Random trees and vertex splitting

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Statistical physics, combinatorics and probability: from discrete to continuous models

Institut Henri Poincare

Outline

- Introduction and background
- Definition of the vertex splitting model
- Vertex degree distribution
- Correlations
- Structure functions and the Hausdorff dimension
- Open problems

Random graphs: two main approaches

1. Equilibrium statistical mechanics

   $\mathcal{T}$= Set of graphs, μ a probability measure on $\mathcal{T}$

   $$\mu(T) = Z^{-1} e^{-\beta E(T)}$$

2. Growing trees $T_n \mapsto T_{n+1}$, time discrete

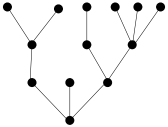
   Stochastic growth rules induce a probability measure on $\mathcal{T}_t$, the trees that can arise in t steps

- Sometimes (1) is more natural
- Sometimes (2) is more natural
- Sometimes (1) and (2) are known to be equivalent

Galton-Watson trees

- p_n = probability of having n descendents,

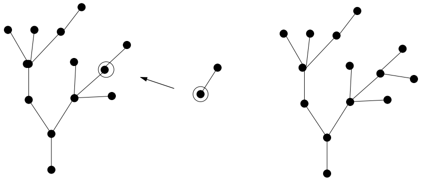
$$\sum_n p_n = 1, \quad m = \sum_n n p_n$$



- $m < 1$ subcritical, $m > 1$ supercritical, $m = 1$ critical
- n generations at time $t = n - 1$ if no extinction
- Well understood

Preferential attachment trees

- In each timestep one new edge is attached to a preexisting tree



- Probability of attaching to a vertex v of degree k

$$P_v = \frac{w_k}{\sum_k n_k w_w}, \quad w_k \geq 0,$$

$n_k =$ the number of vertices of degree k.

- Growth rule induces a probability measure on $\mathcal{T}_t$.

Local trees

- Weight factor of a tree T

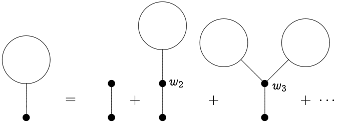
$$W(T) = \prod_{i \in T} w_{\sigma(i)}$$

$\sigma(i) =$ degree of the vertex i

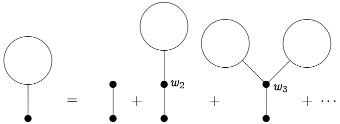
- Partition functions

$$Z_N = \sum_{T:|T|=N} W(T), \quad Z = \sum_N \zeta^N Z_N, \quad |\zeta| < \zeta_0$$

- Generating function $g(z) = \sum_n w_n z^{n-1}$, radius of convergence ρ
- Main equation



Generic trees



- Algebraically

$$Z(\zeta) = \zeta g(Z(\zeta)) = \zeta \sum_{i=0}^{\infty} w_{i+1} Z^i(\zeta)$$

- Define $Z_0 = \lim_{\zeta \to \zeta_0} Z(\zeta)$
- If $Z_0 < \rho$ then we say that the trees are *generic*.
- $Z(\zeta) - Z_0 \sim \sqrt{\zeta_0 - \zeta}$

- Define

$$\mu(T) = Z_0^{-1}\zeta_0^{|T|}\prod_{i\in T} w_{\sigma(i)}$$

Probability measure

- This measure is the same as the one obtained from a Galton-Watson process with

$$p_n = \zeta_0 w_{n+1} Z_0^{n-1}$$

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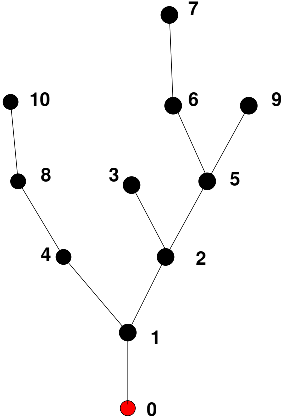
Another equivalence

- Preferential attachment trees $\approx$ Causal trees
- Weight proportional to the number of *causal labelings*



- More branchings, more ways to grow

Properties of generic trees

- Let $V_T(R)$=volume of a ball of radius R centered on the root

$$\langle V_T(R)\rangle \sim R^{d_H}, \quad R\to\infty \quad \text{defines } d_H$$

- Let $p_T(t)$ = probability that a random walker is back at the root after t steps on T

$$\langle p_T(t)\rangle \sim t^{-d_s/2}, \quad t\to\infty \quad \text{defines } d_s$$

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- Averages taken w.r.t. a measure on infinite trees

$$\nu_N = Z_N^{-1} \prod_{i \in T} w_{\sigma(i)}$$

$$\nu_N \to \nu_\infty \quad \text{as} \quad N \to \infty$$

- Properties:

  (i) $d_H = 2$

  (ii) $d_s = 4/3$

  (iii) There is a unique infinite simple path whose outgrowths are critical GW-trees

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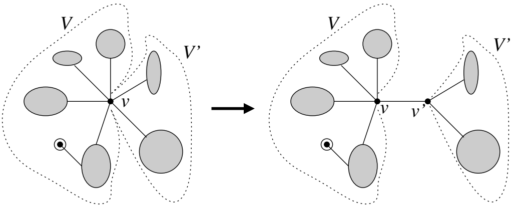
Preferential attachment trees

- In general many infinite simple paths
- $d_H = \infty$ in many cases (all cases?)
- Can calculate vertex degree distribution - and fluctuations. Independent of the initial tree.
- Broad distribution of sizes of subtrees, depends on the initial tree

The vertex splitting model

F. David, M. Dukes, S. Stefansson and T.J.: J. Stat. Mech. (2009) P04009

- A model of randomly growing rooted, planar trees



- Degree of vertices is bounded by an integer d (we also discuss the case $d = \infty$)
- Nonnegative splitting weights $w_1, w_2, \ldots, w_d$
- $n_j(T)$ = the number of vertices of degree j in a tree T
  p_j = Probability of choosing a vertex $v \in T$ of degree j

$$p_j = \frac{w_j}{\sum_i w_i n_i(T)}$$

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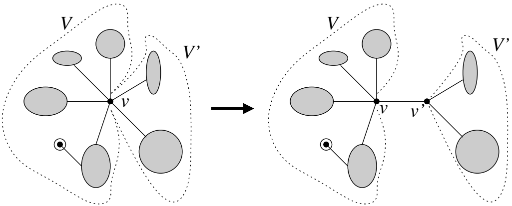

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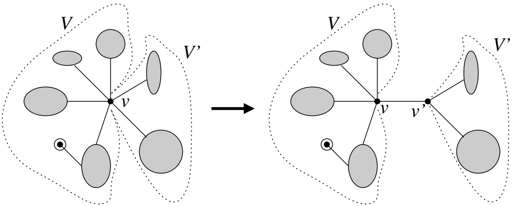

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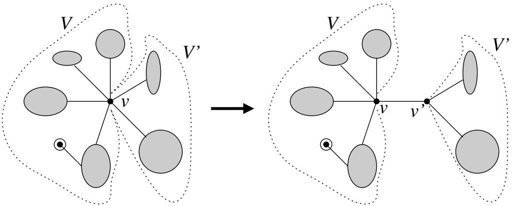

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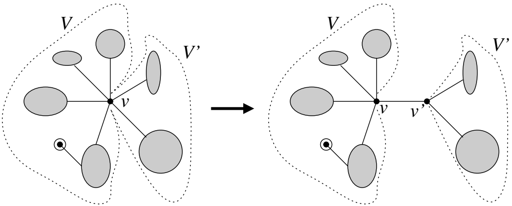

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Splitting rules

- The parameters of the model are

$$\begin{bmatrix} 0 & w_{1,2} & w_{1,3} & \cdots & w_{1,d-1} & w_{1,d} \\ w_{2,1} & w_{2,2} & w_{2,3} & \cdots & w_{2,d-1} & w_{2,d} \\ w_{3,1} & w_{3,2} & w_{3,3} & \cdots & w_{3,d-1} & 0 \\ w_{4,1} & w_{4,2} & w_{4,3} & & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ w_{d,1} & w_{d,2} & 0 & \cdots & 0 & 0 \end{bmatrix}$$

  a symmetric matrix of non–negative partitioning weights

- Split a vertex of degree i into vertices of degree k and $i+2-k$ with probability $w_{k,i+2-k}/w_i$ – all such splittings equally probable
- The splitting weights $w_1, w_2, \ldots, w_d$ are related to the partitioning weights by

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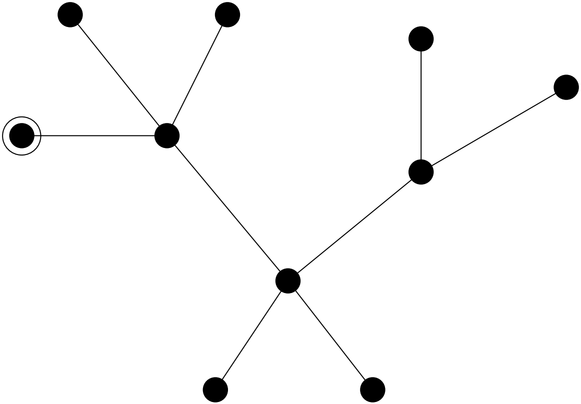
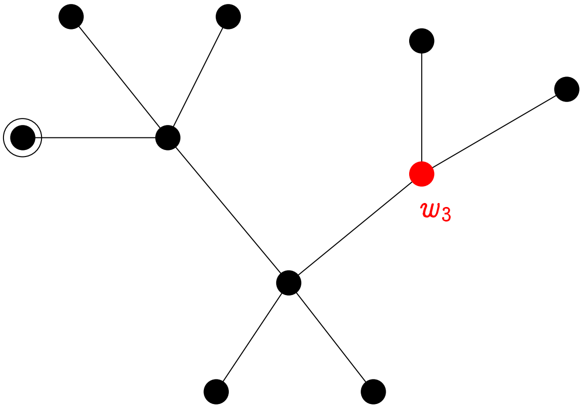
A tree



Vertex splitting rules



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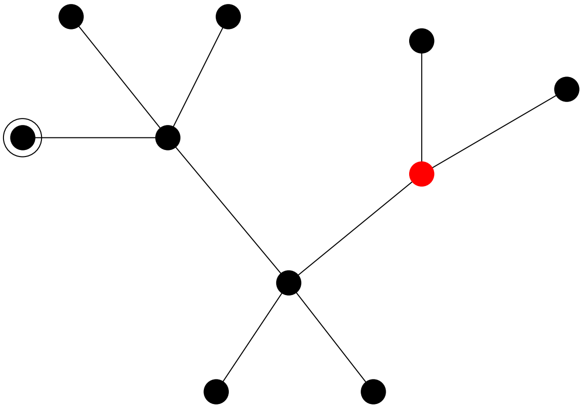

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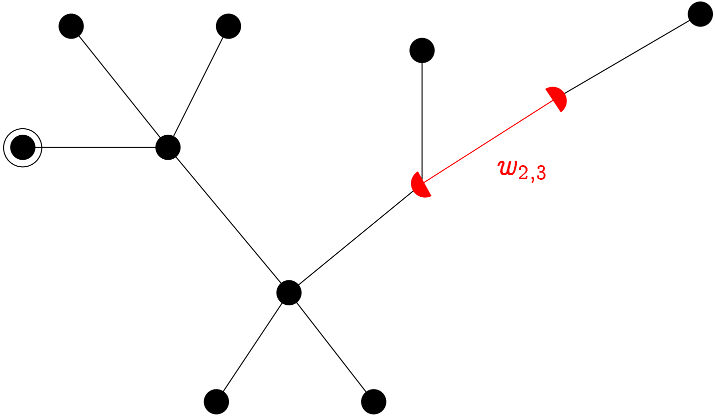

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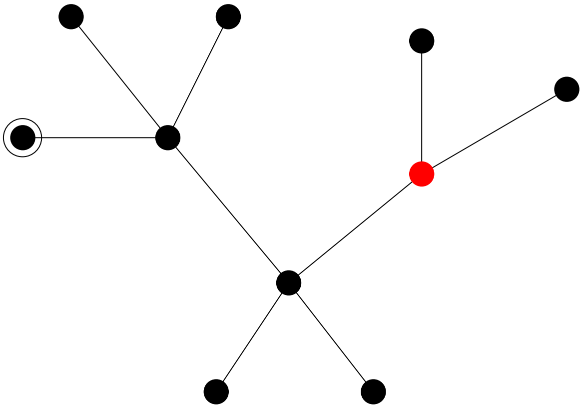

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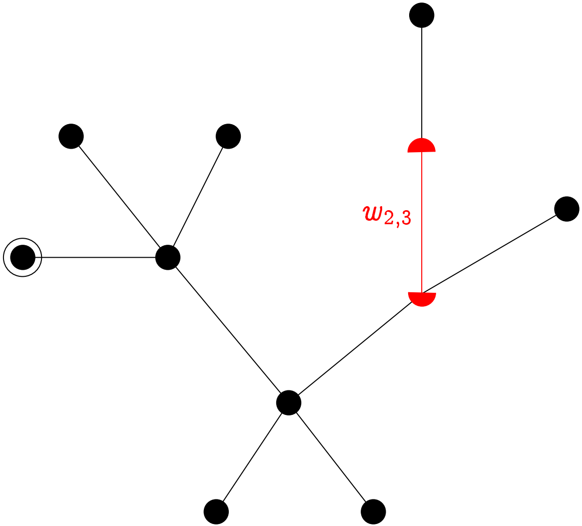

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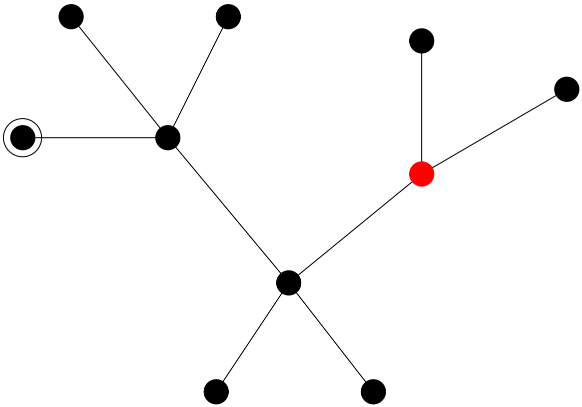

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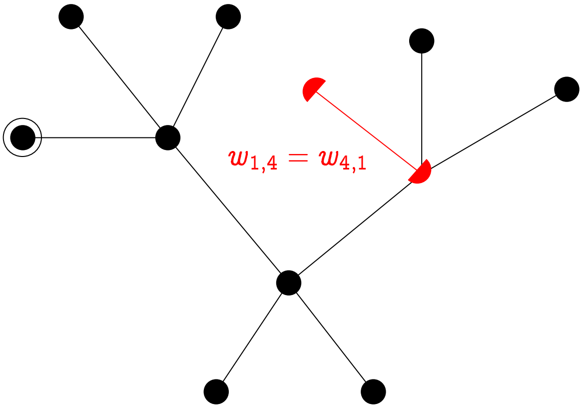

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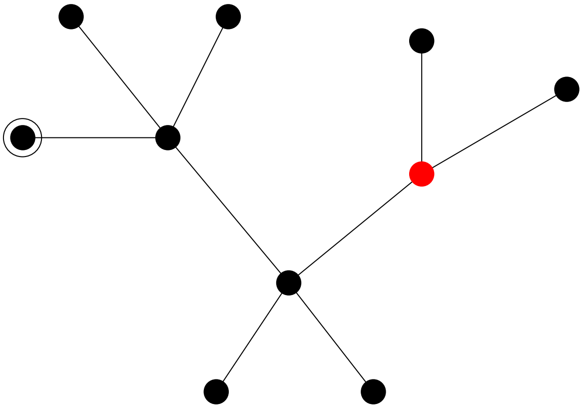

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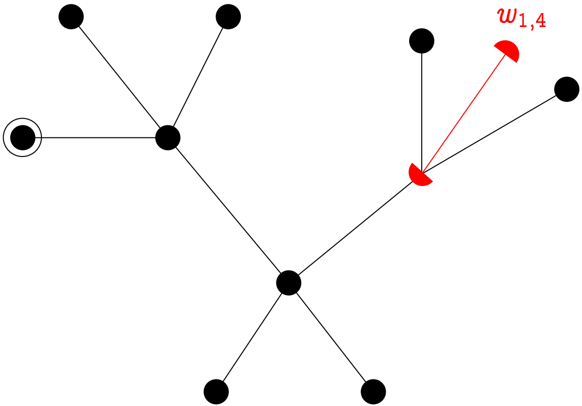

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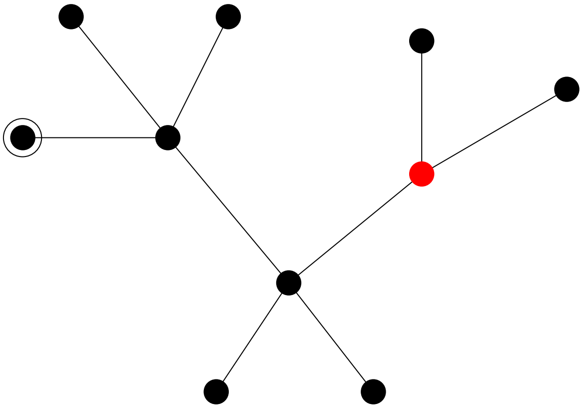

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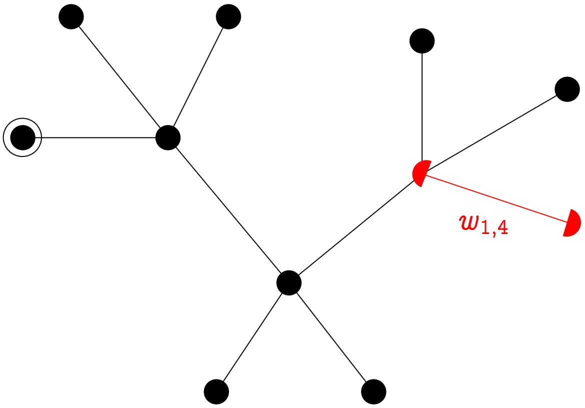
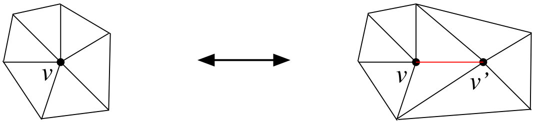


Relation to other models

- Ergodic moves in Monte Carlo simulations of triangulations in 2d-quantum gravity



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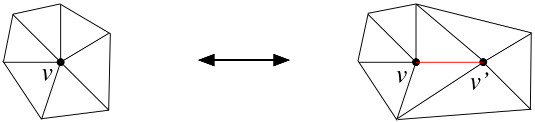

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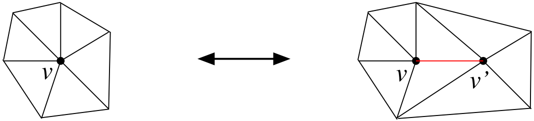

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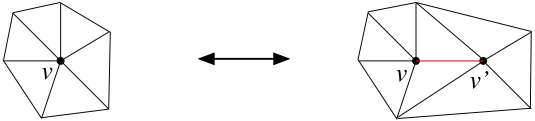

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Main results

- Distribution of vertex degrees in a large tree
- Correlations between the degrees of vertices
- "Shape" of trees – Hausdorff dimension

If we consider linear splitting weights

$$w_i = ai + b.$$

the analysis simplifies due to the Euler relation for trees

$$\sum_{i=1}^{d} n_i(T) = |T|, \quad \sum_{i=1}^{d} i n_i(T) = 2(|T| - 1).$$

The normalization factor $\sum_i w_i n_i(T)$ depends only on the size of the tree T.

Recurrence for generating functions

Let $p_t(n_1, \ldots, n_d)$ be the probability that the tree T at time t has $(n_1(T), \ldots, n_d(T)) = (n_1, \ldots, n_d)$.
The probability generating function

$$\mathcal{H}_t(\mathbf{x}) = \sum_{n_1 + \cdots n_d = t} p_t(n_1, \ldots, n_d) x_1^{n_1} \cdots x_d^{n_d}$$

satisfies the recurrence

$$\mathcal{H}_{t+1}(\mathbf{x}) = \sum_{n_1 + \cdots + n_d = t} \frac{p_t(n_1, \ldots, n_d)}{\sum_{i=1}^{d} n_i w_i} \mathbf{c}(\mathbf{x}) \cdot \nabla (x_1^{n_1} \cdots x_d^{n_d}),$$

where $\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_d(\mathbf{x}))$ with

$$c_i(\mathbf{x}) = \frac{i}{2} \sum_{j=1}^{i+1} w_{j,i+2-j} x_j x_{i+2-j} \quad \text{and} \quad \nabla = \Big(\partial/\partial x_1, \ldots, \partial/\partial x_d\Big).$$

Vertex degree distribution

- Begin with a tree T_0 at time 0
- At time $t > 0$ we have a tree T_t with $n_i(T_t)$ vertices of degree i
- Let $\bar{n}_{t,i}$ denote the average of $n_i(T)$ over all trees that can arise at time t, i.e.

$$\overline{n}_{t,k} = \sum_{n_1+\ldots+n_d=t} p_t(n_1, \ldots, n_d) n_k = \partial_k \mathcal{H}_t(\mathbf{x})|_{\mathbf{x}=\mathbf{1}},$$

- Define

$$\rho_{t,i} = \frac{\bar{n}_{t,i}}{t}$$

and we will use the notation

$$\rho(t) = (\rho_{t,1}, \ldots, \rho_{t,d})$$

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- The recurrence for $\mathcal{H}_t$ gives rise to a recurrence for $\rho(t)$.

$$\mathcal{H}_{t+1}(\mathbf{x}) = \frac{1}{\mathcal{W}(t)}\mathbf{c}(\mathbf{x}) \cdot \nabla\mathcal{H}_t(\mathbf{x}).$$

$$\Longrightarrow$$

$$\rho_{t+1,k} = \frac{t}{\mathcal{W}(t)}\left(-w_k\rho_{t,k} + \sum_{i=k-1}^{d} iw_{k,i+2-k}\rho_{t,i}\right) + t(\rho_{t,k} - \rho_{t+1,k}).$$

- Under mild conditions on the $w_{i,j}$ the limits

$$\lim_{t\to\infty} \rho_{t,i} = \rho_i$$

exist and are the unique positive solution to the linear equations

$$\rho_k = -\frac{w_k}{w_2}\rho_k + \sum_{i=k-1}^{d} i\frac{w_{k,i+2-k}}{w_2}\rho_i.$$

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Perron-Frobenius

Theorem. Let A be a matrix with nonnegative matrix elements such that all the matrix elements of A^p are positive (A primitive) for some integer p. Then the eigenvalue of A with the largest absolute value is positive and simple. The corresponding eigenvector can be taken to have positive entries.

Iterating the recurrence equation for $\rho(t)$ we find

$$\rho(t) = \frac{1}{t} \prod_{i=1}^{t-1} \left(1 + \frac{1}{(2a+b)i - 2a} B \right) \rho_0$$

where B is a matrix with nonnegative entries except on the diagonal. If B is primitive and diagonalizable, then $\rho(t)$ converges to the normalized Perron-Frobenius eigenvector of B.

Examples

- $d = 3$ The matrix B is diagonalizable and

$$\rho_1 = \rho_3 = 2/7, \quad \rho_2 = 3/7$$

if the partitioning weights are chosen to be uniform, i.e.

$$w_{i,j} = w_{i+j-2} \frac{2}{(i+j-2)(i+j-1)}.$$

- $d = 4$ Can again solve explicitly with uniform partitioning weights and get ρ_i's which vary with a and b.
- $d = \infty$ Do not have a proof of convergence but can solve the equation for the ρ_i's

$$\rho_k \sim \frac{1}{k!} 2^{k-1} k^{-1-x}, \quad x = b/a$$

$$\rho_k = \frac{1}{e(k-1)!}, \quad a = 0.$$

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General splitting weights

- Use mean field theory for the normalization factor $\sum_i n_i(T) w_i \to t \sum_i \rho_i w_i$.
  Equation for a steady state vertex distribution

$$\rho_k = -\frac{w_k}{w}\rho_k + \sum_{i=k-1}^{d} i \frac{w_{k,i+2-k}}{w}\rho_i,$$

  subject to the constraints

$$\rho_1 + \ldots + \rho_d = 1, \quad w_1\rho_1 + \ldots + w_d\rho_d = w.$$

- There is a unique positive solution by the Perron-Frobenius theorem.
- For $d = 3$ and uniform partitioning weights we find

$$\rho_3 = \frac{7\alpha - \sqrt{\alpha\,(\alpha + 24\,\beta + 24)}}{6(2\alpha - \beta - 1)}$$

  where $\alpha = w_2/w_1$ and $\beta = w_3/w_1$.

Comparison with simulations

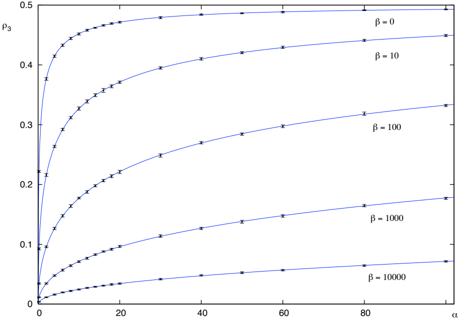

FIGURE 4. The value of ρ_3 as given in (2.45) compared to results from simulations. Each point is calculated from 20 trees on 10000 vertices.

A comparison of the theoretical prediction with simulations in the case d=3 and uniform partitioning weights.

$$\alpha = \frac{w_2}{w_1}, \quad \beta = \frac{w_3}{w_1}$$

Correlations

In a typical infinite tree, what is the proportion of edges whose endpoints have degrees j and k ?

Let $n_{j,k}$ = number of such edges in a finite tree of size t, where the vertex of degree j is closer to the root

Let $\rho_{j,k} = \lim_{t\to\infty} \frac{n_{j,k}}{t}$. Then (for linear splitting weights)

$$\begin{aligned}\rho_{jk} &= -\frac{w_j + w_k}{w_2}\rho_{jk} + (j-1)\frac{w_{j,k}}{w_2}\rho_{j+k-2} \\ &\quad +(j-1)\sum_{i=j-1}^{d}\frac{w_{j,i+2-j}}{w_2}\rho_{ik} + (k-1)\sum_{i=k-1}^{d}\frac{w_{k,i+2-k}}{w_2}\rho_{ji}.\end{aligned}$$

assuming the existence of the limit.

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assuming the existence of the limit.

Explicit solutions

Can solve in simple cases and find nontrivial correlations
Take $d = 3$, linear splitting weights and uniform partitioning weights. Then $\rho_1 = \rho_3 = 2/7$ and $\rho_2 = 3/7$. Let $y = w_3/w_2$.
Then the solutions to the correlation equation are

$$\rho_{21} = \frac{4(3-y)}{7(11-2y)}, \qquad \rho_{31} = \frac{10}{7(11-2y)},$$

$$\rho_{22} = \frac{4y^2 - 12y + 105}{7(2y+7)(11-2y)}, \qquad \rho_{32} = \frac{2(-8y^2 + 18y + 63)}{7(2y+7)(11-2y)},$$

$$\rho_{23} = \frac{2(-4y^2 + 20y + 21)}{7(2y+7)(11-2y)}, \qquad \rho_{33} = \frac{8(3y-14)}{7(2y+7)(2y-11)}.$$

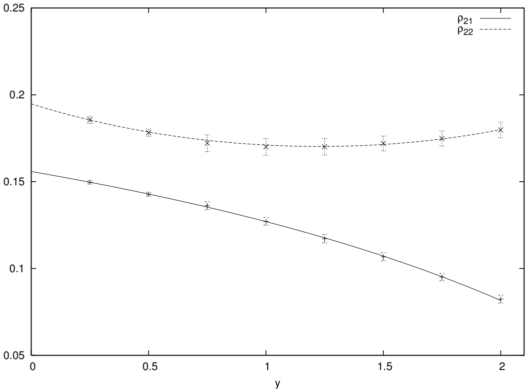
Sum rules

The following sum rules hold:

$$\begin{aligned}
\rho_{21} + \rho_{31} &= \rho_1 &&= 2/7 \\
\rho_{22} + \rho_{32} &= \rho_2 &&= 3/7 \\
\rho_{23} + \rho_{33} &= \rho_3 &&= 2/7, \\
\rho_{21} + \rho_{22} + \rho_{23} &= \rho_2 &&= 3/7 \\
\rho_{31} + \rho_{32} + \rho_{33} &= 2\rho_3 &&= 4/7.
\end{aligned}$$

These relations show that there are only two independent link densities, e.g. ρ_{21} and ρ_{22}.

Comparison with simulations

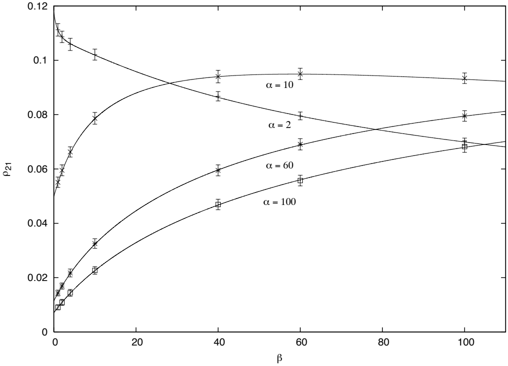


Nonlinear splitting weights

Taking $d = 3$ and general nonlinear splitting weights

$$\rho_{21} = \frac{1}{3}\,\frac{(3+\beta)\,(7\,\alpha-\gamma)}{(2\,\alpha-\beta-1)\,(3\,\alpha+2\,\beta+\gamma+6)}$$

where $\alpha = w_2/w_1$, $\beta = w_3/w_1$ and $\gamma = \sqrt{\alpha\,(\alpha+24\,\beta+24)}$.



Comparison with simulations

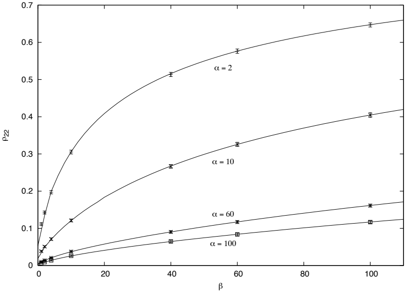


FIGURE 17. The solution (5.5) for the density ρ_{22} plotted as a function of β for a few values of α. Each datapoint is calculated from simulations of 100 trees on 10000 vertices.

$$
\begin{aligned}
\rho_{22} \quad = \quad & \frac{16}{3}\Big(284\,\alpha^2\beta^4\gamma - 177\,\alpha^5\beta\gamma + 3564\,\alpha^3 + 18\,\alpha^6\gamma + 161\alpha\,\beta^5\gamma - 873\,\gamma + 11979\,\alpha^2\beta^3 \\
& -2259\,\alpha^5 - 39\,\alpha^6\beta - 207\,\alpha^5\gamma + 6516\,\alpha^2\beta^4 - 5205\,\alpha^5\beta - 1419\,\alpha^4\beta\gamma + 996\,\alpha\beta^5 \\
& -5994\,\alpha^4 - 892\,\alpha^4\beta^2\gamma + 1543\,\alpha^2\beta^5 - 18\,\alpha^7 - 668\,\alpha^3\beta^4 + 324\,\alpha^2\gamma + 909\,\alpha\beta^3\gamma \\
& -2600\,\alpha^5\beta^2 - 975\,\alpha^3\beta^3 + 222\,\alpha\beta^6 - 1533\,\alpha^3\beta^2\gamma + 10206\,\alpha^2\beta^2 - 11799\,\alpha^4\beta \\
& -5300\,\alpha^4\beta^3 - 1521\,\alpha^3\beta\gamma + 1899\,\alpha^2\beta^2\gamma + 1059\alpha^2\,\beta^3\gamma + 1269\,\alpha^3\beta^2 + 3240\alpha^2\,\beta \\
& +756\,\alpha\beta^3 + 4860\,\alpha^3\beta + 6\,\beta^6\gamma - 11703\,\alpha^4\beta^2 + 1728\alpha^2\,\beta\gamma - 162\,\alpha^3\gamma + 486\alpha\,\beta^2\gamma \\
& +18\,\beta^4\gamma + 1530\,\alpha\beta^4 + 624\alpha\,\beta^4\gamma - 772\,\alpha^3\beta^3\gamma - 9\,\alpha^6 + 24\,\beta^5\gamma\Big)\Big/\Big((3\,\alpha + 2\,\beta + \gamma + 6) \\
& \times\left(11\,\alpha^2 + 25\,\alpha\beta + 5\,\alpha\gamma + 3\,\beta\gamma + 12\,\alpha + 4\,\beta^2\right)(-\alpha + \gamma)\,(1 - 2\,\alpha + \beta)\,(7\,\alpha + 2\,\beta + \gamma)^2\Big)
\end{aligned}
$$

Subtree probabilities

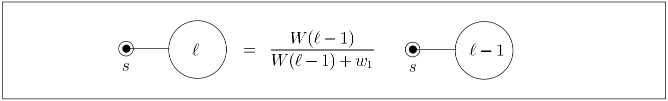
- Label vertices in the tree by their time of creation
- Use linear weights
- Derive expressions for the probabilistic structure of the tree as seen from the vertex created at a given time
- Average over the creation time
- Introduce a scaling assumption
- Extract the Hausdorff dimension
- Get results which agree with simulations

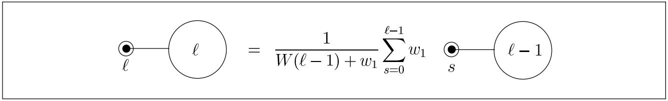
- Begin with a tree consisting of a single vertex at time $t = 0$
- In a tree of size ℓ let $p_R(\ell; s)$ be the probability that the vertex created at time $s \leq \ell$ is the root
- We find

$$p_R(\ell; s) = \frac{1}{W(\ell-1)+w_1} W(\ell-1) p_R(\ell-1; s), \quad s < \ell$$

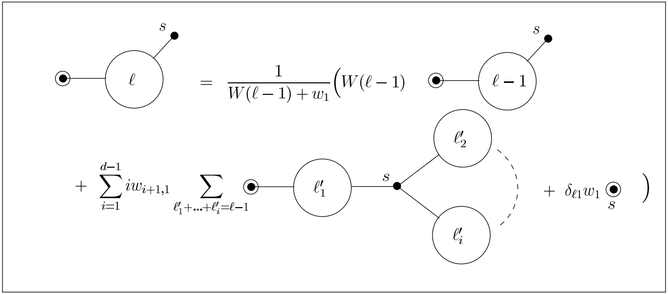
$$p_R(\ell; \ell) = \frac{1}{W(\ell-1)+w_1} \sum_{s=0}^{\ell-1} w_1 p_R(\ell-1; s), \quad s = \ell$$

$W(\ell) = (2a+b)\ell - a$ is a normalization factor.

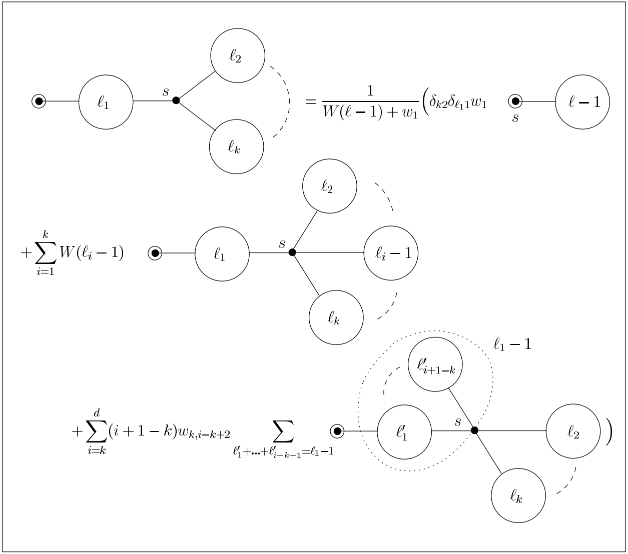




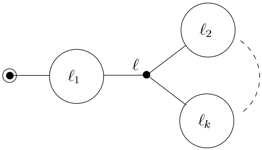
Let $p_k(\ell_1, \ell_2, \ldots, \ell_k; s)$ be the probability that the vertex v created at time s has degree k, the root subtree has ℓ_1 links and the other subtrees incident on v have size $\ell_2, \ldots, \ell_k$. Denote the sum of the ℓ_i's by ℓ. Then for $k = 1$ and $s < \ell$



and for $k > 1$ and $s < \ell$

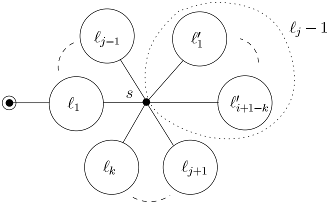


Finally $k > 1$ and $s = \ell$



$$= \frac{1}{W(\ell-1)+w_1} \sum_{s=0}^{\ell-1}\sum_{j=2}^{k}\sum_{i=k-1}^{d-1} w_{k,i-k+2} \sum_{\ell'_1+\ldots+\ell'_{i+1-k}=\ell_j-1}$$

$\times$



We average over s to get simpler recursions:

$$p_R(\ell+1) = \frac{\ell+1}{\ell+2} p_R(\ell).$$

$$p_1(\ell+1) \tag{3.11}$$
$$= \frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big[W(\ell)p_1(\ell) + \sum_{i=1}^{d-1} i w_{i+1,1} \sum_{\substack{\ell'_1+\ldots+\ell'_i \\ =\ell}} p_i(\ell'_1,\ldots,\ell'_i) + 2\delta_{\ell 0} w_1\Big].$$

$$p_k(\ell_1,\ldots,\ell_k)$$
$$= \frac{\ell+1}{\ell+2}\frac{1}{W(\ell)+w_1}\Big[\delta_{k2}\delta_{\ell_1 1} w_1 p_R(\ell) + \sum_{i=1}^{k} W(\ell_i-1) p_k(\ell_1,\ldots,\ell_i-1,\ldots,\ell_k)$$
$$+ \sum_{i=k}^{d} (i-k+1)\, w_{k,i-k+2} \sum_{\substack{\ell'_1+\ldots+\ell'_{i+1-k} \\ =\ell_1-1}} p_i(\ell'_1,\ldots,\ell'_{i+1-k},\ell_2,\ldots,\ell_k) \tag{3.12}$$
$$+ \sum_{j=2}^{k}\sum_{i=k-1}^{d} w_{k,i-k+2} \sum_{\substack{\ell'_1+\ldots+\ell'_{i+1-k} \\ =\ell_j-1}} p_i(\ell_1,\ldots,\ell_{j-1},\ell'_1,\ldots,\ell'_{i+1-k},\ell_{j+1},\ldots,\ell_k)\Big]$$

Finally we define the "two point functions" that are needed to calculate the Hausdorff dimension:

$$q_{ki}(\ell_1, \ell_2) = \sum_{\ell'_1+\ldots+\ell'_{k-i}=\ell_1} \sum_{\ell''_1+\ldots+\ell''_i=\ell_2} p_k(\ell'_1, \ldots, \ell'_{k-i}, \ell''_1, \ldots, \ell''_i),$$

which is the probability that i trees of total volume ℓ_1, none of which contains the root, are attached to a vertex of order k in a tree of total volume $\ell = \ell_1 + \ell_2$. There are $d(d-1)/2$ such functions, $1 \leq i \leq k-1$.

The two point functions satisfy the recursion relation

$$\begin{aligned} q_{ki}(\ell_1,\ell_2) \;&=\; \frac{\ell+1}{\ell+2}\,\frac{1}{W(\ell)+w_1}\Big[\\ &\sum_{j=k-1}^{d} w_{k,j+2-k}\Big((j-i)q_{ji}(\ell_1-1,\ell_2)+iq_{j,j-(k-i)}(\ell_1,\ell_2-1)\Big) \\ &+\Big(W(\ell_1-1)+(k-i-1)(w_2-w_3)\Big)q_{ki}(\ell_1-1,\ell_2) \\ &+\Big(W(\ell_2-1)+(i-1)(w_2-w_3)\Big)q_{ki}(\ell_1,\ell_2-1) \\ &+\delta_{k2}\delta_{\ell_1 1}w_1 p_R(\ell_2)+\delta_{i1}\delta_{\ell_2 1}w_{k,1}\sum_{\ell'_1+\ldots+\ell'_{k-1}=\ell_1} p_{k-1}(\ell'_1,\ldots,\ell'_{k-1})\Big] \end{aligned}$$

An almost closed system of linear equations.

Hausdorff dimension

- Let T be a tree with ℓ edges and v, w vertices of T.
- Denote the graph distance between v and w by $d_T(v, w)$.
- We define the radius of T as

$$R_T = \frac{1}{(2\ell)} \sum_{v \in T} d_T(r, v)\, \sigma(v),$$

- We define the Hausdorff dimension of the tree, d_H, by the scaling law for large trees

$$\langle R_T \rangle \sim \ell^{1/d_H} \qquad \ell \to \infty$$

This definition is different from the one we wrote down earlier for infinite trees but is expected to be equivalent.

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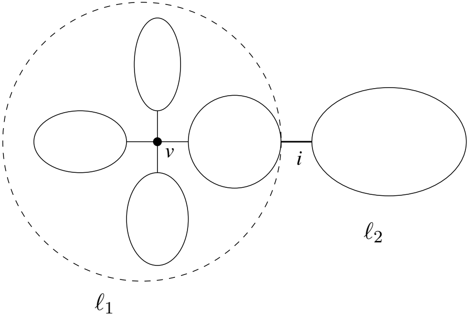
Combinatorics



- Cutting the tree at an edge i we get two subtrees of size ℓ_1 and ℓ_2
- One can prove the following identity:

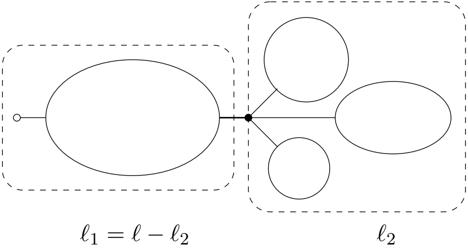
$$\sum_w d_T(v,w)\sigma(w) \;=\; \sum_i (2\ell_2(v;i)+1)$$

valid for any vertex v. We use it for $v = r$.

- The identity implies:

$$\langle R_T \rangle = \frac{1}{2\ell} \sum_T P(T) \sum_i (2\ell_2(r;i) + 1)$$

$$= \frac{\ell + 1}{2\ell} \sum_{\ell_2=0}^{\infty} (2\ell_2 + 1) \sum_{k=1}^{d} q_{k,k-1}(\ell - \ell_2; \ell_2)$$



- We use a scaling assumptions about the q functions

$$q_{ki}(\ell_1, \ell - \ell_1) = \ell^{-\rho}\omega_{ki}(\ell_1/\ell) + O(\ell^{\rho+1})$$

- Inserting into the recurrence equation for q_{ki} keeping leading order terms in ℓ^{-1} gives

$$(2-\rho)\overline{\omega}_{ki} \quad = \quad \frac{1}{w_2}\sum_{j=k-1}^{d} w_{k,j+2-k}\Big((j-i)\overline{\omega}_{ji} + i\overline{\omega}_{j,j-(k-i)}\Big) - \frac{w_k}{w_2}\overline{\omega}_{ki}.$$

- This is a Perron-Frobenius type equation. Gives ρ in principle.
- Can solve in simple cases and prove some bounds in more general cases.

Hausdorff dimension

Linear weights and $d = 3$

$$d_H = \frac{3(1 + \sqrt{1 + 16y})}{8y}, \quad y = w_3/w_2$$

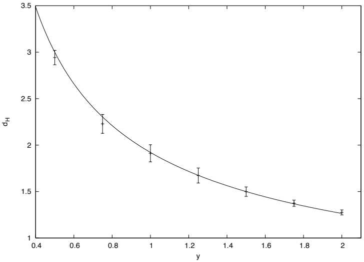


FIGURE 13. Equation (4.25) compared to simulations. The Hausdorff dimension, d_H, is plotted against $y = w_3/w_2$. The leftmost datapoint is calculated from 50 trees on 50000 vertices and the others are calculated from 50 trees on 10000 vertices.

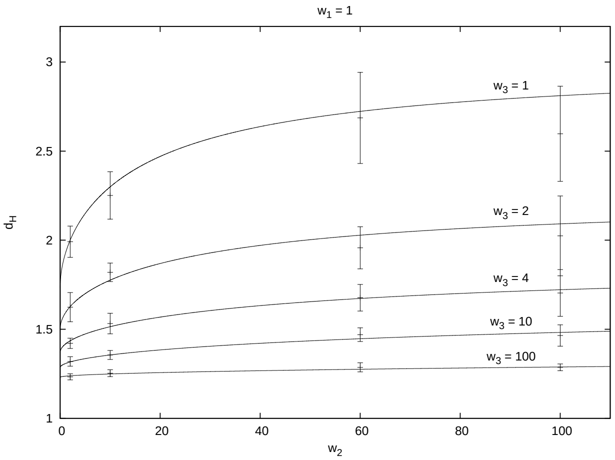
Hausdorff dimension

General solution for $d = 3$

$$d_H = \frac{(w_{2,2} - 2w_{3,1}) + \sqrt{(w_{2,2} - 2w_{3,1})^2 + 8w_{3,1}(w_{2,1} + 3w_{3,2})}}{(w_{2,2} - 2w_{3,1}) + \sqrt{(w_{2,2} - 2w_{3,1})^2 + 16w_{3,1}w_{3,2}}}.$$

$$w_{3,2} = w_3/3$$

$$w_{3,1} = w_{2,2} = w_2/3$$



Open problems

- Description in terms of equilibrium statistical mechanics
- The infinite volume limit
- A continuum limit
- Spectral properties
- Properties of the finite volume measures