A combinatorial approach to total positivity via path enumeration.

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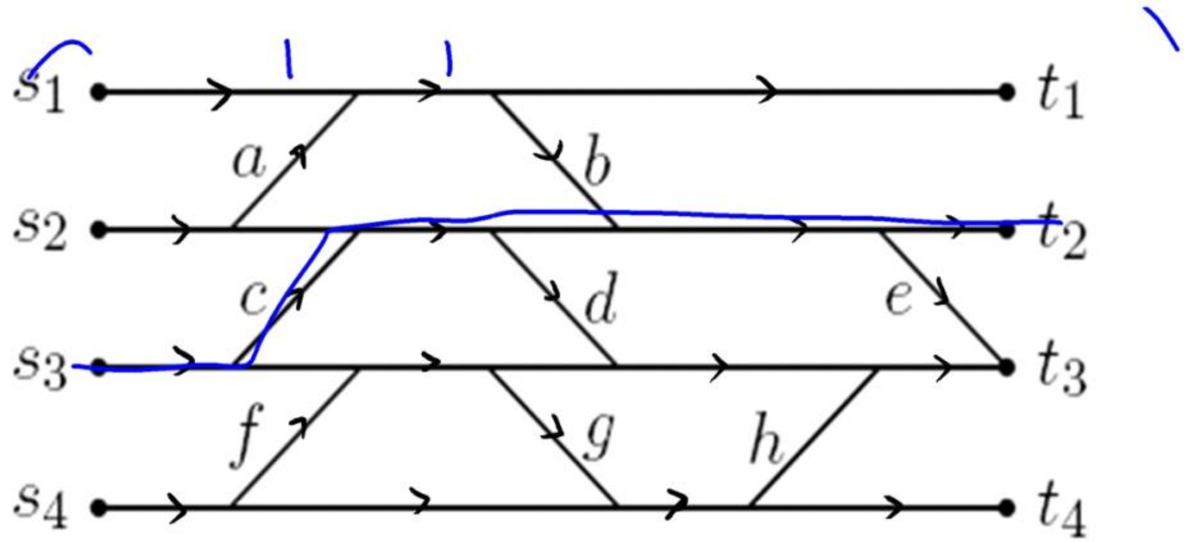
OUTLINE:

1) A classical result of Lindström, Gessel-Viennot.

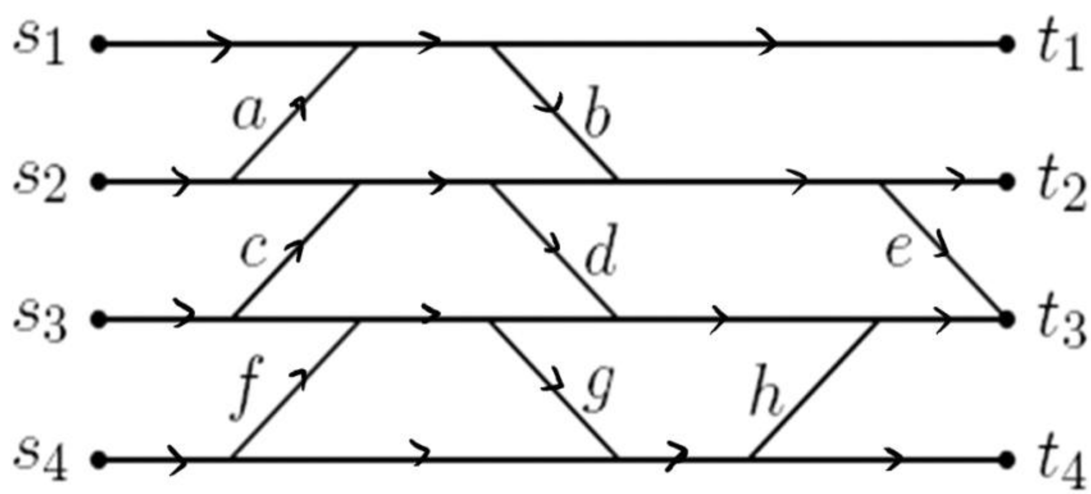
2) Some basic properties of matrices via L, G-V.

3) Positivity in matrices and the Grassmannian



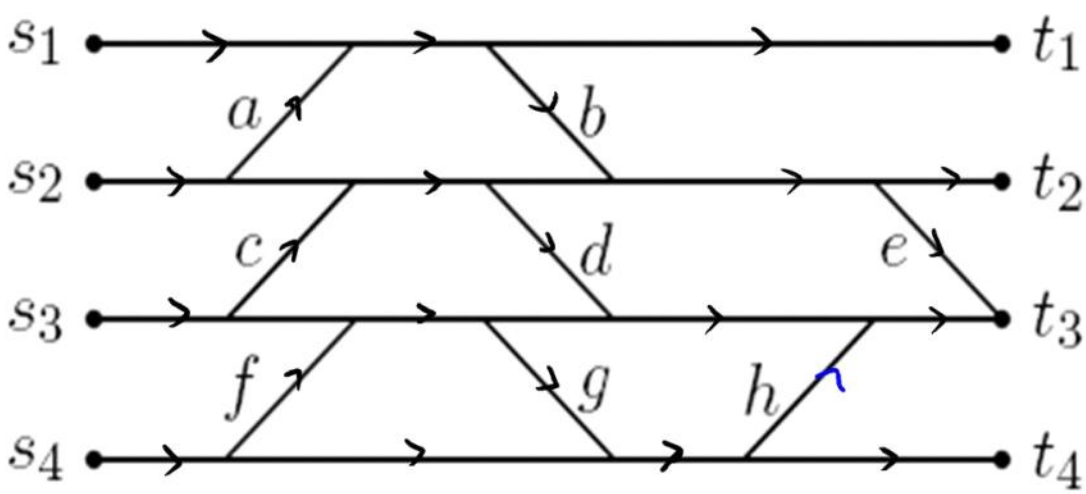
- In this acyclic network, unlabeled edges have weight 1.
- The weight of a path is the product of the weights of its edges.

N:



We can define the weight matrix

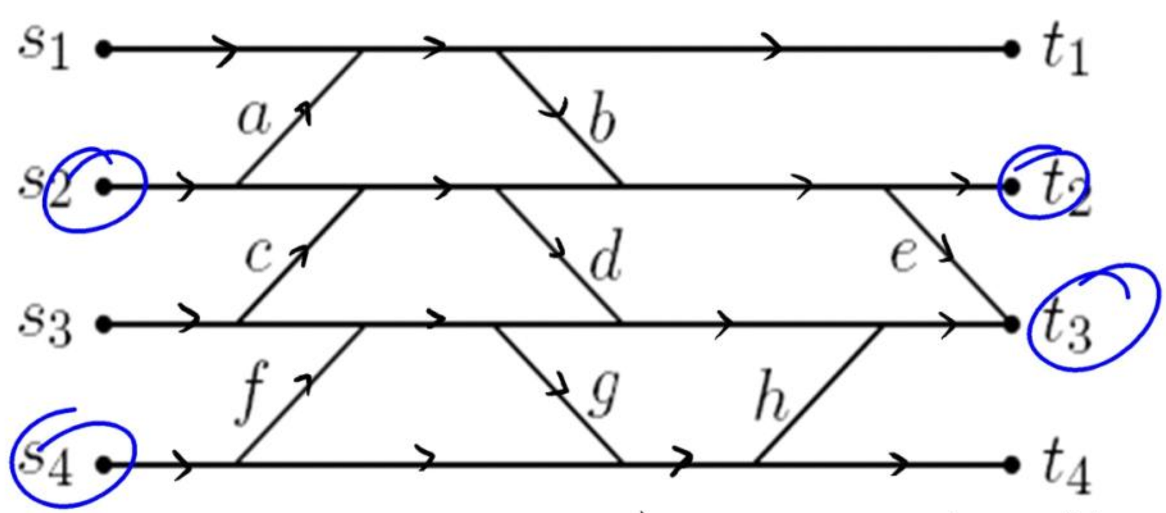
by $$x_{ij}(N) = \sum_{\substack{\text{paths } P \text{ from} \\ s_i \text{ to } t_j}} wt(P) .$$



$$S \downarrow \begin{bmatrix} 1 & b & be & 0 \\ a & 1+ab & d+e+abe & 0 \\ 0 & c & 1+cd+ce+gh & g \\ 0 & 0 & f+h+fgh & 1+fg \end{bmatrix}$$

Theorem [Lindström, Gessel-Viennot]:

Suppose N is a (planar) acyclic network. Then each square minor $\Delta_{I,J}$ of the weight matrix of N enumerates the families of non-intersecting paths from the sources in I to the sinks in J.



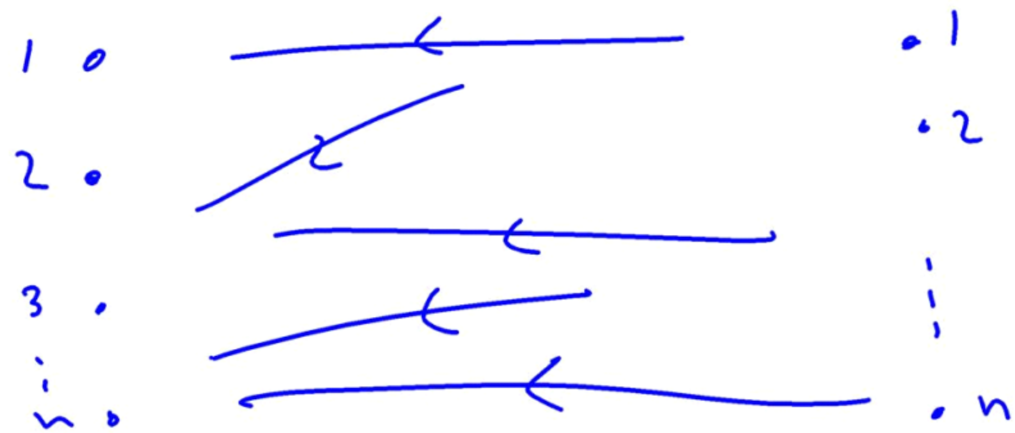
Egs. $\Delta_{1234,1234}(x(N)) = \det(x(N)) = 1$

$\Delta_{24,12}(x(N)) = 0$

$\Delta_{24,23}(x(N)) = (f + h + fgh)(1+ab)$

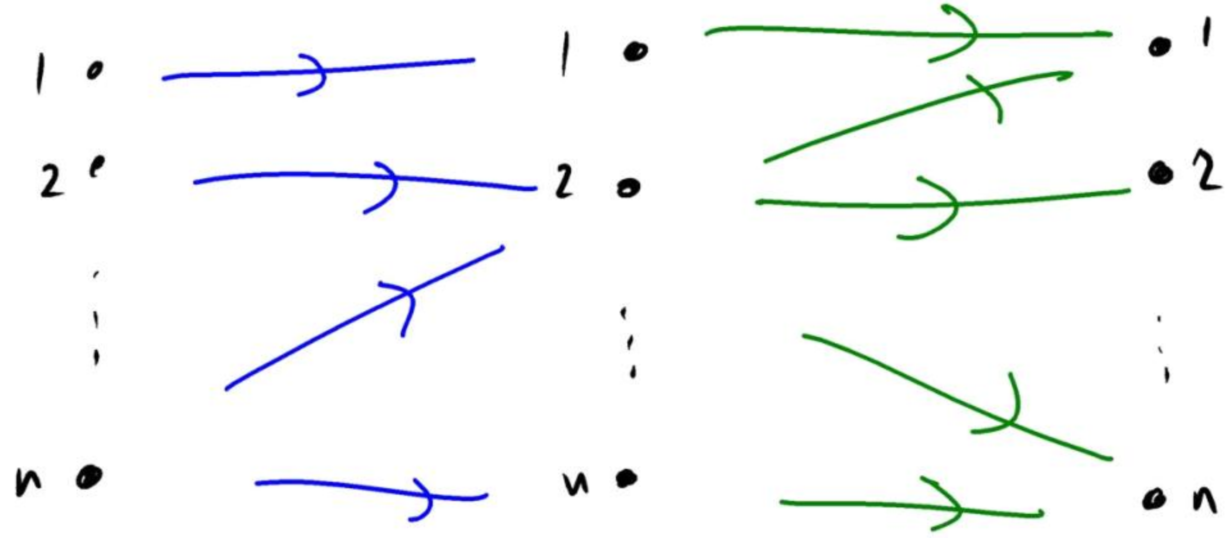
Some simple facts about matrix determinants via LGV.

1) $\det M^{tr} = \det M$. $n \times n$



2) For $n \times n$ matrices A and B,

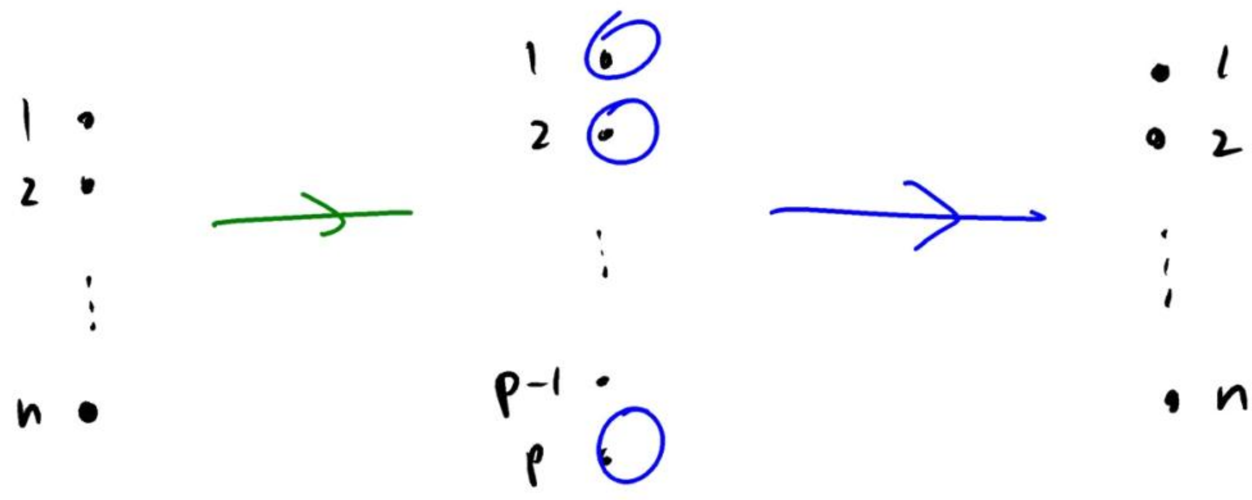
$$\det(AB) = (\det A)(\det B)$$



3) Cauchy Binet expansion: $n \leq p$

$A = n \times p$ matrix, $B = p \times n$ matrix.

$$\det(AB) = \sum_{\substack{Z = n\text{-subset} \\ \text{of } [p]}} \left(\det \Delta_{[n],Z}(A)\right)\left(\det \Delta_{Z,[n]}(B)\right)$$



Next: using these ideas to talk about total positivity in matrices and in the Grassmannian

matrices	Grassmannian
Def: A matrix is totally positive (non-neg) if all of its square minors are >0 (≥ 0).	**Def**: A point in $Gr(k,n)$ is totally positive (non-neg) if it can be written with all Plücker coordinates >0 (≥ 0).

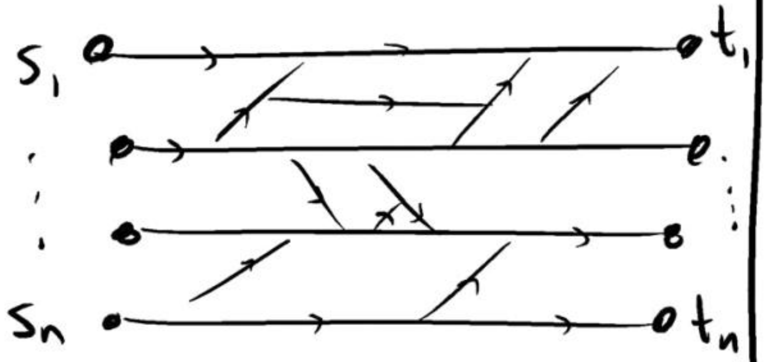
Aside on Grassmannians:

Think of points in $Gr(k,n)$, i.e. k-subspaces of $\mathbb{R}^n$, as rowspans of $k \times n$ matrices.

$$\begin{array}{cc} & n \\ k & \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix} \end{array}$$

The $k \times k$ minors of any representation give the (projective) Plücker coords.

matrices

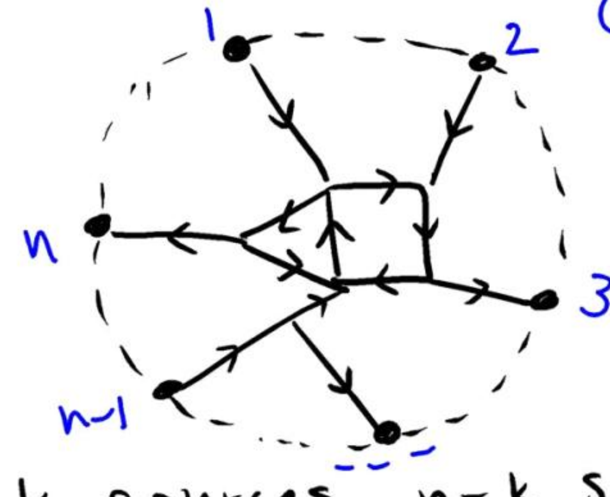


acyclic

weight matrix

$$X_{ij} = \sum_{P:\, s_i \to t_j} wt(P)$$

Grassmannian $Gr(k,n)$



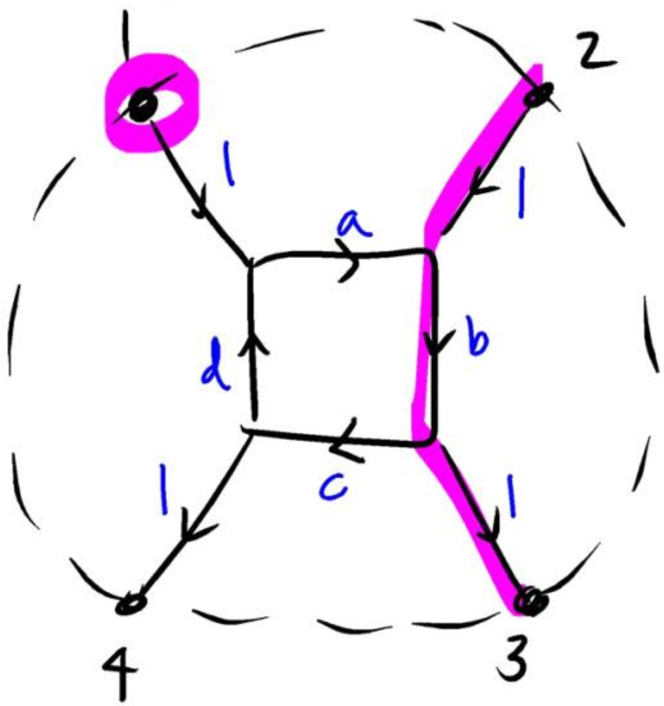
k sources, n−k sinks

"weight matrix"

$$X_{ij} = \pm \sum_{P:\, i \to j} (-1)^{wind(P)}\, wt(P)$$

Sources 1,2 sinks 3,4

Eg:



"weight matrix"

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & -\frac{ab}{1+X} & -\frac{abc}{1+X} \\ 2 & 0 & 1 & \frac{b}{1+X} & \frac{bc}{1+X} \end{array}$$

Δ_{13}

$$X_{13} = ab - abX + abX^2 - abX^3 + \dots$$

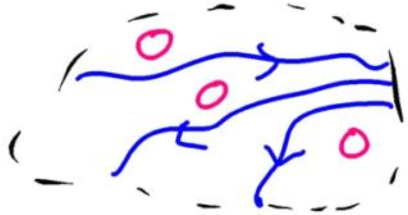
matrices	Grassmannian
acyclic planar network w/ pos edge weights $\Big\downarrow$ THM (L, G-V) TNN matrix	

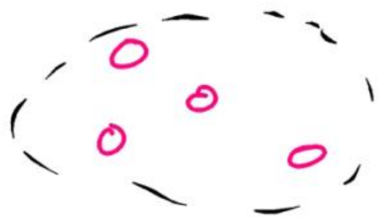
matrices	Grassmannian
acyclic planar network w/ pos edge weights	circular planar network with pos edge weights
$\Big\downarrow$ THM (L, G-V)	$\Big\downarrow$? yes
TNN matrix	point in TNN Grassmannian

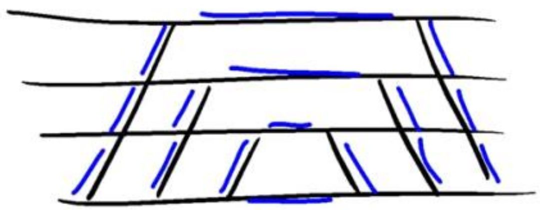
Theorem (Postnikov):

All Plücker coordinates for the "weight matrix" of a circular planar network are subtraction-free rational expressions in the edge weights.

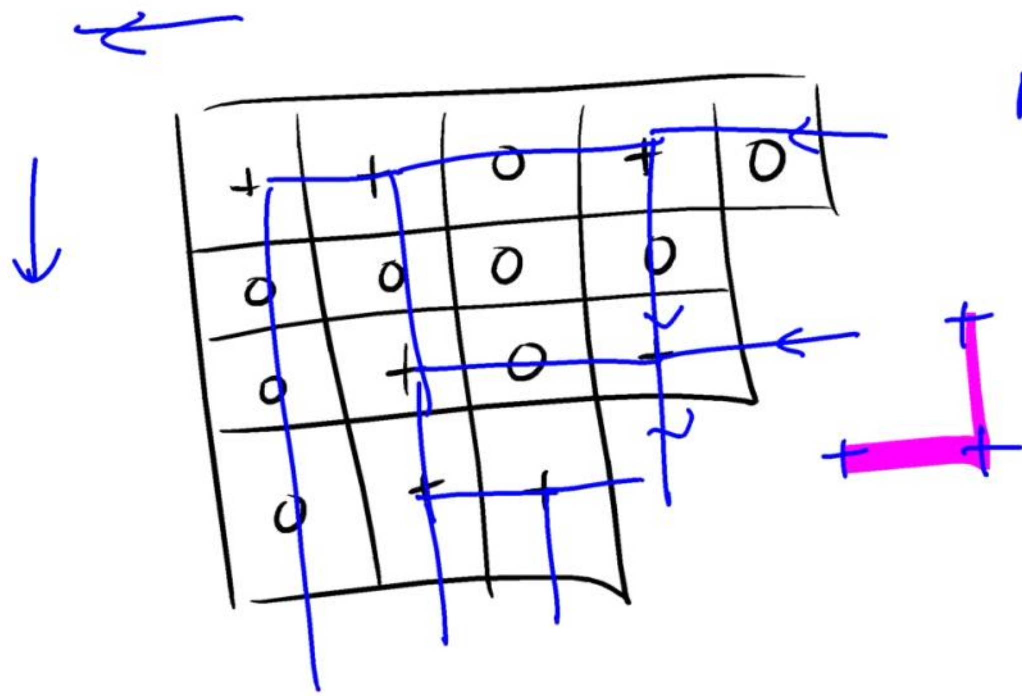
Thm (KT): The Plücker coordinates (ie $k \times k$ minors) are generating functions for certain path families in the network

$\Delta_{[k],J} = \sum$  sinks $J \setminus [k]$

$\sum$  cycles

matrices	Grassmannian
Every TNN matrix is the weight matrix of some network.	Every point in the TNN Grassmannian is the "weight matrix" of some network.
(Brenti)	(Postnikov, KT)
	?

In the Grassmannian, we can find nice networks using Le-diagrams.



partition determines the source set

Le-diagrams index the cells of the TNN Grassmannian → one for each possible vanishing pattern of Plücker coordinates.

A special subset of these, called permutation tableaux, are key in work by Corteel-Williams on the PASEP model.