Critical percolation on the complete graph

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Bernoulli bond percolation

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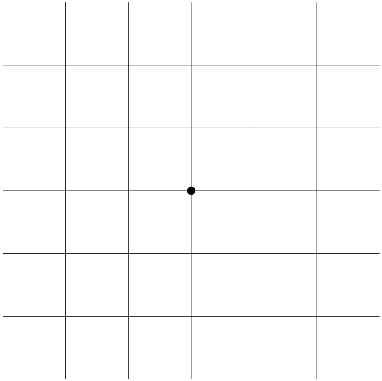

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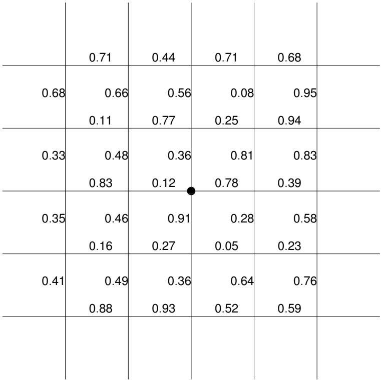

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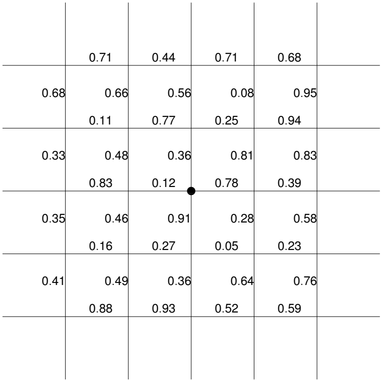

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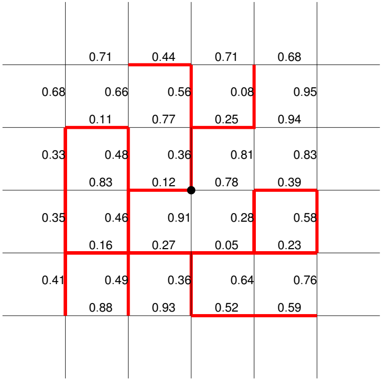

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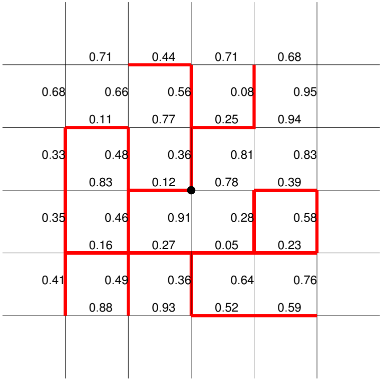
- Here $p = 0.6$.



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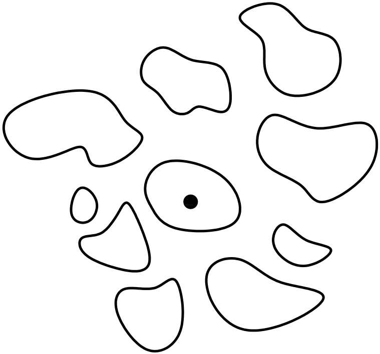

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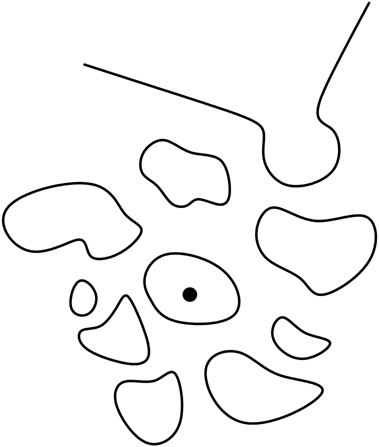

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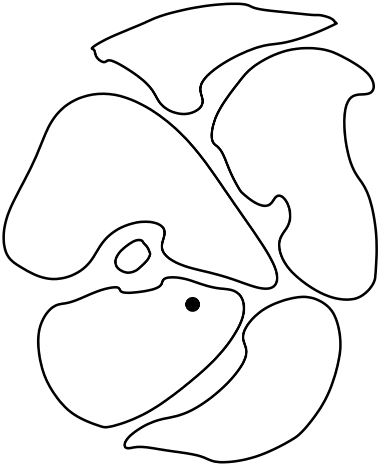
- $p = 1/2$: there is no infinite component* but $|\mathcal{C}_p|$ only has polynomial tails.

  (Critical case)



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In the remainder of this talk, I will focus on rescaled critical percolation on the complete graph.
A caveat : only “static” critical percolation, not the percolation process. (E.g., no “critical exponent” analogues.)

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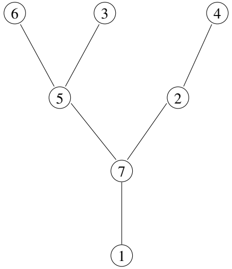
- For $p = 1/n$, a.a.s. the largest component of $G_{n,p}$ has size $\Theta(n^{2/3})$ and there may be many components of this order.

A quick comment : graph surplus.

A tree is a connected graph with no cycles. A tree always has one more vertex than it has edges.



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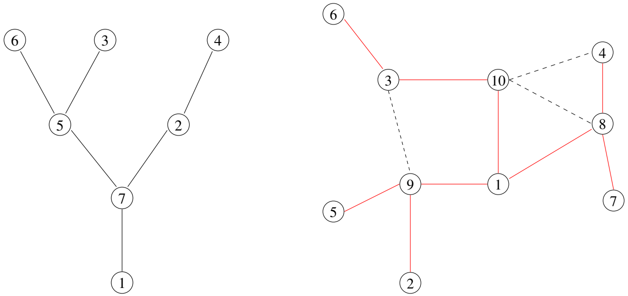

A connected graph which is not a tree has surplus equal to the number of edges more than a tree that it has. The graph on the right has surplus 3.

Convergence of component sizes of $G_{n,1/n}$.

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$$(C_1^{(n)}/n^{2/3}, C_2^{(n)}/n^{2/3}, \ldots) \to (L_1, L_2, \ldots), \quad \textit{and}$$
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What are the limit sequences ?

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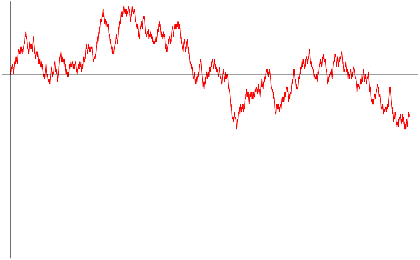

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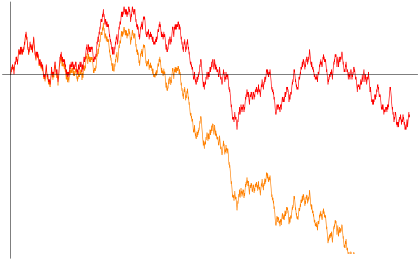

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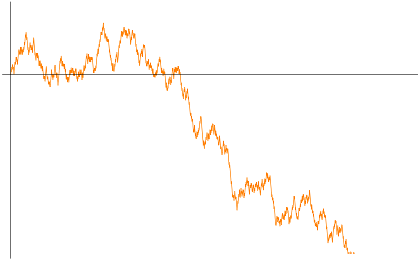

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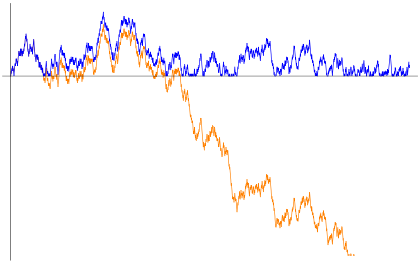

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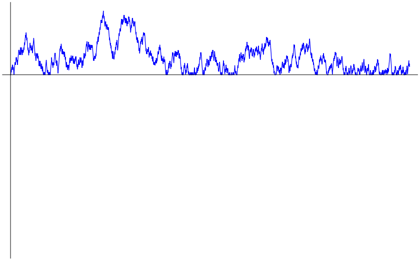

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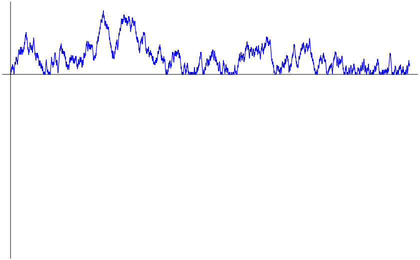

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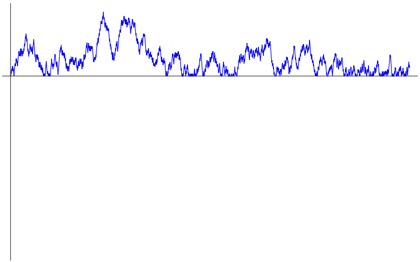

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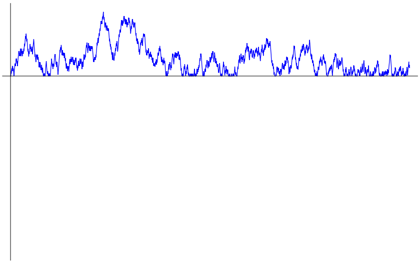

- Then S_i is distributed as the number of marks under the excursion with length L_i.

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- Large random trees are rather well understood and there is a very well-developed limit theory, which I will now briefly review.
- In order to understand the structure of the big clusters in the critical case $p = 1/n$, we have developed a corresponding theory for graphs with surplus edges.

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This probability does not depend on the precise choice of the tree T.

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- Now let $\mathcal{C}$ be a component of $G_{n,1/n}$ with m vertices, and suppose we condition $\mathcal{C}$ to be a tree.

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- Now let $\mathcal{C}$ be a component of $G_{n,1/n}$ with m vertices, and suppose we condition $\mathcal{C}$ to be a tree.
- By the preceding calculation, $\mathcal{C}$ is distributed as a *uniform random* tree on m labelled vertices ; any spanning tree of the vertex set is equally likely. (There are m^{m-2} such possible trees.)

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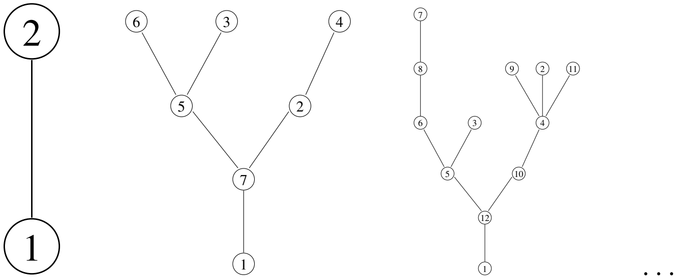
These trees are well understood ; they are distributed as Poisson Galton-Watson trees conditional on their size. In particular, they have height $\Theta(\sqrt{m})$.

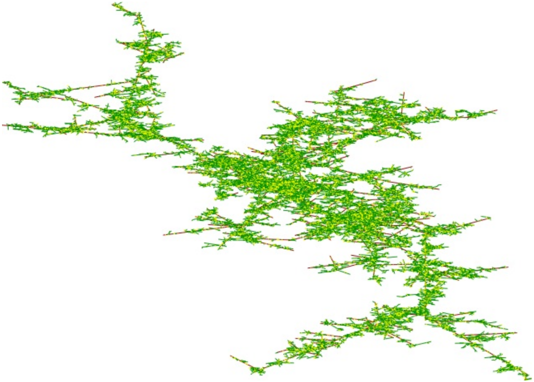
Digression : convergence of large random trees.

Suppose now we imagine letting m get large. Take the tree and let its edges all have length $\frac{1}{\sqrt{m}}$. (This is like looking at the tree from further and further away, while adding more and more vertices.)



The continuum random tree.

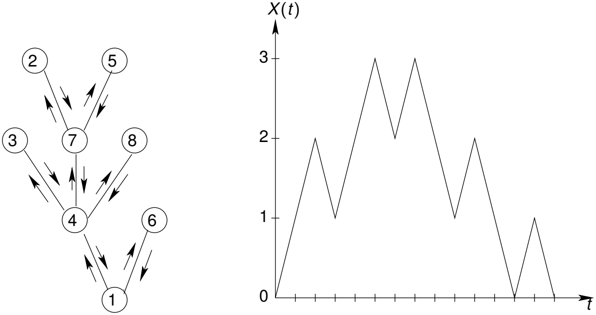
A very deep and beautiful result due to David Aldous says that as $m \to \infty$, there is a limiting object, called the Brownian continuum random tree.



[Picture by Grégory Miermont.]

The Harris walk.

The Harris walk is an excursion encoding heights in a finite tree.



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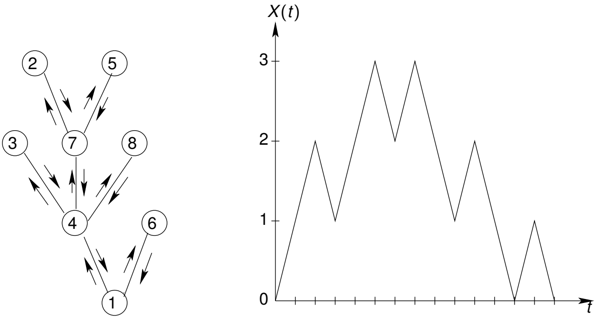


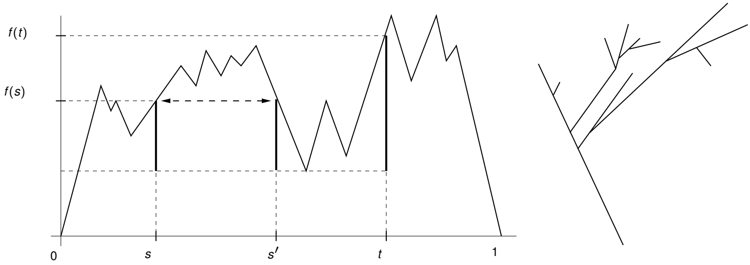
[Picture by Marie Albenque.]

When the tree is uniformly random on $1, 2, \ldots, m$, this essentially looks like a random walk with $2m$ steps, conditioned to stay positive and return to zero at time $2m$.

Rescaled, its limit is a Brownian excursion.

Trees coded by continuous functions.

Given *any* continuous excursion, we can define an associated tree (metric space), by gluing together points of the excursion that (a) have the same height and (b) can see each other.



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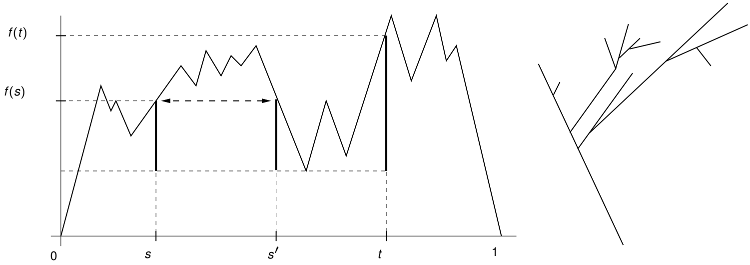

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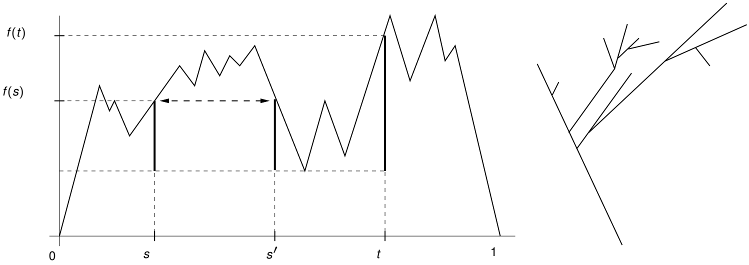

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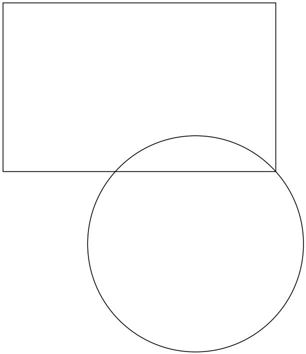
This is the limit of a uniform random tree : what kind of limit ?

Metric space convergence.

Given sets S, T in a metric space (M, d), let

$$b(S, T) = \inf\{r \ : \ T \subset b_r(S)\}.$$



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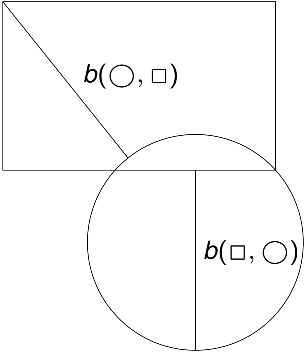
Given sets S, T in a metric space (M, d), let

$$b(S, T) = \inf\{r \ : \ T \subset b_r(S)\}.$$



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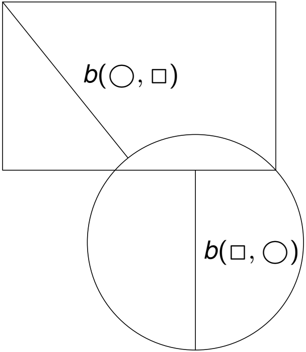

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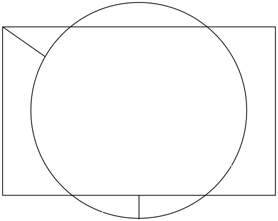

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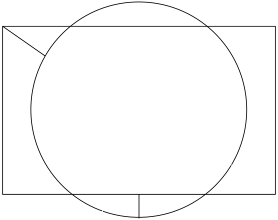


The *Gromov-Hausdorff* distance between metric spaces (S, d_1) and (T, d_2) is

$$\inf\{d_H(S, T)\},$$

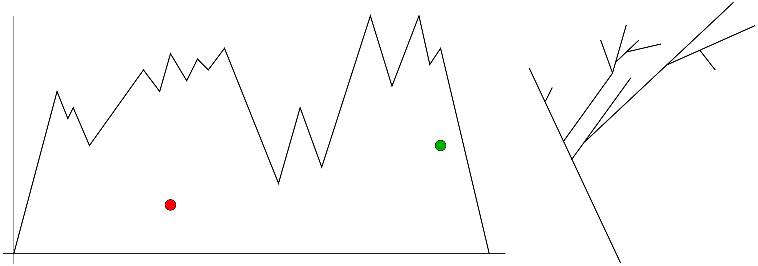
where the infemum is over all metric spaces (M, d) containing both $(S, d_1$ and (T, d_2) as subspaces.

Metric space convergence.

The uniform random tree on $1, \ldots, m$, with distances rescaled by $1/\sqrt{m}$, converges in distribution to the Brownian continuum random tree, with respect to the Gromov-Hausdorff distance.

Handling the surplus edges.

We saw how to associate a tree to a continuous excursion by gluing together points of the excursion that (a) have the same height and (b) can see each other.



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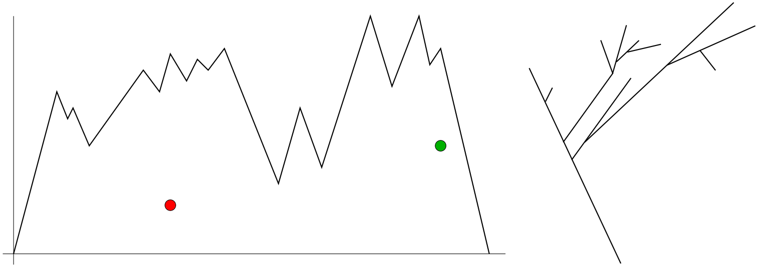

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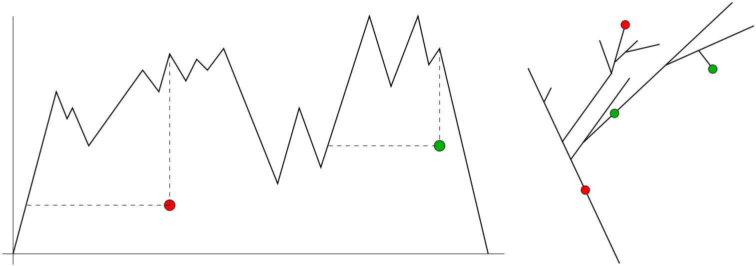

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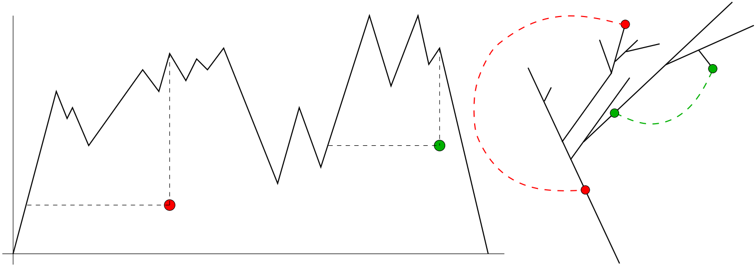

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We obtain a new metric space by identifying these points.

Limiting random graph : what kind of limit.

Let $\mathcal{C}^{(n)} = (\mathcal{C}_1^{(n)}, \mathcal{C}_2^{(n)}, \ldots)$ be the size-ordered sequence of components of $G_{n,1/n}$.

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Here convergence is with respect to the metric

$$d(\mathcal{C}, \mathcal{M}) = \left(\sum_{i=1}^{\infty} d_{GH}(\mathcal{C}_i, \mathcal{M}_i)^4 \right)^{1/4}.$$

Proof idea.

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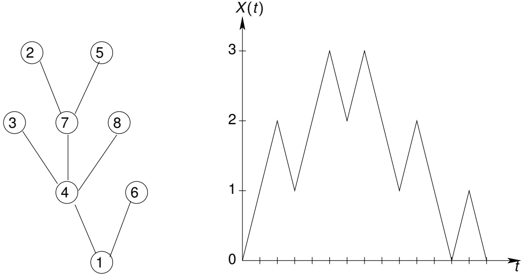
- Then $\mathcal{C}$ is distributed as a *uniform random* connected graph on *m* labelled vertices with *s* surplus edges (I am equally likely to pick each of connected graphs with surplus *s* on those labels.)

Proof idea.

The key turns out to be to study a component of $G_{n,1/n}$ conditioned on its size but *not* on its excess. Can we find a similar limiting object ?

A canonical spanning tree.

Consider the Harris walk of a spanning tree of $\{1, \ldots, m\}$.



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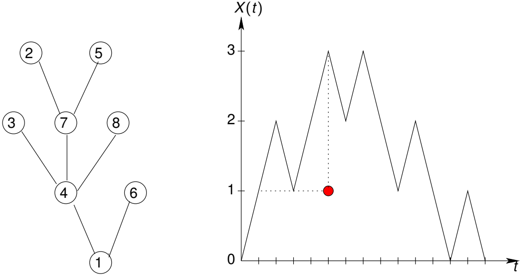

A mark at a lattice point under the excursion corresponds to a unique possible surplus edge.

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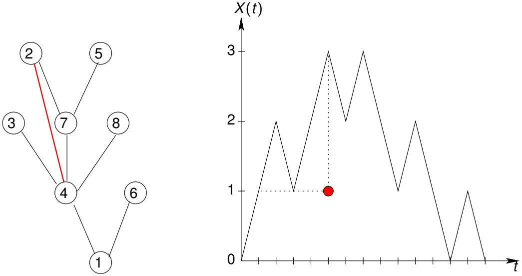

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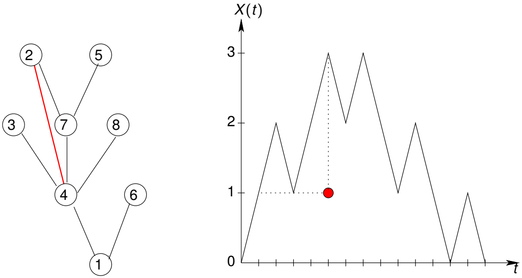

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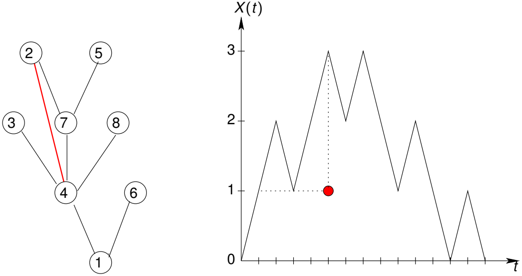

For every graph G on $1, \ldots, m$, there is some tree T so that G can be obtained from T by adding such marks.

Furthermore, for distinct trees T, T', the graphs that can be obtained by adding such edges are disjoint.

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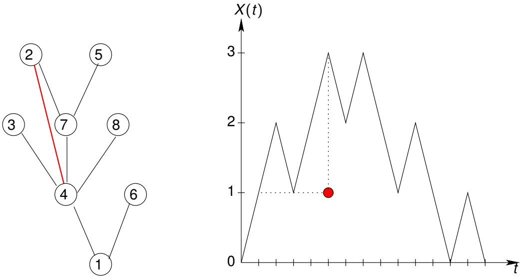

In other words, any connected graph G on $1, \ldots, m$ will have a "canonical" spanning tree T, and G can be obtained from T by adding a subset of the marks under the Harris walk for T.

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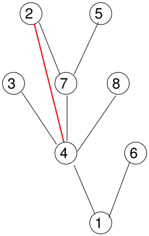

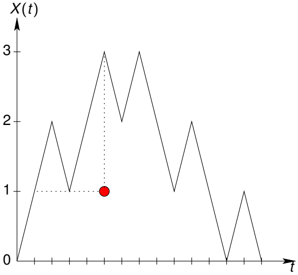

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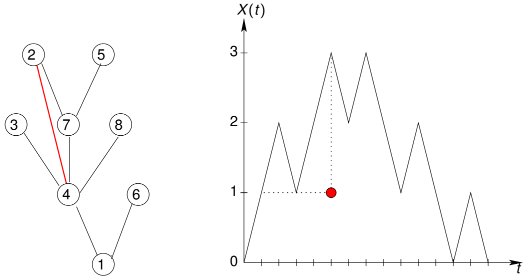

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In $G_{n,1/n}$, conditional upon T, each possible surplus edge (mark) should be present with probability $1/n$.

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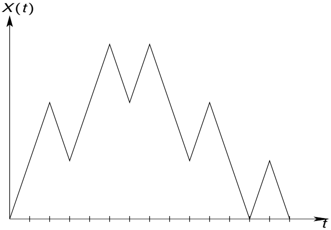
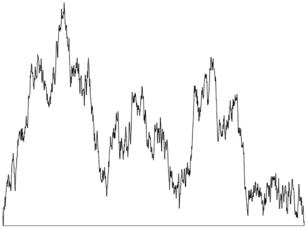
The result : a uniformly random connected component of $G_{n,1/n}$, conditioned upon having size m.

The depth-first walk for a uniform random tree.

When time is scaled by $1/m$, space is scaled by $1/\sqrt{m}$, the limit of Harris walk of a *uniform* tree is *Brownian excursion*.





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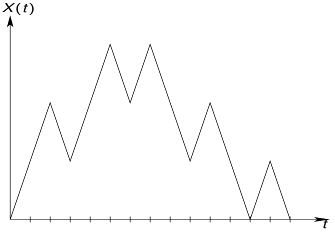

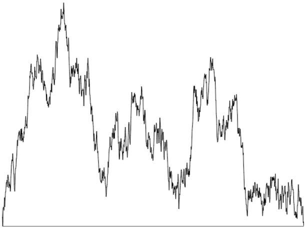

The *area* $a(T)$ is then $\Theta(m^{3/2})$, so if $m = n^{2/3}$ then $a(T) = \Theta(n)$ and

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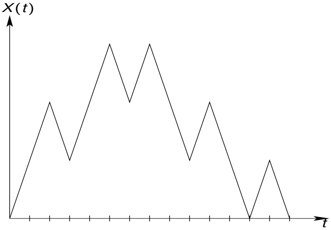

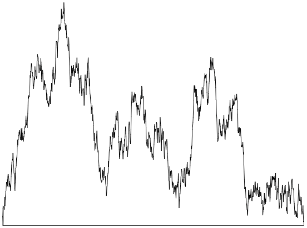

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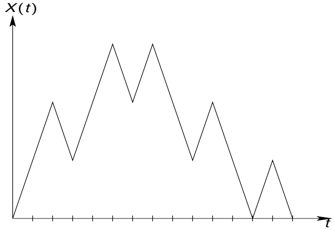

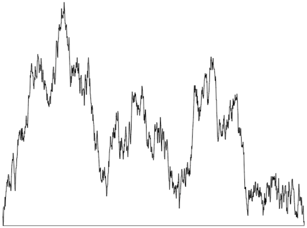

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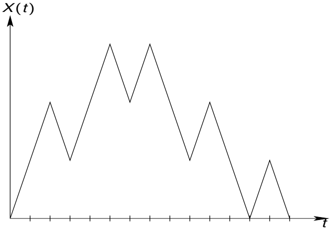

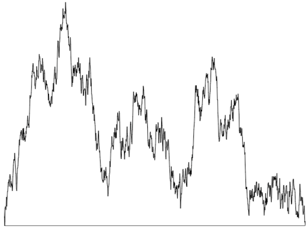

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This tilt can be shown to be exactly the effect of the quadratic drift. (Using Girsanov's theorem.)

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