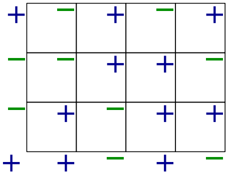
The critical Ising model on isoradial graphs : an approach via dimers

Cédric Boutillier (UPMC Paris)

joint work with Béatrice de Tilière (Neuchâtel)

September 7, 2009

The Ising model



- G graph. ex: rectangular box of $\mathbb{Z}^2$
- spin configuration $\sigma : G \to \{-1, +1\}$
- energy: $H(\sigma) = - \sum_{e=(v,w)} J_e \sigma_v \sigma_w$
- probability of a spin configuration:

$$P(\sigma) = \frac{1}{Z} \exp(-H(\sigma))$$

Questions: $Z = ?$, $\langle \sigma_v \sigma_w \rangle = ?$

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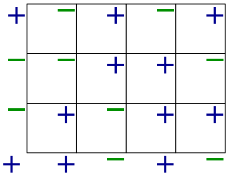
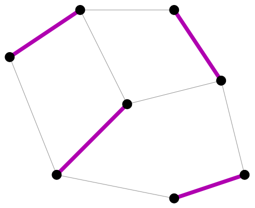


Several techniques to study/solve the Ising model

- Random cluster model (percolation)
- Transfer matrix
- Free fermion interpretation

  Fisher: correspondence with **dimers models**

Dimer models



- a **dimer configuration** or a **perfect matching** $\mathcal{C}$ of a graph $\mathcal{G}$ is a subset of edges such that every vertex is incident with exactly one edge of $\mathcal{C}$.
- weight $w : E(\mathcal{G}) \to \mathbb{R}_+^*$
- probability:

$$P(\mathcal{C}) = \frac{1}{\mathcal{Z}} \prod_{e \in \mathcal{C}} w_e$$

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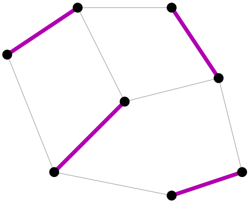


$$P(\mathcal{C}) = \frac{1}{\mathcal{Z}} \prod_{e \in \mathcal{C}} w_e$$

Adjacency matrix:

$$K_{u,v} = \begin{cases} w_e & \text{if } e : u \to v \\ -w_e & \text{if } e : v \to u \end{cases}$$

Theorem (Kasteleyn)

if $\mathcal{G}$ is planar, there exists an orientation such that

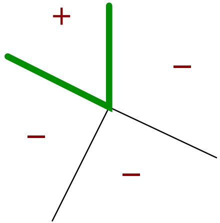
$$\mathcal{Z} = \mathrm{Pf}\, K = \sqrt{\det K},$$

$$P\left[(v_1, v_2), \ldots, (v_{2k-1}, v_{2k}) \in \mathcal{C}\right] = \left(\prod_{i=1}^{k} K_{v_{2i-1}, v_{2i}}\right) \mathrm{Pf}(K^{-1}_{v_i, v_j}).$$

K called Kasteleyn matrix

Fisher correspondence

Fisher 1966: correspondence between Ising on G and dimers on a decorated graph G_D.



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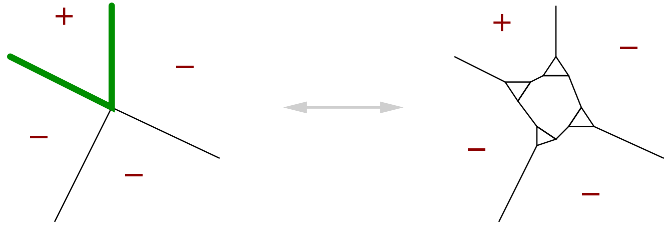

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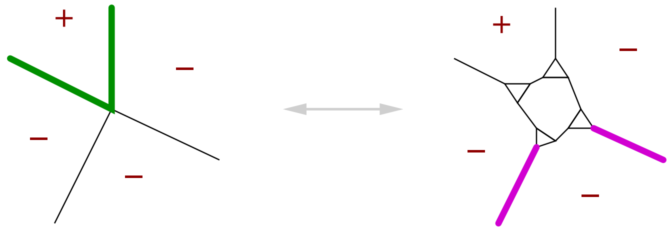

presence of a “long” edge in the dimer configuration ⇔ lack of this piece of Ising contour

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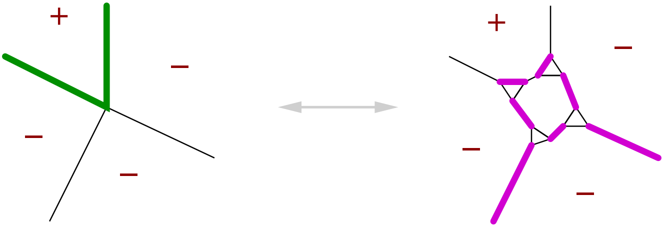

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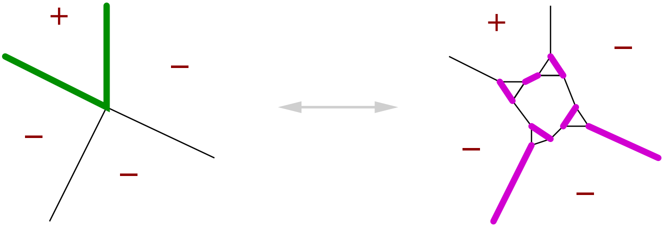

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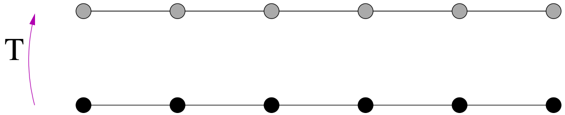
Transfer matrix T

- $2^n \times 2^n$ matrix: columns and rows indexed by spin config. on a row.



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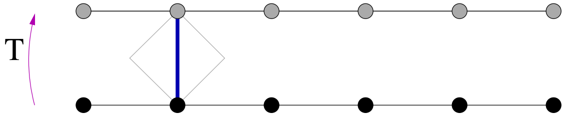

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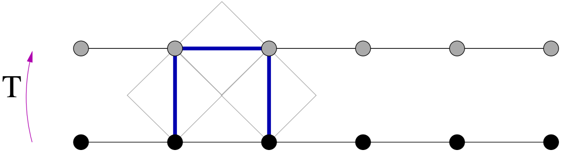

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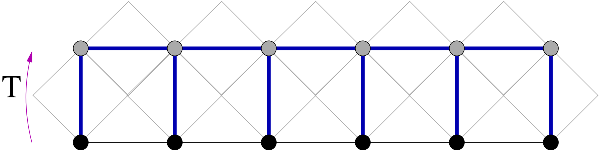

- combination of local operators building the graph edge by edge
- for periodic boundary conditions:

$$Z = \mathrm{tr}\, T^n \simeq \lambda_{\max}^n$$

- If parameters J are different on each row:

$$Z = \mathrm{tr}\, T_1 \cdots T_n \simeq ??$$

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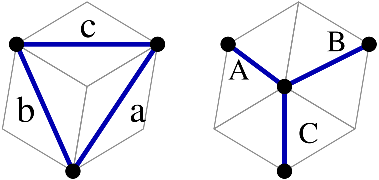
miracle: if the Boltzmann weights satisfy the *star-triangle relation*



$$abc = ABC + \frac{1}{ABC}$$
$$\frac{a}{bc} = \frac{A}{BC} + \frac{BC}{A}$$
$$\vdots$$
$$[a, \ldots, A, \cdots = \exp J]$$

then $T_1, \ldots, T_n$ commute and

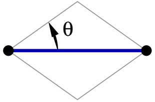
$$Z \simeq \lambda^{(1)}_{\max} \cdots \lambda^{(n)}_{\max}$$

INTEGRABILITY

Parametrization of the star-triangle relation

Isoradial graphs: convenient representation

edge $\leftrightarrow$ rhombus



1-parameter family of interaction constants $J(\theta)$:

$$\sinh(2J(\theta)) = \frac{\mathrm{sn}\left(\frac{2K(k)}{\pi}\theta | k\right)}{\mathrm{cn}\left(\frac{2K(k)}{\pi}\theta | k\right)}, \quad k^2 \in \mathbb{R} \quad \text{(Baxter)}$$

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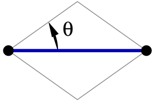

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"self-duality" (Kramer-Wannier) :

$$k = 0, \quad J(\theta) = \frac{1}{2} \log\left(\frac{1 + \sin\theta}{\cos\theta}\right)$$

critical inverse temperature for square ($\theta = \frac{\pi}{4}$), honeycomb ($\theta = \frac{\pi}{3}$), triangular ($\theta = \frac{\pi}{6}$) lattices.

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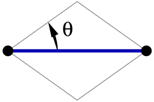

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Critical Ising model on isoradial graphs

Statistical Mechanics on isoradial graphs

- **Ising**: Baxter, Costas-Santos, Mercat, Smirnov-Chelkak
- **Electrical network, random walk, spanning trees**: star-triangle transformation for *conductances* $c(\theta) = \tan\theta$. Kenyon gave a local formula for the Green function.
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- describe a Gibbs measure for the critical Ising model

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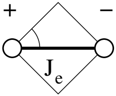
- give a "local" expression for the probability of cylinders
  simplest example:

$\mathbb{P}\Bigg($ 

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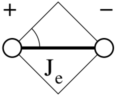
$\mathbb{P}\Bigg($ 

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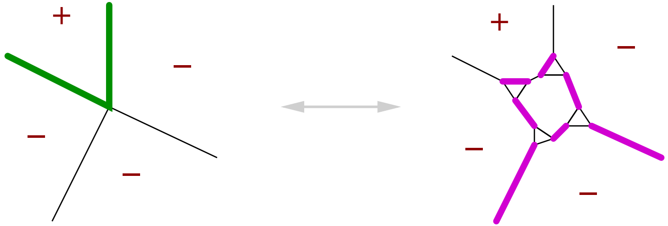
- **Tool**: *dimer models*

Fisher correspondence for isoradial critical Ising



missing piece of Ising contour ⇔ "long" dimer

dimer weights w_e:

  1 for short edges, coth $J(\theta) = \cot\frac{\theta}{2}$ for long edges

For finite planar graphs: Kasteleyn matrix K, K^{-1}.
For an infinite Fisher graph ?

Theorem (CB, B. de Tilière)

- *The inverse of the Kasteleyn matrix on the Fisher graph G_D has the following integral representation:*

$$K^{-1}_{v,w} = \frac{1}{(2\pi)^2} \oint_{C_{vw}} f_v(\lambda) f_w(-\lambda) \mathrm{Exp}_{\mathbf{vw}}(\lambda) \log(\lambda) \mathrm{d}\lambda$$

where C_{vw} is a contour avoiding $\mathbb{R}^+ \vec{vw}$.

$$\mathrm{Exp}_{\mathbf{v},\mathbf{w}}(\lambda) = \prod_{j=0}^{n} \frac{\lambda + e^{i\beta_j}}{\lambda - e^{i\beta_j}} \frac{\lambda + e^{i\gamma_j}}{\lambda - e^{i\gamma_j}} \quad \textit{discrete harmonic}$$

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- *The following expressions:*

$$\mathbb{P}[(v_1, v_2), \dots (v_{2k-1}, v_{2k})] = \left(\prod_{j=1}^{k} K_{v_{2j-1}, v_{2j}} \right) \mathrm{Pfaff}\left(K^{-1}_{v_i, v_j}\right)$$

define a Gibbs measure for the dimer model, and thus for the Ising model.

Idea of the proof

$$K_{vw}^{-1} = \frac{1}{(2\pi)^2} \oint_{C_{vw}} f_v(\lambda) f_w(-\lambda) \mathrm{Exp}_{\mathbf{vw}}(\lambda) \log(\lambda) \mathrm{d}\lambda$$

- $f_w(-\lambda) Exp_{v,w}(\lambda)$ is in the kernel of K. So $\left(K \cdot K^{-1}\right)_{vw} = 0$ if $w \neq v$.
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- The expression with the Pfaffian satisfy compatibility relations on cylinders.

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- **Why is it a probability measure?**

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- This expression depends only on the geometry of a path between v and w. Locality.
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- Take a large piece of G_D containing $(v_1, v_2), \ldots, (v_{2k-1}, v_{2k})$, and complete to make the graph periodic $\tilde{G}_D$.

$$\left(\prod_{j=1}^{k} K_{v_{2j-1}, v_{2j}} \right) \mathrm{Pfaff}\left(K_{v_i, v_j}^{-1} \right)$$

is the same on G_D and $\tilde{G}_D$.

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- We are left to understand what happens for $\mathbb{Z}^2$-periodic G_D.

End of the proof

- In the periodic case, K^{-1} defines a Gibbs measures for dimers (Pfaffian process).
- For a general isoradial graph G, The coefficient $K^{-1}_{v,w}$ does not depend much on the graph: can be computed in a periodic graph coinciding with G on a large ball.
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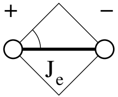
- **Computation:**

$P\Big($  $\Big)$

$$= 1 - K_{v,w} \,\mathrm{Pfaff} \begin{bmatrix} 0 & K^{-1}_{v,w} \\ K^{-1}_{w,v} & 0 \end{bmatrix}$$

$$= 1 - K_{v,w} K^{-1}_{w,v} = \frac{1}{4} - \frac{\theta_e}{2\pi \sin \theta_e}$$

Free energy

Theorem (Baxter; CB., B de Tilière)

Let G be a periodic isoradial graph with N sites in the fundamental domain. The free energy per site F_{Ising} is given by

$$F_{Ising} = -\frac{\log 2}{2} - \frac{1}{N} \sum_{e \in f.d.} \frac{\theta_e}{\pi} \log \theta_e + \frac{1}{\pi} \Big(L(\theta_e) + L(\frac{\pi}{2} - \theta_e) \Big)$$

where $L(\theta) = -\int_0^\theta \ln 2 \sin t \mathrm{d}t$ *is the Lobachevsky function.*

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Sketch of the proof: by deformation (following Kenyon)

- from the weight preserving correspondence, relate F_{Ising} and F_{dimers}.
- "Flatten" the graph by deforming the chains until $\theta_e = 0$ or $\frac{\pi}{2}$.

- When the graph is flat: independent copies of 1d lattices
- Control evolution along the deformation:

$$\frac{\partial \mathrm{Pfaff}\, A}{\partial A_{i,j}} = (\mathrm{Pfaff}\, A)(A^{-1})_{j,i}$$

$$\frac{\partial \log \mathrm{Pfaff}\, K}{\partial \alpha} = \sum_{i<j} \frac{\partial K_{i,j}}{\partial \alpha} K^{-1}_{j,i} = \sum_{e \in \mathrm{f.d}} \mathbb{P}[e] \frac{\partial \log w_e}{\partial \alpha}$$

- Integrate along the deformation

Asymptotics

Theorem (CB., B. de Tilière)

Let x and y be two vertices of type "2". Then

$$K^{-1}_{x,y} = \frac{1}{2\pi}\Im\left(\frac{e^{i\frac{\alpha+\alpha'}{2}}}{x-y}\right)(1+o(1)).$$

As a consequence,

$$\mathrm{Corr}[e_x, e_y] = -\frac{\sin\theta\sin\theta'}{4\pi^2(x-y)^2}(1+o(1)).$$

Dimers on periodic planar graphs

Kasteleyn: finite graph on the torus

- Partition function, and correlations:

$$Z = \frac{1}{2}\sum_{j=1}^{4} \pm \mathrm{Pfaff}\, K_j$$

$$\mathbb{P}[e_1, \ldots, e_k] = \frac{1}{2}\sum_{j=1}^{4} \pm \left(\prod_{l=1}^{k} (K_j)_{v_{2l-1}, v_{2l}}\right) \mathrm{Pfaff}\left((K_j^{-1})_{v_l, v_{l'}}\right)$$

  K_j Kast. matrix with $\pm$ on edges crossing non trivial cycles.
- if $\mathbb{Z}_m \times \mathbb{Z}_n$ periodicity, K_j^{-1} by discrete Fourier transform:

$$K_j^{-1}(w_{x,y}, v_{0,0}) = \frac{1}{(2\pi)^2} \sum_{\pm z^m = \pm w^n = 1} z^x w^y \frac{Q_{v,w}(z,w)}{P(z,w)}.$$

Lemma: $P(z,w) = \det K(z,w) = c \det \Delta(z,w)$. Harnack curve (of genus 0).
The 4 K_j^{-1} converge to the same integral: Pfaffian process in the limit

If you want to know more about dimers. . .

October 5-10

3 mini-courses
+ 4 introductory talks
+ workshop

organized here

registration: `http://ipht.cea.fr/statcomb2009/dimers/`