INTERMITTENCY ON CATALYSTS

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§ INTRODUCTION

In this talk we consider the Parabolic Anderson Model on $\mathbb{Z}^d$, $d \geq 1$:

$$\begin{aligned} &\frac{\partial}{\partial t} u(x,t) = \kappa \Delta u(x,t) + \gamma \xi(x,t) u(x,t), \\ &u(\cdot, 0) \equiv 1, \end{aligned}$$

where $\kappa \in (0, \infty)$ is the diffusion constant, $\gamma \in (0, \infty)$ is the coupling constant, Δ is the discrete Laplacian, and ξ is an $\mathbb{N}$-valued random field.

The PAM is the parabolic analogue of the Schrödinger equation in a random potential and has been studied intensively since 1990.

INTERPRETATION

Consider a system of two types of particles, catalyst A and reactant B, such that:

– A-particles perform an autonomous dynamics given by ξ, with $\xi(x,t)$ the number of A-particles at site x at time t.

– B-particles perform independent simple random walks at rate $2d\kappa$ and split into two at a rate that is equal to γ times the number of A-particles present at the same location.

Then

$u(x,t)$	$=$	average number of B-particles at site x at time t given the evolution of the A-particles.

A systematic study of the PAM for time-independent random fields ξ has been carried out since 1990:

Gärtner & Molchanov
Gärtner & dH
Gärtner, König & Molchanov
Biskup & König
Mörters & Sidorova
van der Hofstad, Mörters & Sidorova
+ ...

The focus of these papers is on the height, shape and location of the dominant peaks in the u-field in the limit of large t.

Until 2004, the only time-dependent example studied was where ξ consists of independent Brownian noises:

Carmona & Molchanov
Carmona, Koralov & Molchanov
Carmona, Molchanov & Viens
Cranston, Mountford & Shiga
Greven & dH

§ THREE TYPES OF CATALYST

In this talk we consider three different choices for the catalyst:

(1) Independent simple random walks.
(2) Symmetric exclusion process.
(3) Voter model.

Choice (1) was first considered in:

Kesten & Sidoravicius
Gärtner & Heydenreich

We study the annealed Lyapunov exponents

$$\lambda_p = \lim_{t\to\infty} \frac{1}{pt} \log \mathbb{E}([u(0,t)]^p), \quad p \in \mathbb{N},$$

where the expectation is over the ξ-field.

In particular, we investigate the dependence of λ_p on the diffusion constant κ. It turns out that there is a critical dimension at which the behavior changes.

The system is said to be intermitttent if $\lambda_1 < \lambda_2 < \cdots$

§ FEYNMAN-KAC REPRESENTATION

The starting point of the analysis is the formula

$$\boxed{u(0,t) = E_0^X\Big(\exp\Big[\gamma\int_0^t \xi(X(s),t-s)\,ds\Big]\Big),}$$

where X is simple random walk on $\mathbb{Z}^d$ with step rate $2d\kappa$, and the expectation is taken w.r.t. X given $X(0) = 0$.

Consequently, studying λ_p amounts to doing a large deviation analysis for a random field and a random walk together.

§ INDEPENDENT SIMPLE RANDOM WALKS

For this case ξ has state space $(\mathbb{N} \cup \{0\})^{\mathbb{Z}^d}$ and generator

$$(Lf)(\xi) = \frac{1}{2d} \sum_{x \sim y} \xi(x)[f(\xi^{x \curvearrowright y}) - f(\xi)],$$

where $\xi^{x \curvearrowright y}$ is the configuration obtained from ξ by moving a particle from x to y.

We start from the Poisson distribution with intensity $\rho \in (0, \infty)$, which is an equilibrium.

Let $G_d = \int_0^\infty p_t(0,0)dt$ be the Green function at the origin of simple random walk on $\mathbb{Z}^d$ stepping at rate 1. The following dichotomy holds:

THEOREM 1:

$\lambda_p = \infty$ *if and only if* $p \geq 1/\gamma G_d$.

Thus, for recurrent random walk no λ_p is finite, while for transient random walk only those with small enough p are.

THEOREM 2: *Assume* $p < 1/\gamma G_d$. *Then:*

(i) $\kappa \mapsto \lambda_p(\kappa)$ *is continuous, strictly decreasing and convex on* $[0, \infty)$.

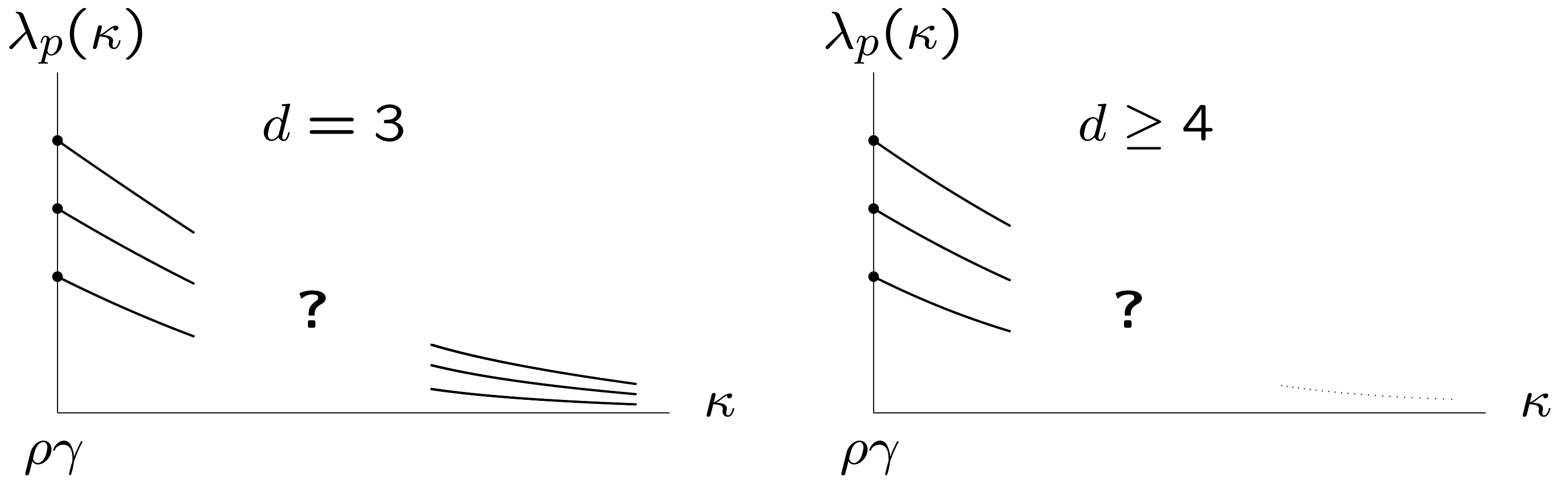
(ii) For $\kappa = 0$*:*

$$\lambda_p(0) = \rho\gamma \, \frac{(1/\gamma G_d)}{(1/\gamma G_d) - p}.$$

(iii) For $\kappa \to \infty$*:*

$$\lim_{\kappa \to \infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma] = \rho\gamma^2 G_d + 1_{d=3}\,(2d)^3(\rho\gamma^2 p)^2\,\mathcal{P}_3,$$

$$\mathcal{P}_3 = \sup_{\substack{f \in H^1(\mathbb{R}^3) \\ \|f\|_2 = 1}} \left[\|(-\Delta_{\mathbb{R}^3})^{-1/2} f^2\|_2^2 - \|\nabla_{\mathbb{R}^3} f\|_2^2 \right].$$



$\kappa \mapsto \lambda_p(\kappa)$ for $p = 1, 2, 3$ when $p < 1/\gamma G_d$

Remarkable: $\mathcal{P}_3$ is the variational problem for the polaron model analyzed in Lieb (1977) and in Donsker and Varadhan (1983).

Thus, the system is intermittent for

$d \geq 3$ small κ
$d = 3$ large κ.

CONJECTURE 3:

In $d = 3$*, the curves are distinct.*

CONJECTURE 4:

In $d \geq 4$*, the curves merge successively.*

§ SYMMETRIC EXCLUSION PROCESS

For this case ξ has state space $\{0,1\}^{\mathbb{Z}^d}$ and generator

$$(Lf)(\xi) = \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)\,[f(\xi^{x\leftrightarrow y}) - f(\xi)],$$

where $\xi^{x\leftrightarrow y}$ is the configuration obtained from ξ by interchanging the states at x and y, and $p(\cdot,\cdot)$ is a symmetric random walk kernel.

We start from the Bernoulli distribution with density $\rho \in (0,1)$, which is an equilibrium.

THEOREM 5:

$\lambda_p \in [\rho\gamma, \gamma]$ *and* $\kappa \to \lambda_p(\kappa)$ *is continuous, strictly decreasing and convex on* $[0, \infty)$*.*

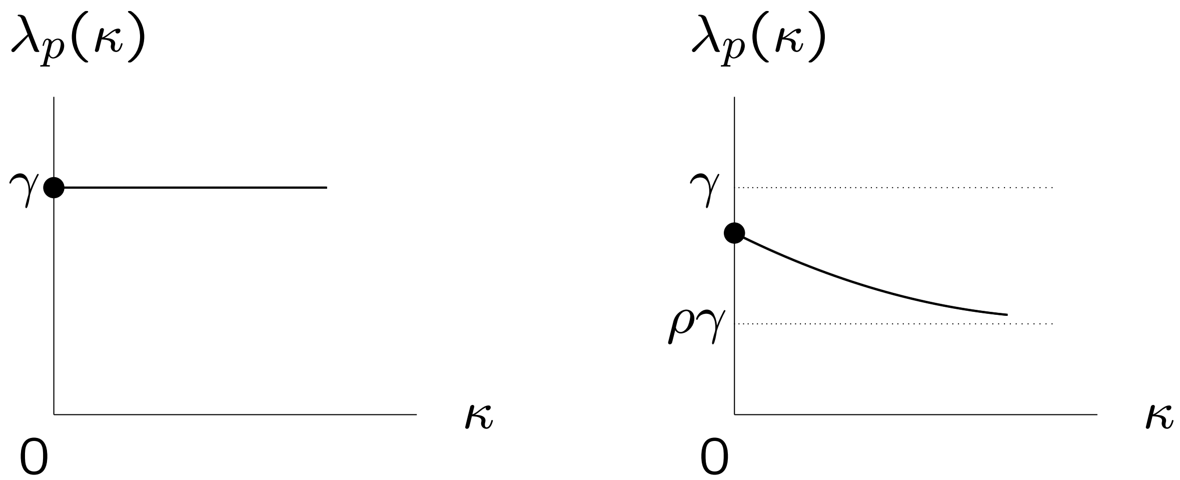
THEOREM 6:

(i) If $p(\cdot,\cdot)$ *is recurrent, then* $\lambda_p(\kappa) = \gamma$ *for all* p *and* κ*.*
(ii) If $p(\cdot,\cdot)$ *is transient, then*
(a) $\lambda_p(\kappa) \in (\rho\gamma, \gamma)$ *for all* p *and* κ*.*
(b) $p \mapsto \lambda_p(0)$ *is strictly increasing.*
(c) $\lim_{\kappa\to\infty} \lambda_p(\kappa) = \rho\gamma$*.*

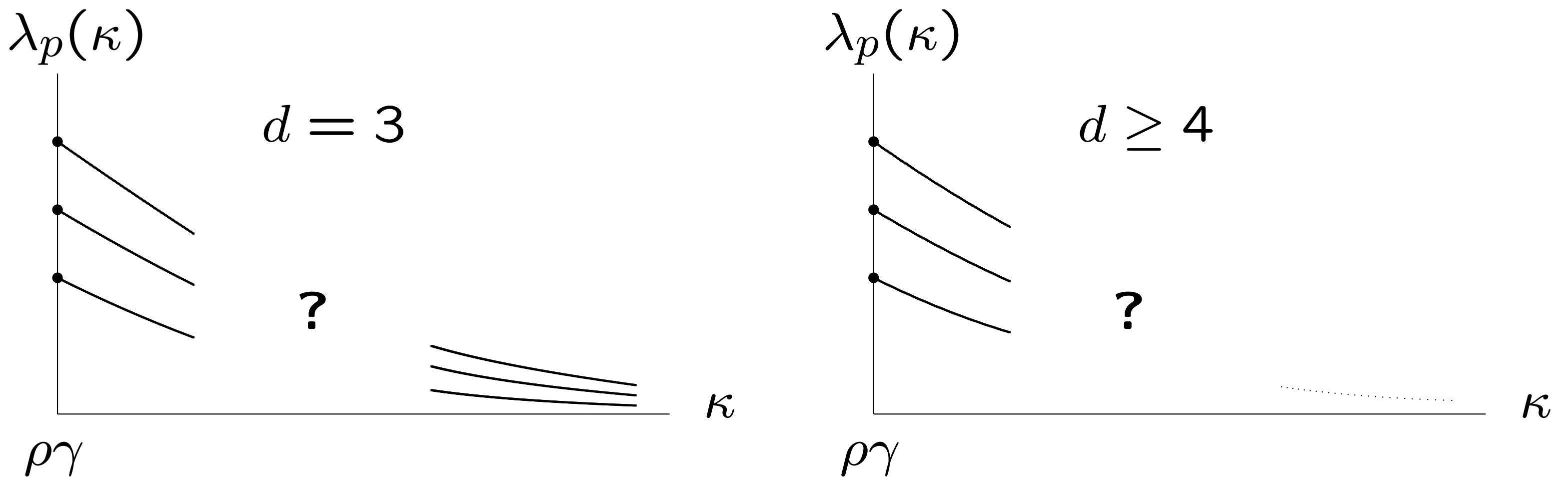
THEOREM 7:

If $p(\cdot,\cdot)$ *is simple random walk in* $d \geq 3$*, then*

$$\lim_{\kappa\to\infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma]$$
$$= \rho(1-\rho)\gamma^2 G_d + 1_{d=3}\,(2d)^3\,[\rho(1-\rho)\gamma^2 p]^2\,\mathcal{P}_3.$$



$\kappa \mapsto \lambda_p(\kappa)$ for recurrent and transient random walk



$\kappa \mapsto \lambda_p(\kappa)$ for $p = 1, 2, 3$ for simple random walk

§ VOTER MODEL

For this case ξ has state space $\{0,1\}^{\mathbb{Z}^d}$ and generator

$$(Lf)(\xi) = \sum_{\{x,y\}\subset\mathbb{Z}^d} p(x,y)\,[f(\xi^{x\to y}) - f(\xi)],$$

where $\xi^{x\to y}$ is the configuration obtained from ξ by imposing on y the state of x, and $p(\cdot,\cdot)$ is a random walk kernel.

We start from the Bernoulli distribution with density $\rho \in (0,1)$, which is not an equilibrium, or from the non-Bernoulli equilibrium distribution.

We expect similar behavior as for symmetric exclusion, but so far only partial results have been obtained.

CONJECTURE 8:

(i) If $p(\cdot,\cdot)$ *is not-strongly transient, then* $\lambda_p = \gamma$ *for all* p*.*
(ii) If $p(\cdot,\cdot)$ *is strongly transient, then* $\lambda_p \in (\rho\gamma, \gamma)$ *for all* p*.*

THEOREM 9:

The conjecture is true when $p(\cdot,\cdot)$ *has zero mean and finite variance, in which case the separation is between* $1 \leq d \leq 4$ *and* $d \geq 5$*.*

THEOREM 10:

$\lambda_p \in [\rho\gamma, \gamma]$ *and* $\kappa \to \lambda_p(\kappa)$ *is continuous on* $[0, \infty)$ *and strictly decreasing at least near* 0*.*

THEOREM 11:

(a) $p \mapsto \lambda_p(0)$ *is strictly increasing.*
(b) $\lim_{\kappa \to \infty} \lambda_p(\kappa) = \rho\gamma$*.*

CONJECTURE 12:

If $p(\cdot,\cdot)$ *is simple random walk in* $d \geq 5$*, then*

$$\lim_{\kappa\to\infty} 2d\kappa[\lambda_p(\kappa) - \rho\gamma]$$
$$= \rho(1-\rho)\gamma^2 \frac{G_d^*}{G_d} + 1_{d=5}(2d)^5 \left[\rho(1-\rho)\gamma^2 \frac{1}{G_d}\, p\right]^2 \mathcal{P}_5,$$

where

$$G_d = \int_0^\infty p_t(0,0)\, dt,$$
$$G_d^* = \int_0^\infty t\, p_t(0,0)\, dt,$$

and $\mathcal{P}_5$ *is given by a variational formula analogous to* $\mathcal{P}_3$.

CONCLUSION

Detailed results have been otained for three classical choices of catalyst.

For reversible dynamics (IRW + SE) a detailed analysis can be carried through. For non-reversible dynamics (VM) some aspects remain to be clarified.

There is a degree of universality in the qualitative behavior of the three models, with a special role for the critical dimension.