Boundary CFT and applications to loop models

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WORK WITH JÉRÔME DUBAIL AND HUBERT SALEUR

Summary

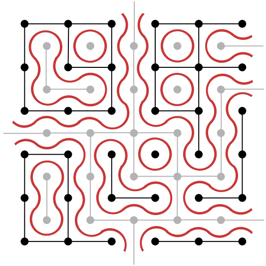
CRITICAL PHENOMENA IN TWO DIMENSIONS

- Scale and conformally invariant models
- Percolation, Ising model, self-avoiding walk…
- CFT, SLE, combinatorics…

FROM DISCRETE MODELS TO CONTINUUM LIMIT

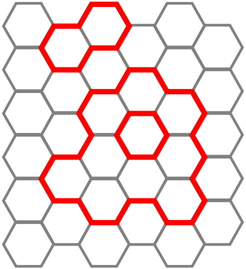
- In CFT, from lattice algebras to Virasoro algebra
- Exact solutions from Yang-Baxter integrability
- In SLE, from discrete holomorphicity to proof of conformal invariance

DENSE LOOPS AND THE Q-STATE POTTS MODEL



- Weight $n = \sqrt{Q}$ per loop
- Conformally invariant for $-2 < n \leq 2$

Dilute loops and the O(n) vector model



- Weight $n \in]-2,2]$ per loop and x per monomer
- Conformally invariant for $x = x_{\rm c} = \frac{1}{\sqrt{2+\sqrt{2-n}}}$

THE Q-STATE POTTS MODEL

- Defined on graph $G = (V, E)$
- Spins $\sigma_i = 1, 2, \ldots, Q$ on each vertex $i \in V$
- Nearest-neighbour coupling J_e for edge $e = (ij) \in E$

$$Z_G(Q, \mathbf{J}) = \sum_{\{\sigma_i\}} \prod_{e \in E} \exp\left[J_e \delta(\sigma_i, \sigma_j)\right]$$

FORTUIN-KASTELEYN'S *random cluster* REPRESENTATION

- We have $\mathrm{e}^{J\delta(\sigma_i,\sigma_j)} = 1 + v_e \delta(\sigma_i, \sigma_j)$ with $v_e = \mathrm{e}^{J_e} - 1$
- Expand, and let v_e terms define edge subset $A \subseteq E$

$$Z_G(Q, \mathbf{v}) = \sum_{A \subseteq E} Q^{C(A)} \prod_{e \in A} v_e$$

- Here $C(A) = \#$ connected components (clusters) in A

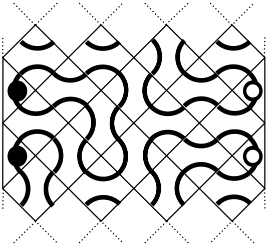
If G is planar, representation as *loop model*

- N loops live on medial lattice $\mathcal{M}$
- Vertices of $\mathcal{M}$ are edge mid-points of G
- Use Euler relation $C = (N + |V| - |E|)/2$

$$Z_G(Q, \mathbf{v}) = Q^{|V|/2} \sum_{A \subseteq E} Q^{N/2} \prod_{e \in A} \frac{v_e}{\sqrt{Q}}$$

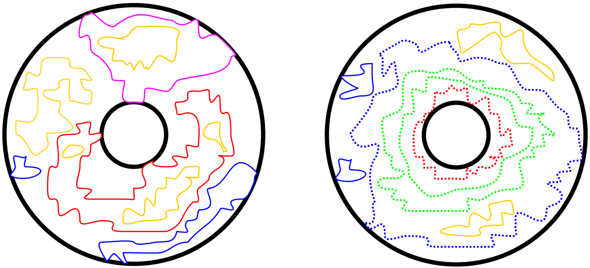
- Selfdual for $v_e \equiv v = \sqrt{Q}$
- In fact conformally invariant
- Just weight $n = \sqrt{Q}$ per loop

Conformal boundary loop model



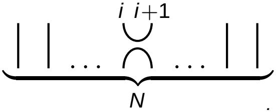
- Example for dense loops
- Modified weight for loops touching one or more boundaries

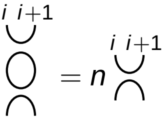
Annular geometry

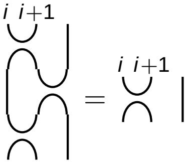


- n, n_1, n_2, n_{12} for loop touching no/1st/2nd/both boundaries
- Can also distinguish homotopy
- Conformally invariant for *any* real value of these weights

Temperley-Lieb (TL) algebra

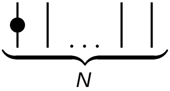
- No distinguished boundaries
- Generators $e_i =$ 

- Relations: 

and 

- Algebraically: $e_i^2 = n e_i$ and $e_i e_{i\pm1} e_i = e_i$

One-boundary Temperley-Lieb (1BTL) algebra

- Extra generator $b_1 =$ 

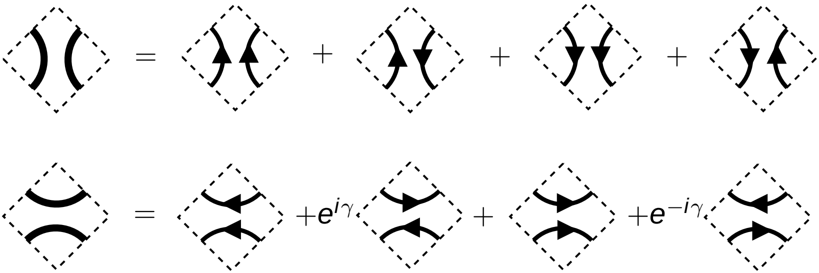
- Relations: $b_1^2 = b_1$ and $e_1 b_1 e_1 = n_1 e_1$

Two-boundary Temperley-Lieb (2BTL) algebra

- Introduce similarly b_2 acting on right boundary
- Weight n_{12} imposed by taking a quotient

Relation to six-vertex and height models

- Orient each loop; distribute $n = 2\cos\gamma$ over orientations



- Summing over loop splittings gives six-vertex model
- Dual height model leads to a (deformed) free bosonic field

From scale invariance to conformal invariance

- Critical phenomena are scale invariant
- Leads to the renormalisation group approach

- For short-ranged interactions, take space-dependent scale factor
- Angle preserving transformation
- Linked to analytic maps in 2D (Cauchy-Riemann)
- Gives infinite number of generators (from Laurent series)

Virasoro algebra

- Generators L_n (and also $\bar{L}_n$) satisfy

$$L_n L_m - L_m L_n = (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}$$

- Central charge c enters also correlation functions, finite-size corrections...

Representation theory

- Physical operators (magnetisation, energy...) are primaries ϕ
- Definition: $L_n \, \phi = 0$ for $n > 0$ (highest weight)
- L_0 is the dilatation operator
- $L_0 \, \phi = h \, \phi$, where h is a critical exponent

Characters

- $\{L_n\}$ acting on ϕ generates a representation $\mathcal{V}(c,h)$
- Inner product defined by $L_n^\dagger = L_{-n}$
- The character of $\mathcal{V}(c,h)$:

$$\chi_{(c,h)}(q) = \mathrm{Tr}\, q^{L_0 - c/24} = \frac{q^{h-c/24}}{P(q)}$$

  where $\frac{1}{P(q)} \equiv \prod_{n=1}^{\infty} \frac{1}{1-q^n}$ is the generating function of integer partitions. $\mathcal{V}(c,h)$ is not always irreducible.
- $q = \exp(2\pi \mathrm{i} L/N)$ for a system of size $L \times N$

Minimal models

- Definition: finite number of primaries, no negative norms
- Only possible for $m \geq 2$, $1 \leq r < m$, $1 \leq s \leq m$, all integers, in:

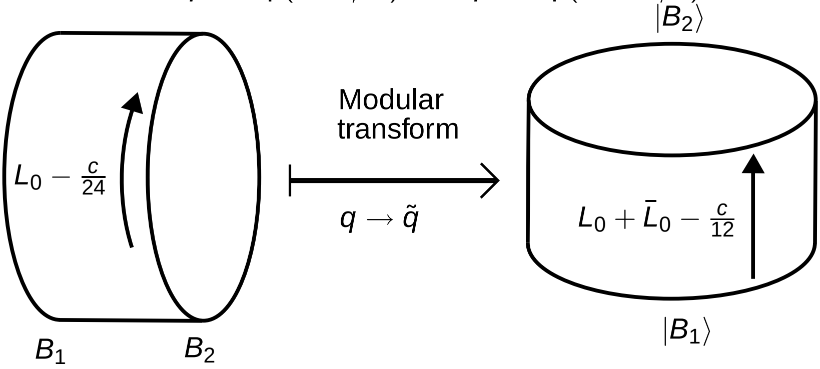
$$
\begin{aligned}
c &= 1 - \frac{6}{m(m+1)} \\
h_{r,s} &= \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}
\end{aligned}
$$

CFT of dense loops

- Loop weight $n = 2\cos\gamma$. Gives above CFT when $\gamma = \frac{\pi}{m+1}$, and $m > 0$ (not necessarily integer).

Boundary conformal field theory

- Define theory in upper half plane
- Using analytic continuation in lower half plane, one has now a *single* Virasoro algebra $\{L_n\}$
- Related to bulk CFT by modular transform of the annulus
- We have $q = \exp(-\pi L/N)$ and $\tilde{q} = \exp(-2\pi N/L)$



Renormalisation group flows in bulk CFT...

- Example: Ising model on triangular lattice
  - $H = -J_1 \sum_{\langle ij \rangle} S_i S_j + J_2 \sum_{\langle ijk \rangle} S_i S_j S_k$
  - Flow from tricritical to critical point (universality...)
- Zamolodchikov: c decreases along RG flows

...and in boundary CFT

- Example: Conformally invariant boundary conditions of Ising model: free or fixed ($++++\ldots$ or $----\ldots$)
  - Subtle: Brascamp-Kunz bc's ($+-+-\ldots$) flow to free
- Define g-factor $g_B = \langle B | 0 \rangle$ of the boundary condition B
  - Makes sense both in discrete model and in CFT
- Affleck-Ludwig: g decreases along boundary RG flows

Coulomb gas method

- Free field limit of heights (dual to oriented loops)
  - New ingredient: non-conservation of arrows at boundary
- Result: dominant critical exponent for any *sector*
  - Sector labeled by number of non-contractible loops and whether these can/cannot touch the boudaries

Combinatorial analysis of Markov trace

- Goal is to give correct weight to non-contractible loops
- Result: non-trivial amplitude/multiplicity for each sector

Example for dense loops [Bauer-Saleur 1989]

$$
\begin{aligned}
Z &= \sum_{\ell \geq 0} U_\ell \left(\frac{n}{2}\right) K_\ell(q) \\
K_\ell(q) &= \frac{q^{-c/24}}{P(q)} \left(q^{h_{1,1+\ell}} - q^{h_{-1,1+\ell}}\right)
\end{aligned}
$$

- $U_\ell(x)$ is ℓ'th order Chebyshev polynomial of 2nd kind
- ℓ is interpreted as the number of non-contractible loops

- We have extended this to the 1BTL and 2BTL cases…
- …and also to the model of dilute loops

Partition function for dense 1BTL model

$$
\begin{aligned}
n &= 2\cos\gamma \\
n_1 &= \frac{\sin(r+1)\gamma}{\sin r\gamma} \\
Z &= \frac{q^{-c/24}}{P(q)}\left(\sum_{j\geq 0}\frac{\sin(r+2j)\gamma}{\sin r\gamma}q^{h_{r,r+2j}} + \sum_{j\geq 1}\frac{\sin(-r+2j)\gamma}{\sin -r\gamma}q^{h_{-r,-r+2j}}\right)
\end{aligned}
$$

- 1st/2nd term: nearest non-contractible loop does/does not touch the boundary

Partition function for dense 2BTL model

- Obtained from lengthy computation of modular transforms
- Similar structure, now with 4 sectors and a tricky term involving n_{12}

Boundary chromatic polynomial

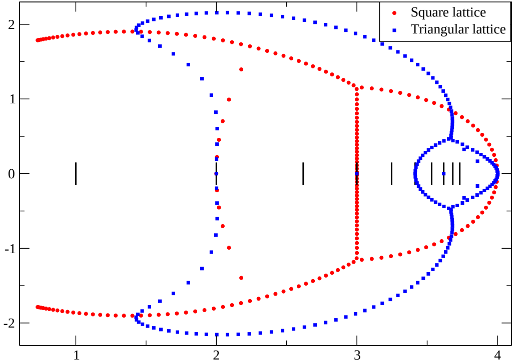
- Graph colouring with $Q = n^2$ (resp. $Q_1 = nn_1$) colours at bulk (resp. boundary) vertices
- Number of colourings = chromatic polynomial = special case of $Z_{\rm Potts}(Q, Q_1) = Z_{\rm 1BTL}(n, n_1)$
- Phase diagram inferred from accumulation points of $Z = 0$

Beraha-Kahane-Weiss theorem

- Isolated zeros when amplitude of dominant term vanishes
- Curves of zeros when two leading exponents coincide
- As $Q \uparrow 4$, ground state has $\ell \uparrow \infty$

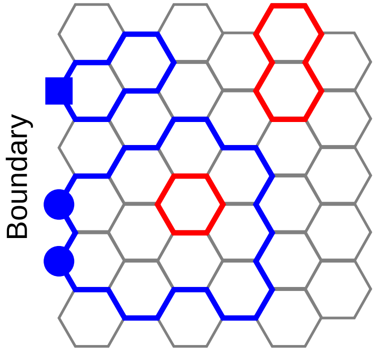
Numerical check for $Q_1 = Q - 2$

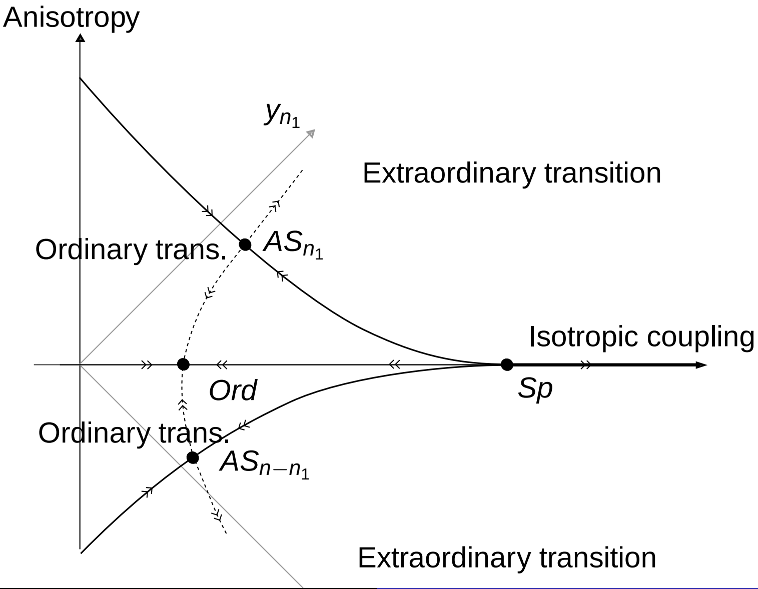
- Annulus of width $L = 2$ spins
- Chromatic zeros of square and triangular lattices in complex Q plane



Dilute model with surface anisotropy

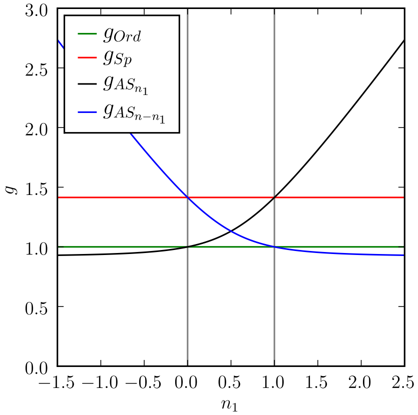
- Different weights for boundary monomers of n_1-type and $(n - n_1)$-type loops
- New integrable points when the former (latter) stand at a special (ordinary) surface transition




Anisotropy
y_{n_1}
Extraordinary transition
Ordinary trans.
AS_{n_1}
Isotropic coupling
Ord
Sp
Ordinary trans.
AS_{n-n_1}
Extraordinary transition

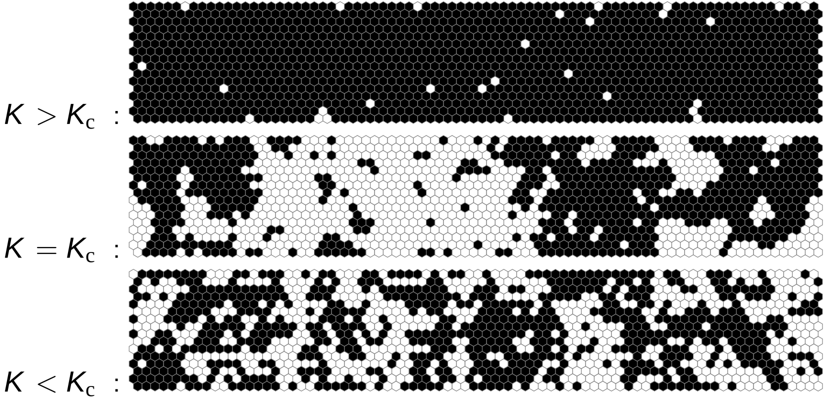
Phase diagram

- Consistent with exactly computed boundary entropies



Combinatorial and probabilistic applications

- Probability of having ≥ 1 Ising cluster crossing the annulus: $P_{\rm c}(\tau) = \frac{\eta(i\tau)\eta(i\tau/12)^2}{\eta(i\tau/2)^2\eta(i\tau/6)}$ for $K = K_{\rm c}$, with $\tau = N/L$
- $\sqrt{3}/4$ crossings per unit length for $\tau \gg 1$



Critical percolation

- Exact boundary entropy $g(n = 1, n_1)$ in finite size, in terms of refined ASM (alternating sign matrices)
- Probabilities for number of clusters wrapping annulus, refined according to whether they touch no/one/both rims
  - For instance in a square geometry:

j	$\sum_{\alpha,\beta} P_j^{\alpha\beta}$	$P_j^{\rm bb}$	$P_j^{\rm ub} = P_j^{\rm bu}$	$P_j^{\rm uu}$
0	0.636454001888			
1	0.361591025956	0.277067148156	0.041313949815	0.0018959781702
2	0.001954814340	0.001895978170	0.000029339472	0.0000001572261
3	0.000000157814	0.000000157226	0.000000000294	0.0000000000002