Excited against the tide......

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Thanks!

Self-interacting random walks:

- A n.n. RW path $\vec{\eta}_n$ is a sequence $\{\eta_i\}_{i=0}^n$ for which $\eta_i = (\eta_i^{[1]}, \ldots, \eta_i^{[d]}) \in \mathbb{Z}^d$ and $|\eta_{i+1} - \eta_i| = 1$ for each i.
- Notation: $p^{\vec{\eta}_i}(y, x)$ is conditional probability that the walk steps from $\eta_i = y$ to x, given the history $\vec{\eta}_i = (\eta_0, \ldots, \eta_i)$.
- $$Q(\vec{X}_n = (x_0, x_1, \ldots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{x}_i}(x_i, x_{i+1}).$$
- Q assumed to be translation invariant w.r.t. starting point.

Self-interacting random walks include:

- simple random walk
- annealed RWRE
- reinforced random walks
- (annealed) cookie random walks

Properties of interest:

- recurrence/transience
- LLN: existence of $v := \lim_{n\to\infty} \frac{X_n}{n}$, Q-a.s.
- CLT: $\frac{X_n - nv}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)$

How do these properties change as we vary some parameter(s) of the model?

The model:

- site-percolation λ-cookie environment $\omega \in \{0,1\}^{\mathbb{Z}^d}$, i.e. cookies at $\{x : \omega_x = 1\}$
- right drift (parameter β) when eat a cookie, left drift (μ) otherwise

Given ω, the ERWD $\{X_n\}_{n \geqslant 0}$ has law $\mathbb{Q}_\omega$ defined by

- $\mathbb{Q}_\omega(X_0 = o) = 1$ and

$p^o_\omega(o, \eta_1)$

$$= \frac{1 + (\beta I_{\{\omega_o = 1\}} - \mu(1 - I_{\{\omega_o = 1\}}))e_1 \cdot \eta_1}{2d}, \qquad \text{and}$$

$p^{\vec{\eta}_i}_\omega(\eta_i, \eta_{i+1})$

$$= \frac{1 + (\beta I_{\{\omega_{\eta_i} = 1\}} I_{\{\eta_i \notin \vec{\eta}_{i-1}\}} - \mu(1 - I_{\{\omega_{\eta_i} = 1\}} I_{\{\eta_i \notin \vec{\eta}_{i-1}\}}))e_1 \cdot (\eta_{i+1} - \eta_i)}{2d}.$$

Annealed ERWD

Annealed measure

$$Q(\cdot, \star) = \int_{\star} \mathbb{Q}_{\omega}(\cdot) d\mathbb{Q}.$$

Under Q, interested in $v^{[1]}(d, \beta, \mu, \lambda)$ defined by

$$v^{[1]} = \lim_{n \to \infty} \frac{X_n^{[1]}}{n}.$$

v exists Q-a.s. for $d \geqslant 6$, by a theorem of Bolthausen, Sznitman and Zeitouni (2003).

Under annealed measure: reparameterisation $\beta^* = \beta\lambda - \mu(1 - \lambda)$

Theorem: (H, '09)

$\nu^{[1]}(d, \beta, \mu, \lambda)$ is continuous in $(\beta, \mu, \lambda) \in [0,1]^3$ when $d \geqslant 6$ and when $d \geqslant 12$, is strictly increasing:

- in $\beta \in [0,1]$ for each $\mu, \lambda \in (0,1]$
- in $\lambda \in [0,1]$ for each $\mu, \beta \in (0,1]$

(Weaker results for monotonicity in μ).

if e.g. $\mu = 0$, we get monotonicity in $\beta, \lambda \in [0,1]$ for $d \geqslant 9$.

Excited random walk ($\lambda = 1$, $\mu = 0$)

- Benjamini and Wilson '03
- Kozma
- Zerner
- Berard and Ramirez
- Basdevant and Singh
- Others

Theorem: (v.d. Hofstad, H.)
For $d \geqslant 9$, $\nu(\beta)$ is increasing in $\beta \in [0, 1]$.

Strategy

- Speed exists ($d \geqslant 6$) by Bolthausen, Sznitman and Zeitouni '03
- Show that speed formula (from expansion technique with v.d. Hofstad) converges

$$v^{[1]} = \frac{(\beta + \mu)\lambda - \mu}{d} + \sum_{m=2}^{\infty} \sum_{x,y} (y^{[1]} - x^{[1]}) \pi_m(x, y)$$

- differentiate speed formula, show that "leading" term dominates

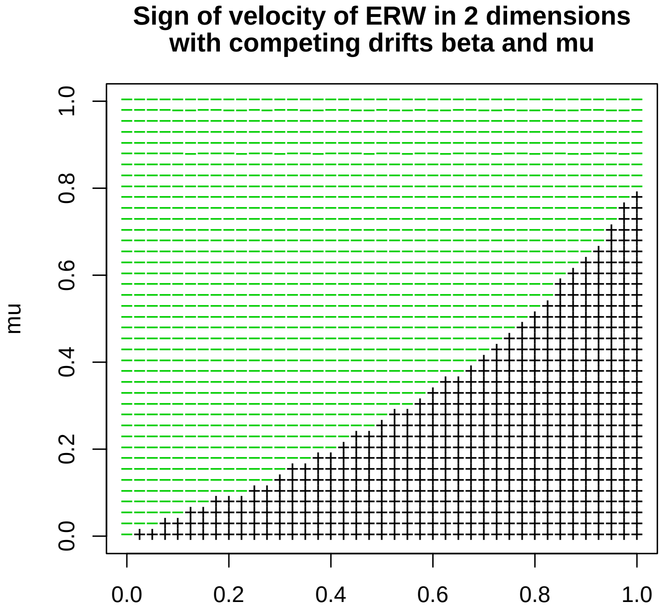
More interesting

Conjecture:

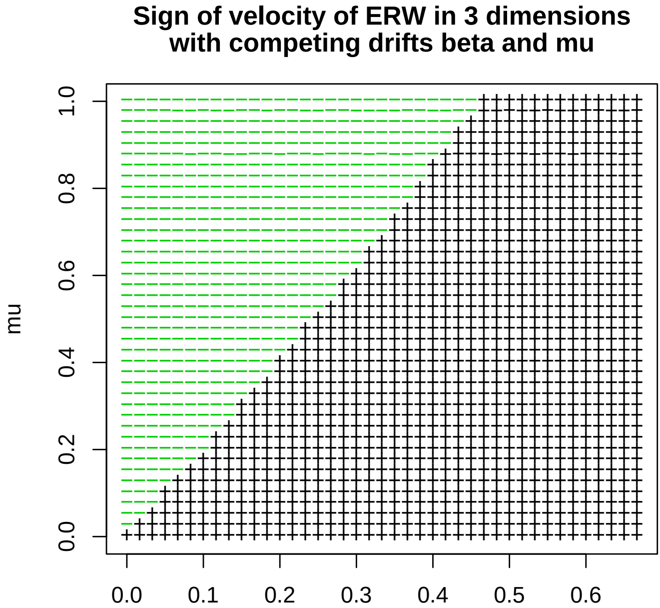
- For all $d \geqslant 2$, $(\mu, \beta, \lambda) \in [0,1]^3$, $\nu^{[1]}$ exists and is monotone increasing in β for fixed μ, λ and decreasing in μ for fixed β, λ respectively.
- For each $d \geqslant 3$ and $\mu \in [0,1]$ and all λ sufficiently large, $\exists!$ $\beta_0(\mu, d, \lambda) \in [0,1]$ such that $\nu(d, \mu, \beta_0, \lambda) = 0$. The same is true if the roles of λ and β are reversed.

Theorem: (H.) True in high dimensions.

Simulations $\lambda = 1$, $d = 2$



Simulations $\lambda = 1$, $d = 3$



Strategy for zero-speed result

Show that:

- ($d \geqslant 2$) for each $\mu > 0$, speed** < 0 if $\lambda\beta$ is small
- ($d \geqslant 9$) for each $\mu > 0$, speed > 0 if $\lambda\beta$ is large

Then apply Theorem on continuity and monotonicity of velocity in high dimensions

Expansion overview

For self-interacting random walks we have

$$Q(\vec{X}_n = (x_0, x_1, \dots, x_n)) = \prod_{i=0}^{n-1} p^{\vec{x}_i}(x_i, x_{i+1}).$$

With v.d. Hofstad we investigate the *two-point function*

$$c_n(x) = Q(X_n = x).$$

- Write

$$c_{n+1}(x) = \sum_y p^o(y) c_n(x-y) + \sum_{m=2}^{n+1} \sum_y \pi_m(y) c_{n+1-m}(x-y).$$

  Here $\sum_x c_n(x) = 1$.
- derive bounds on the lace expansion coefficients
- analyse the recursion relation, using the bounds on the lace expansion coefficients (and induction)

Who cares?

Taking the Fourier transform, get

$$\hat{c}_{n+1}(k) = \hat{p}^{o}(k)\hat{c}_n(k) + \sum_{m=2}^{n+1} \hat{\pi}_m(k)\hat{c}_{n+1-m}(k),$$

where

$$\hat{c}_n(k) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} c_n(x) = E\big[e^{ik\cdot X_n}\big].$$

Under strong* assumptions on π_m, can inductively prove

$$\hat{c}_n(kn^{-1}) = e^{ik\cdot \nu + e_n(k)}, \qquad \hat{c}_n(kn^{-\frac{1}{2}})e^{-ik\nu\sqrt{n}} = e^{-\frac{1}{2}k^t \Sigma k + \epsilon_n(k)}.$$

*The good news is that ν and Σ are described in terms of the expansion coefficients π_m.

Theorem: Speed formula

If $\lim_{n\to\infty} \sum_{m=2}^{n} \sum_{x} x\pi_m(x)$ exists and $n^{-1}X_n \xrightarrow{Q} v$, then

$$v = \sum_x x p^o(x) + \sum_{m=2}^{\infty} \sum_x x\pi_m(x).$$

speed formula proof

- Summing recursion over x:

$$1 = 1 + \sum_{m=2}^{n+1} \sum_x \pi_m(x).$$

Thus $\sum_x \pi_m(x) = 0$.

- Multiply recursion by $x = y + (x - y)$ and sum over x

$$\sum_x x c_{n+1}(x) = \sum_y y p^o(y) + \sum_x x c_n(x) + \sum_{m=2}^{n+1} \sum_y y \pi_m(y).$$

i.e.

$$E[X_{n+1} - X_n] = E[X_1] + \sum_{m=2}^{n+1} \sum_y y \pi_m(y).$$

speed proof cont.

If

$$\lim_{n\to\infty} E[X_{n+1} - X_n] = \tilde{v}$$

then since $X_n = \sum_{m=1}^{n}(X_m - X_{m-1})$, we have also

$$\lim_{n\to\infty} E[n^{-1}X_n] = \tilde{v}.$$

If $n^{-1}X_n \overset{Q}{\to} v$, by bounded convergence we get

$$\lim_{n\to\infty} E[n^{-1}X_n] = v,$$

so $v = \tilde{v}$. □

Variance formula (symmetric case)

Suppose that $E[X_n] = 0$ for each n and for each $i, j \in \{1, 2, \ldots, d\}$,

$$\lim_{n\to\infty} \frac{E[X_n^{[i]} X_n^{[j]}]}{n} = \Sigma_{ij}, \quad \text{and} \quad \sum_{m=2}^{\infty} \sum_{y} y^{[i]} y^{[j]} \pi_m(y) < \infty,$$

Then

$$\Sigma_{ij} = E[X_1^{[i]} X_1^{[j]}] + \sum_{m=2}^{\infty} \sum_{y} y^{[i]} y^{[j]} \pi_m(y).$$

The expansion coefficients.

$$\pi_m^{(N)}(y) := \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{x}_1^{(0)}} \sum_{\vec{x}_{j_1+1}^{(1)}} \cdots \sum_{\vec{x}_{j_N+1}^{(N)}} I_{\{x_{j_N+1}^{(N)}=y\}}$$

$$\times p^o(x_1^{(0)}) \prod_{k=1}^{N} Q^{\vec{x}_{j_{k-1}+1}^{(k-1)}}(\vec{X}_{j_k} = \vec{x}_{j_k})\Delta_{j_k}^{(k)}.$$

For ERWD,

$$\Delta^{(n)} = \frac{(\beta+\mu)\lambda e_1 \cdot (x_{j_n+1}^{(n)} - x_{j_n}^{(n)})}{2d}\left[I_{\{x_{j_n}^{(n)}\notin\vec{x}_{j_{n-1}}^{(n-1)}\circ\vec{x}_{j_n-1}^{(n)}\}} - I_{\{x_{j_n}^{(n)}\notin\vec{x}_{j_n-1}^{(n)}\}}\right]$$

$$|\Delta^{(n)}| \leqslant \frac{(\beta+\mu)\lambda}{2d} I_{\{x_{j_n+1}^{(n)}=x_{j_n}^{(n)}\pm e_1\}} I_{\{x_{j_n}^{(n)}\in\vec{x}_{j_{n-1}}^{(n-1)}\}}.$$

Define

$$\pi_m^{(N)}(x,y) = \sum_{\vec{j}\in\mathcal{A}_{m,N}} \sum_{\vec{x}_1^{(0)}} \sum_{\vec{x}_{j_1+1}^{(1)}} \cdots \sum_{\vec{x}_{j_N+1}^{(N)}} I_{\{x_{j_N}^{(N)}=x,\,x_{j_N+1}^{(N)}=y\}} \cdots.$$

Reformulation of speed formula

Since

$$\sum_{y \in \mathbb{Z}^d} \pi_m^{(N)}(x, y) = 0,$$

$$\sum_{y \in \mathbb{Z}^d} y \pi_m(y) = \sum_{x, y \in \mathbb{Z}^d} (y - x) \pi_m(x, y),$$

so that

$$v = \frac{(\beta + \mu)\lambda - \mu}{d} + \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} \sum_{x, y \in \mathbb{Z}^d} (y - x) \pi_m^{(N)}(x, y).$$

Does speed formula converge?

- $\mathbb{P}_d$ is law of simple symmetric random walk in d dimensions, Then

$$G_d^{*i}(x) = \sum_{k=0}^{\infty} \frac{(k+i-1)!}{(i-1)!k!} \mathbb{P}_d(X_k = x), \quad \text{for } i \geqslant 1.$$

  Note that $G_d^{*i}(x) < \infty$ if and only if $d > 2i$.
- $G_d^{*i} := G_d^{*i}(o)$. For $i \geqslant 0$, let $q_d = (d-1)/d$

$$\mathcal{E}_i(d) = q_d^{-(i+1)} G_{d-1}^{*(i+1)} - 1.$$

Yes, when $d \geqslant 6$

Define

$$a_d = \frac{d}{(d-1)^2} G_{d-1}^{*2}.$$

$2a_d < 1$ when $d \geqslant 6$.

Proposition:

- $\sum_{x,y\in\mathbb{Z}^d} \sum_m |\pi_m^{(1)}(x,y)| \leqslant (\beta+\mu)\lambda d^{-1}\mathcal{E}_0(d)$,
- $N \geqslant 2$,

$$\sum_{x,y\in\mathbb{Z}^d} \sum_m |\pi_m^{(N)}(x,y)| \leqslant d^{-1}(d-1)^{-1} G_{d-1}\mathcal{E}_1(d) \big((\beta+\mu)\lambda\big)^N a_d^{N-2}.$$

Consequences:

- continuity of speed as a function of (λ, β, μ) for free. $(d \geqslant 6)$
- $d \geqslant 9$, for any μ, speed is positive for $\lambda\beta$ large enough.
- $d \geqslant 9$, for any β, μ, speed is negative for λ small enough.

bounds in terms of SRW

Lemma: For all $u \in \mathbb{Z}^d$, $\vec{\eta}_m$, and $i \in \mathbb{Z}_+$,

$$\sum_{j=0}^{\infty} \frac{(j+i)!}{j!} Q^{\vec{\eta}_m}(X_j = u) \leqslant i! q_d^{-(i+1)} G_{d-1}^{*(i+1)}, \qquad \text{etc.}$$

Given $\vec{\eta}_m$ and $\vec{z}_{j+1}$, define

$$\Delta(\vec{z}_{j+1}) = \left(p^{\vec{\eta}_m \circ \vec{z}_j}\left(z_j, z_{j+1}\right) - p^{\vec{z}_j}\left(z_j, z_{j+1}\right)\right) I_{\{z_0 = \eta_m\}}.$$

Lemma: For any $\vec{\eta}_m$,

$$\sum_{j=0}^{\infty} (j+1) \sum_{\vec{z}_{j+1}} |\Delta(\vec{z}_{j+1})| Q^{\vec{\eta}_m}(\vec{X}_j = \vec{z}_j) \leqslant m(\beta + \mu)\lambda a_d, \qquad \text{etc.}$$

Partial derivatives of speed formula

Let $\varphi^{(N)}_{\beta,m}(x,y) = \frac{\partial}{\partial\beta}\pi^{(N)}_m(x,y)$ etc. Then

$$\left|\frac{\partial v^{[1]}}{\partial\beta} - \frac{\lambda}{d}\right| \leqslant \sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sum_{x,y}|\varphi^{(N)}_{\beta,m}(x,y)|$$

$$\left|\frac{\partial v^{[1]}}{\partial\lambda} - \frac{\beta+\mu}{d}\right| \leqslant \sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sum_{x,y}|\varphi^{(N)}_{\lambda,m}(x,y)|$$

$$\left|\frac{\partial v^{[1]}}{\partial\mu} - \frac{-(1-\lambda)}{d}\right| \leqslant \sum_{N=1}^{\infty}\sum_{m=2}^{\infty}\sum_{x,y}|\varphi^{(N)}_{\mu,m}(x,y)|,$$

Derivatives of formula components

$$\frac{\partial}{\partial \beta} p^{\vec{\eta}_m}(\eta_m, x) = \frac{\lambda I_{\{\eta_m \notin \vec{\eta}_{m-1}\}}}{2d} \left(I_{\{x-\eta_m = e_1\}} - I_{\{x-\eta_m = -e_1\}} \right),$$

$$\frac{\partial}{\partial \lambda} p^{\vec{\eta}_m}(\eta_m, x) = \frac{(\beta + \mu) I_{\{\eta_m \notin \vec{\eta}_{m-1}\}}}{2d} \dots$$

$$\frac{\partial}{\partial \mu} p^{\vec{\eta}_m}(\eta_m, x) = \frac{\lambda I_{\{\eta_m \notin \vec{\eta}_{m-1}\}} - 1}{2d} \dots$$

and

$$\frac{\partial}{\partial \beta} \left(p^{\vec{\eta}_m}(\eta_m, x) - p^{\vec{x}_n \circ \vec{\eta}_m}(\eta_m, x) \right)$$

$$= \frac{\lambda}{2d} I_{\{\eta_m \notin \vec{\eta}_{m-1}, \eta_m \in \vec{x}_{n-1}\}} \left(I_{\{x-\eta_m = e_1\}} - I_{\{x-\eta_m = -e_1\}} \right).$$

The other terms are similar.

Monotonicity results

- Proceed as before using these slightly different bounds. Get

$$\left|\frac{\partial \nu^{[1]}}{\partial \beta} - \frac{\lambda}{d}\right| \leqslant \lambda \cdot \mathsf{stuff}(d)$$

$$\left|\frac{\partial \nu^{[1]}}{\partial \lambda} - \frac{\beta+\mu}{d}\right| \leqslant (\beta+\mu) \cdot \mathsf{stuff}(d)$$

$$\left|\frac{\partial \nu^{[1]}}{\partial \mu} - \frac{-(1-\lambda)}{d}\right| \leqslant \mathsf{stuff}(d),$$

- stuff(d) is order d^{-2}
- need $2a_d < 1$ for "stuff" to converge
- "stuff" involves G_{d-1}^{*i} for $i = 1, 2, 3$, so need $d \geqslant 8$
- Then need d large enough to beat constants, e.g. $(\beta + \mu)\lambda \leqslant 2$
- μ derivative not informative when $\lambda \approx 1$

Non-positive speeds:

Lemma: For each $d \geqslant 2$ and $\mu > 0$, the speed** is negative for $\lambda\beta$ sufficiently small.

Corollary: Fix $d \geqslant 9$, and $\mu \in [0, 1]$. For each λ sufficiently large, can find a $\beta_0(\mu, d, \lambda)$ so that the speed is 0. For each $d \geqslant 12$ $\beta_0(\mu, d, \lambda)$ is unique. The same is true with the roles of λ and β reversed.

sketch proof of lemma:

Fix $d \geqslant 2$ and $\mu > 0$.
Prove that $\limsup_{n\to\infty} n^{-1}X_n^{[1]} < \frac{1}{3}E[X_3^{[1]}]$, Q-almost surely:

- Explicitly write down

$$\mathbb{Q}_\omega(X_{n+3}^{[1]} - X_n^{[1]} = 3|\vec{X}_n = \vec{x}_n)$$

$$\mathbb{Q}_\omega(X_{n+3}^{[1]} - X_n^{[1]} = 2|\vec{X}_n = \vec{x}_n)$$

$$\mathbb{Q}_\omega(X_{n+3}^{[1]} - X_n^{[1]} = 1|\vec{X}_n = \vec{x}_n)$$

  also -1,-2,-3 (and 0)
- the first two increase if you switch on a cookie
- so does the sum of all three
- reverse is true for negative terms

sketch proof cont.

- Take expectations w.r.t. $\mathbb{Q}$, get quantities bounded by $Q(X_3^{[1]} = j)$
- By coupling, X_n is left of walk with environmental regeneration every 3 steps
- the latter has speed $\frac{1}{3}E[X_3^{[1]}]$
  - continuous in $(\beta, \lambda) \in [0, 1]^2$
  - $< -\epsilon(d, \mu)$ when $\beta\lambda = 0$.

□

Other models?

- excitement in two coordinates with $(\beta^{[1]}, \beta^{[2]})$: monotonicity of $\nu^{[1]}$ in $\beta^{[2]}$?
- once-reinforced random walk on a tree?
- variance of a random walk with partial once-reinforcement?
- certain models of RWRE in high dimensions
- once reinforced random walk in high dimensions?????
(requires a tremendous advance in our analysis of the recursion equation)