Asymptotic behavior in $\mathbb{Z}^d$ of the critical two-point functions for long-range statistical-mechanical models in high dimensions

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(Joint work with L.-C. Chen)

1. Motivation

The 2-point function $G_p(x)$

$$\text{e.g.,}\quad G_p^{\mathrm{SAW}}(x) = \underbrace{\sum_{\omega:o\to x} p^{|\omega|}\prod_{j=1}^{|\omega|} D(\omega_j-\omega_{j-1})}_{G_p^{\mathrm{RW}}(x)} \underbrace{\prod_{0\le s<t\le|\omega|}(1-\delta_{\omega_s,\omega_t})}_{\text{self-avoidance}},$$

where $D(x)$ is the $\mathbb{Z}^d$-symmetric 1-step distribution.

The (model-dependent) critical point p_{c} $(=1$ for RW)

$$\chi_p = \sum_{x\in\mathbb{Z}^d} G_p(x) \begin{cases} <\infty & \text{iff } p<p_{\mathrm{c}}, \\ \nearrow\infty & \text{as } p\nearrow p_{\mathrm{c}}. \end{cases}$$

1. Motivation

Finite-range 1-step distributions on $\mathbb{Z}^d$

$$D(x) = \frac{h(x/L)}{\sum_{y\in\mathbb{Z}^d\setminus\{o\}} h(y/L)} \quad (x\in\mathbb{Z}^d\setminus\{o\}), \qquad D(o)=0,$$

where $h:[-1,1]^d\to\mathbb{R}_+$ is bounded, piecewise-cont., $\mathbb{Z}^d$-symm.

e.g., $h(x)=\mathbb{1}_{\{|x|\le 1\}},\ L=1 \implies$ the nearest-neighbor model.

Known results for finite-range models ([HHS:03], [H:08], [S:07])

$$G_{p_c}(x) \underset{|x|\uparrow\infty}{\sim} \frac{{}^\exists A}{p_c}\frac{a_d}{\sigma^2|x|^{d-2}} \qquad (d>d_c \text{ and } d\vee L\gg 1),$$

where $G_1^{\mathrm{RW}}(x)\sim\dfrac{a_d}{\sigma^2|x|^{d-2}}$, $a_d=\dfrac{d\Gamma(\frac{d-2}{2})}{2\pi^{d/2}}$, $\sigma^2=\sum_{x\in\mathbb{Z}^d}|x|^2D(x)$.

Question: What if $D(x)\approx|x|^{-d-\alpha}$? (n.b., $\sigma^2=\infty$ if $\alpha\le 2$.)

2. Models

Random walk and self-avoiding walk

$$G_p^{\mathrm{SAW}}(x) = \underbrace{\sum_{\omega:o\to x} p^{|\omega|} \prod_{j=1}^{|\omega|} D(\omega_j - \omega_{j-1})}_{G_p^{\mathrm{RW}}(x)} \prod_{0\le s<t\le|\omega|} (1-\delta_{\omega_s,\omega_t}).$$

Bond percolation

$$\mathbf{n} = \{n_b\}_{b\in\mathbb{B}_{\mathbb{Z}^d}}, \qquad n_{u,v} = \begin{cases} 1 & \text{with probab., } pD(v-u), \\ 0 & \text{with probab., } 1-pD(v-u), \end{cases}$$

$$p = \mathbb{E}_p^{\mathrm{perc}}\Big[\big|\{x\in\mathbb{Z}^d : n_{o,x}>0\}\big|\Big],$$

$$x \underset{\mathbf{n}}{\longleftrightarrow} y \overset{\mathrm{def}}{\iff} x = y \text{ or } {}^{\exists}\text{a path of positive bonds from } x \text{ to } y,$$

$$G_p^{\mathrm{perc}}(x) = \mathbb{P}_p^{\mathrm{perc}}(o\longleftrightarrow x) \equiv \mathbb{P}_p^{\mathrm{perc}}\Big(\big\{\mathbf{n}\in\{0,1\}^{\mathbb{B}_{\mathbb{Z}^d}} : o \underset{\mathbf{n}}{\longleftrightarrow} x\big\}\Big).$$

2. Models

The (ferromagnetic) Ising model

- The Hamiltonian on $\Lambda \subset \mathbb{Z}^d$:

$$H_\Lambda(\varphi) = -\sum_{\{u,v\}\in\mathbb{B}_\Lambda} J_{u,v}\varphi_u\varphi_v \quad (\varphi = \{\varphi_x\}_{x\in\Lambda} \in \mathcal{S}_\Lambda \equiv \{\pm 1\}^\Lambda),$$

where $J_{u,v} \geq 0$ $(u, v \in \mathbb{Z}^d)$ is the spin-spin coupling.

- The 2-point function:

$$\langle \varphi_o\varphi_x\rangle_\Lambda = \frac{\sum_{\varphi\in\mathcal{S}_\Lambda} \varphi_o\varphi_x\, e^{-\beta H_\Lambda(\varphi)}}{\sum_{\varphi\in\mathcal{S}_\Lambda} e^{-\beta H_\Lambda(\varphi)}} \underset{\Lambda\uparrow\mathbb{Z}^d}{\longrightarrow} G_p^{\text{Ising}}(x),$$

where $\beta \in (0,\infty)$ is the inverse temperature and

$$p = \sum_{x\in\mathbb{Z}^d} \tanh(\beta J_{o,x}), \qquad D(x) = \frac{\tanh(\beta J_{o,x})}{\sum_{y\in\mathbb{Z}^d} \tanh(\beta J_{o,y})}.$$

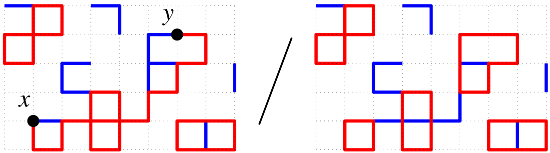
2. Models

The (ferromagnetic) Ising model

- The random current representation ([GHS:70]):

  Let $\mathbf{n} = \{n_b\}_{b\in\mathbb{B}_\Lambda} \in \mathbb{Z}_+^{\mathbb{B}_\Lambda}$, $\quad \partial\mathbf{n} = \Big\{v \in \Lambda : \sum_{b\ni v} n_b \text{ is odd}\Big\}$.

$$\frac{\displaystyle\sum_{\varphi\in\mathcal{S}_\Lambda} \varphi_x\varphi_y\, e^{-\beta H_\Lambda(\varphi)}}{\displaystyle\sum_{\varphi\in\mathcal{S}_\Lambda} e^{-\beta H_\Lambda(\varphi)}} = \frac{\displaystyle\sum_{\partial\mathbf{n}=\{x\}\triangle\{y\}} \prod_{b\in\mathbb{B}_\Lambda} \frac{(\beta J_b)^{n_b}}{n_b!}\, \mathbb{1}_{\{x \underset{\mathbf{n}}{\longleftrightarrow} y\}}}{\displaystyle\sum_{\partial\mathbf{n}=\varnothing} \prod_{b\in\mathbb{B}_\Lambda} \frac{(\beta J_b)^{n_b}}{n_b!}}$$

$=$ 

2. Models

The long-range 1-step distribution with $\alpha > 0$ and $L \in [1, \infty)$

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d \setminus \{o\}} h(y/L)}, \qquad h(x) \underset{|x|\uparrow\infty}{\sim} |x|^{-d-\alpha},$$

such that $^\exists v_\alpha = O(L^{\alpha \wedge 2})$, $^\exists \epsilon > 0$,

$$1 - \hat{D}(k) = v_\alpha |k|^{\alpha \wedge 2} \times \begin{cases} 1 + O(|k|^\epsilon) & (\alpha \neq 2), \\ \log \frac{1}{|k|} + O(1) & (\alpha = 2). \end{cases}$$

Known results for long-range models ([HHS:08])

For $d > d_c \equiv \begin{cases} 2(\alpha \wedge 2) & \text{(Ising \& SAW)} \\ 3(\alpha \wedge 2) & \text{(percolation)} \end{cases}$ and $L \gg 1$,

$$\hat{G}_p(k) \asymp \frac{1}{p_c - p + p\big(1 - \hat{D}(k)\big)} \qquad \text{uniformly in } p < p_c.$$

3. Results

Theorem (joint work with L.-C. Chen)

Let $\alpha \neq 2$. For RW with $d > \alpha$ and any L, and for the other models with $d > d_c$ and $L \gg 1$,

$$G_{p_c}(x) \underset{|x|\uparrow\infty}{\sim} \frac{A}{p_c} \frac{g_\alpha}{v_\alpha |x|^{d-\alpha\wedge 2}}, \qquad g_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})},$$

where $A^{\mathrm{RW}} = p_c^{\mathrm{RW}} = 1$ and

$$\frac{1}{A} = 1 + \begin{cases} \dfrac{1}{p_c} \lim\limits_{k\to 0} \dfrac{\hat{\pi}_{p_c}(0) - \hat{\pi}_{p_c}(k)}{1 - \hat{D}(k)} & \text{(SAW)}, \\ p_c \lim\limits_{k\to 0} \dfrac{\hat{\pi}_{p_c}(0) - \hat{\pi}_{p_c}(k)}{1 - \hat{D}(k)} & \text{(Ising \& percolation)}. \end{cases}$$

Here, $\pi_p(x)$ is the (model-dependent) lace-expnasion coefficient.

Remark For $\alpha > 2$,

$$1 - \hat{D}(k) \sim \frac{|k|^2}{2d} \sum_{x \in \mathbb{Z}^d} |x|^2 D(x) = \frac{|k|^2}{2d} \sigma^2 \equiv v_\alpha |k|^2,$$

$$\hat{\pi}_{p_c}(0) - \hat{\pi}_{p_c}(k) \sim \frac{|k|^2}{2d} \sum_{x \in \mathbb{Z}^d} |x|^2 \pi_{p_c}(x),$$

hence

$$\frac{g_\alpha}{v_\alpha |x|^{d-\alpha\wedge 2}} = \frac{\frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})}}{\frac{\sigma^2}{2d}|x|^{d-2}} = \frac{\frac{d\Gamma(\frac{d-2}{2})}{2\pi^{d/2}}}{\sigma^2 |x|^{d-2}} \equiv \frac{a_d}{\sigma^2 |x|^{d-2}},$$

$$\lim_{k \to 0} \frac{\hat{\pi}_{p_c}(0) - \hat{\pi}_{p_c}(k)}{1 - \hat{D}(k)} = \frac{1}{\sigma^2} \sum_{x \in \mathbb{Z}^d} |x|^2 \pi_{p_c}(x).$$

This reproves the results of [HHS:03], [H:08], [S:07].

4. Key ideas

Key 1: The lace expansion ([BS:85], [HS:90], [S:07])

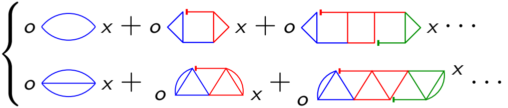
$$G_p(x) = I_p(x) + (K_p * G_p)(x) \equiv I_p(x) + \sum_{y \in \mathbb{Z}^d} K_p(y)\, G_p(x-y),$$

$$I_p(x) = \begin{cases} \delta_{o,x} & \text{(RW \& SAW)}, \\ \pi_p(x) & \text{(Ising \& percolation)}, \end{cases}$$

$$K_p(x) = \begin{cases} pD(x) & \text{(RW)}, \\ pD(x) + \pi_p(x) & \text{(SAW)}, \\ (\pi_p * pD)(x) & \text{(Ising \& percolation)}. \end{cases}$$

Diagrammatic bounds in terms of G_p's:

$|\pi_p(x)| \le$



(percolation),

(Ising & SAW).

4. Key ideas

Key 2: Convolution bounds ([HHS:03])

For $a \geq b > 0$ with $a + b > d$,

$$\sum_{z \in \mathbb{Z}^d} \frac{1}{(|x-z| \vee 1)^a} \frac{1}{(|z-y| \vee 1)^b} \leq \frac{{}^\exists C}{(|x-y| \vee 1)^{(a \wedge d + b) - d}}.$$

Assume that ${}^\exists\theta \ll 1$ such that

$$p \leq 2, \qquad G_p(x) \leq \frac{2\theta}{|x|^{d-\alpha\wedge 2}} \qquad (x \neq o).$$

Then, for $d > d_{\mathrm{c}}$,

$$|\pi_p(x)| \leq \frac{O(\theta)^2}{|x|^{d+\alpha\wedge 2+\rho}} \qquad (x \neq o),$$

where $\rho = \begin{cases} 2(d - d_{\mathrm{c}}) & \text{(Ising \& SAW)}, \\ d - d_{\mathrm{c}} & \text{(percolation)}. \end{cases}$

4. Key ideas

Key 3: Approximation by $S \equiv G^{\mathrm{RW}}$: $\quad {}^\exists\lambda$, ${}^\exists\mu$ s.t., $G_p \simeq \lambda S_\mu$?

$$G_p = \lambda S_\mu + G_p * \delta - \delta * \lambda S_\mu.$$

Since $\quad S_p = \delta + pD * S_p \quad$ and $\quad G_p = \delta + (I_p - \delta) + K_p * G_p$,

$$G_p = \underbrace{\lambda I_p}_{A/p_{\mathrm{c}}} * S_\mu + G_p * \Big(\underbrace{\delta - \mu D - \lambda(\delta - K_p)}_{E}\Big) * S_\mu.$$

Choose $\mu = 1 - \lambda\big(1 - \hat{K}_p(0)\big)$ and

$$\boxed{\text{CS}} \quad \lambda = \left(1 - \hat{K}_p(0) + \lim_{k\to 0} \frac{\hat{K}_p(0) - \hat{K}_p(k)}{1 - \hat{D}(k)}\right)^{-1},$$

so that

$$\boxed{\text{CS}} \quad \hat{E}(0) = \lim_{k\to 0} \frac{\hat{E}(0) - \hat{E}(k)}{1 - \hat{D}(k)} = 0.$$

Key 4: Analysis of $S \equiv G^{\mathrm{RW}}$

$$S_p(x) = \delta_{o,x} + (pD * S_p)(x) \underset{p\le 1}{\le} \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{-ik\cdot x}}{1-\hat{D}(k)}.$$

For finite-range models (and the long-range models with $\alpha > 2$),

$$\frac{1}{1-\hat{D}(k)} = \int_0^\infty e^{-t(1-\hat{D}(k))}\,\mathrm{d}t \simeq \int_0^\infty e^{-t\frac{\sigma^2}{2d}|k|^2}\,\mathrm{d}t \longrightarrow \text{Gaussian}.$$

For the long-range models with index $\alpha < 2$,

CS
$$\frac{1}{1-\hat{D}(k)} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1}\, e^{-t(1-\hat{D}(k))^{2/\alpha}}\,\mathrm{d}t$$
$$\simeq \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\frac{\alpha}{2}-1}\, e^{-tv_\alpha^{2/\alpha}|k|^2}\,\mathrm{d}t \longrightarrow \text{Gaussian}.$$

5. Key ideas

Key 5: Verification of the assumed bound on $G_p(x)$

$$F_p = p \vee \sup_{x \neq o} f_p(x), \qquad f_p(x) = \frac{G_p(x)}{\theta / |x|^{d-\alpha \wedge 2}},$$

$$\text{(n.b., } f_p^{\mathrm{RW}}(x) \leq 1 \text{ with } \theta = O(L^{-\kappa}) \text{ for some } \kappa > 0.)$$

(1) F_p is continuous in $p \in [1, p_c)$.

(2) $F_p \leq 3$ implies $F_p \leq 2$ for every $p \in [1, p_c)$, if $\theta \ll 1$.

For (1), by the Simon-Lieb ineq., for finite-range models,

$$pD(x) \leq G_p(x) \leq e^{-{}^{\exists}C|x|} \quad \text{for every } p < p_c.$$

For the long-range models ([ACCN:88], [AN:86] for $d = \alpha = 1$),

$$\boxed{\text{CS}} \quad pD(x) \leq G_p(x) \leq \frac{{}^{\exists}C}{|x|^{d+\alpha}} \quad \text{for every } p < p_c.$$

5. Conclusion

What we have achieved so far

Let $\alpha \neq 2$. For RW with $d > \alpha$ and any L, and for the other models with $d > d_c$ and $L \gg 1$,

$$G_{p_c}(x) \underset{|x|\uparrow\infty}{\sim} \frac{A}{p_c} \frac{g_\alpha}{v_\alpha |x|^{d-\alpha\wedge 2}}, \qquad g_\alpha = \frac{\Gamma(\frac{d-\alpha\wedge 2}{2})}{2^{\alpha\wedge 2}\pi^{d/2}\Gamma(\frac{\alpha\wedge 2}{2})},$$

where $A^{\mathrm{RW}} = p_c^{\mathrm{RW}} = 1$ and

$$\frac{1}{A} = 1 + \begin{cases} \dfrac{1}{p_c} \lim\limits_{k\to 0} \dfrac{\hat{\pi}_{p_c}(0) - \hat{\pi}_{p_c}(k)}{1 - \hat{D}(k)} & \text{(SAW)}, \\ p_c \lim\limits_{k\to 0} \dfrac{\hat{\pi}_{p_c}(0) - \hat{\pi}_{p_c}(k)}{1 - \hat{D}(k)} & \text{(Ising \& percolation)}. \end{cases}$$

What we like to include for completion

Prove results for $\alpha = 2$, to see if there is a log correction.