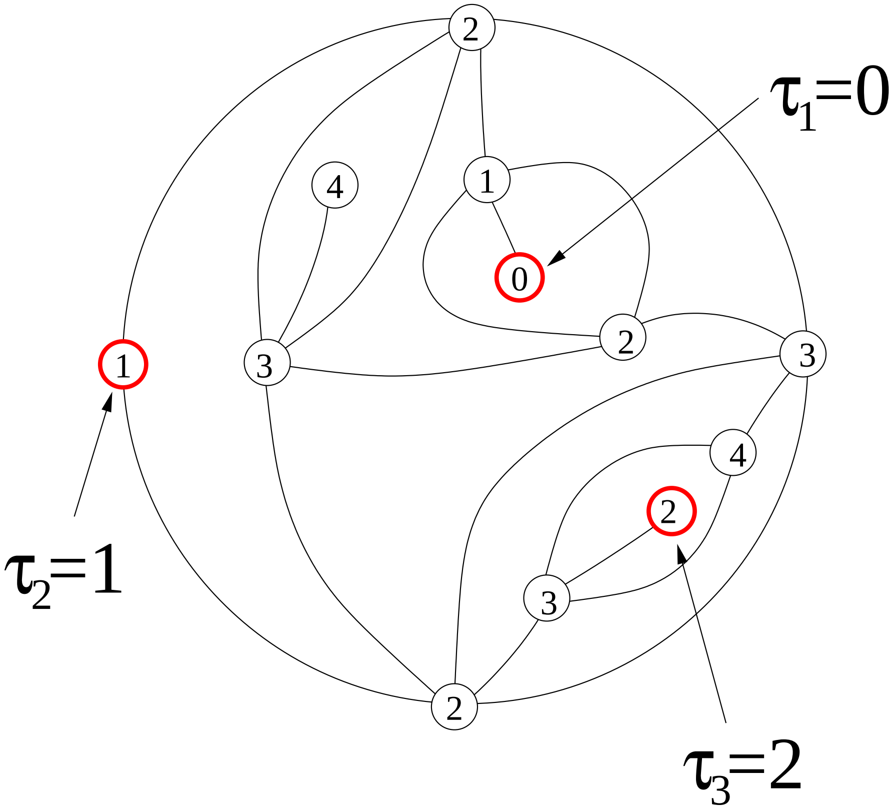
we choose delays so that:

- $\diamond$ $|\tau_i - \tau_j| < d(v_i, v_j)\ \forall i \neq j$ (cond. 1)

$\longrightarrow$ ensures that $\ell(v_i) = \tau_i$

- $\diamond$ $\tau_i - \tau_j = d(v_i, v_j) \bmod 2$ (cond. 2)

$\longrightarrow$ ensures that the parity of $\ell_j(v)$ is independent of j so that again $|\ell(v) - \ell(v')| = 1$ for v and v' neighbors



2
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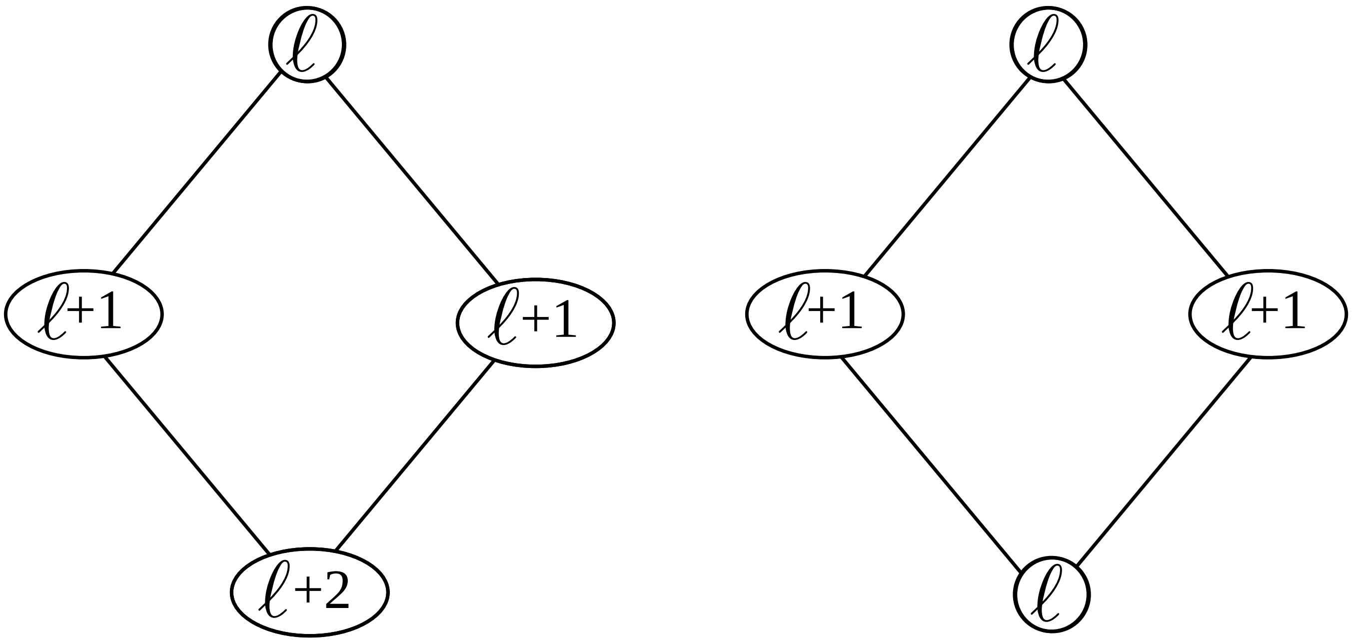
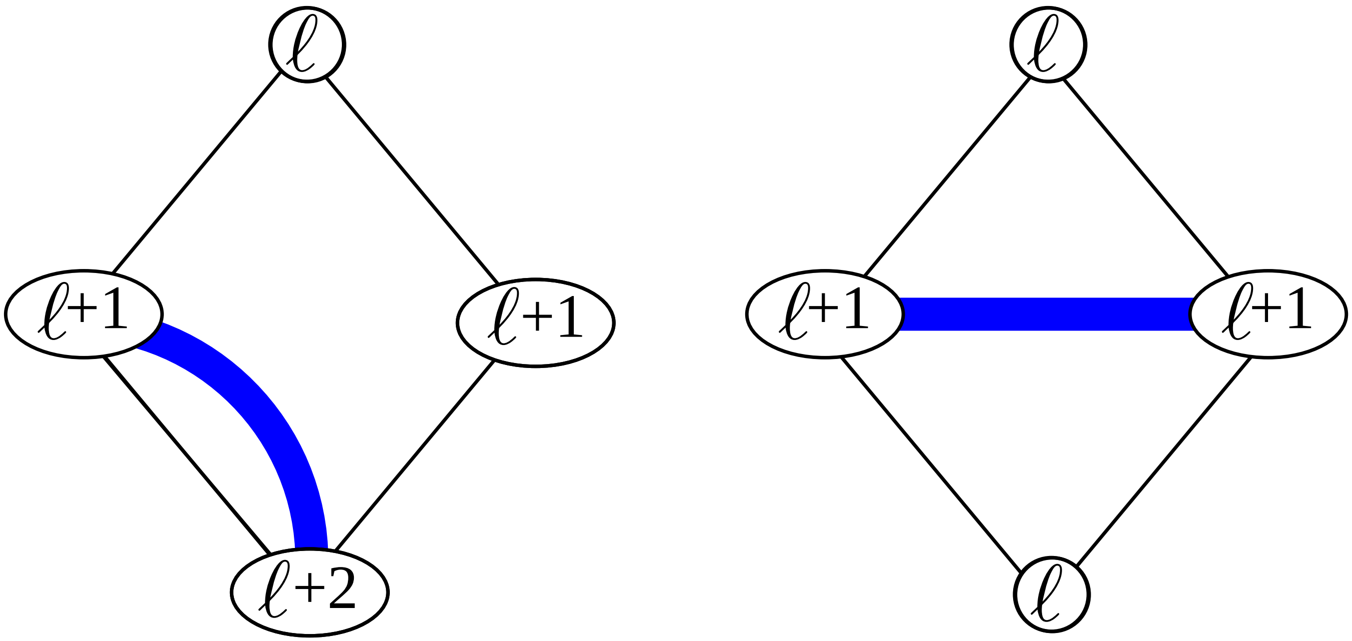
$\tau_1=0$
$\tau_2=1$
$\tau_3=2$

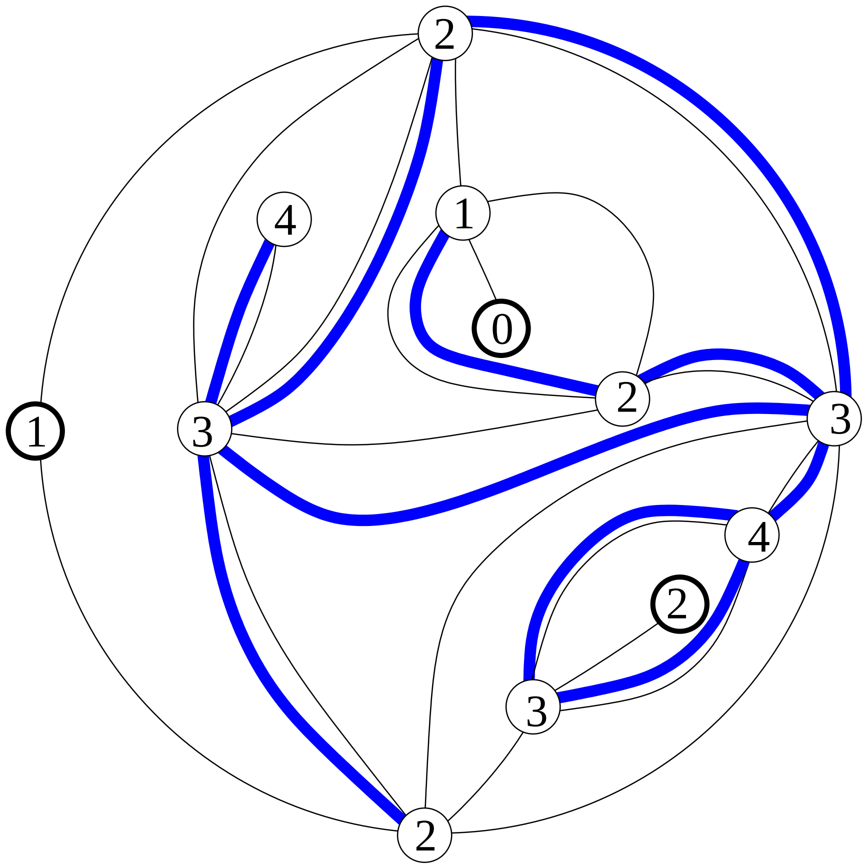
faces → edges



faces → edges



same rules as in Schaeffer's bijection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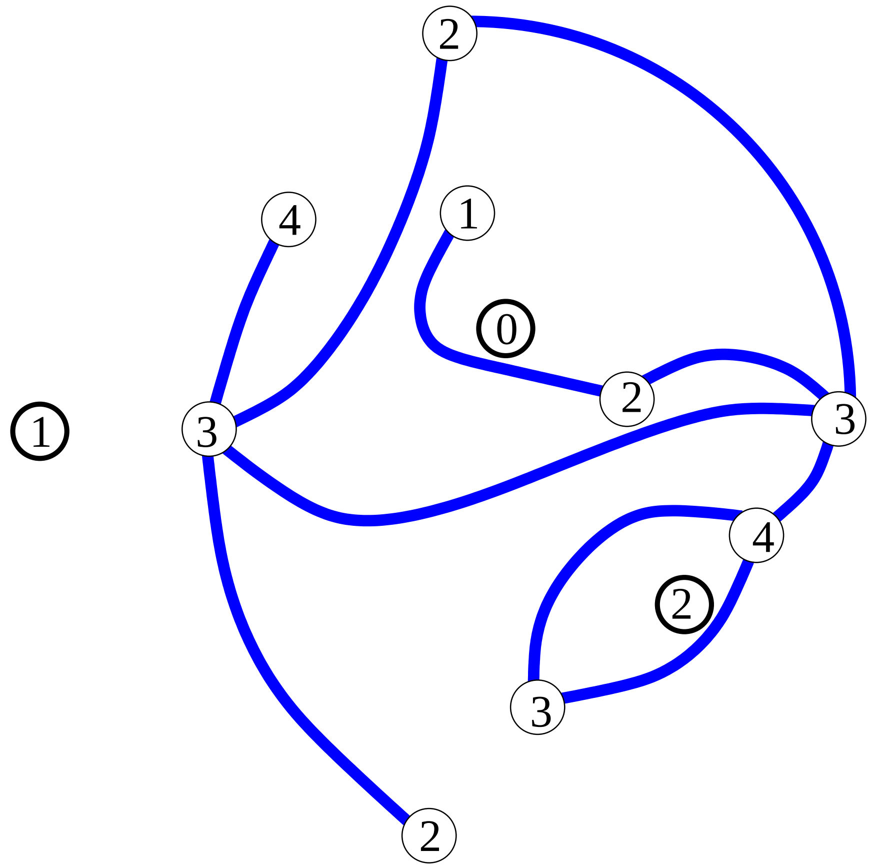
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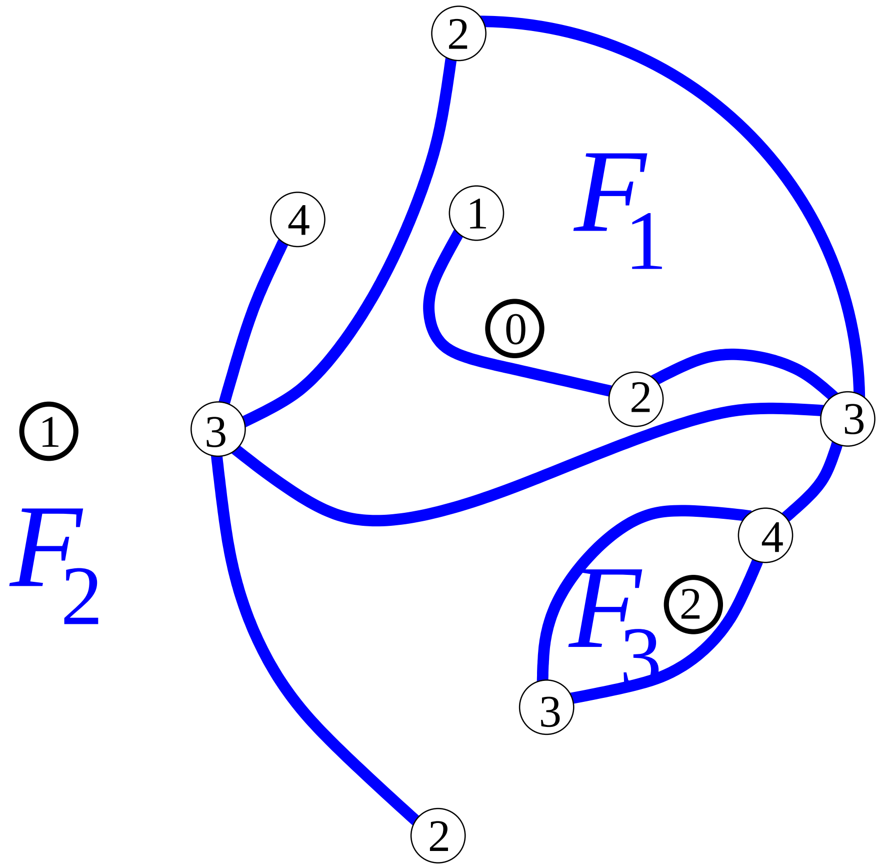
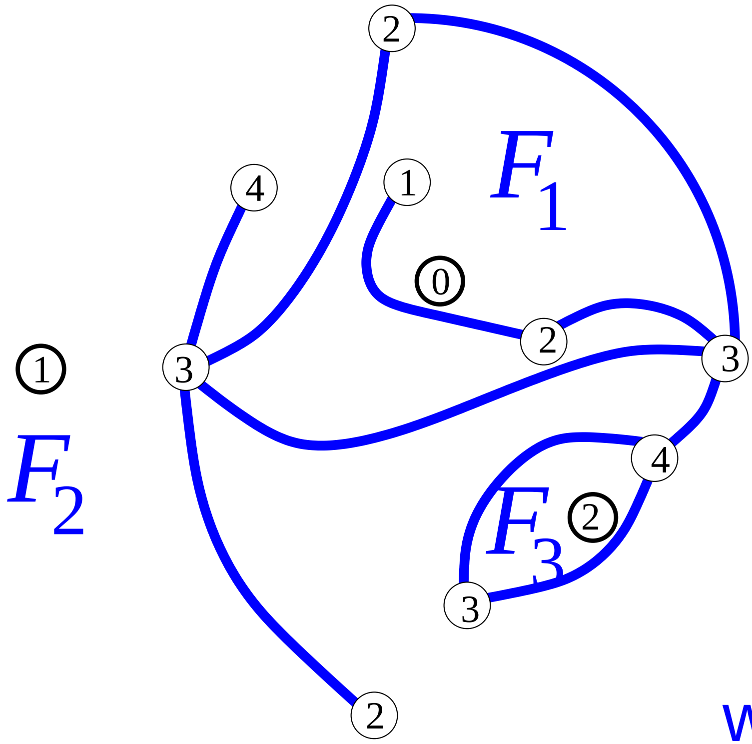


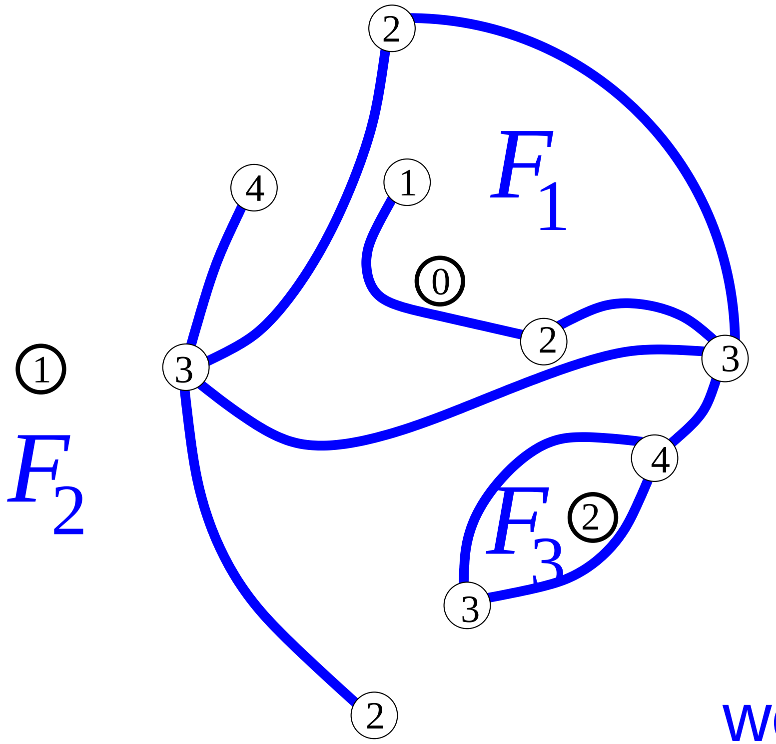
end up with a planar well-labeled map with p faces

well-labeled maps



well-labeled:

well-labeled maps

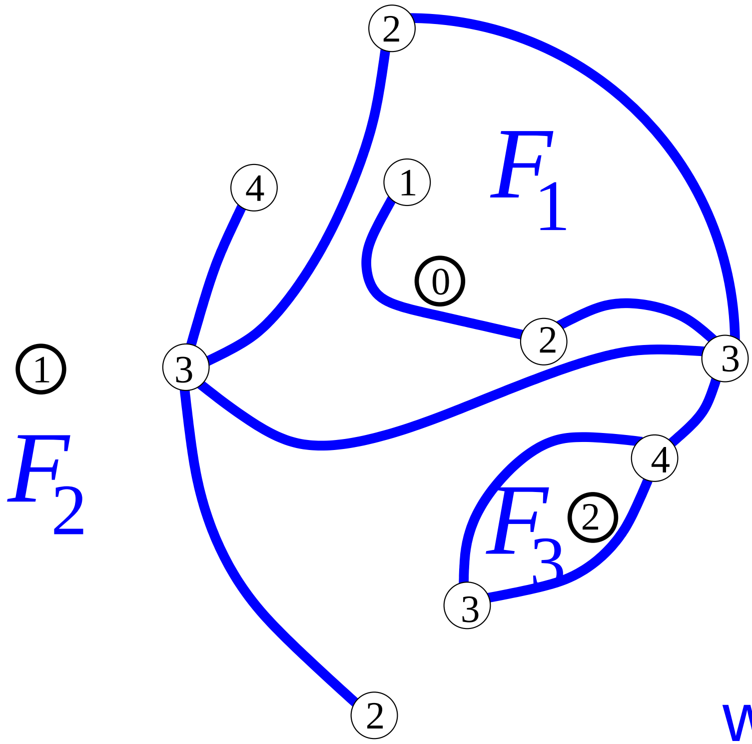


well-labeled:

◇ labels vary by at most 1 between neighbors

$|\ell(v) - \ell(v')| \leq 1$ if v and v' are neighbors on the map

well-labeled maps



well-labeled:

◇ labels vary by at most 1 between neighbors

$|\ell(v) - \ell(v')| \leq 1$ if v and v' are neighbors on the map

◇ $\min_{v \text{ incident to } F_i} \ell(v) = 1 + \tau_i$

bijection: for fixed given delays

p-pointed quadrangulations

with marked vertices satisfying

◇ $d(v_i, v_j) > |\tau_i - \tau_j| \; \forall \; i \neq j$

◇ $d(v_i, v_j) = \tau_i - \tau_j \mod 2$

$$\Updownarrow$$

well-labeled maps with p faces

with labels satisfying

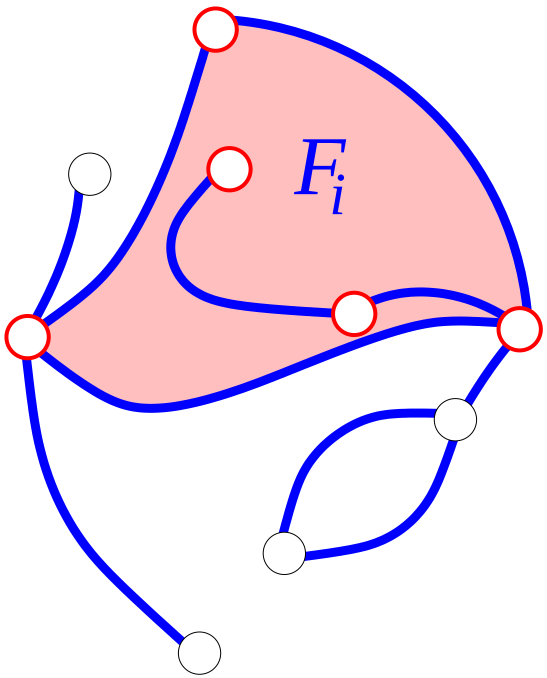
◇ $|\ell(v) - \ell(v')| \leq 1$ if v and v' are neighbors

◇ $\min\limits_{v \text{ incident to } F_i} \ell(v) = 1 + \tau_i$

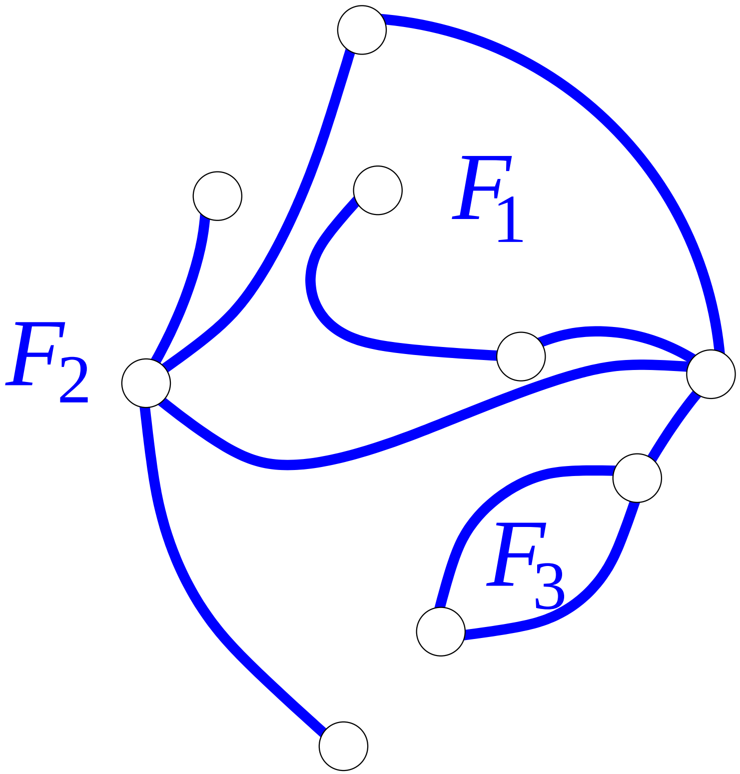
this coding keeps track of some of the distances:

if v is incident to F_i, then the minimum of $\ell_j = d(v, v_j) + \tau_j$ is reached for $j = i$ and therefore:

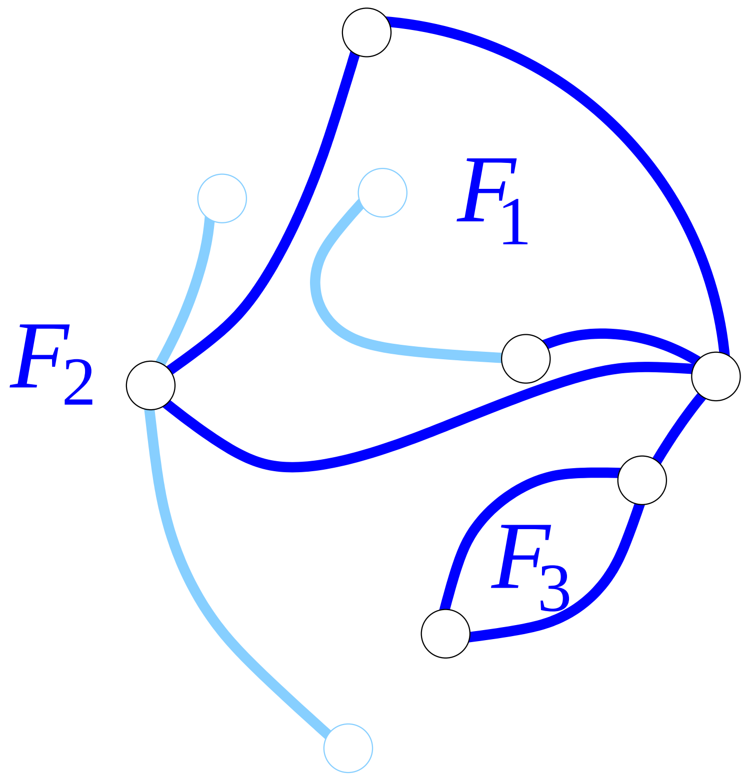
$$d(v, v_i) = \ell(v) - \tau_i$$



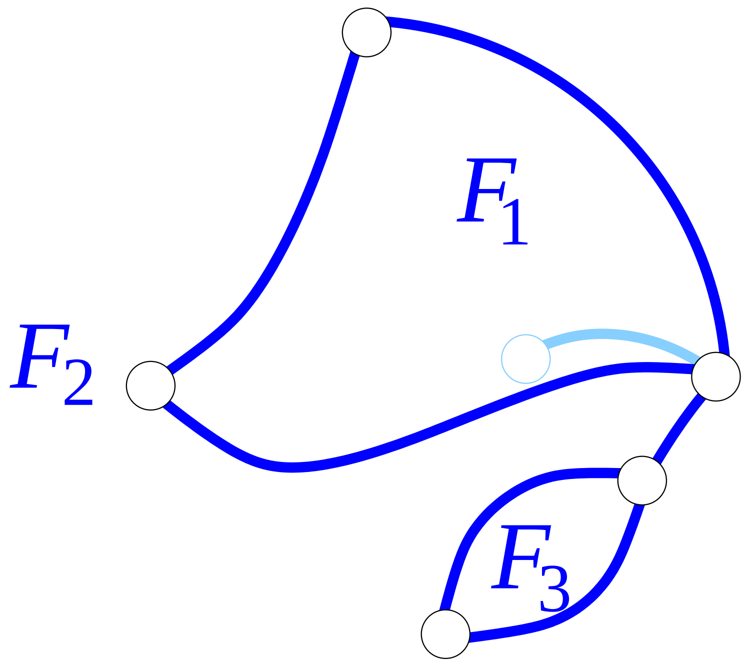
planar maps are classified according to their backbone



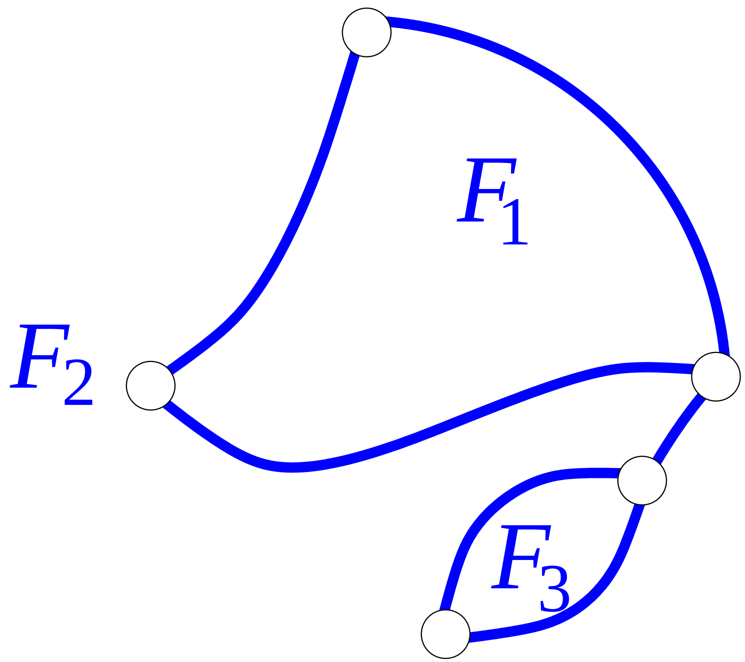
planar maps are classified according to their backbone



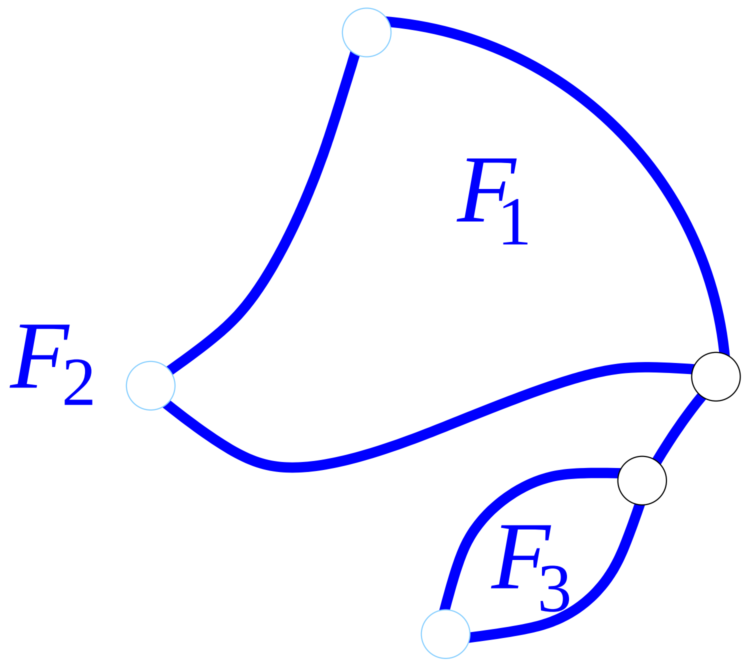
planar maps are classified according to their backbone



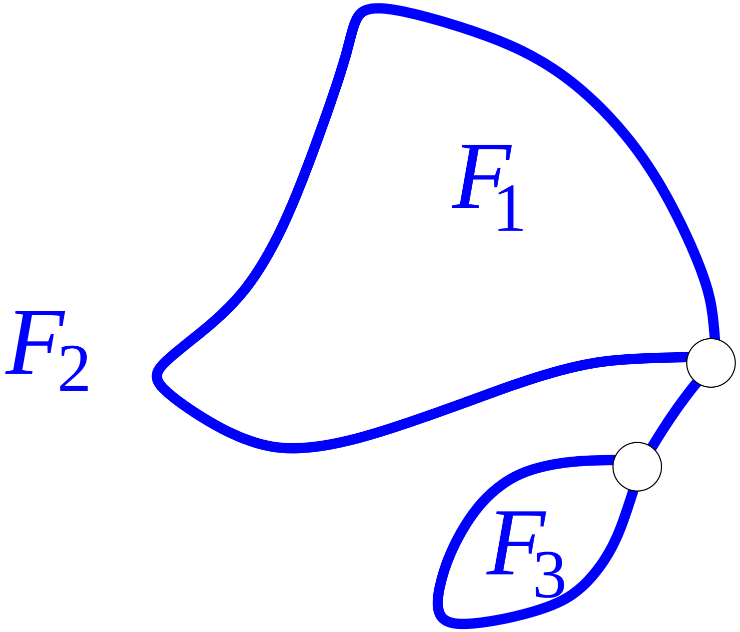
planar maps are classified according to their backbone



planar maps are classified according to their backbone



planar maps are classified according to their backbone

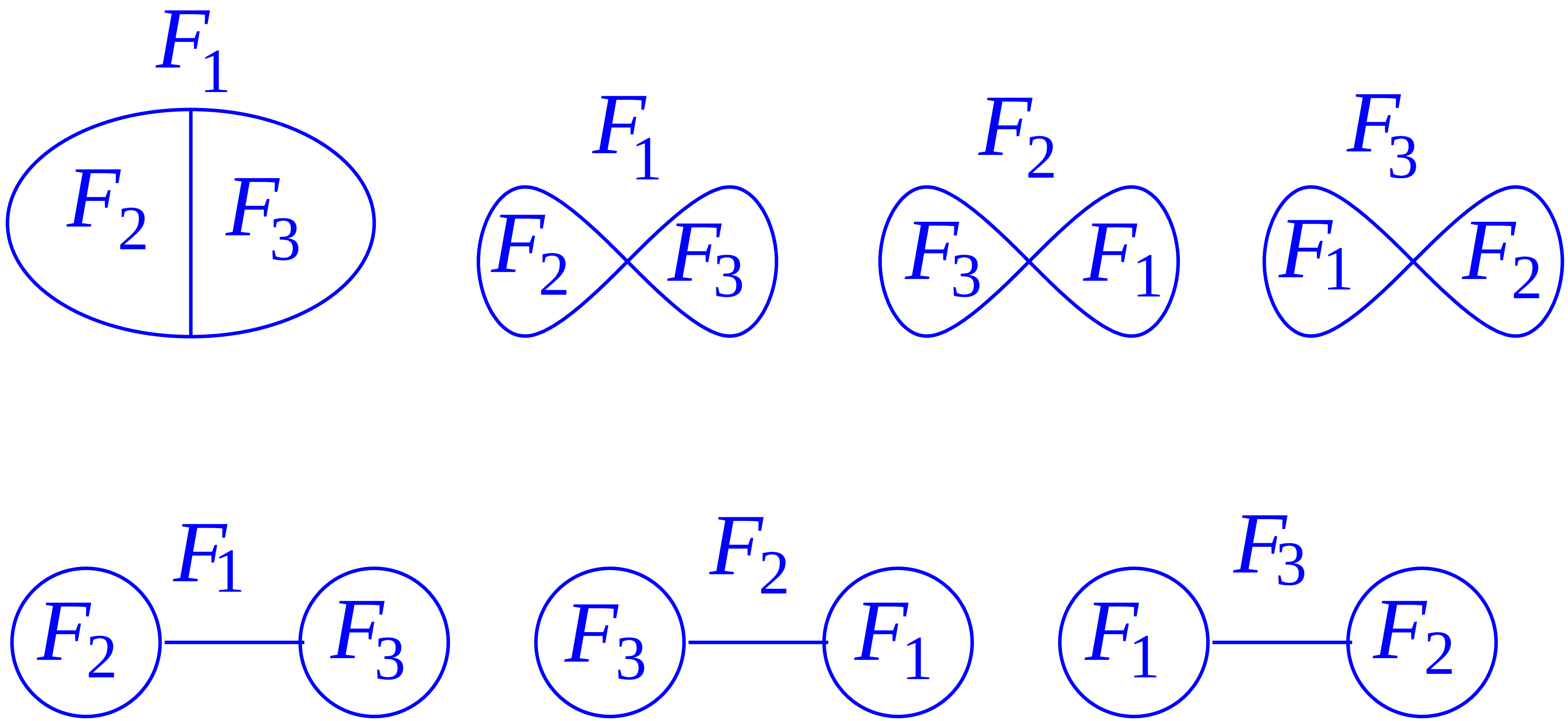


all vertices have degree $\geq 3 \Rightarrow$ finite number of backbones

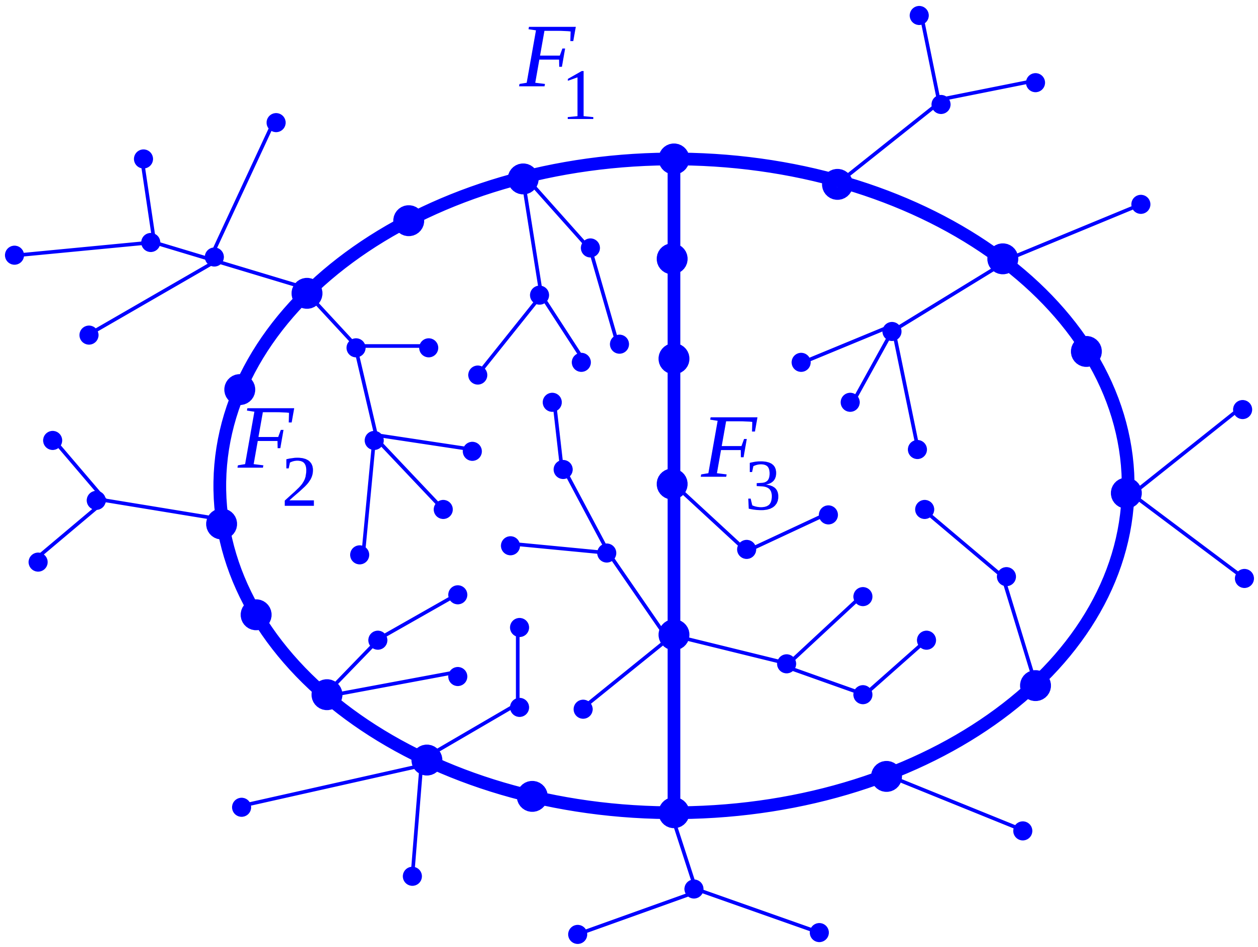
case of 3 marked vertices

planar maps with 3 (distinguished) faces

$\longrightarrow$ seven possible backbones



map $=$ backbone $+$ attached trees



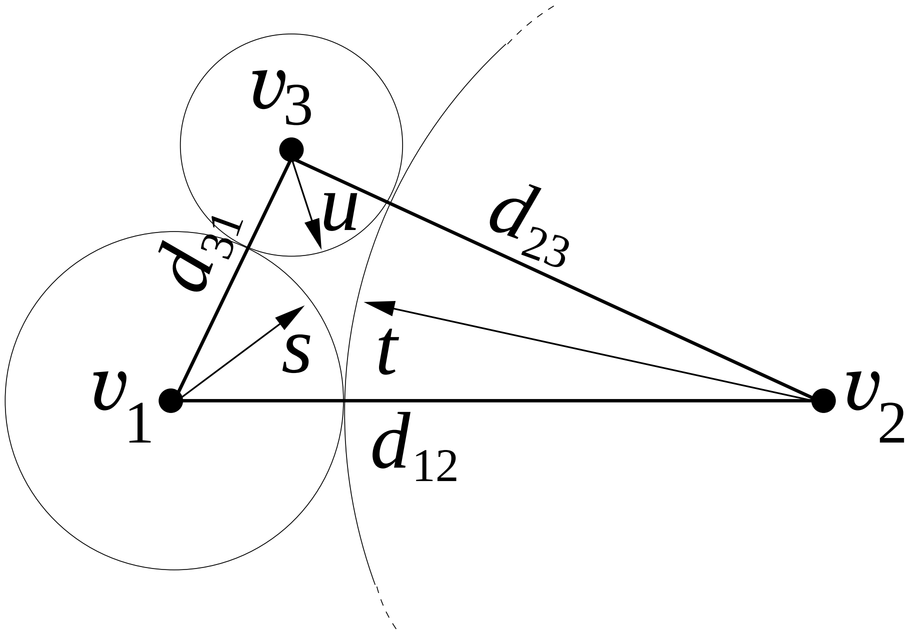
distance parametrization

for 3 points, we can use the following parametrization:

$$d_{12} \equiv d(v_1, v_2) = s + t$$

$$d_{23} \equiv d(v_2, v_3) = t + u$$

$$d_{31} \equiv d(v_3, v_1) = u + s$$



with s, t, u integers, $s, t, u \geq 0$ and at most one may vanish

choice of delays

idea: relate the delays to the distances, namely choose:

$$\tau_1 = -s\ , \quad \tau_2 = -t\ , \quad \tau_3 = -u$$

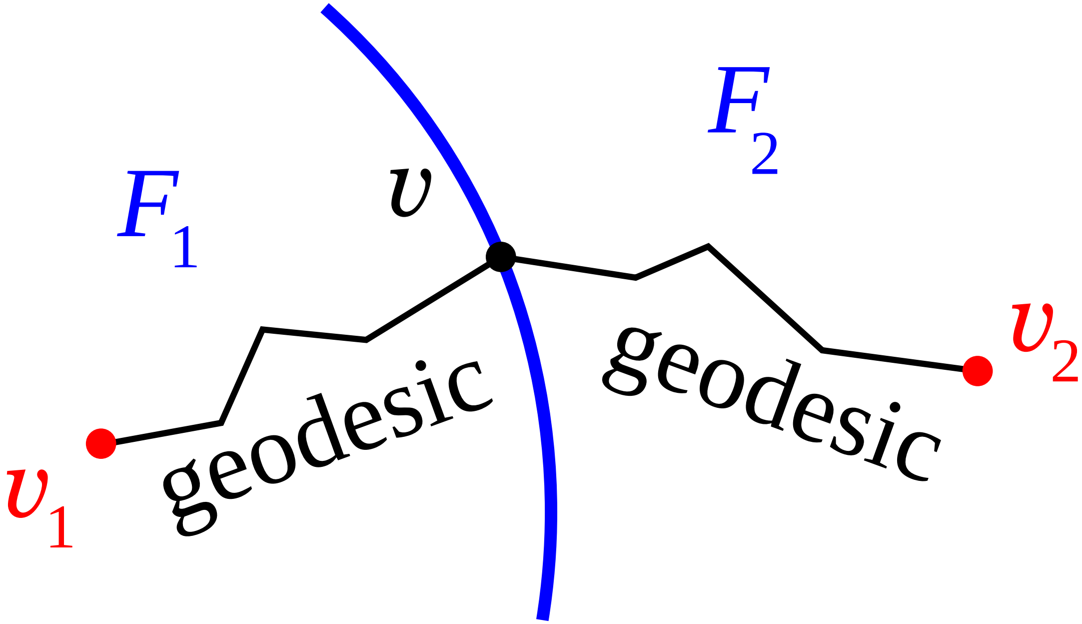
◇ $\tau_1 - \tau_2 = -s + t = s + t \bmod 2 = d(v_1, v_2) \bmod 2$

◇ $|\tau_1 - \tau_2| = |d_{23} - d_{31}| \leq d_{12}$ (triangular inequalities) and equality only if the 3 vertices are “aligned": for instance $d_{23} - d_{31} = d_{12}$ only if v_1 lies on a geodesic path between v_2 and v_3 (i.e. $s = 0$)

• assume that the 3 vertices are not aligned $\Leftrightarrow s, t, u > 0$
• treat later the case of aligned vertices (includes the case when two vertices are immediate neighbors)

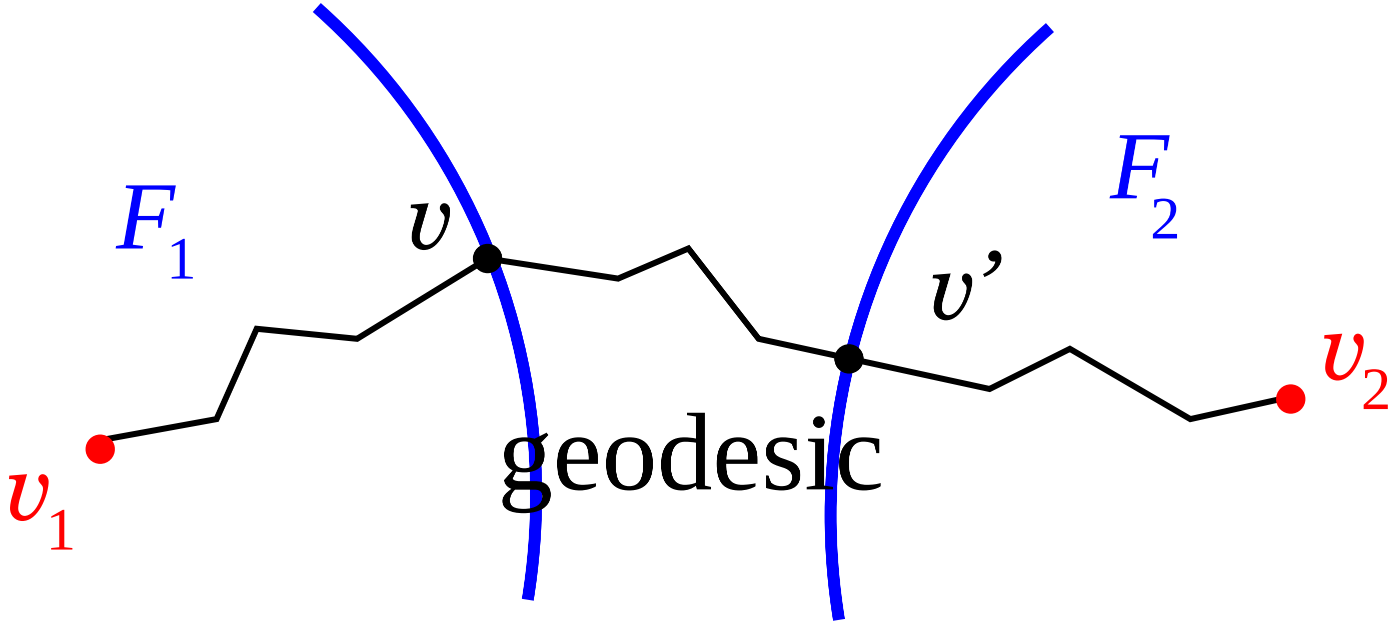
new constraint on labels:

◇ vertex on the boundary between two faces



length $= \ell(v) - \tau_1 + \ell(v) - \tau_2 = 2\ell(v) + (s+t) \geq s + t = d_{12}$

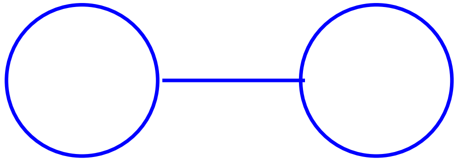
$\Rightarrow \ell(v) \geq 0$ for vertices on boundaries of the well-labeled map

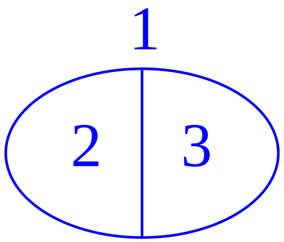


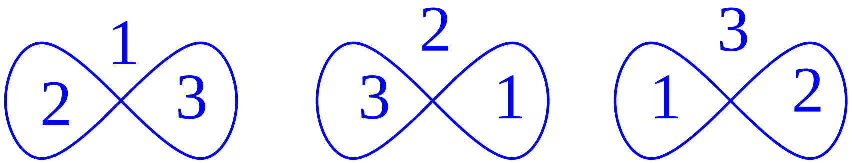
$$s + t = d_{12} = \ell(v) - \tau_1 + \ell(v') - \tau_2 + d(v, v')$$
$$= \ell(v) + \ell(v') + d(v, v') + s + t$$

$\Rightarrow \ell(v) + \ell(v') + d(v, v') = 0$
$\Rightarrow v = v'$ and $\ell(v) = 0$

F_1 and F_2 must have a common boundary (+ permutations)
$\exists$ label 0 on each boundary

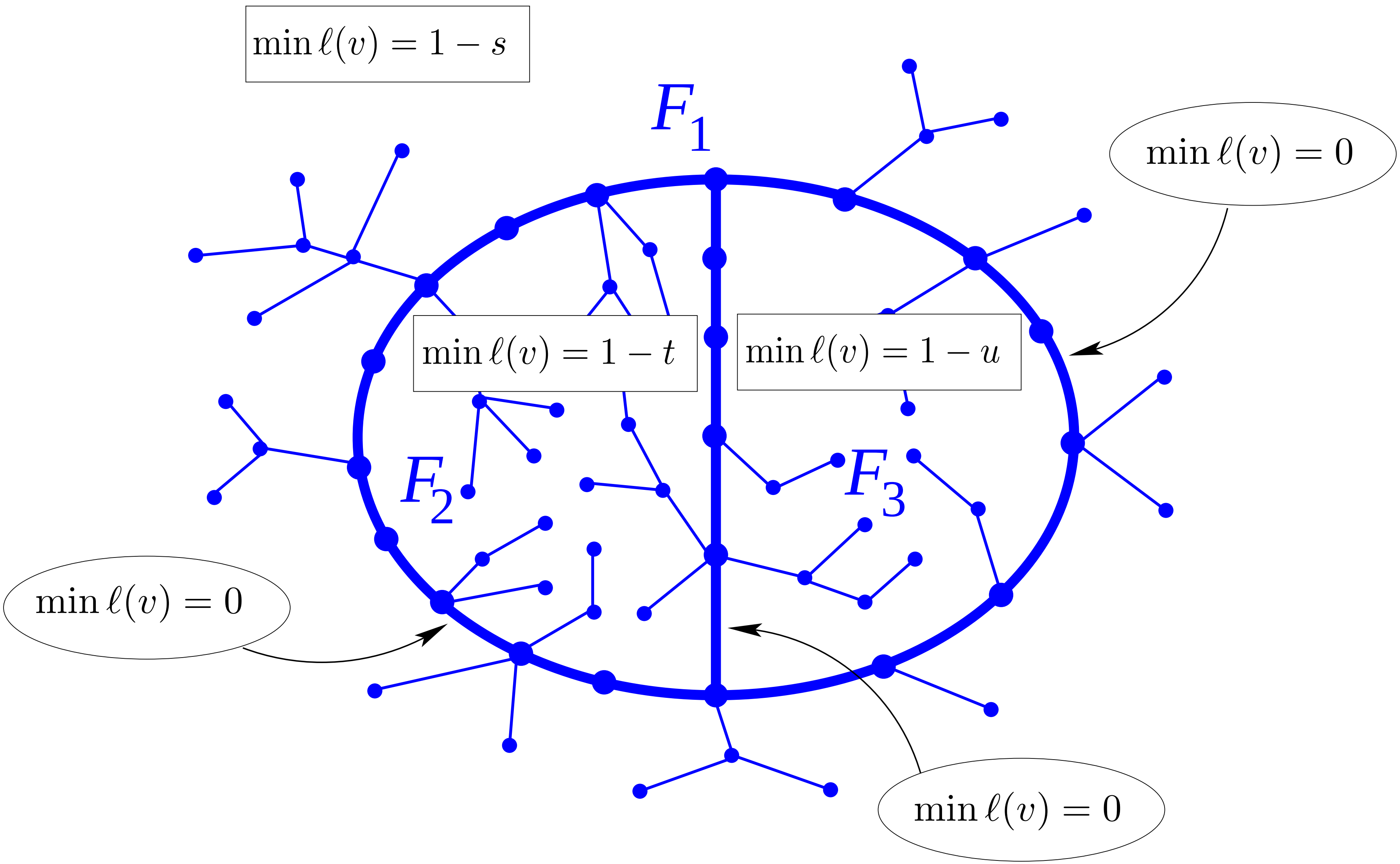
rules out backbones of type: 

the only possible backbones are those of type 

and those of type 

in practice, the latter can be viewed as degenerate cases of the former when one of the boundaries reduces to a single vertex.

rules on labels



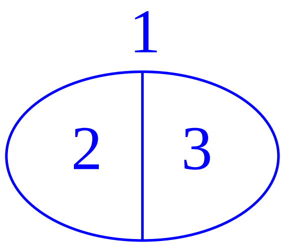
bijection:

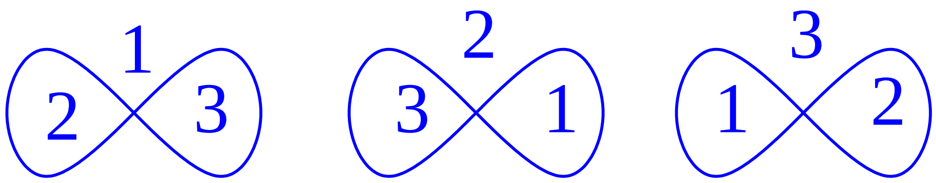
triply-pointed quadrangulations

with marked vertices at prescribed pairwise distances
$d_{12} = s + t$, $d_{23} = t + u$ and $d_{31} = u + s$ with $s, t, u > 0$

$\Updownarrow$

well-labeled maps with 3 faces

with a backbone 

or its degenerate versions 

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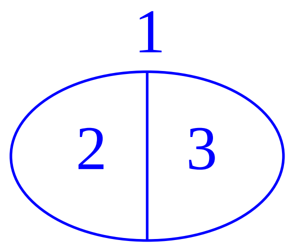
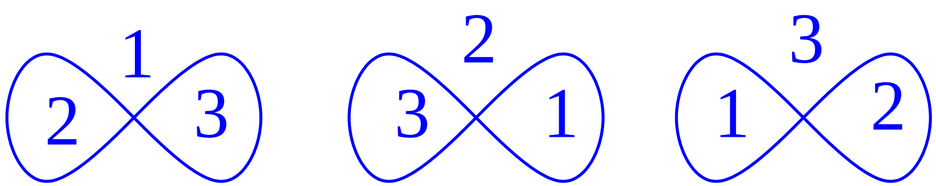
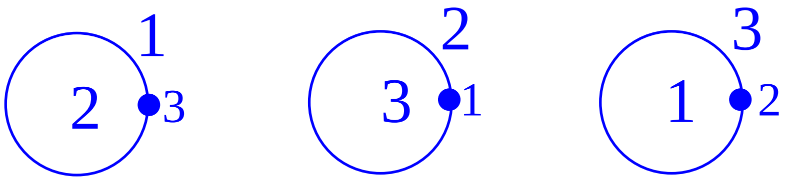
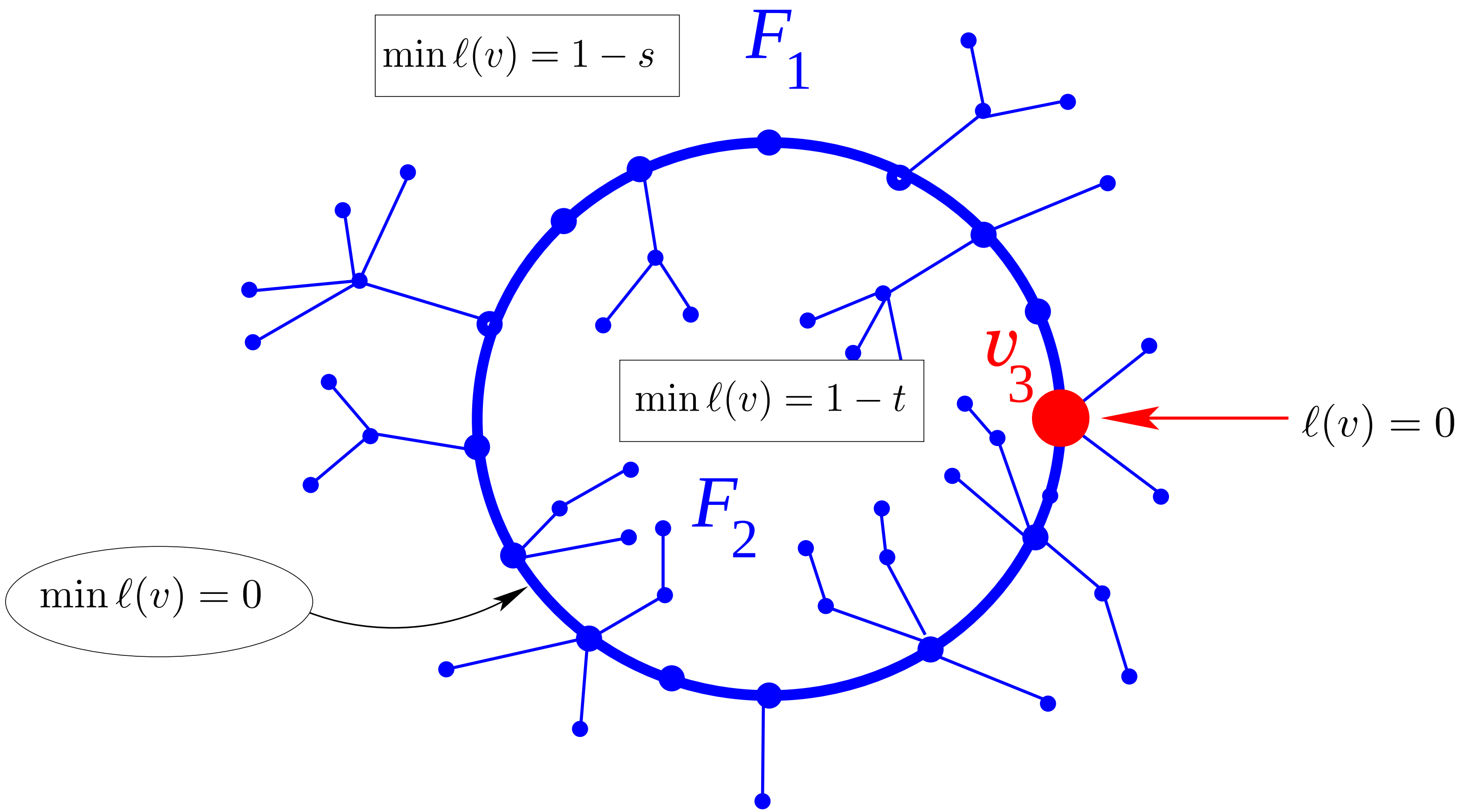
with a backbone 

or its degenerate versions 

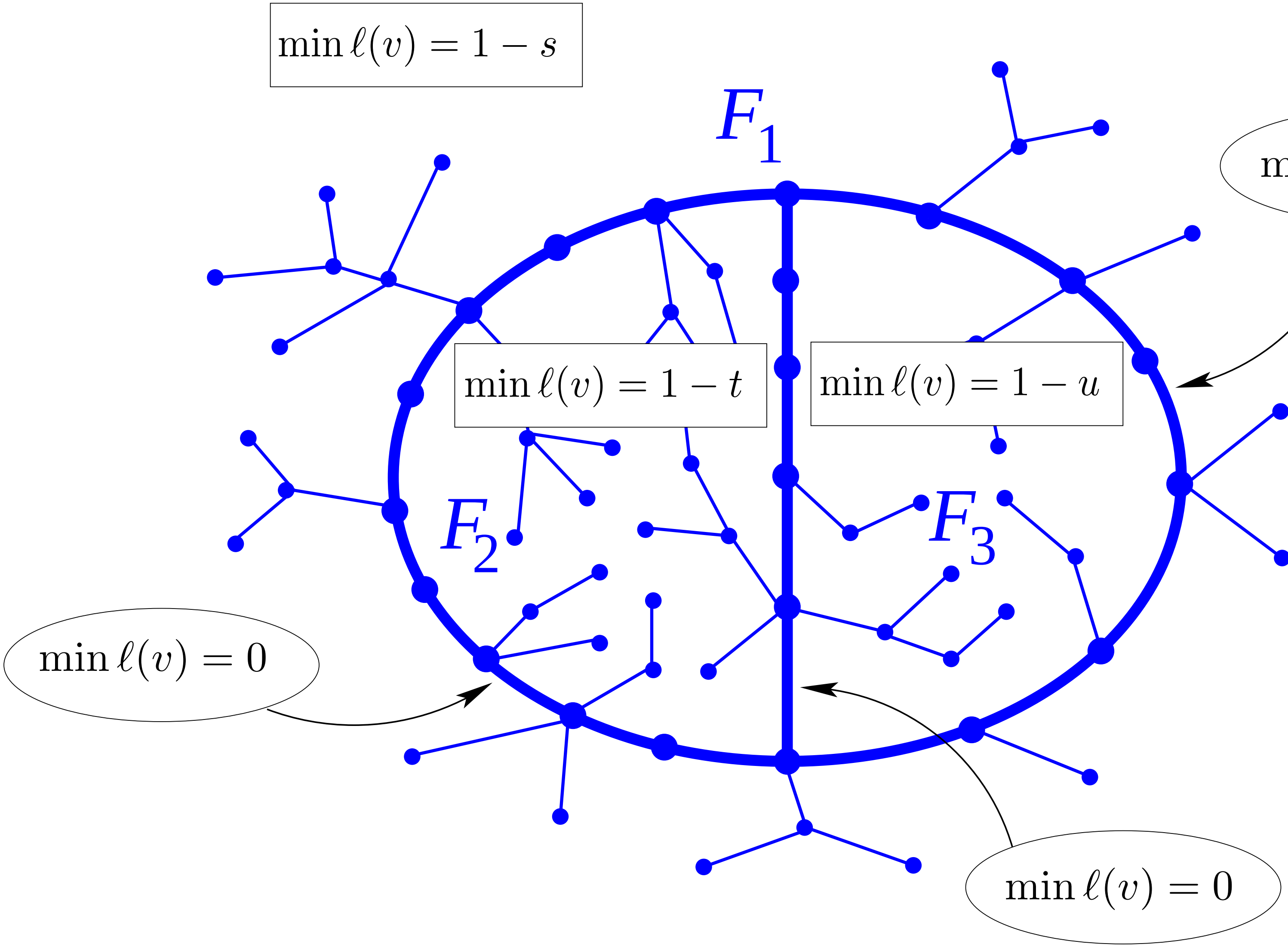
or its degenerate versions 

aligned case

if v_3 lies between v_1 and v_2 ($d_{31} = s$, $d_{23} = t$, $d_{12} = s + t$) apply Miermont's construction on v_1 and v_2 only, with delays $\tau_1 = -s$ and $\tau_2 = -t$



enumeration of well-labeled maps



reminder: the generating function for well-labeled trees planted at a label ℓ and with the condition:

$$\min_{v \in \text{ tree}} \ell(v) \geq 1$$

is given by

$$R_\ell = R \frac{[\ell][\ell+3]}{[\ell+1][\ell+2]} \quad \text{where } [\ell] \equiv \frac{1-x^\ell}{1-x}$$

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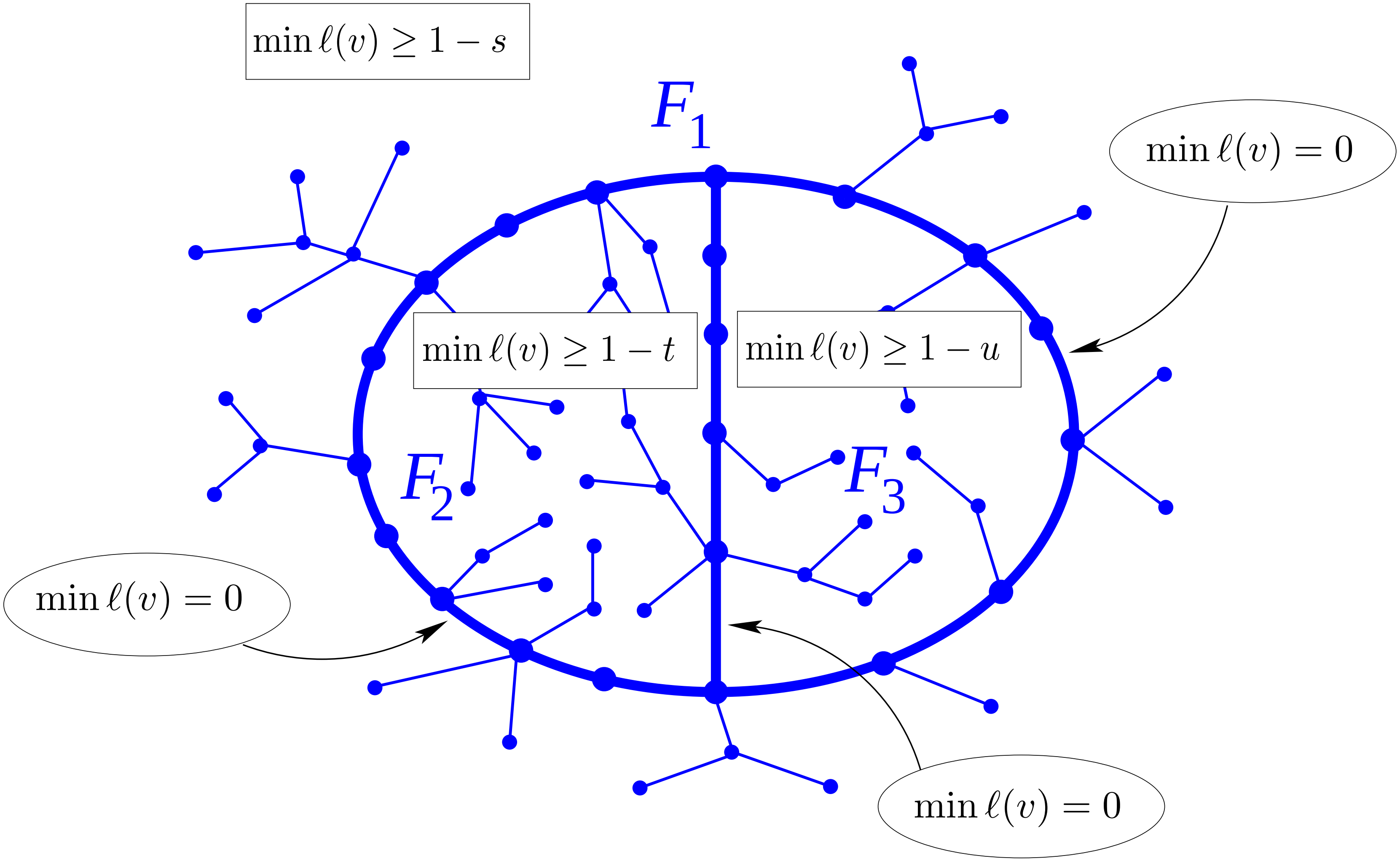
$$R_\ell = R \frac{[\ell][\ell+3]}{[\ell+1][\ell+2]} \quad \text{where } [\ell] \equiv \frac{1-x^\ell}{1-x}$$

if we wish instead: $\min_{v \in \text{ tree}} \ell(v) \geq 1 - s$

this generating function is nothing but: $R_{\ell+s}$

as obtained by a simple shift by s of all labels

consider the generating function $F_{s,t,u}(g)$



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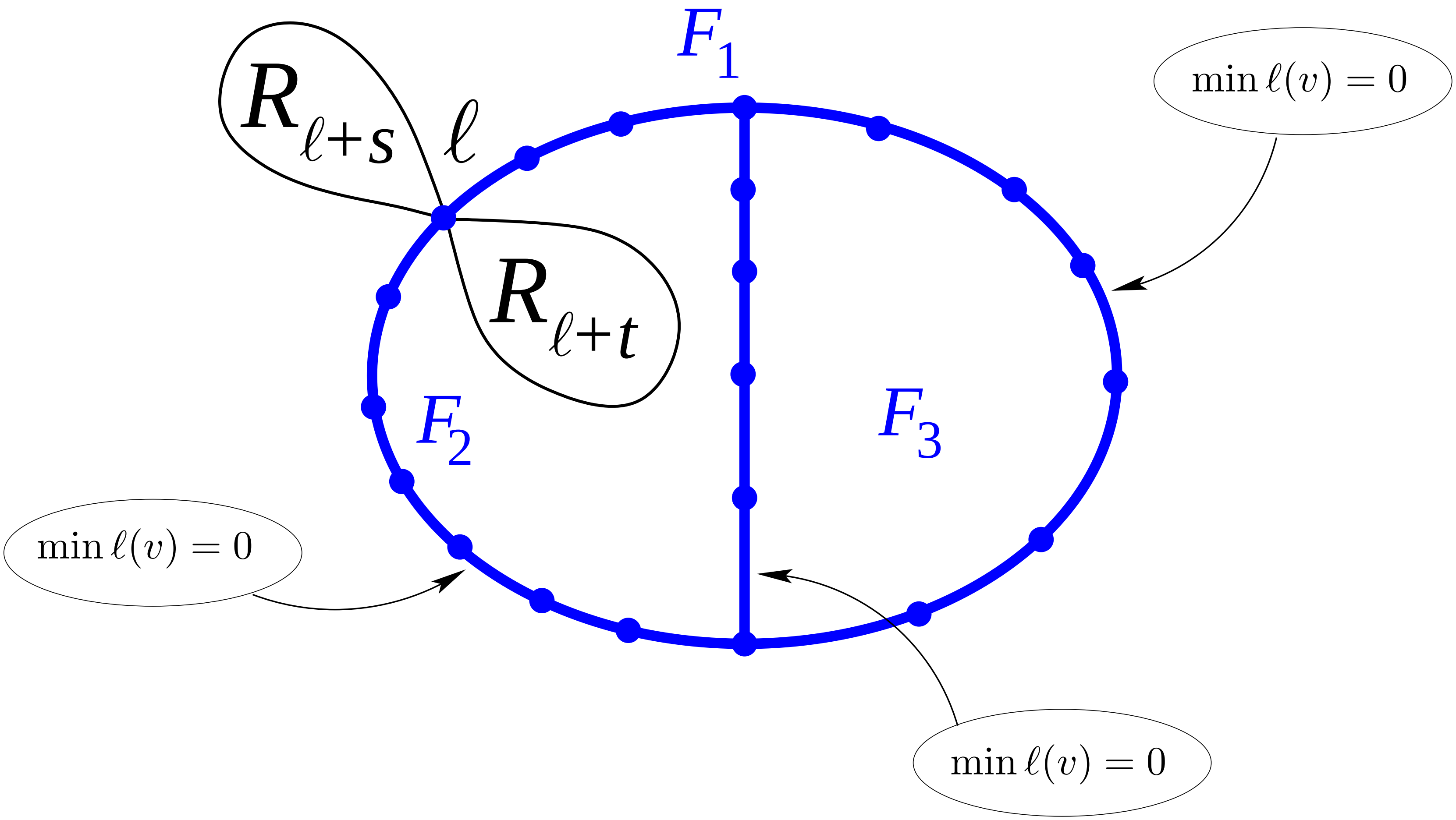

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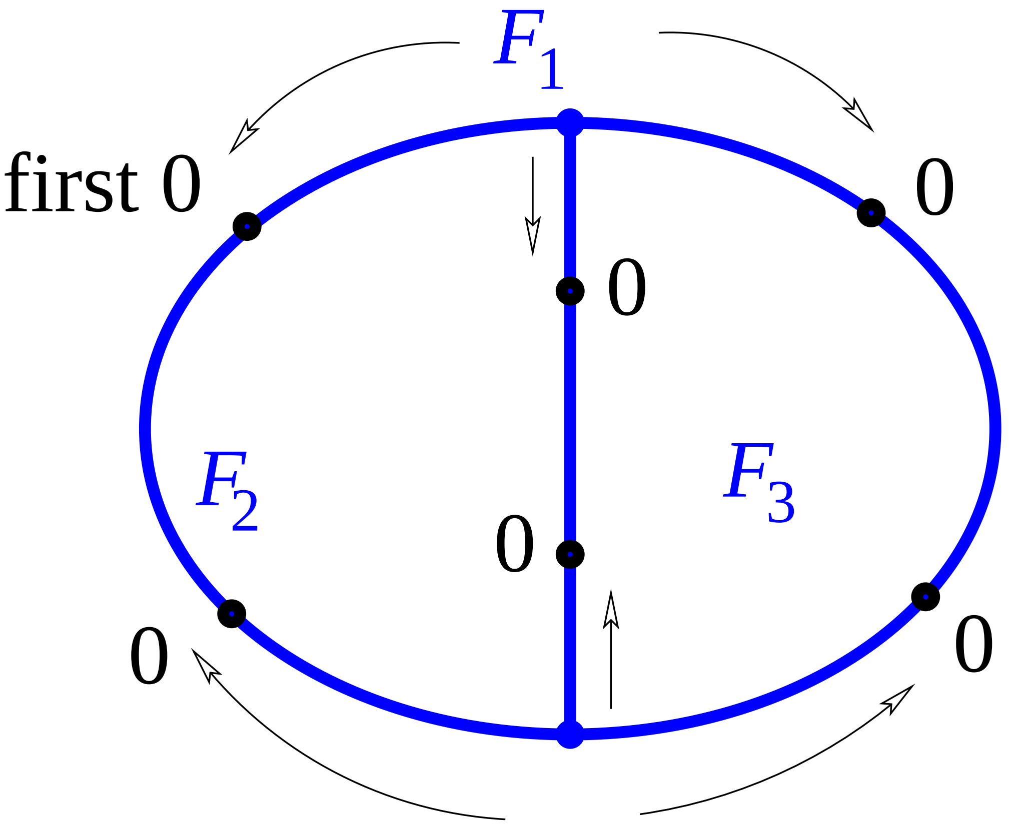

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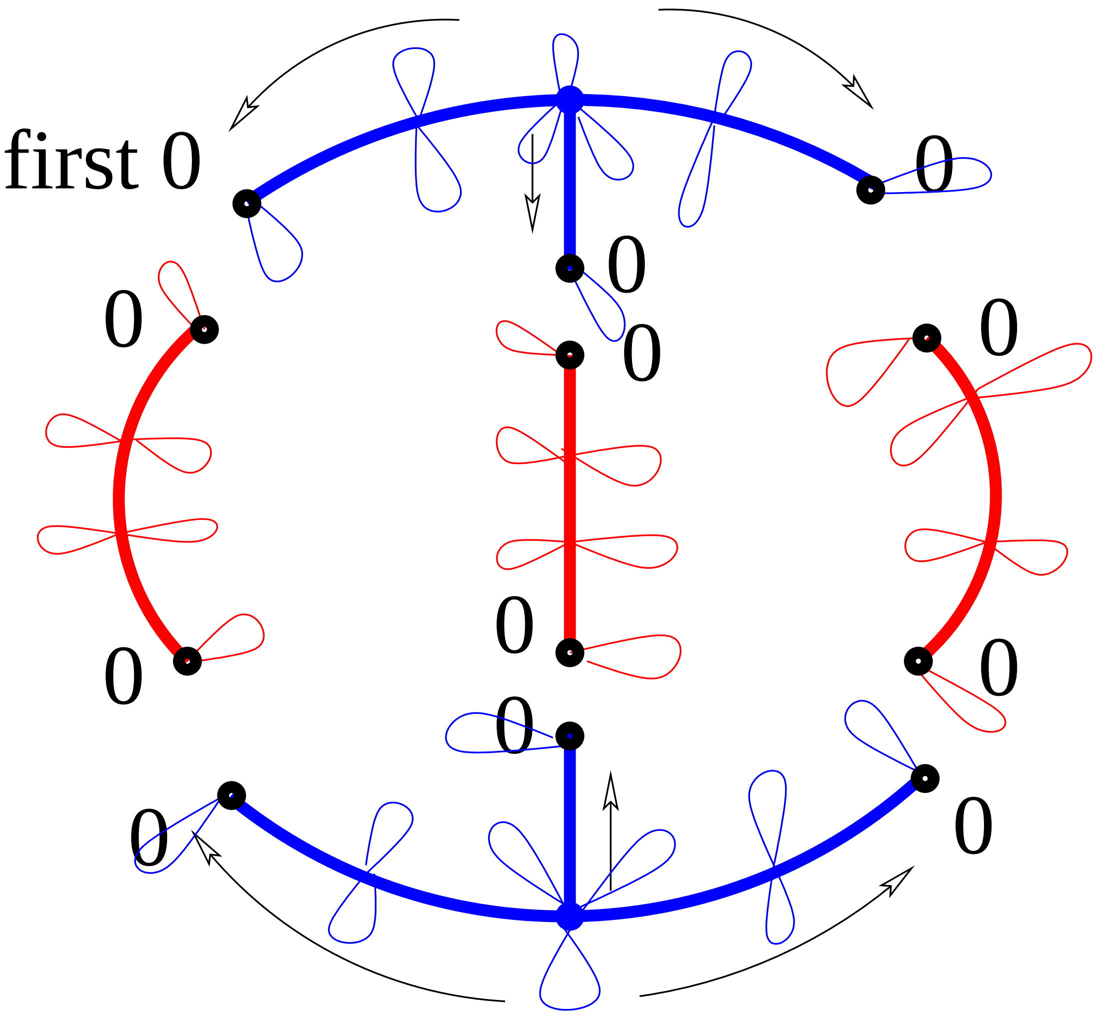

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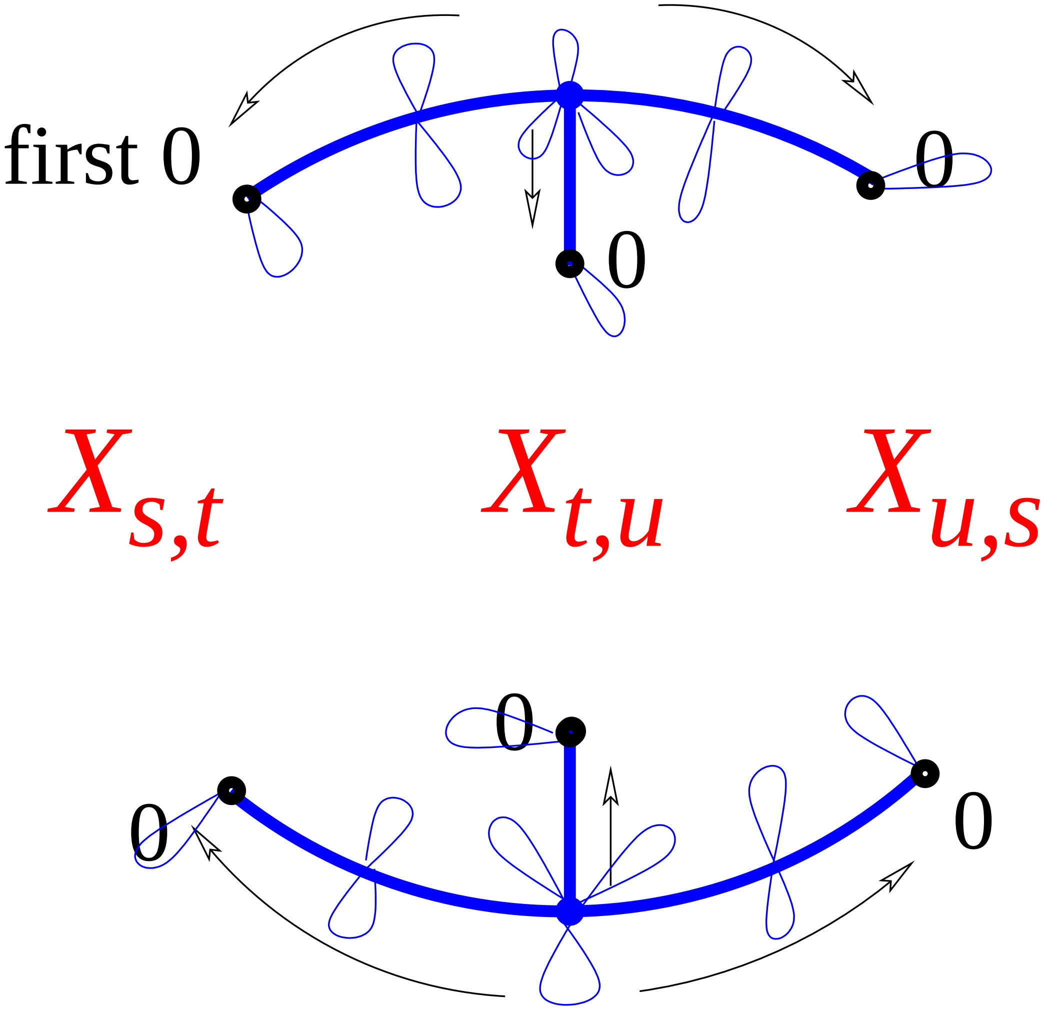


consider the generating function $F_{s,t,u}(g)$

$$Y_{s,t,u}$$

$$X_{s,t} \qquad X_{t,u} \qquad X_{u,s}$$

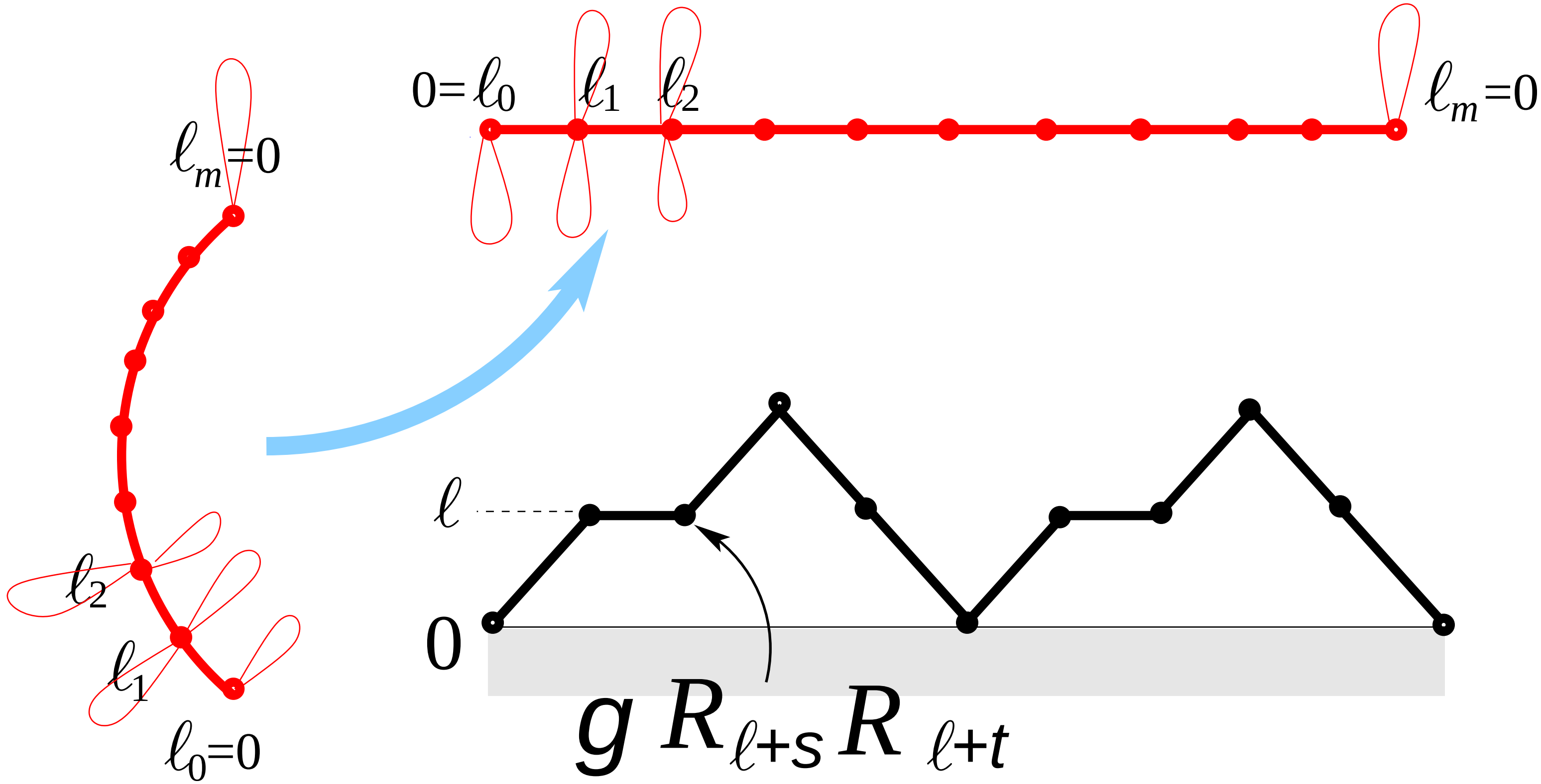
$$Y_{s,t,u}$$

$$F_{s,t,u}(g) = X_{s,t}X_{t,u}X_{u,s}(Y_{s,t,u})^2$$

$Y_{s,t,u}$

$X_{s,t}$ $X_{t,u}$ $X_{u,s}$

$Y_{s,t,u}$



$$X_{s,t} = \sum_{m\geq 0} \sum_{\substack{\text{motzkin paths of length } m \\ \mathcal{M}=(0=\ell_0,\ell_1,\ldots,\ell_m=0)}} \prod_{k=0}^{m-1} g\,R_{\ell_k+s}\,R_{\ell_k+t}$$

recursion relation:

$$X_{s,t} = 1 + gR_sR_t\, X_{s,t}\left(1 + gR_{s+1}R_{t+1}\, X_{s+1,t+1}\right)$$

recursion relation:

$$X_{s,t} = 1 + gR_sR_t\, X_{s,t}\left(1 + gR_{s+1}R_{t+1}\, X_{s+1,t+1}\right)$$

solution:

$$X_{s,t} = \frac{[3][s+1][t+1][s+t+3]}{[1][s+3][t+3][s+t+1]}$$

recursion relation:

$$X_{s,t} = 1 + gR_sR_t\,X_{s,t}\left(1 + gR_{s+1}R_{t+1}\,X_{s+1,t+1}\right)$$

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similarly, recursion relation for $Y_{s,t,u}$:

$$Y_{s,t,u} = 1 + g^3 R_sR_tR_uR_{s+1}R_{t+1}R_{u+1} \times X_{s+1,t+1}X_{t+1,u+1}X_{u+1,s+1}Y_{s+1,t+1,u+1}$$

recursion relation:

$$X_{s,t} = 1 + gR_sR_t\,X_{s,t}\left(1 + gR_{s+1}R_{t+1}\,X_{s+1,t+1}\right)$$

solution:

$$X_{s,t} = \frac{[3][s+1][t+1][s+t+3]}{[1][s+3][t+3][s+t+1]}$$

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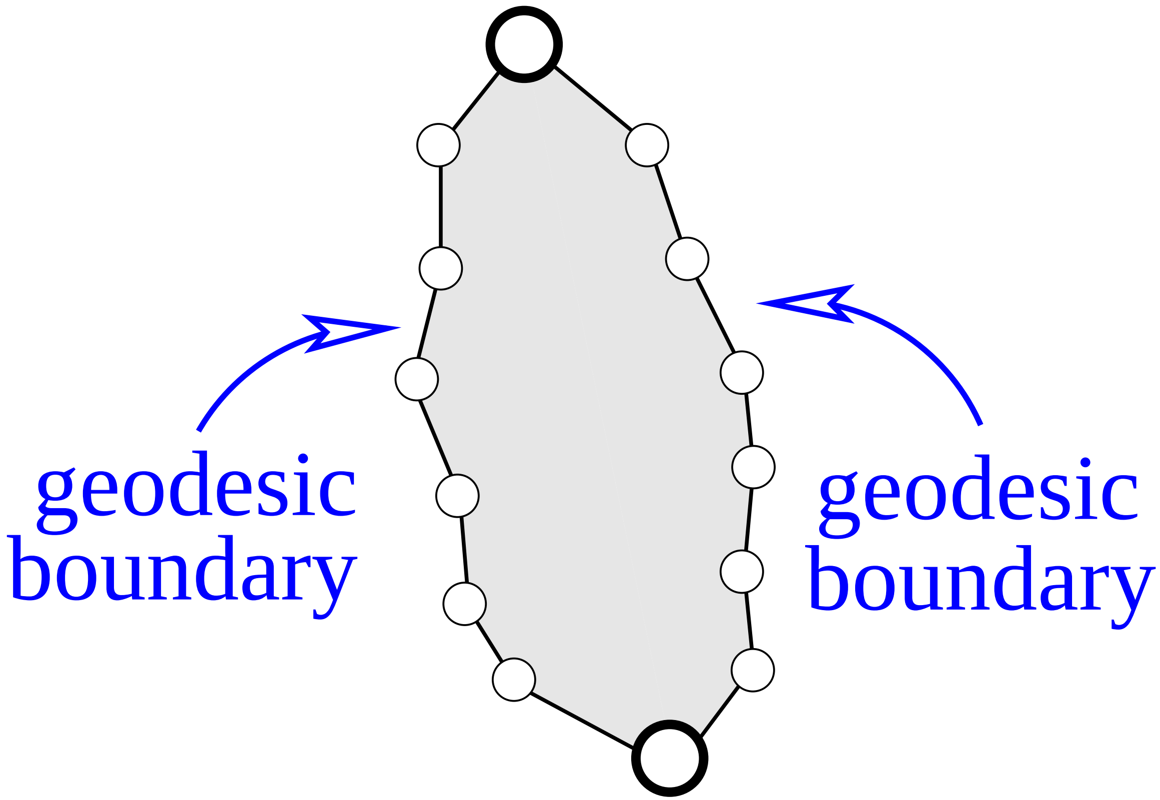
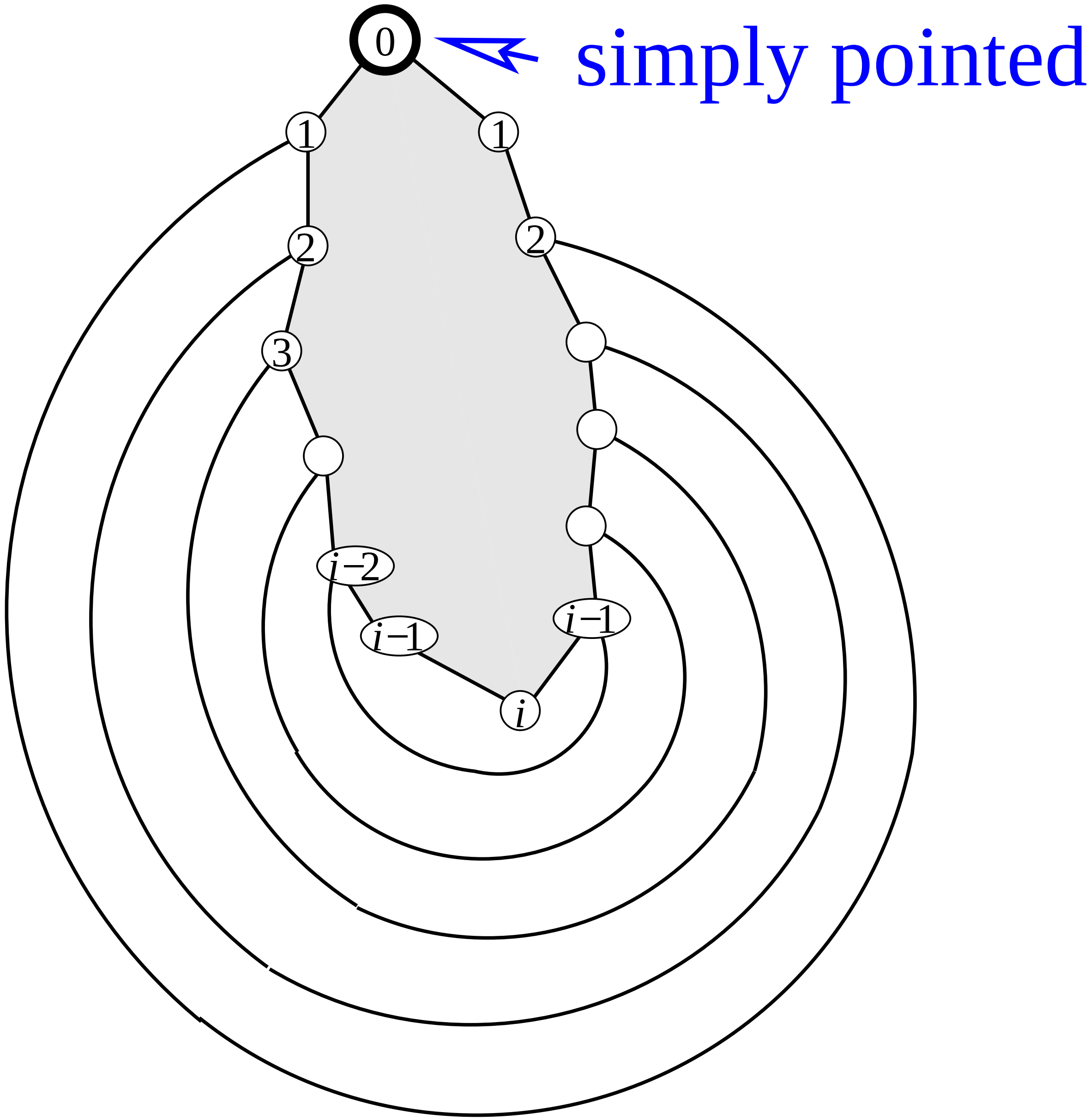
$$Y_{s,t,u} = 1 + g^3 R_sR_tR_uR_{s+1}R_{t+1}R_{u+1} \\ \times X_{s+1,t+1}X_{t+1,u+1}X_{u+1,s+1}Y_{s+1,t+1,u+1}$$

solution:

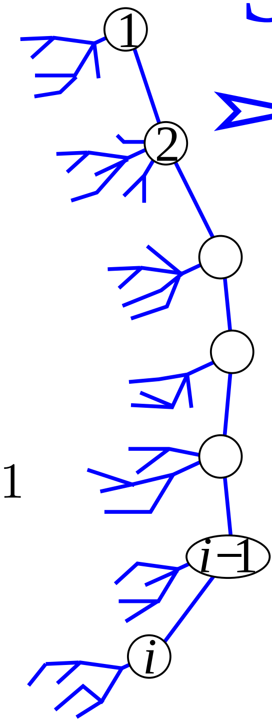
$$Y_{s,t,u} = \frac{[s+3][t+3][u+3][s+t+u+3]}{[3][s+t+3][t+u+3][u+s+3]}$$


geodesic
boundary
geodesic
boundary

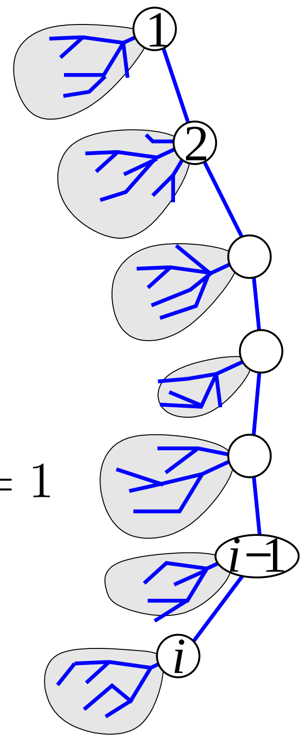

simply pointed
0
1
1
2
2
3
i−2
i−1
i−1
i

Schaeffer

➢ well−labeled (spine) tree

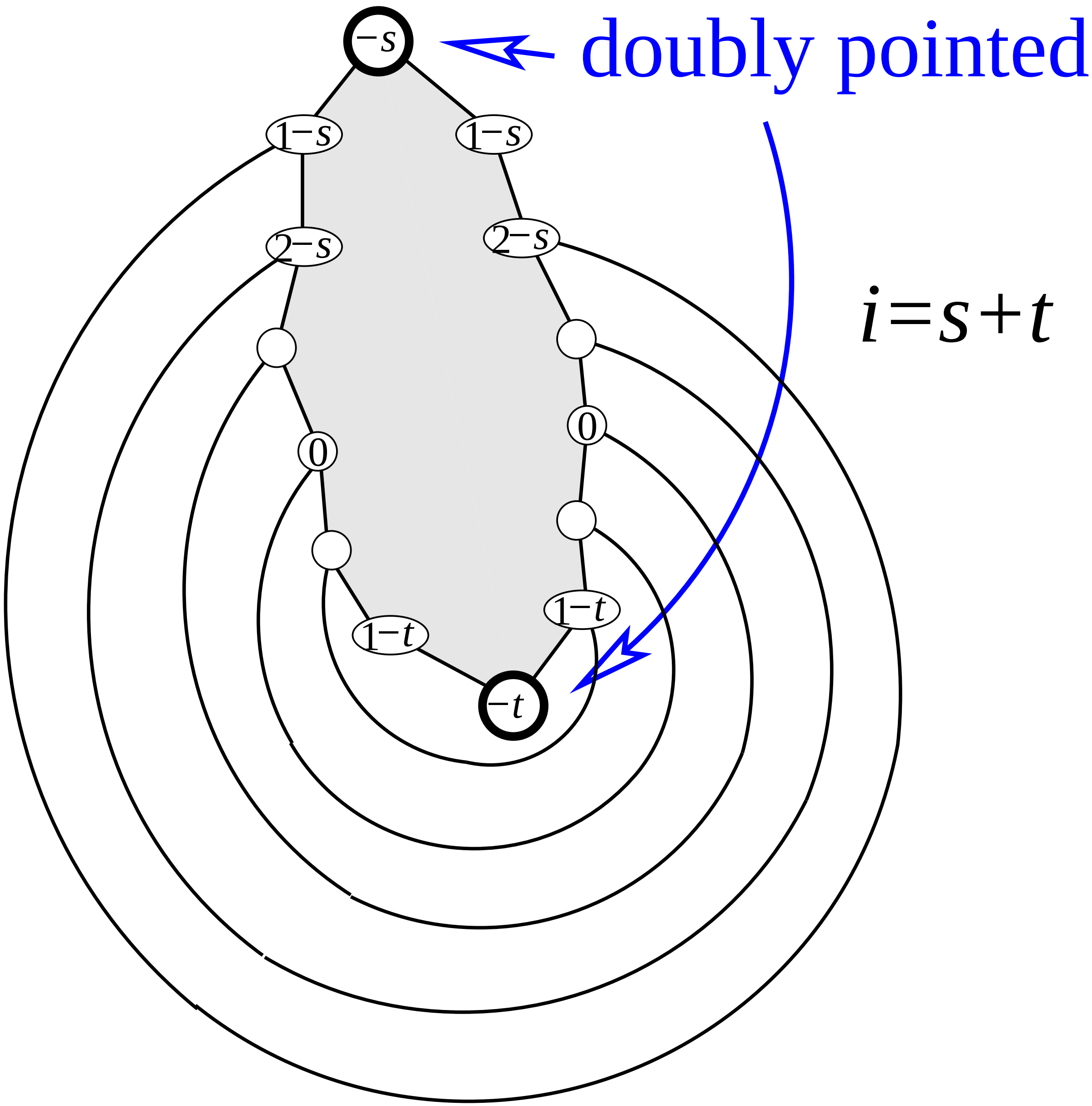


$\min \ell(v) = 1$



$\min \ell(v) = 1$

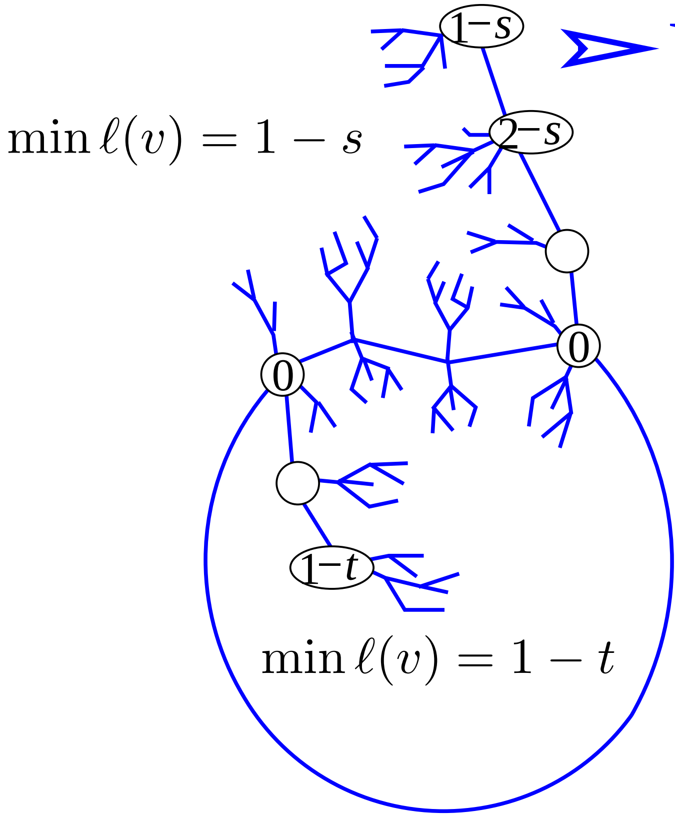
$$\prod_{m=1}^{i} R_m = Z_i$$

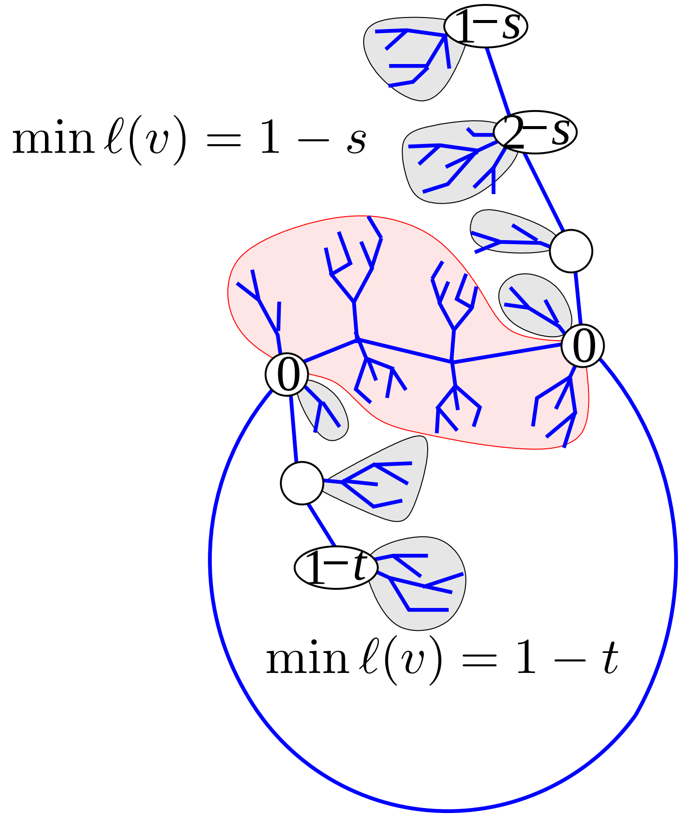


−s
doubly pointed
1−s
1−s
2−s
2−s
i=s+t
0
0
1−t
1−t
−t

Miermont

- well−labeled map with 2 faces





$$\prod_{m=1}^{s} R_m \prod_{m=1}^{t} R_m = X_{s,t}\, Z_s\, Z_t$$

$$X_{s,t} Z_s Z_t = Z_{s+t}$$

namely

$$X_{s,t} = \frac{Z_{s+t}}{Z_s Z_t}$$

with

$$Z_i = \frac{[1][i+3]}{[3][i+1]}$$

so that

$$X_{s,t} = \frac{[3][s+1][t+1][s+t+3]}{[1][s+3][t+3][s+t+1]}$$

three-point function

$$F_{s,t,u}(g) = X_{s,t}X_{t,u}X_{u,s}(Y_{s,t,u})^2$$

$$= \frac{[3]\left([s+1][t+1][u+1][s+t+u+3]\right)^2}{[1]^3[s+t+1][s+t+3][t+u+1][t+u+3][u+s+1][u+s+3]}$$

and the three-point function for quadrangulations reads

$$Q_{d_{12},d_{23},d_{31}}(g) = \Delta_s\Delta_t\Delta_u F_{s,t,u}(g)$$

with $\Delta_s\, f(s) \equiv f(s) - f(s-1)$, and

$s = (d_{12} - d_{23} + d_{31})/2$

$t = (d_{12} + d_{23} - d_{31})/2$

$u = (-d_{12} + d_{23} + d_{31})/2$

scaling limit

$$g = \frac{1}{12}(1-\epsilon^2)\ , \qquad \ell = L\epsilon^{-1/2} \quad \text{with } \epsilon \to 0$$

replace in any well-balanced combination of $[\cdot]$'s:

$$[\ell] = \frac{1-x^\ell}{1-x} \to \sinh(\alpha L)\ , \qquad \alpha = \sqrt{\frac{3}{2}}$$

$$s = S\epsilon^{-1/2},\ t = T\epsilon^{-1/2},\ u = U\epsilon^{-1/2}$$

$$X_{s,t} \to 3\ , \qquad Y_{s,t,u} \to \epsilon^{-1/2}\,\mathcal{Y}(S,T,U;\sqrt{3/2})$$

with

$$\mathcal{Y}(S,T,U;\alpha) \equiv \frac{1}{3\alpha}\frac{\sinh\alpha S \sinh\alpha T \sinh\alpha U \sinh\alpha(S+T+U)}{\sinh\alpha(S+T)\sinh\alpha(T+U)\sinh\alpha(U+S)}$$

$$F_{s,t,u}(g) \sim \epsilon^{-1} \mathcal{F}(S, T, U; \sqrt{3/2})$$

with $\mathcal{F}(S, T, U; \alpha) =$

$$\frac{3}{\alpha^2} \left(\frac{\sinh(\alpha(S+T+U)) \sinh(\alpha S) \sinh(\alpha T) \sinh(\alpha U)}{\sinh(\alpha(S+T)) \sinh(\alpha(T+U)) \sinh(\alpha(U+S))} \right)^2$$

$$Q_{d_{12},d_{23},d_{31}}(g) \sim \epsilon^{1/2} \mathcal{Q}(D_{12}, D_{23}, D_{31}; \sqrt{3/2})$$

with

$$\mathcal{Q}(D_{12}, D_{23}, D_{31}; \alpha) = \frac{1}{2} \partial_S \partial_T \partial_U \mathcal{F}(S, T, U; \alpha)$$

$S = (D_{12} - D_{23} + D_{31})/2$
$T = (D_{12} + D_{23} - D_{31})/2$
$U = (-D_{12} + D_{23} + D_{31})/2$

three-point function

$$d_{12} = D_{12} n^{1/4} \qquad d_{23} = D_{23} n^{1/4} \qquad d_{31} = D_{31} n^{1/4}$$

$$s = S n^{1/4} \qquad t = T n^{1/4} \qquad u = U n^{1/4}$$

we get the probability density

$$\rho(D_{12}, D_{23}, D_{31}) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} d\xi \, \frac{\xi}{\mathrm{i}} \, e^{-\xi^2} \mathcal{Q}(D_{12}, D_{23}, D_{31}; \sqrt{3/2}\sqrt{-\mathrm{i}\xi})$$

$\rho(D_{12}, D_{23}, D_{31}) dD_{12} dD_{23} dD_{31}$ is the probability that the three marked vertices be at rescaled pairwise distances in the ranges $[D_{12}, D_{12} + dD_{12}]$, $[D_{23}, D_{23} + dD_{23}]$, $[D_{31}, D_{31} + dD_{31}]$, in the ensemble of triply-pointed quadrangulations of fixed large size n

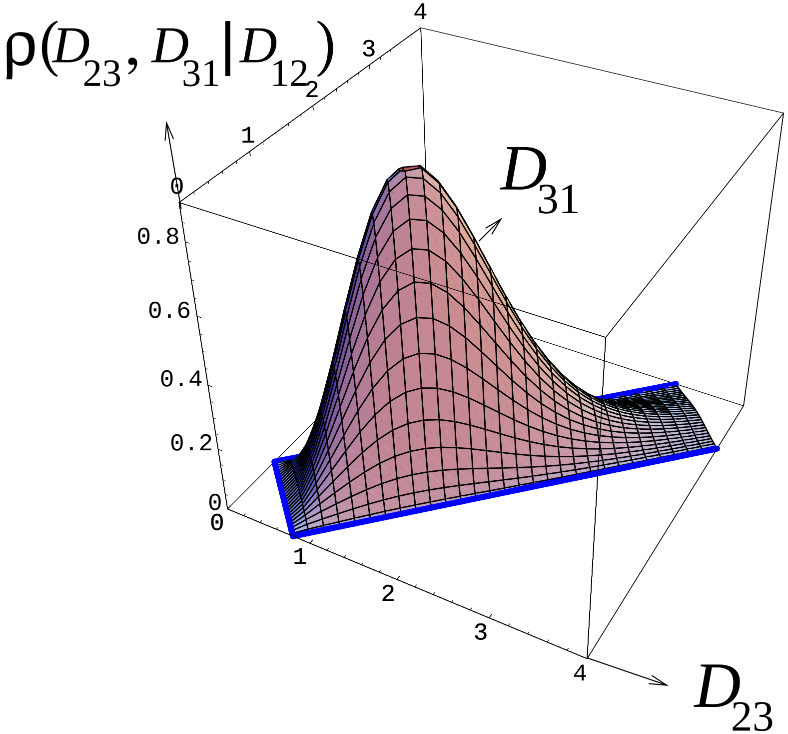
conditional probability density

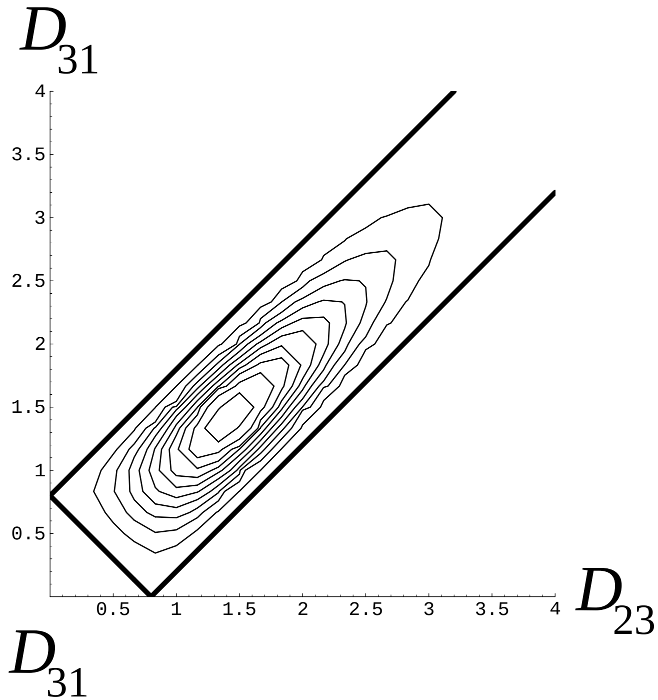
fix one of the distances, say D_{12}, and consider the conditional probability density

$$\rho(D_{23}, D_{31}|D_{12}) \equiv \frac{\rho(D_{12}, D_{23}, D_{31})}{\rho(D_{12})}$$

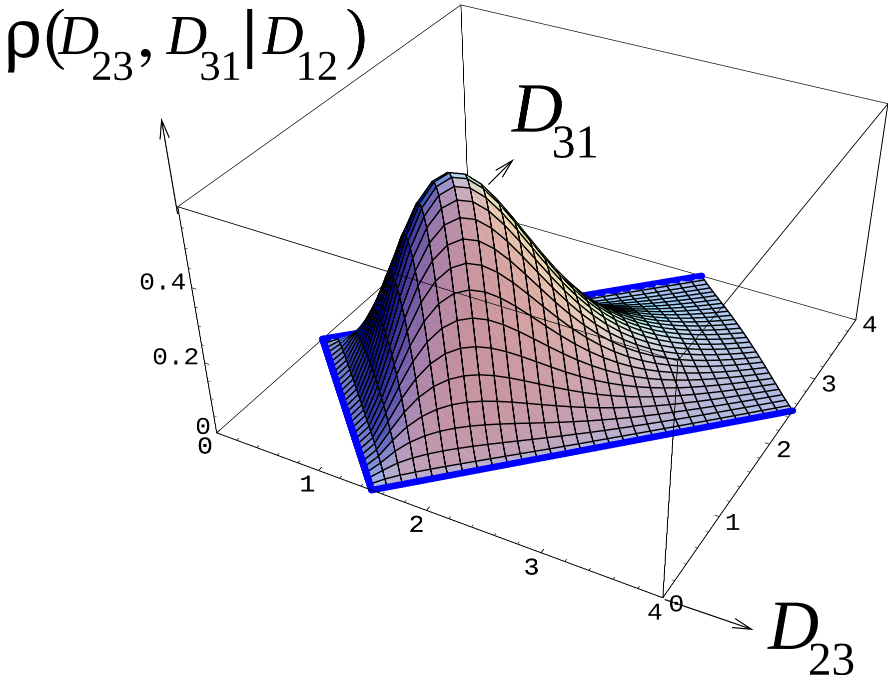
$\rho(D_{23}, D_{31}|D_{12})dD_{23}dD_{31}$ is the probability that the third marked vertex be at rescaled distances in the ranges $[D_{23}, D_{23} + dD_{23}]$ and $[D_{31}, D_{31} + dD_{31}]$ from the first two marked vertices in the ensemble of triply-pointed quadrangulations of fixed large size n, given that the distance between the first two marked vertices is D_{12}

$D_{12} = 0.8$

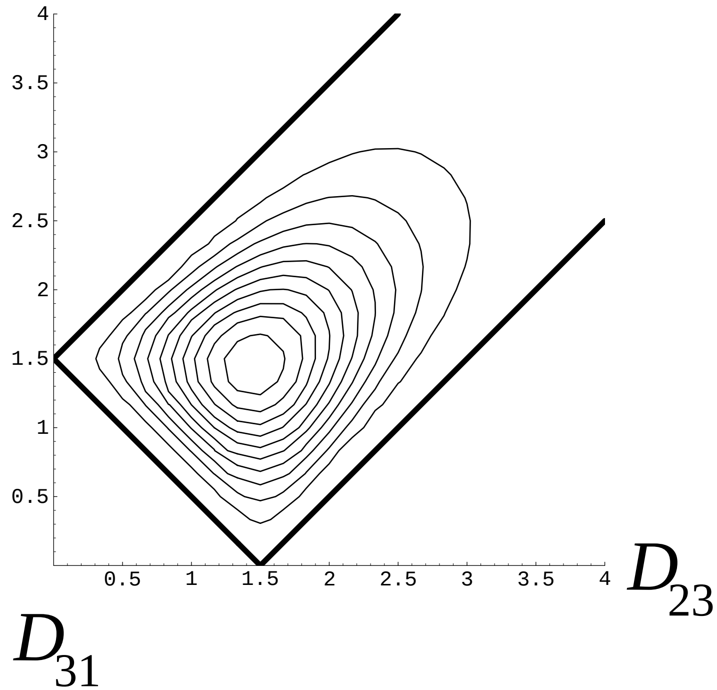

ρ(D23, D31|D12)
D31
D23
0.8
0.6
0.4
0.2
0
1
2
3
4


D31
D23
D31
0.5
1
1.5
2
2.5
3
3.5
4

$D_{12} = 1.5$

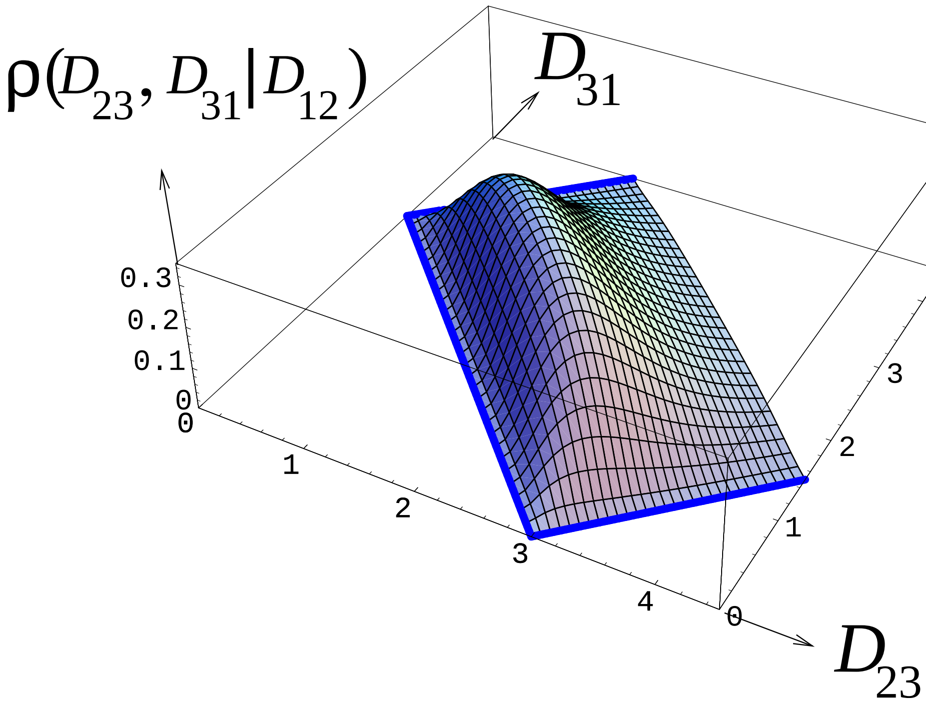


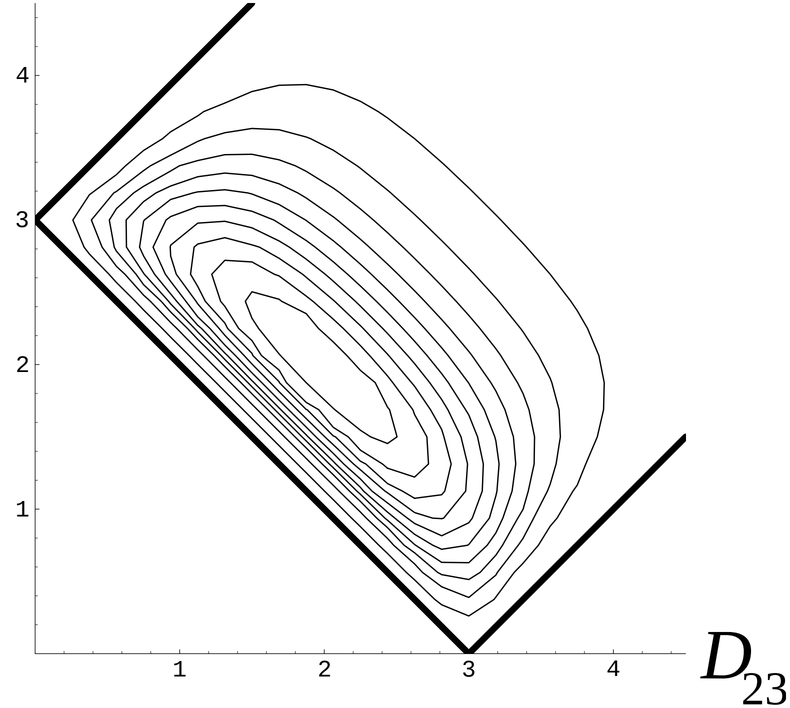
ρ(D23, D31|D12)
D31
D23
0.4
0.2
0
0
1
2
3
4
0
1
2
3
4



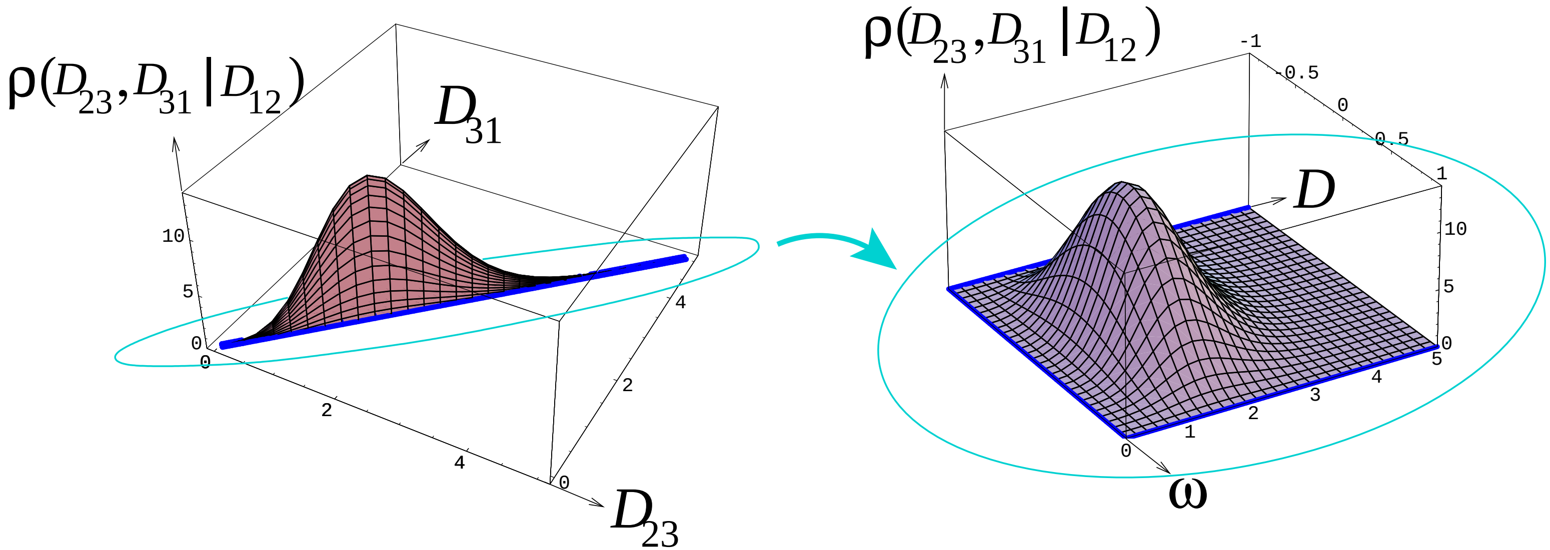
4
3.5
3
2.5
2
1.5
1
0.5
0.5
1
1.5
2
2.5
3
3.5
4
D23
D31

$D_{12} = 3.0$





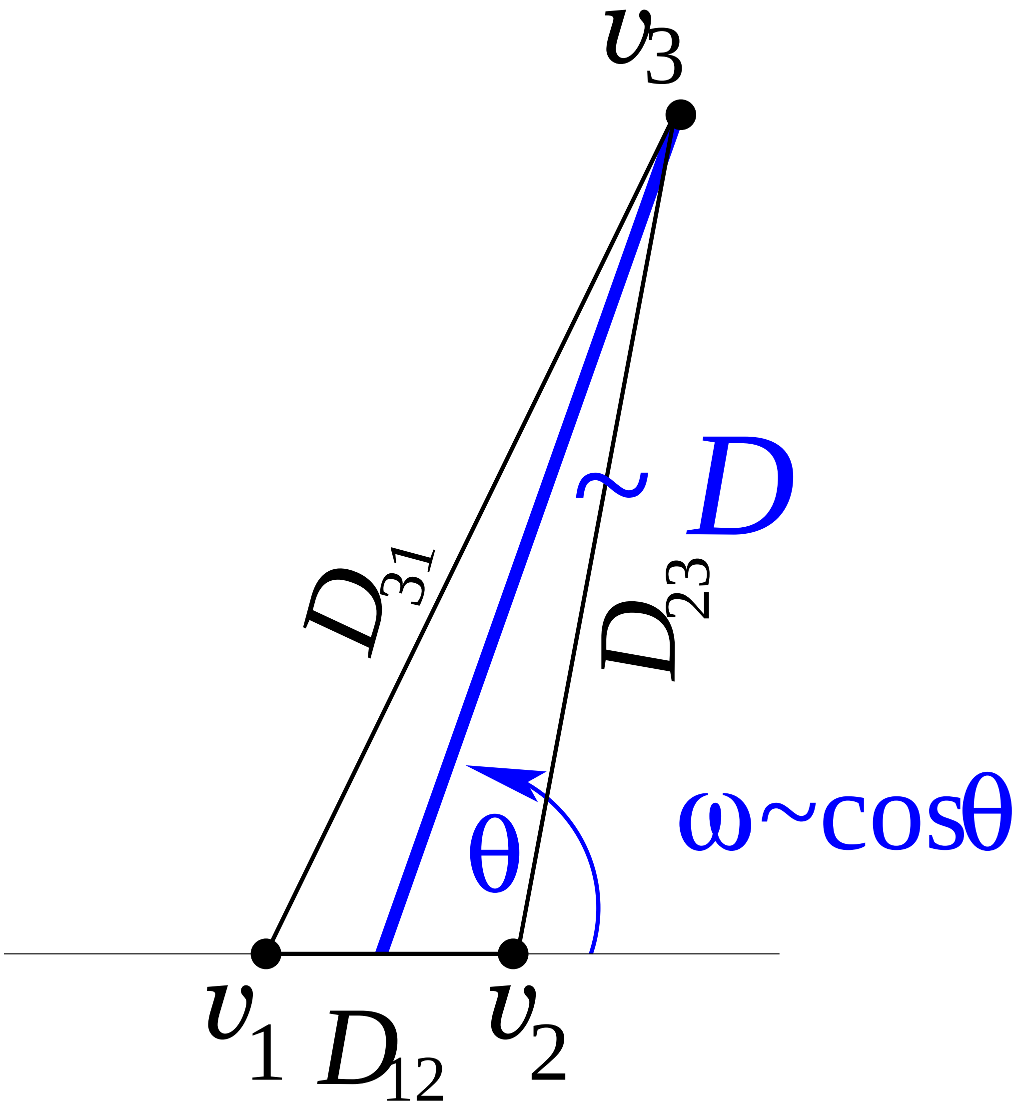
limit of small D_{12}



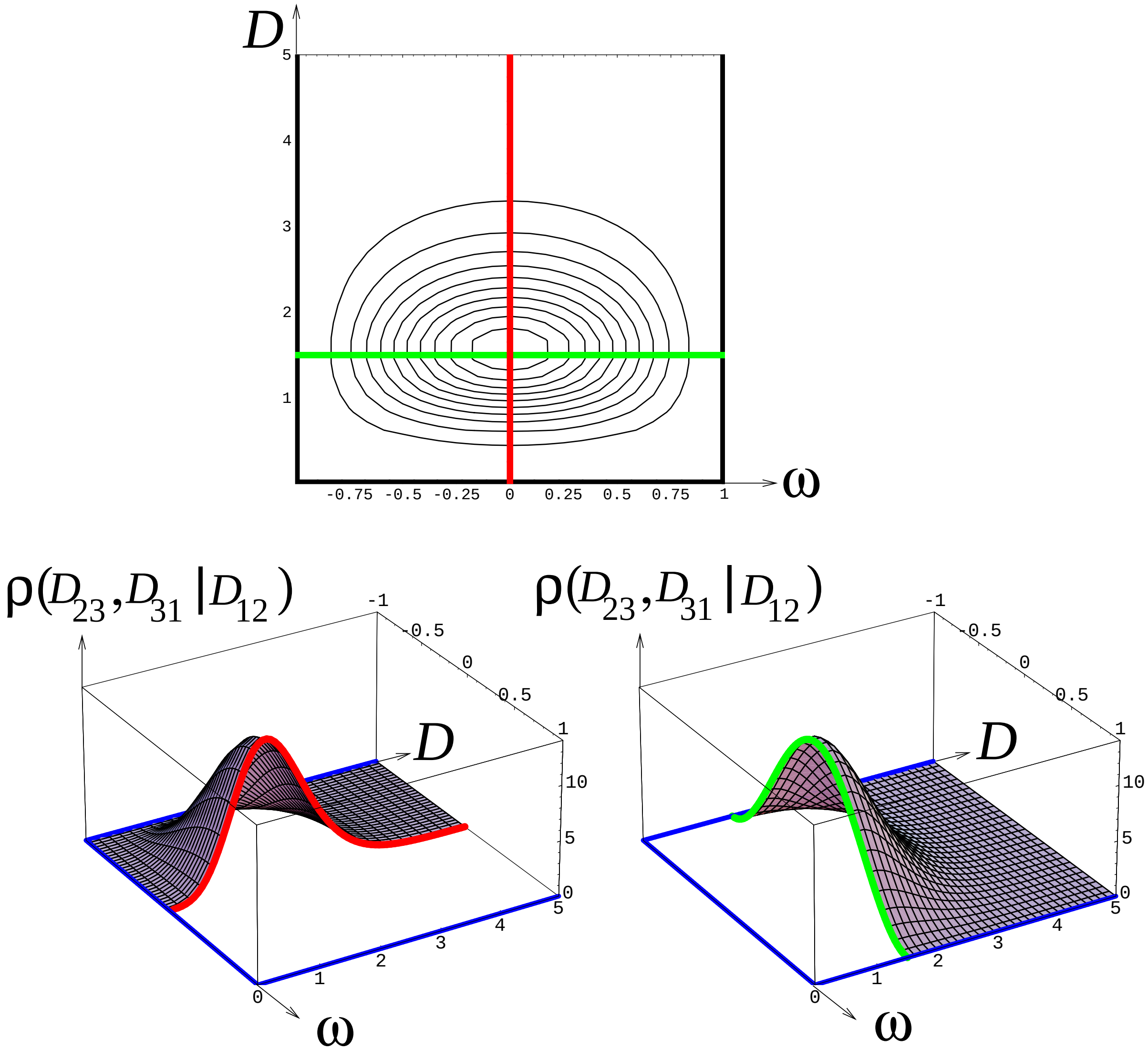
$$\rho(D_{23}, D_{31}|D_{12}) \sim \frac{1}{D_{12}} \times \rho(D) \times \psi(\omega)$$

where $D = (D_{23} + D_{31})/2$, $\omega = (D_{31} - D_{23})/D_{12}$, and

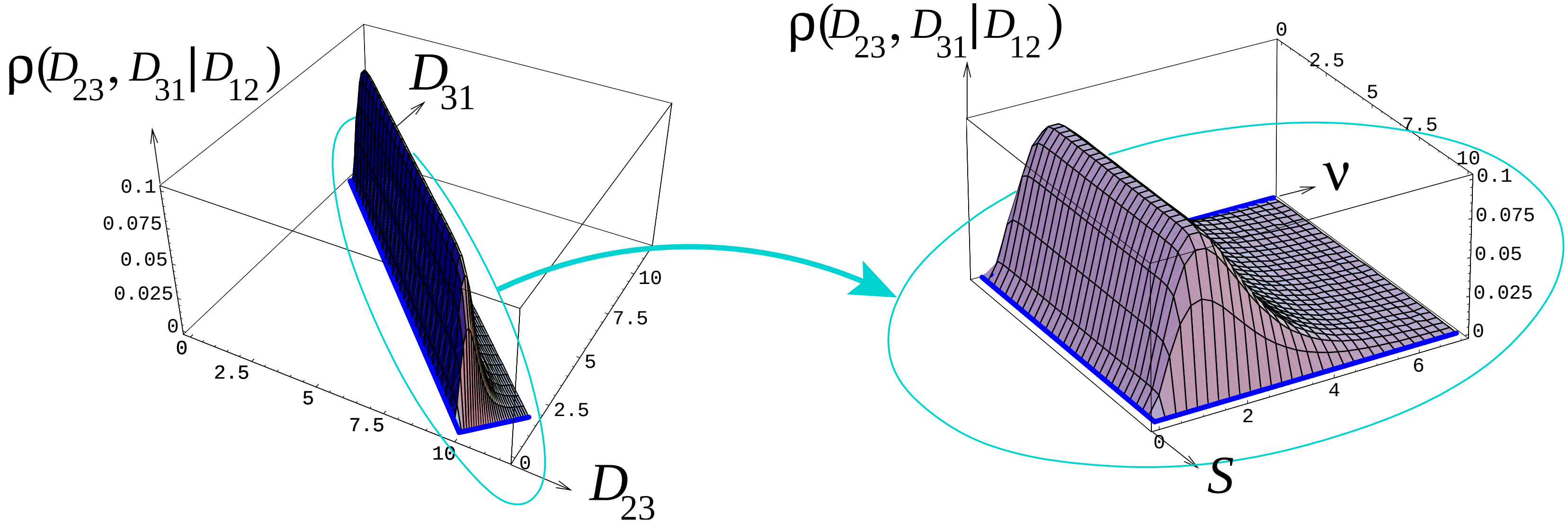
$$\psi(\omega) \equiv \frac{21}{64}(1 - \omega^2)^2(3 - \omega^2)$$



v_3
$\sim D$
D_{31}
D_{23}
θ
ω~cosθ
v_1
D_{12}
v_2


D
5
4
3
2
1
-0.75 -0.5 -0.25 0 0.25 0.5 0.75 1
ω
ρ(D23, D31 | D12)
-1
-0.5
0
0.5
1
D
10
5
0
0
1
2
3
4
5
ω
ρ(D23, D31 | D12)
-1
-0.5
0
0.5
1
D
10
5
0
0
1
2
3
4
5
ω

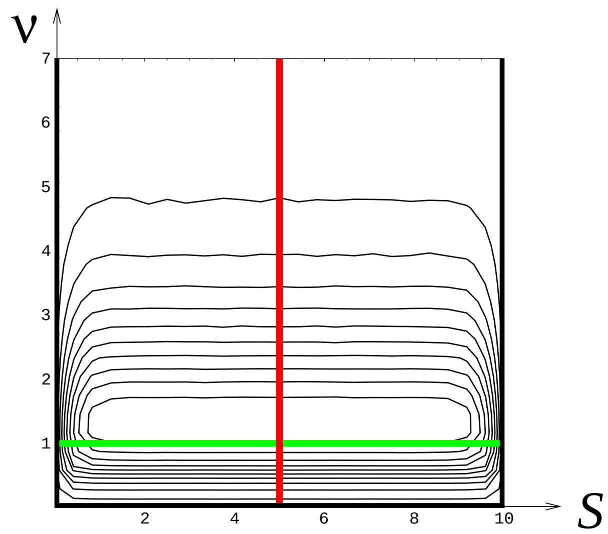
limit of large D_{12}



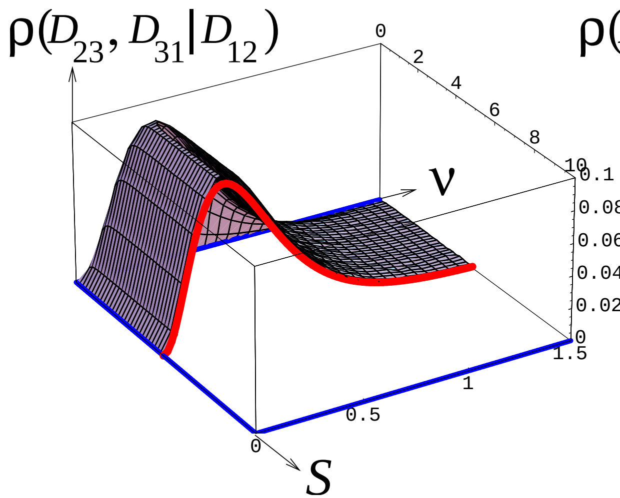
$$\rho(D_{23}, D_{31}|D_{12}) \sim \frac{1}{2D_{12}} \times (9D_{12})^{1/3}\varphi(\nu)$$

where $\nu = (9D_{12})^{1/3}(D_{23} + D_{31} - D_{12})/2$, and

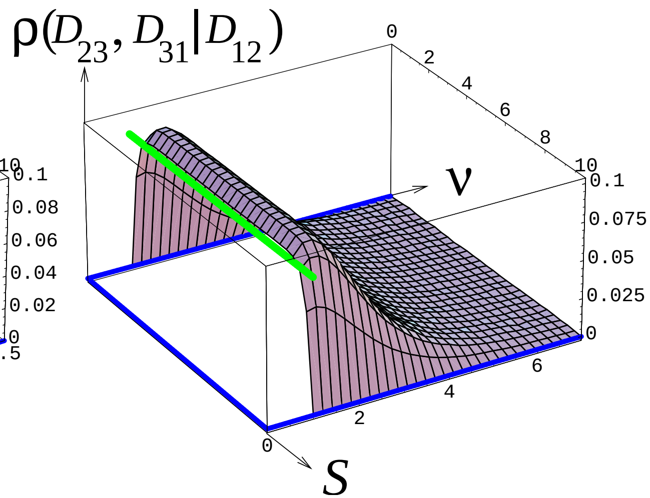
$$\varphi(\nu) \equiv \frac{4}{3}\sinh(\nu/2)^2 \left(11e^{-2\nu} - 8e^{-3\nu}\right)$$



ν
7
6
5
4
3
2
1
2
4
6
8
10
S



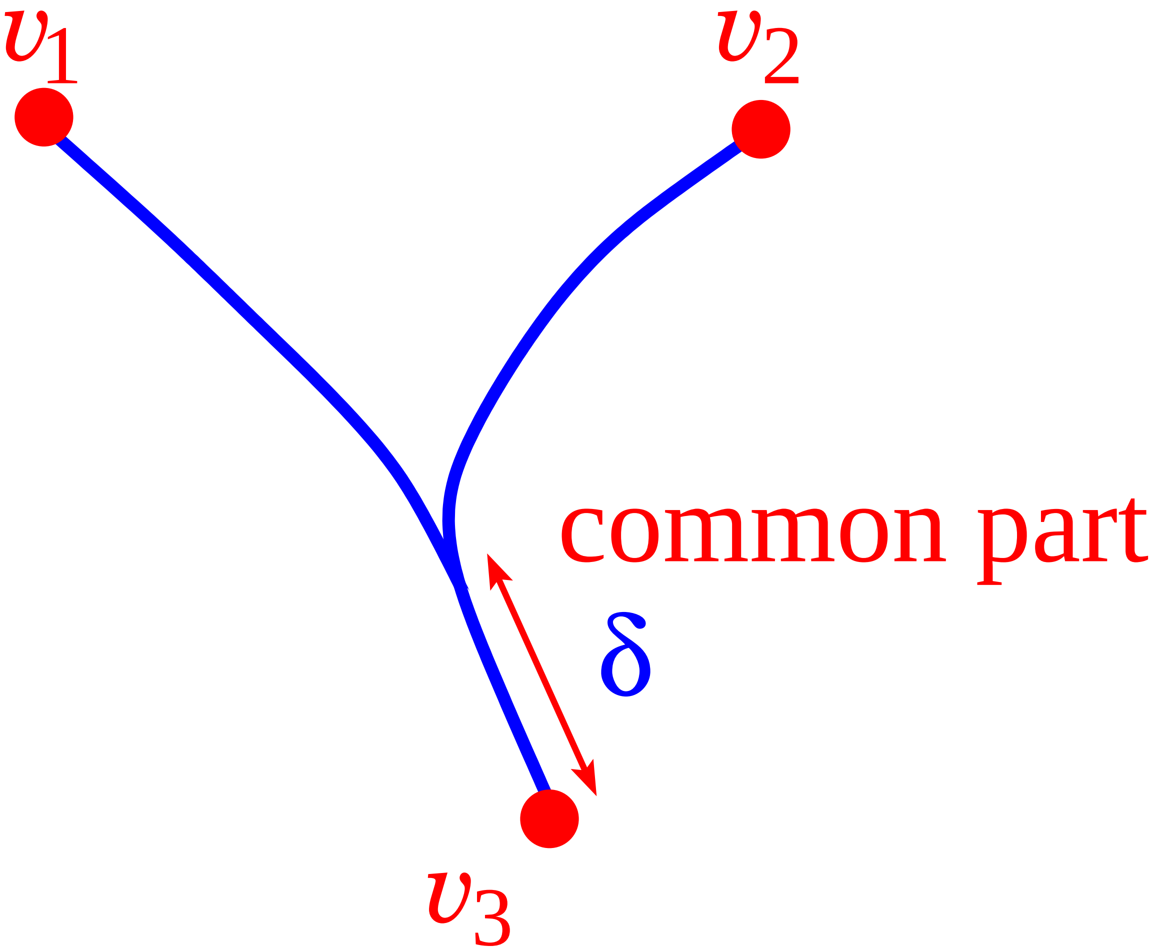
ρ(D23, D31|D12)
0
2
4
6
8
10
ν
0.1
0.08
0.06
0.04
0.02
0
1.5
1
0.5
0
S



ρ(D23, D31|D12)
0
2
4
6
8
10
ν
0.1
0.075
0.05
0.025
0
6
4
2
0
S

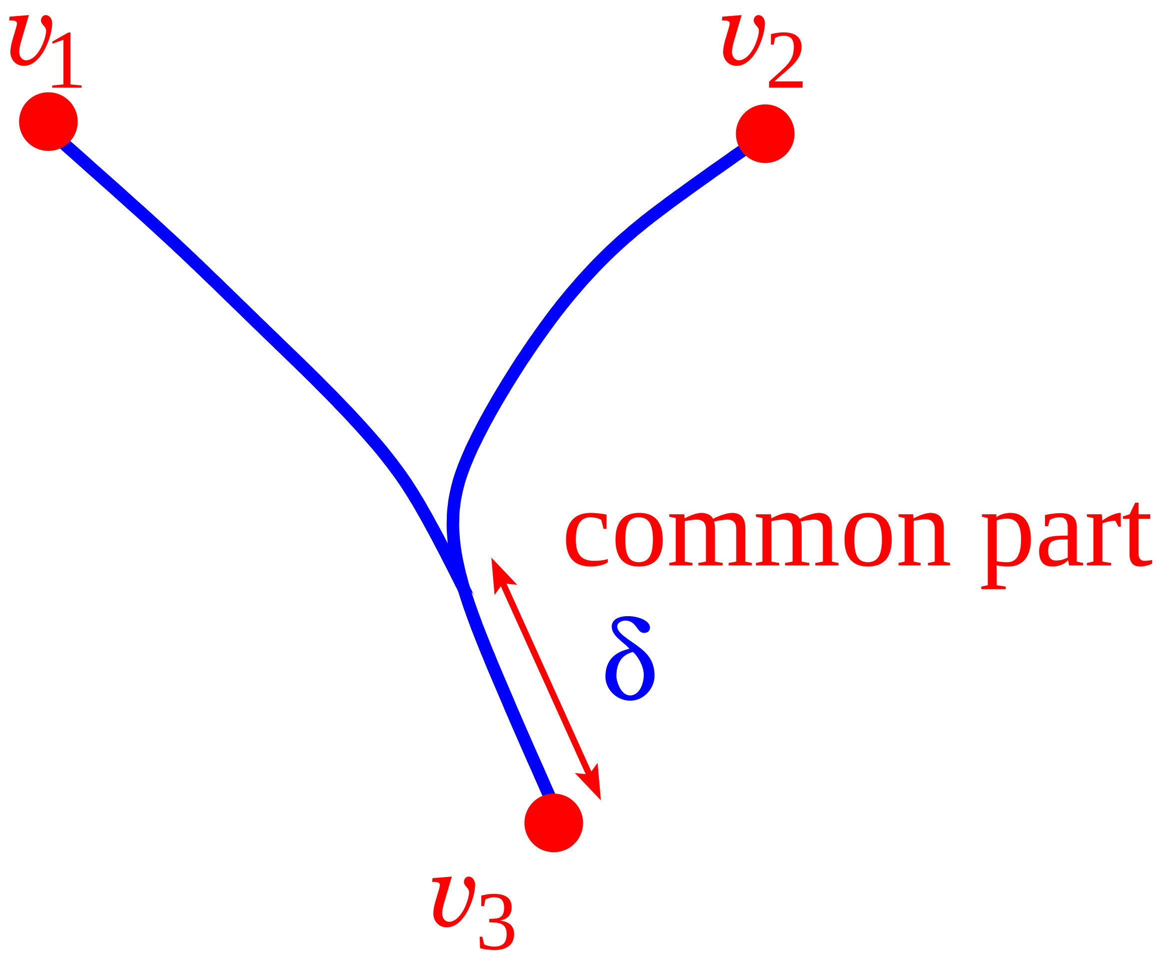
other applications

phenomenon of confluence of geodesics (Le Gall)



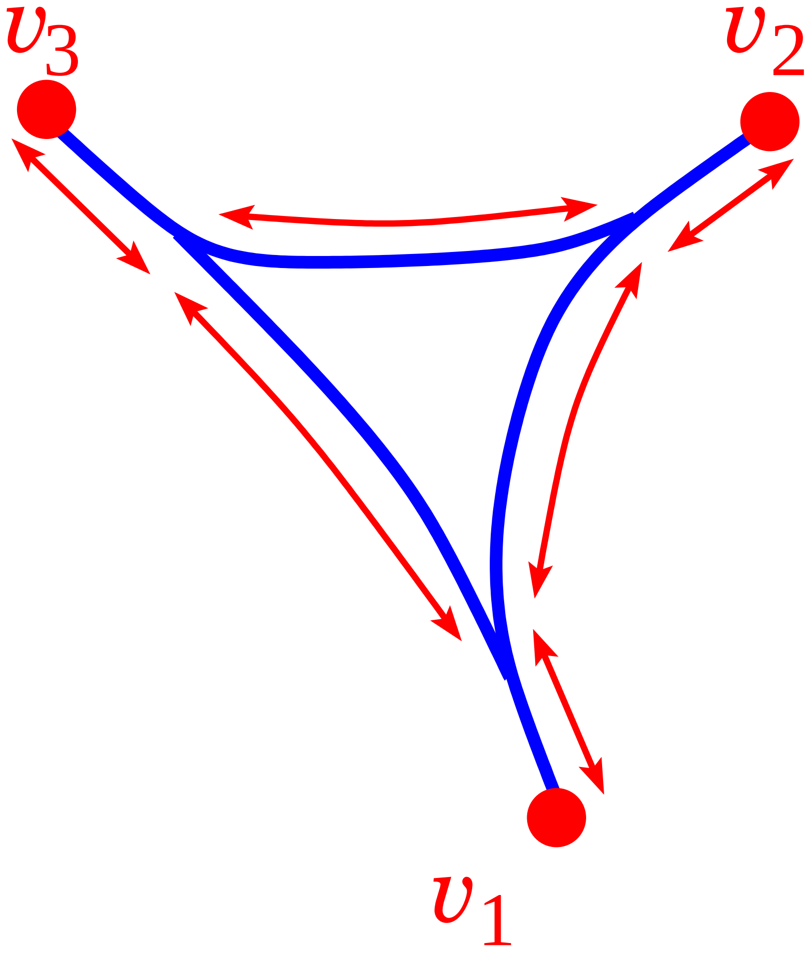
other applications

phenomenon of confluence of geodesics (Le Gall)

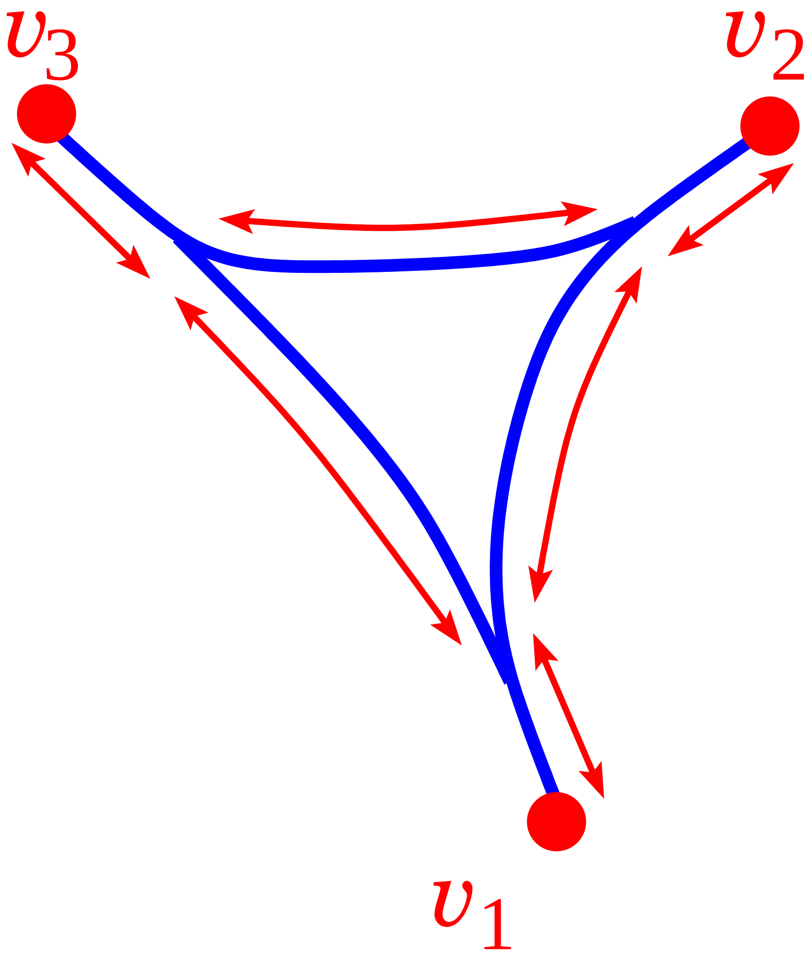


what is the law for the length δ of the common part ?

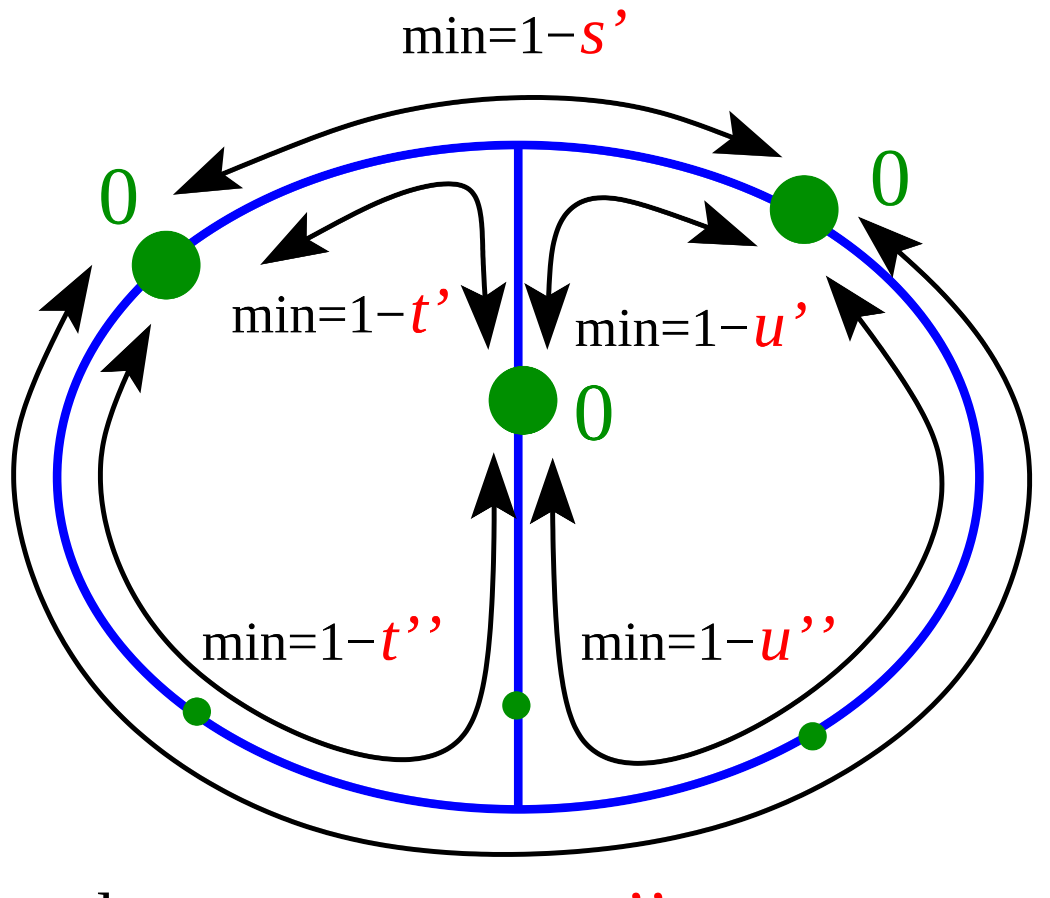
geodesics triangle



geodesics triangle

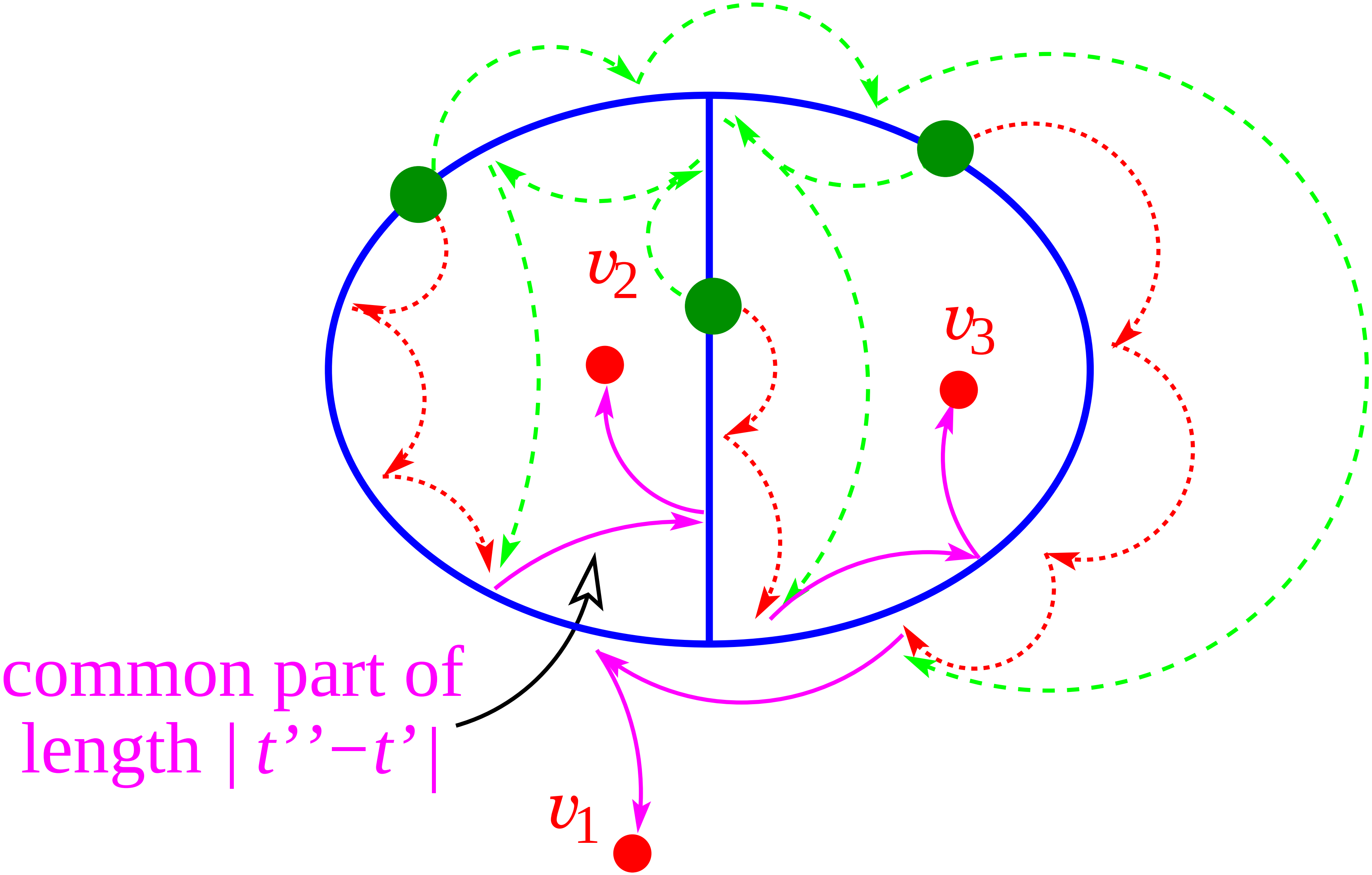


joint law for the 6 lengths and 2 areas ?

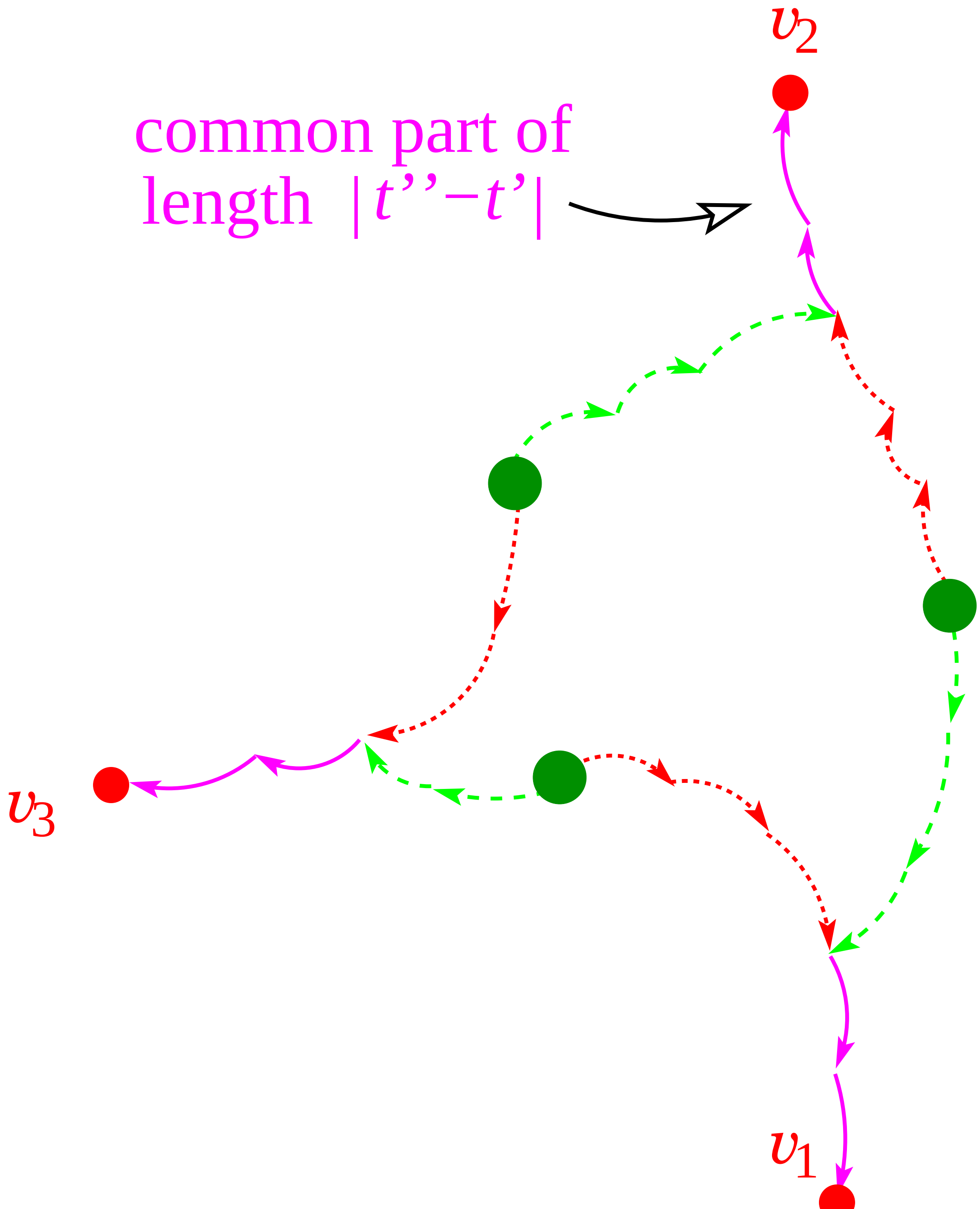


involves

$$X_{s'',t''}\ X_{t'',u''}\ X_{u'',s''}\ Y_{s',t',u'}\ Y_{s'',t'',u''}$$

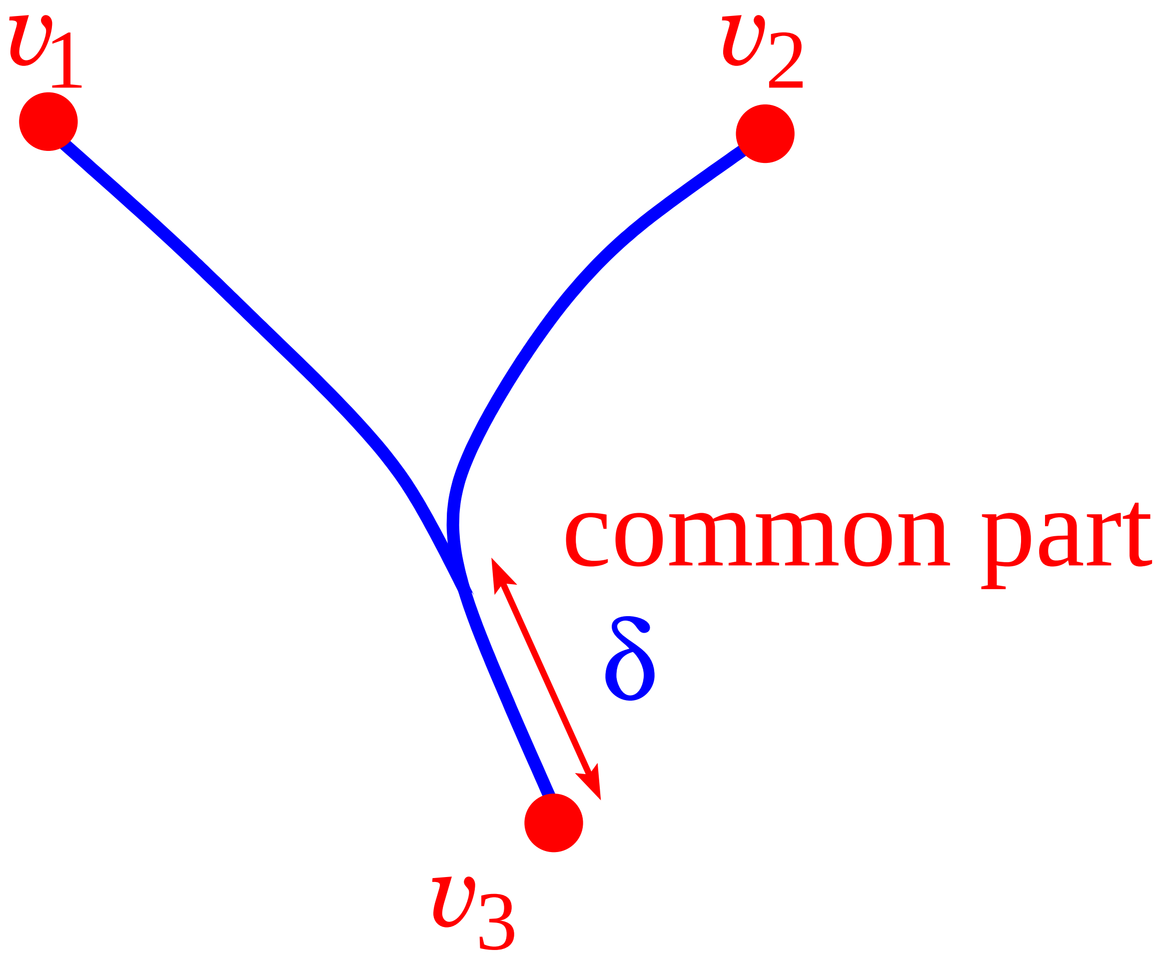


v_2
v_3
v_1
common part of
length | $t''-t'$|

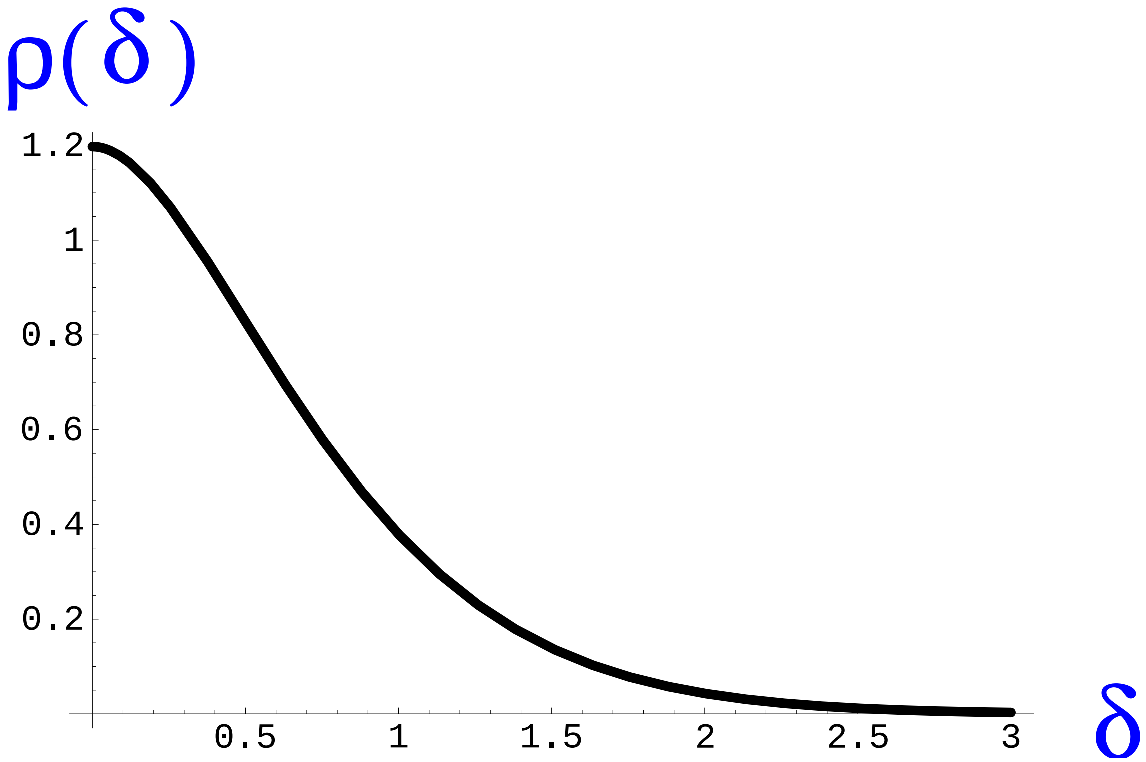


v_2
common part of
length $|t''-t'|$
v_3
v_1

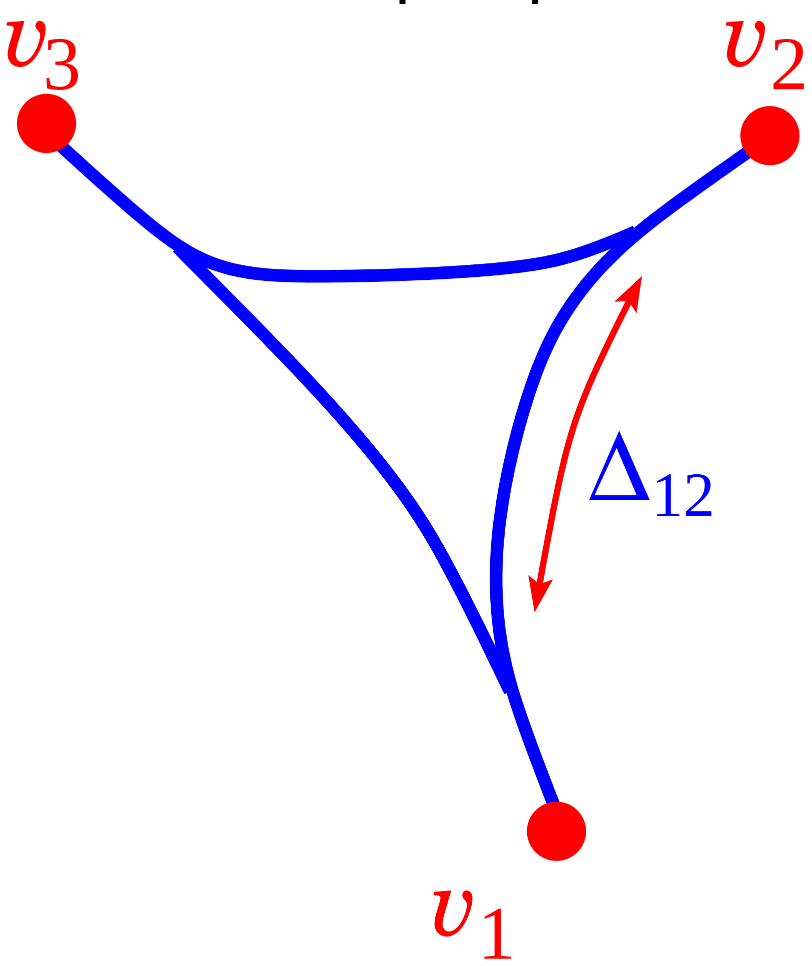
law for the length δ of the common part



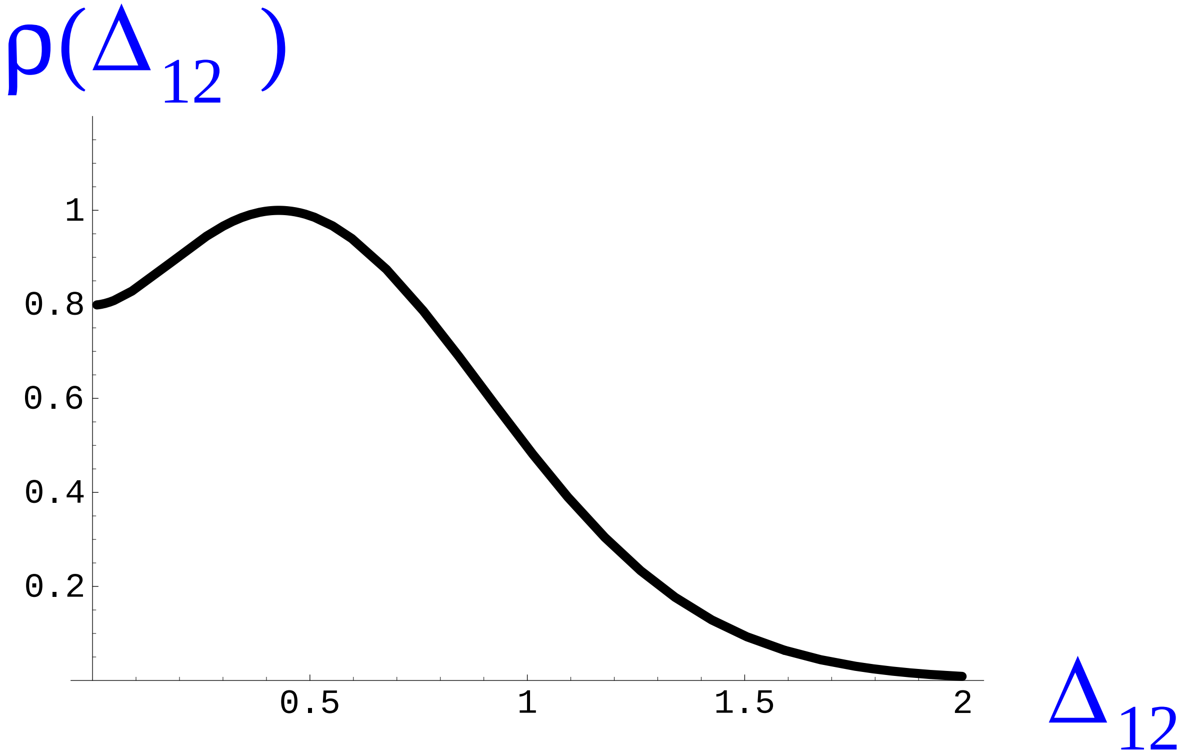
law for the length δ of the common part



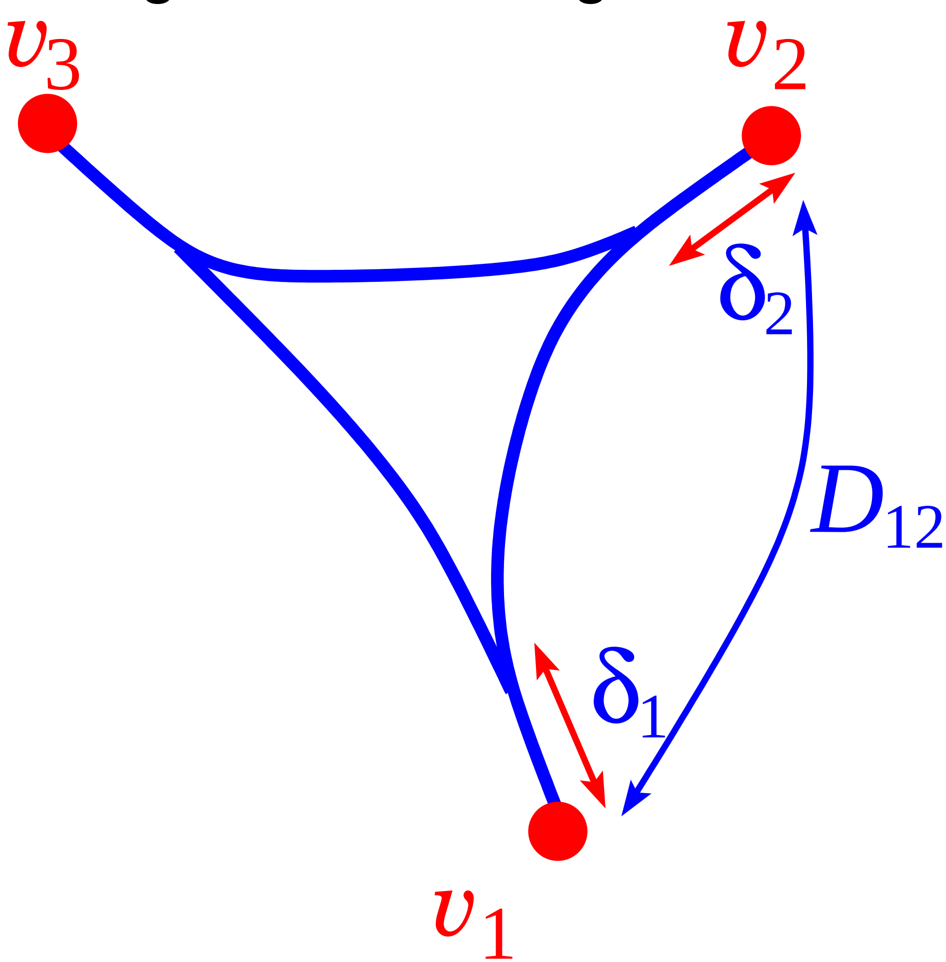
law for the length Δ_{12} of the open part from v_1 to v_2



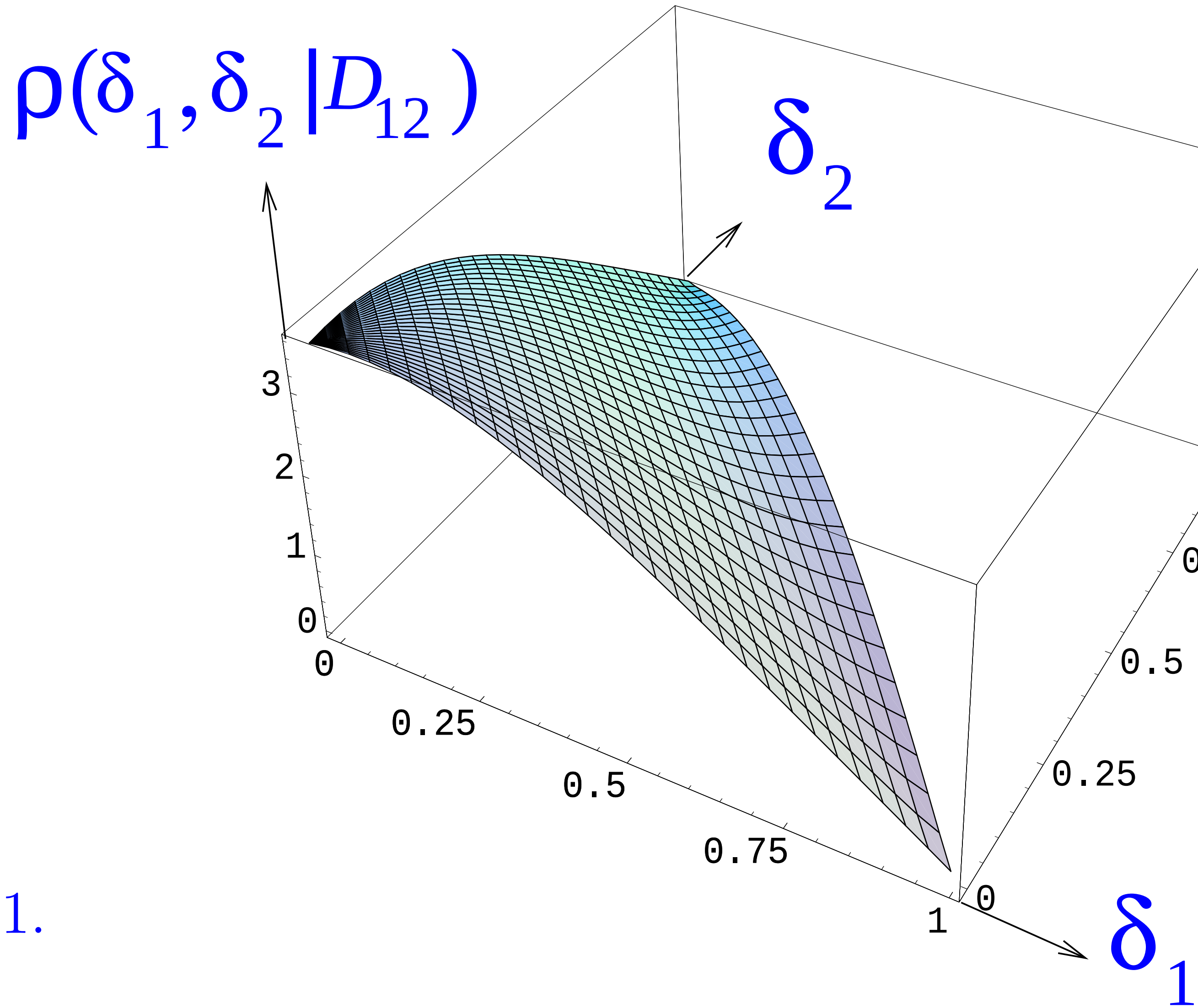
law for the length Δ_{12} of the open part from v_1 to v_2



joint law for the lengths δ_1 and δ_2 given D_{12}

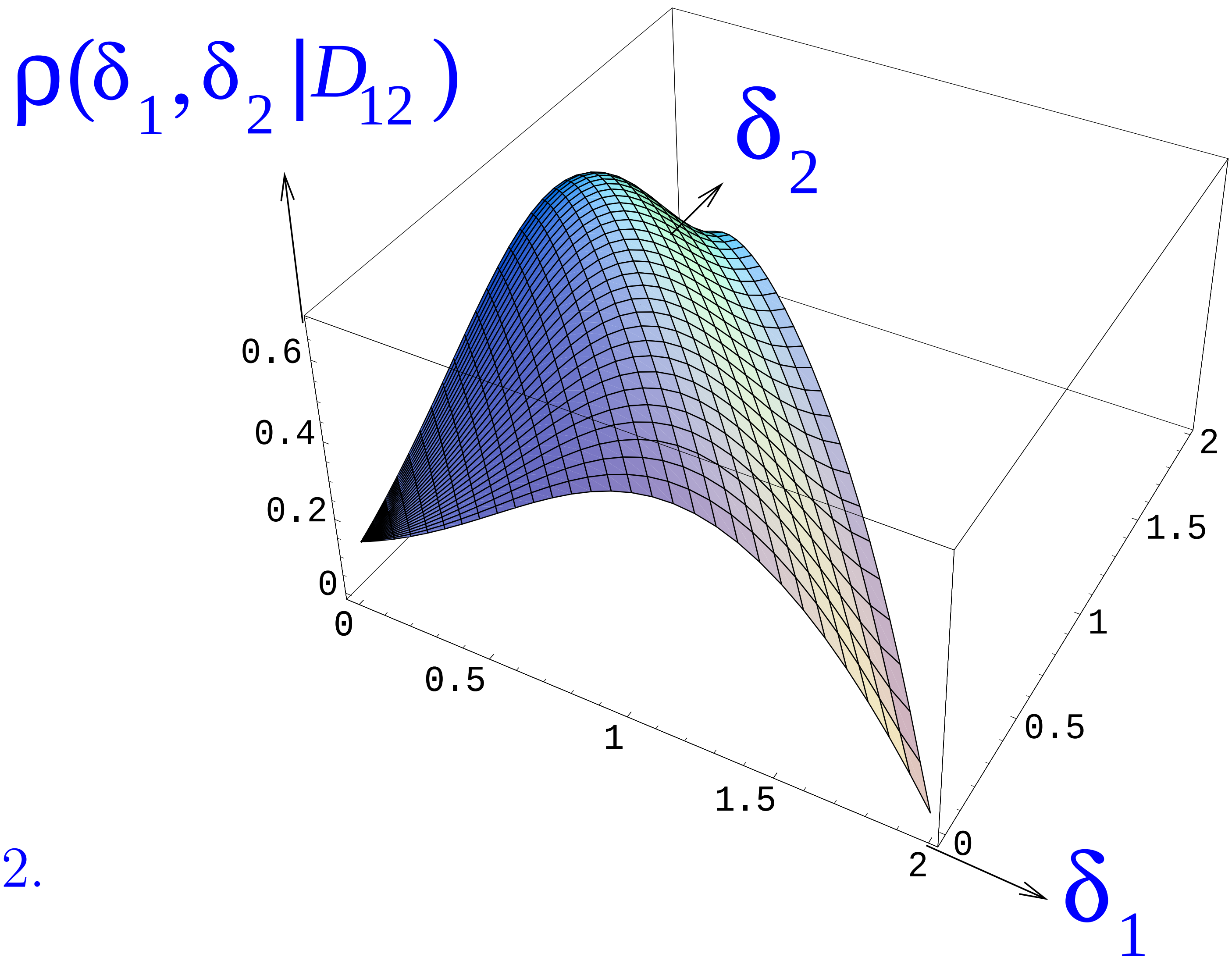


joint law for the lengths δ_1 and δ_2 given D_{12}



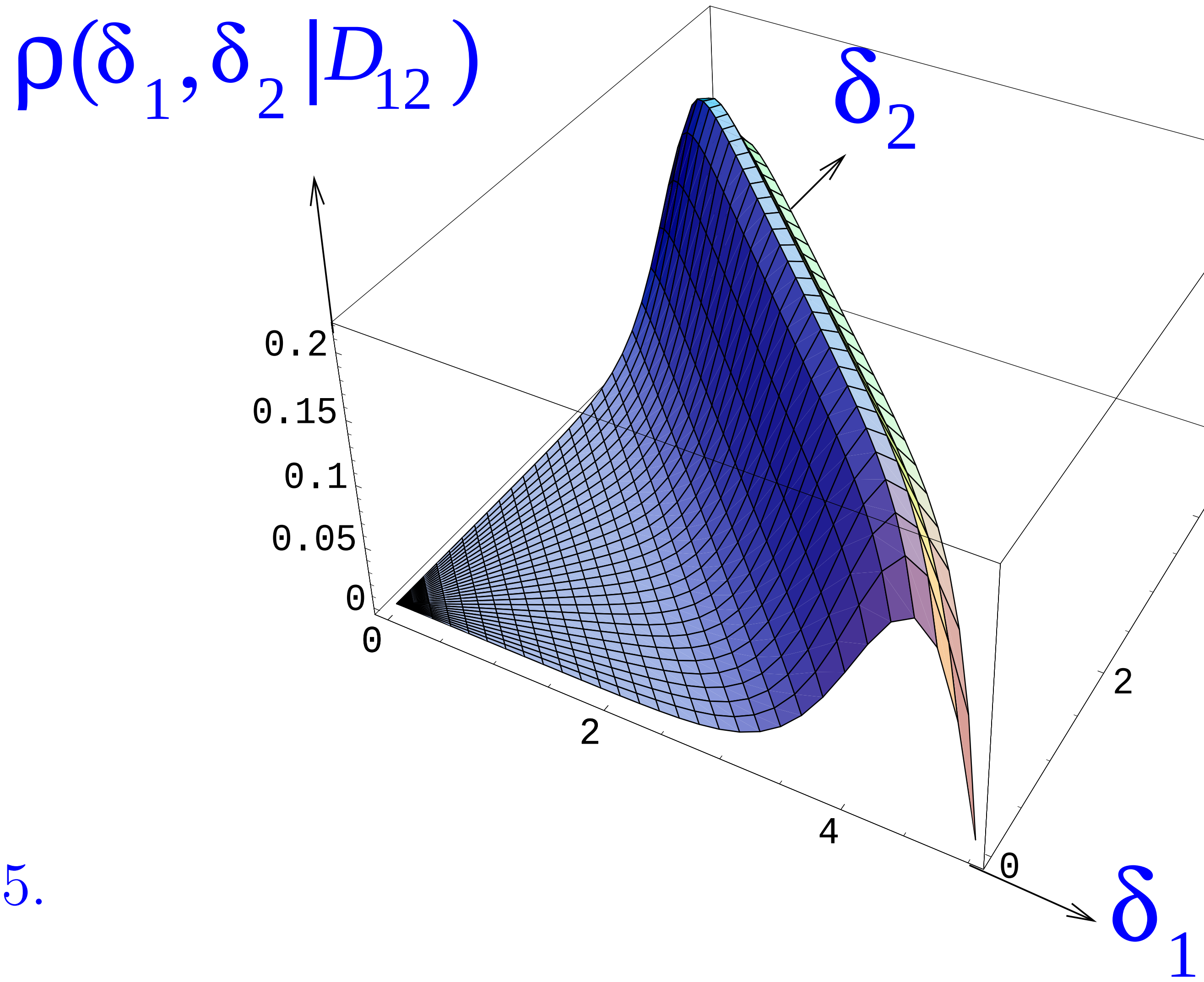
$D_{12} = 1.$

joint law for the lengths δ_1 and δ_2 given D_{12}

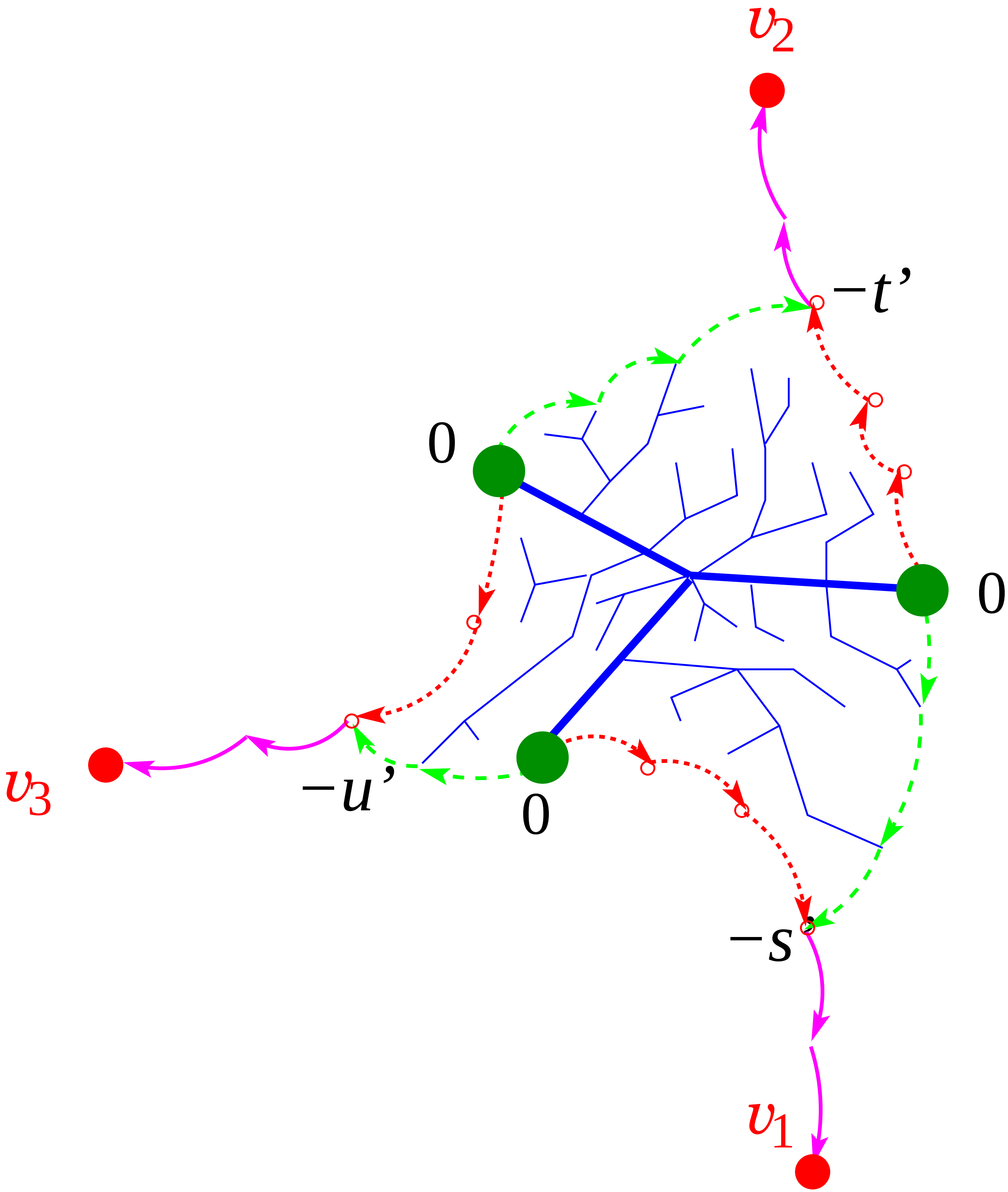


$D_{12} = 2.$

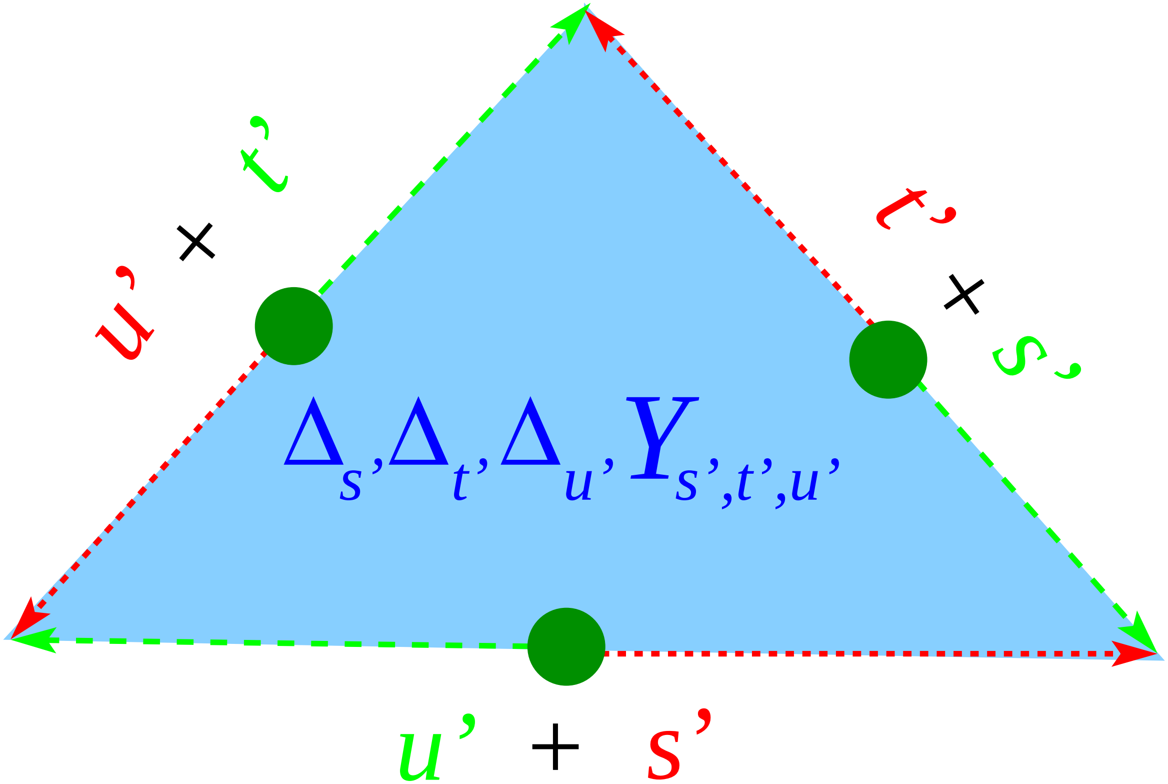
joint law for the lengths δ_1 and δ_2 given D_{12}



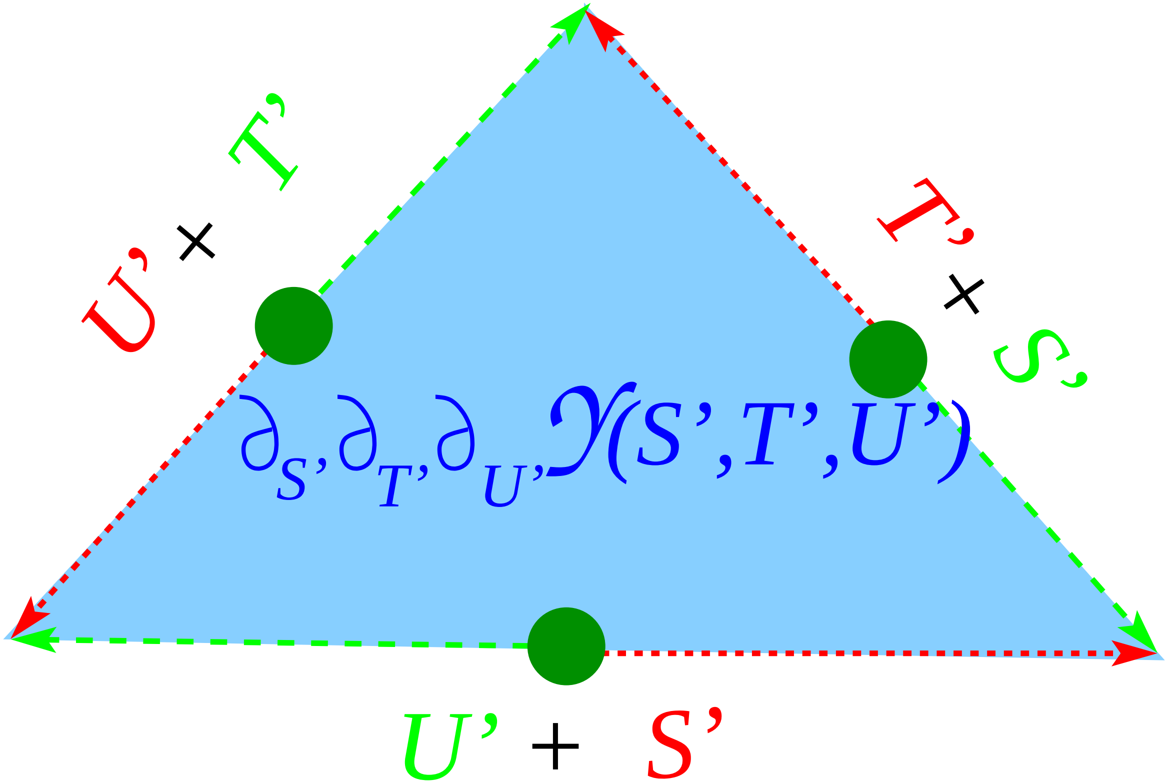
$D_{12} = 5.$



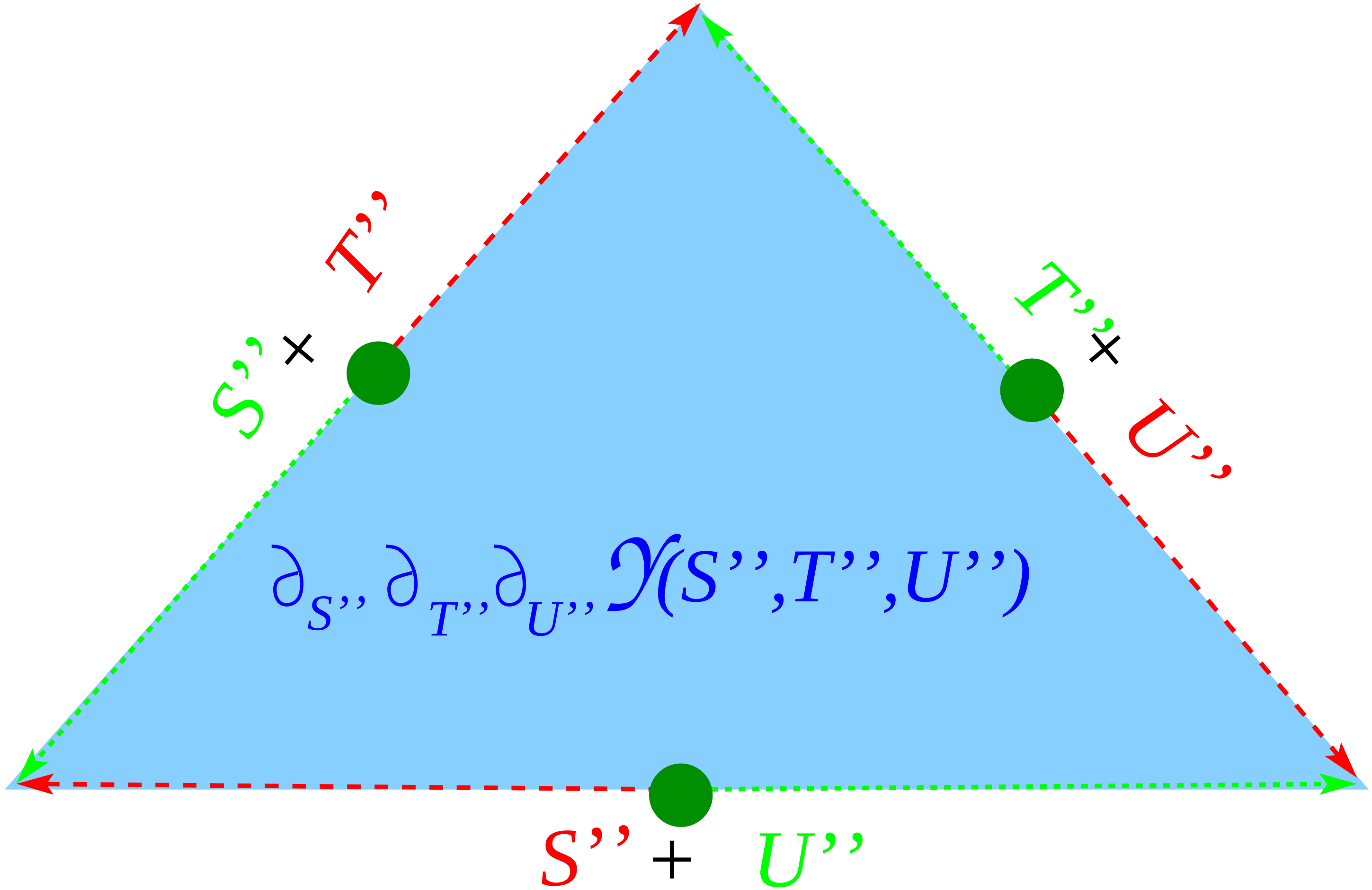
v_2
$-t'$
0
0
v_3
$-u'$
0
$-s'$
v_1



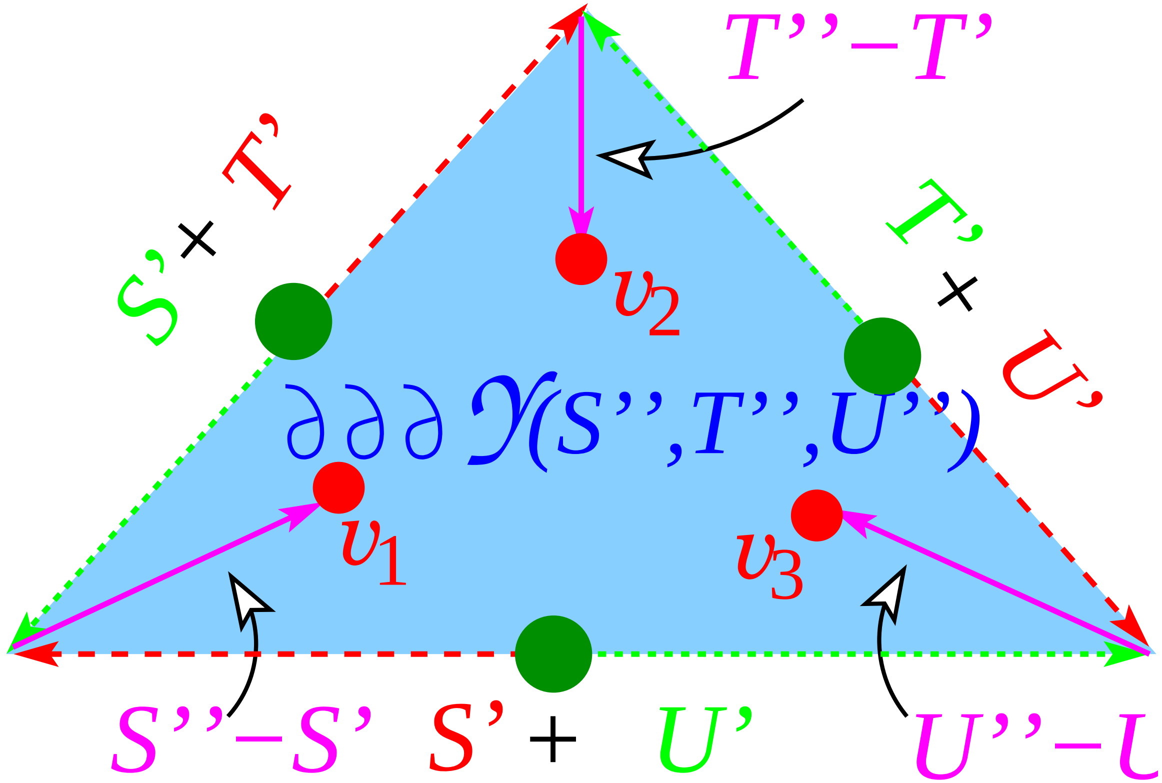
$\Delta_{s'}\Delta_{t'}\Delta_{u'}Y_{s',t',u'} \to$ triangle with geodesic boundaries of side lengths $s' + t'$, $t' + u'$, $u' + s'$



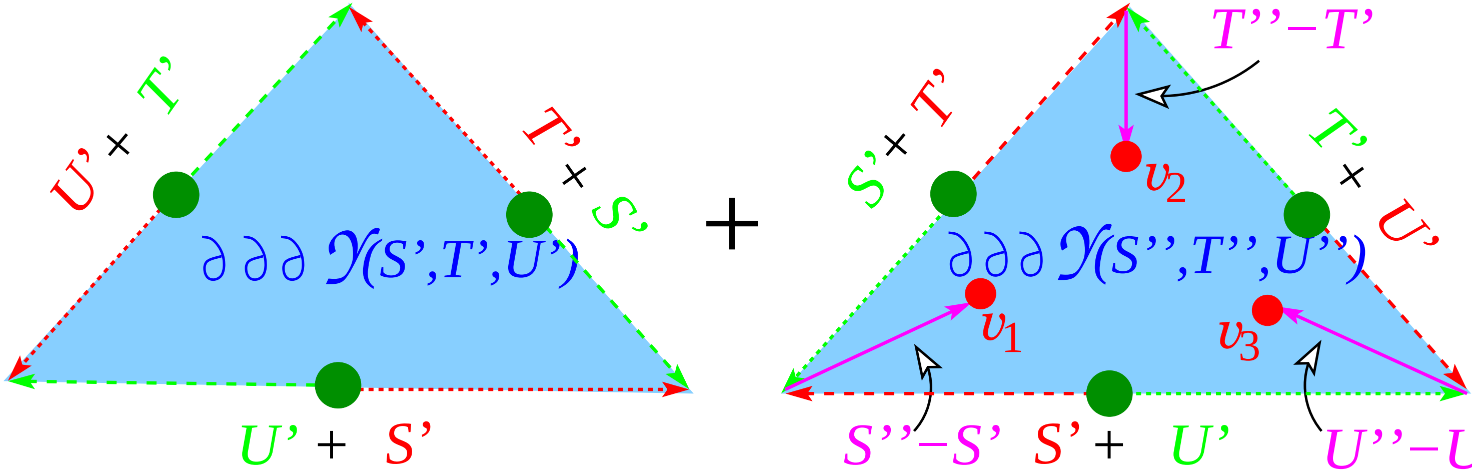
$\partial_{S'}\partial_{T'}\partial_{U'}\mathcal{Y}(S',T',U') \rightarrow$ triangle with geodesic boundaries of side lengths $S'+T'$, $T'+U'$, $U'+S'$



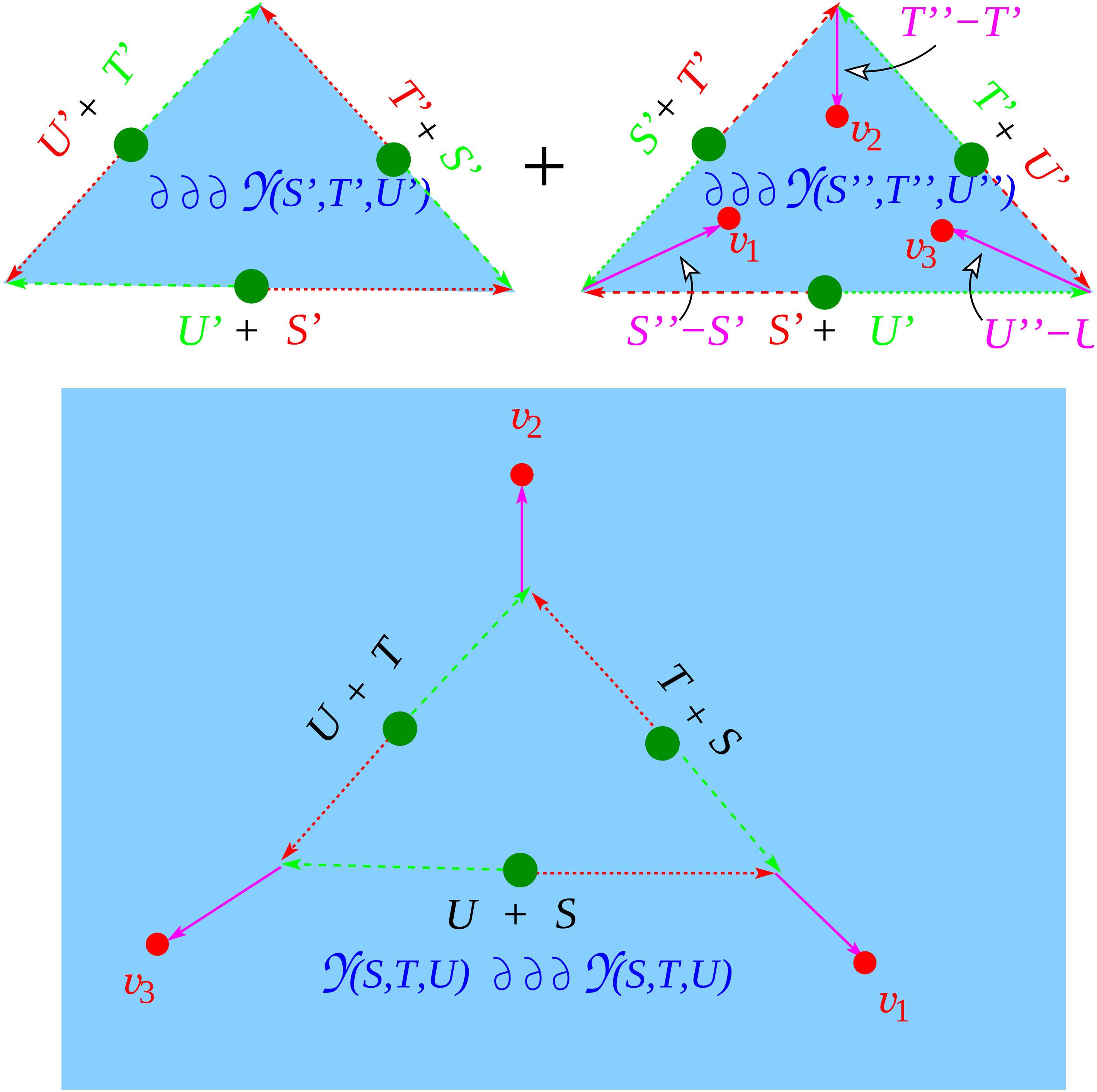
another triangle with geodesic boundaries of different side lengths



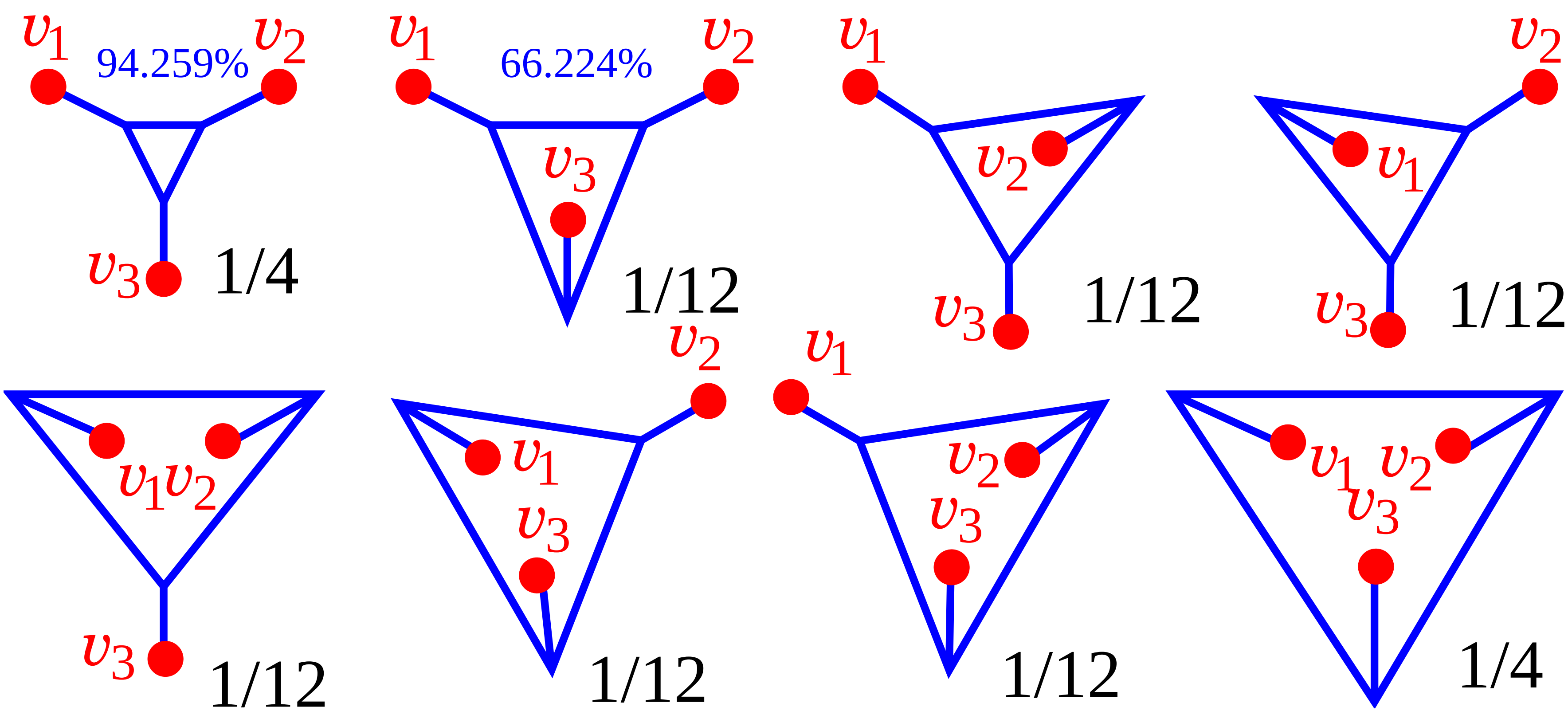
make the side lengths of the two triangles identical (here $S'' > S'$, $T'' > U'$ and $U'' > U'$)



glue the triangles



→ one contribution to the three-point function


v1
94.259%
v2
v3
1/4
v1
66.224%
v2
v3
1/12
v1
v2
v3
1/12
v2
v1
v3
1/12
v1v2
v3
1/12
v2
v1
v3
1/12
v1
v2
v3
1/12
v1 v2
v3
1/4

a word on branching processes

a branching process

a random map is the “superposition" of

a branching process

a random map is the “superposition" of

- a random planar tree

a branching process

a random map is the “superposition" of

- a random planar tree
- integer labels on the tree

a branching process

a random map is the “superposition" of

- a random planar tree
- integer labels on the tree
- boundary condition (*positive labels*)

a branching process

a random map is the “superposition" of

- a random planar tree $\rightarrow$ genealogical tree
- integer labels on the tree
- boundary condition (*positive labels*)

a parent individual gives rise to k children with probability $p(k) = (1-p)p^k, \quad (average\ number\ of\ children\ \frac{p}{1-p})$

a branching process

a random map is the “superposition" of

- a random planar tree $\rightarrow$ genealogical tree
- integer labels on the tree $\rightarrow$ diffusion process in 1D
- boundary condition (*positive labels*)

a parent individual gives rise to k children with probability
$p(k) = (1-p)p^k, \quad (average\ number\ of\ children\ \frac{p}{1-p})$

the child of a parent at position ℓ lives at position $\ell, \ell \pm 1$

a branching process

a random map is the “superposition" of

- a random planar tree $\rightarrow$ genealogical tree
- integer labels on the tree $\rightarrow$ diffusion process in 1D
- boundary condition (*positive labels*) $\rightarrow$ walls, forbidden zone

a parent individual gives rise to k children with probability
$p(k) = (1-p)p^k, \quad (average\ number\ of\ children\ \frac{p}{1-p})$

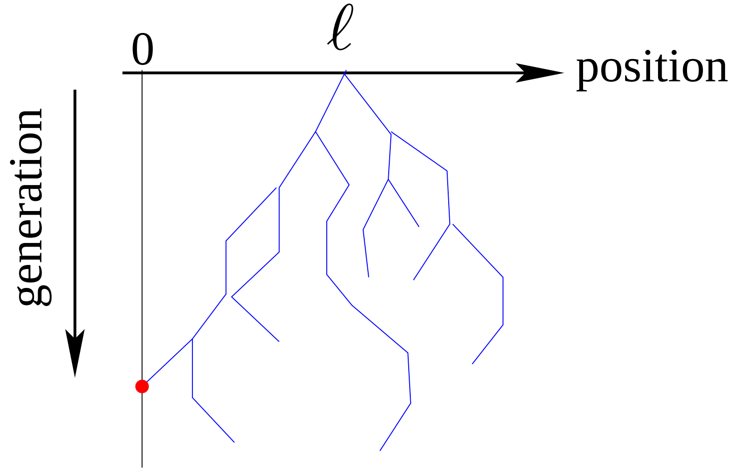
the child of a parent at position ℓ lives at position $\ell, \ell \pm 1$

what is the probability $\mathcal{P}_\ell(p)$ for the population whose germ is at position ℓ to reach position 0 ?

$\mathcal{P}_\ell(p) = 1 - (1-p)R_\ell(g)$ with $g = \frac{p(1-p)}{3}$

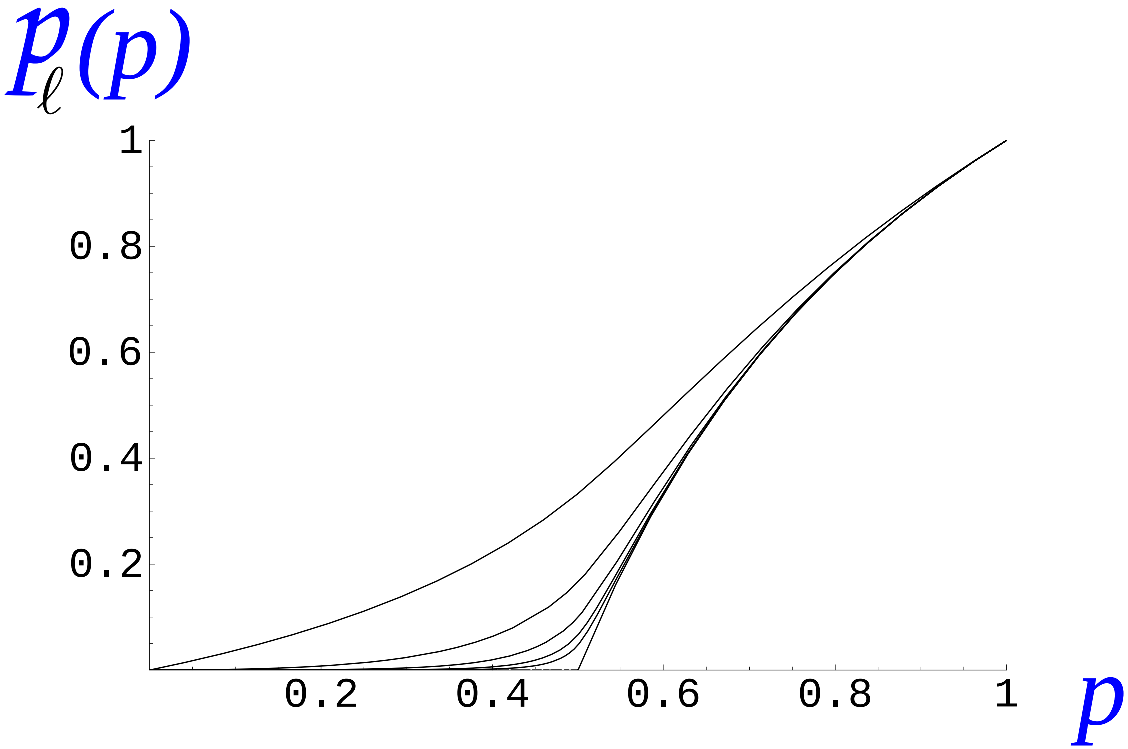
$$\mathcal{P}_\ell(p) = 1 - \frac{1-|2p-1|}{2p} \frac{(1-x^\ell)(1-x^{\ell+3})}{(1-x^{\ell+1})(1-x^{\ell+2})}$$

with $x = \frac{1+2|1-2p|-\sqrt{3|1-2p|}\sqrt{2+|1-2p|}}{1-|1-2p|}$

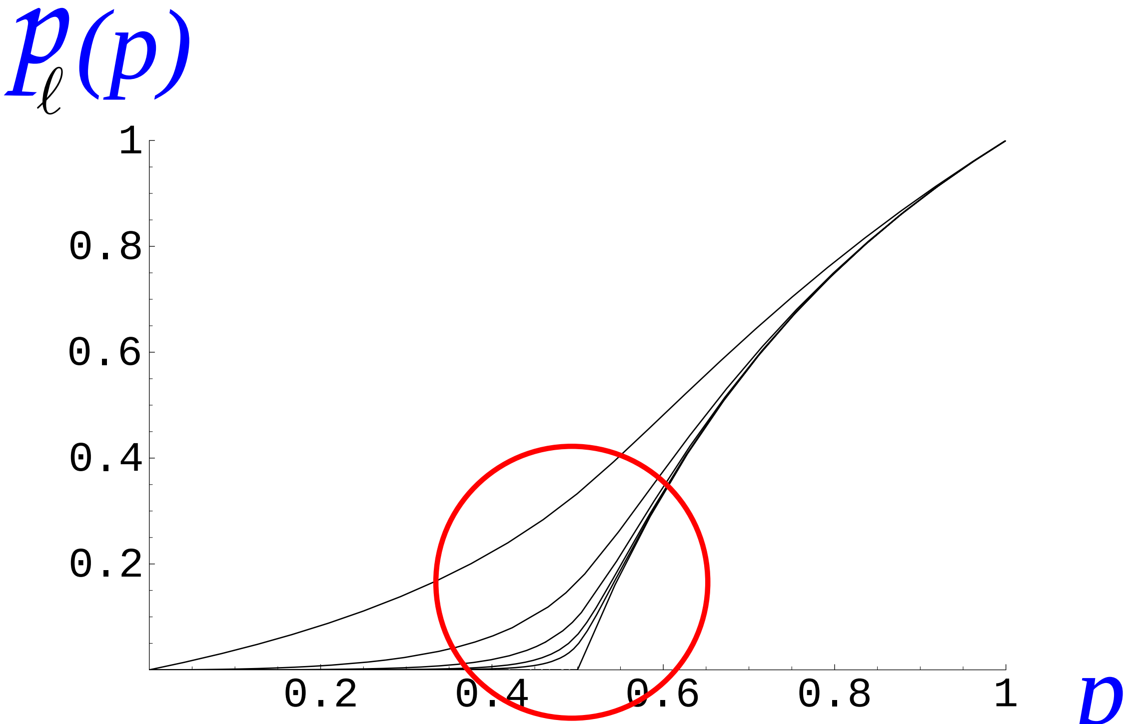


$\mathcal{P}_\ell(p) \overset{\ell\to\infty}{\sim} S(p)$: survival probability

$$S(p) = 1 - \frac{1-|2p-1|}{2p} = \begin{cases} 0 & p \leq \frac{1}{2} \\ \frac{2p-1}{p} & p \geq \frac{1}{2} \end{cases}$$



$p_\ell(p)$
1
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1
p



scaling behavior around $p = \frac{1}{2}$:

$$\mathcal{P}_\ell(p) \sim |2p-1| \left(\frac{3}{\sinh^2(\sqrt{3/2}\ \ell\ |2p-1|^{1/2})} + 1 \right) + (2p-1)$$

THE END