Domino tilings, lattice paths and plane overpartitions

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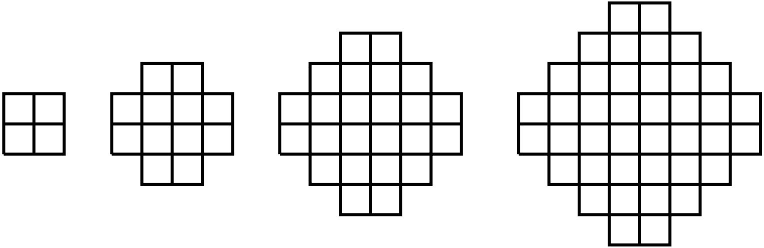
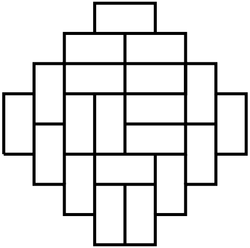
Aztec diamond

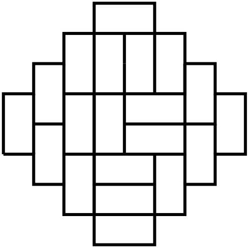


Aztec diamond of order n: 4 staircase of height n glued together.

Domino Tilings

Tile the aztec diamond of order n with $n(n+1)$ dominos.





$2^{\binom{n+1}{2}}$ tilings of the aztec diamond of order n (Elkies et al 92)

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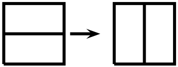

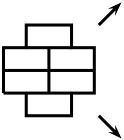

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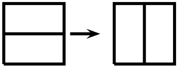

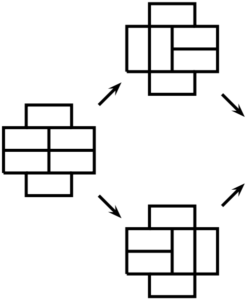

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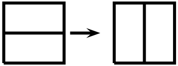

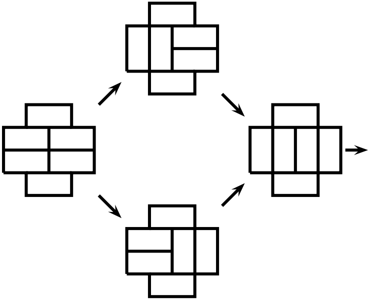

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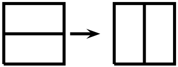

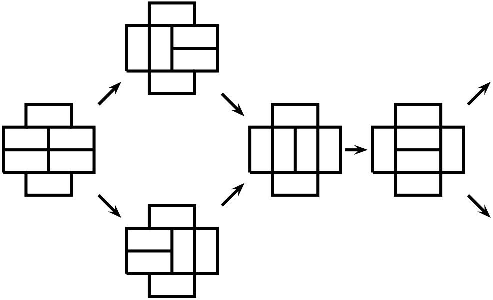

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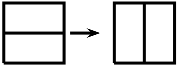

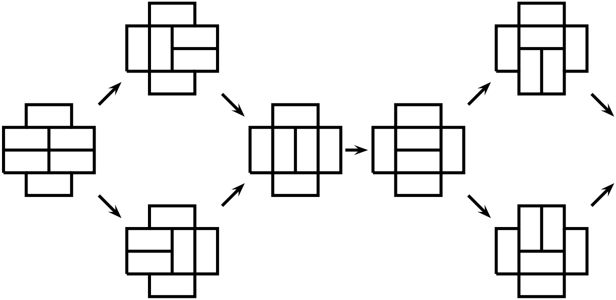

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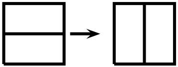

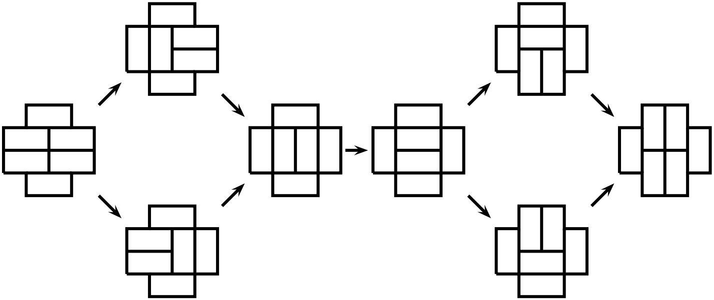

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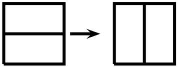
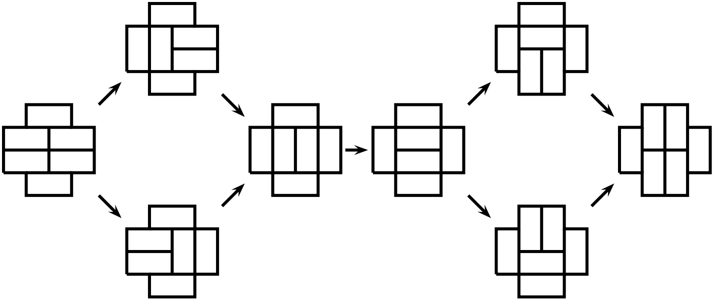




Rank: minimal number of flips from the horizontal tiling

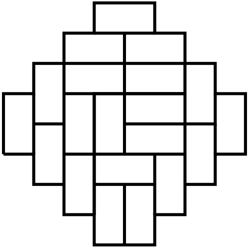
Generating function

Tiling T. Number of vertical dominos : $v(T)$. Rank : $r(T)$.

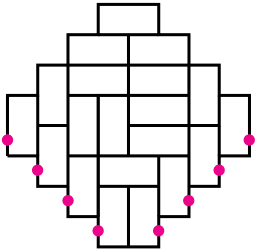
$$A_n(x,q) = \sum_{T \text{ tiling of order } n} x^{v(T)} q^{r(T)} = \prod_{k=0}^{n-1} (1 + xq^{2k+1})^{n-k}.$$

(Elkies et al, Stanley, Benchetrit)

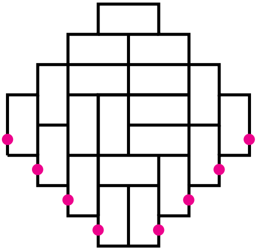
Tilings and lattice paths



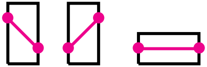
Tilings and lattice paths



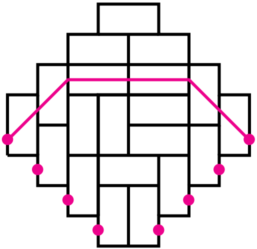
Tilings and lattice paths



Rule



Tilings and lattice paths



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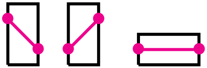
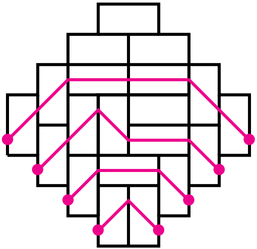


Tilings and lattice paths



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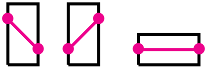
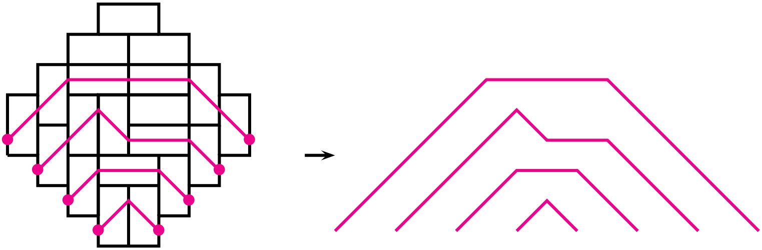


Tilings and lattice paths



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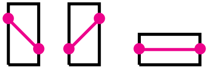


Generating function

- Vertical dominos = North-East and South-East steps
- Rank = height of the paths + constant

Non intersecting paths : Lindström, Gessel-Viennot (70-80s)

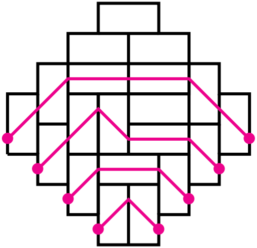
$A_n(x,q)$ = determinant ((x,q)-Schröder numbers)

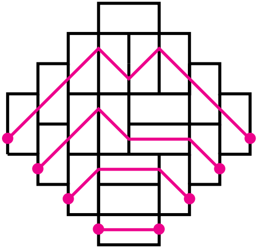
Combinatorics of lattice paths $\Rightarrow$

$$A_n(x,q) = (1+xq)^m A_{n-1}(xq^2,q), \quad A_0(x,q) = 1.$$

$$A_n(x,q) = \prod_{k=0}^{n-1} (1+xq^{2k+1})^{n-k}.$$

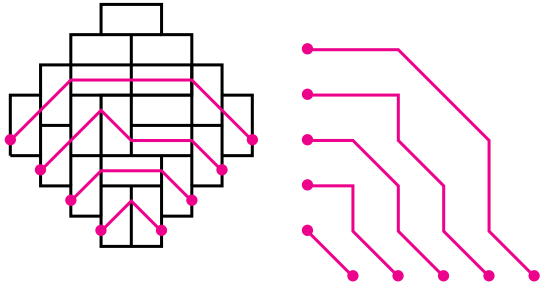
Artic circle



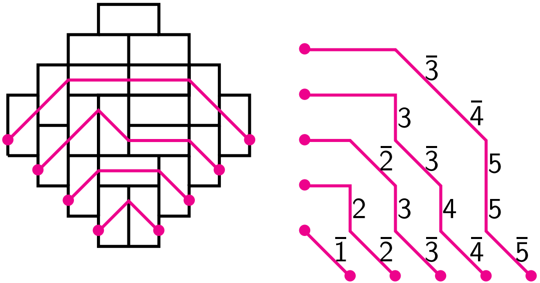


(Johansson 05)

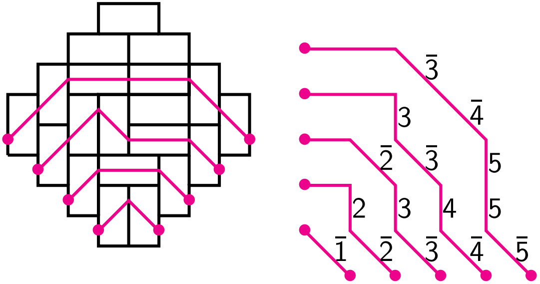
Lattice paths and monotone triangles



Lattice paths and monotone triangles



Lattice paths and monotone triangles



$$
\begin{array}{ccccccccc}
 & & & & \bar{3} & & & & \\
 & & & 3 & & \bar{4} & & & \\
 & & \bar{2} & & \bar{3} & & 5 & & \\
 & 2 & & 3 & & 4 & & 5 & \\
\bar{1} & & \bar{2} & & \bar{3} & & \bar{4} & & \bar{5}
\end{array}
$$

Monotone triangles

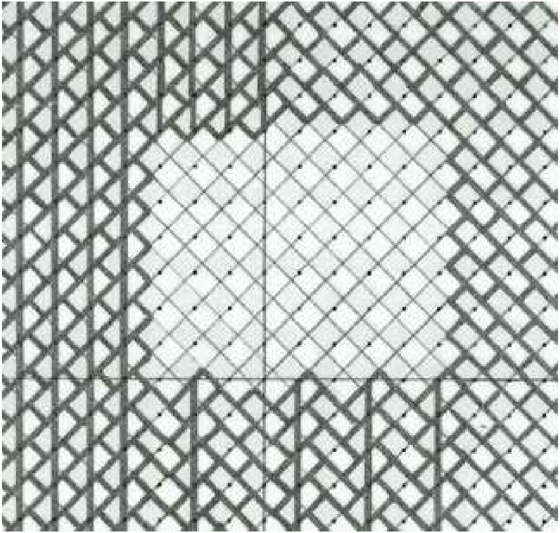
Monotone triangles with weights 2 on the non-diagonal rim hooks

$$
\begin{array}{ccccccccc}
 & & & & \bar{3} & & & & \\
 & & & 3 & & \bar{4} & & & \\
 & & \bar{2} & & \bar{3} & & 5 & & \\
 & 2 & & 3 & & 4 & & 5 & \\
\bar{1} & & \bar{2} & & \bar{3} & & \bar{4} & & \bar{5}
\end{array}
$$

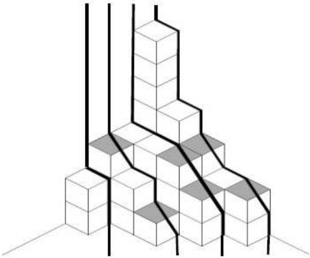
Alternating sign matrices with weight 2 on each -1.

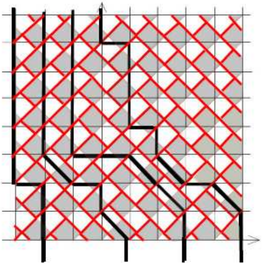
$$
\begin{array}{ccccc}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0
\end{array}
$$

Domino Tilings and plane overpartitions

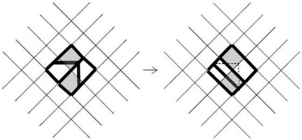


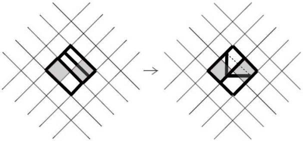
Plane overpartitions





Flips and lattice steps





Plane overpartitions

An overpartition is a partition where the last occurrence of a part can be overlined.
$(\bar{6}, 5, 5, 5, 3, 3, \bar{3}, \bar{1})$

C, Lovejoy (04)

A plane overpartition is a two-dimensional array such that each row is an overpartition and each column is a superpartition.

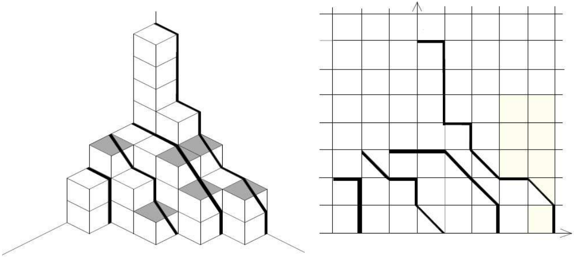
$$\begin{array}{cccc} 5 & 5 & \bar{5} & 3 \\ \bar{5} & 3 & 2 & 2 \\ \bar{5} & \bar{3} & & \\ \bar{5} & & & \end{array}$$

C. Savelief and Vuletic (09)

Generating function :

$$\sum_{\Pi} q^{|\Pi|} = \prod_{i \geq 1} \left(\frac{1+q^i}{1-q^i} \right)^i .$$

Lattice paths and plane overpartitions



Plane overpartitions of shape λ

$$q^{\sum_i i\lambda_i} \prod_{x\in\lambda} \frac{1 + aq^{c_x}}{1 - q^{h_x}}$$

Krattenthaler (96), $a = -q^n$ Stanley content formula

Reverse plane overpartitions included in the shape λ

$$\prod_{x\in\lambda} \frac{1 + q^{h_x}}{1 - q^{h_x}}$$

Related objects

Plane overpartitions are in bijection with super semi-standard young tableaux.

Representation of Lie Superalgebras

Berele and Remmel (85), Krattenthaler (96)

$$
\begin{array}{ccccccc}
5 & \bar{4} & 3 & \bar{3} & 2 & \bar{2} & \bar{1} \\
4 & \bar{4} & \bar{3} & \bar{2} & 1 & 1 & \bar{1} \\
3 & \bar{3} & 2 & 1 & \bar{1} & & \\
\bar{3} & \bar{2} & 1 & & & & \\
2 & & & & & &
\end{array}
\qquad \longleftrightarrow \qquad
\begin{array}{ccccccc}
5 & 3 & 2 & 1 & 1 & \bar{3} & \bar{1} \\
4 & 2 & 1 & \bar{4} & \bar{3} & \bar{2} & \bar{1} \\
3 & 1 & \bar{3} & \bar{2} & \bar{1} & & \\
2 & \bar{4} & \bar{2} & & & & \\
\bar{3} & & & & & &
\end{array}
$$

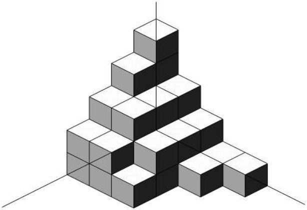
Related objects

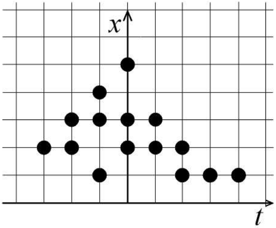
Plane overpartitions are in bijection with diagonally strict partitions where each rim hook counts 2

Vuletic (07), Foda and Wheeler (07, 08)

$$\begin{array}{ccccccc} 5 & \bar{4} & 3 & \bar{3} & 2 & \bar{2} & \bar{1} \\ 4 & \bar{4} & \bar{3} & \bar{2} & 1 & 1 & \bar{1} \\ 3 & \bar{3} & 2 & \bar{2} & \bar{1} & & \\ \bar{3} & 2 & \bar{2} & & & & \\ 2 & & & & & & \end{array} \quad \leftrightarrow \quad \begin{array}{ccccccc} 5 & \bar{4} & 3 & \bar{3} & 2 & \bar{2} & \bar{1} \\ 4 & 4 & 3 & \bar{2} & 1 & 1 & 1 \\ 3 & \bar{3} & 2 & 2 & 1 & & \\ 3 & 2 & 2 & & & & \\ 2 & & & & & & \end{array}$$

Limit shape





Diagonally strict polane partitions weighted by $2^{k(\Pi)}q^{|\Pi|}$
Ronkin function of the polynomial $P(z,w) = z + w + zw$

Vuletic (07)

RSK type algorithms

Generating function of plane overpartitions with at most r rows and c columns

$$\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1+q^{i+j-1}}{1-q^{i+j-1}}.$$

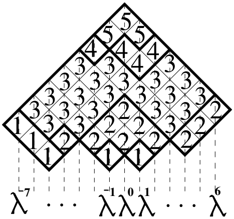
Generating function of plane overpartitions with entries at most n

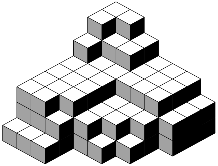
$$\prod_{i=1}^{n}\frac{\prod_{j=1}^{n}(1+aq^{i+j})}{\prod_{j=0}^{i-1}(1-q^{i+j})(1-aq^{i+j})}$$

Generating function of plane overpartitions with at most r rows and c columns and entries at most n?? NICE?

Plane partitions

Interlacing sequences





Rhombus Tilings

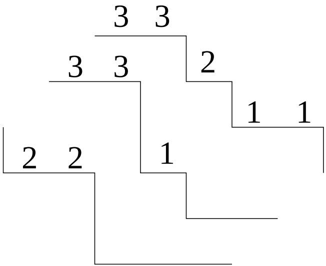
Generating function

$$\sum_{\Pi} q^{|\Pi|} = \prod_{i=1}^{\infty} \left(\frac{1}{1-q^i} \right)^i .$$

Plane partitions

Plane partitions $\leftrightarrow$ Non intersecting paths

$$\begin{array}{ccccc} 3 & 3 & 2 & 1 & 1 \\ 3 & 3 & 1 & & \\ 2 & 2 & & & \end{array}$$



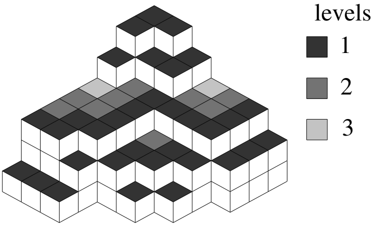
$$\sum_{\Lambda \in \mathcal{P}(a,b,c)} q^{|\Lambda|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-2}}{1 - q^{i+j+k-1}}$$

But...

Plane overpartitions are not a generalization of plane partitions.

$$\sum_{|\Pi|} a^{o(\Pi)} q^{|\Pi|} = \prod_{i=1}^{\infty} \frac{(1+aq^i)^{i-1}}{((1-q^i)(1-aq^i))^{\lfloor (i+1)/2 \rfloor}}.$$

Plane (over)partitions



$$A_\Pi(t) = (1-t)^{10}(1-t^2)^2(1-t^3)$$

$$\sum_{\Pi \in \mathcal{P}(r,c)} A_\Pi(t) q^{|\Pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}.$$

Vuletic (07) + Mac Donald case

$t=0$: plane partitions, $t=-1$: plane overpartitions

Hall-Littlewood functions

Column strict plane partitions $\leftrightarrow$ Plane partition

Knuth (70)

$$\left(\begin{array}{lcl} 4444 & & 4433 \\ 2221 & , & 3322 \\ 111 & & 111 \end{array}\right) \leftrightarrow \begin{array}{l} 4444 \\ 443 \\ 443 \\ 22 \end{array}$$

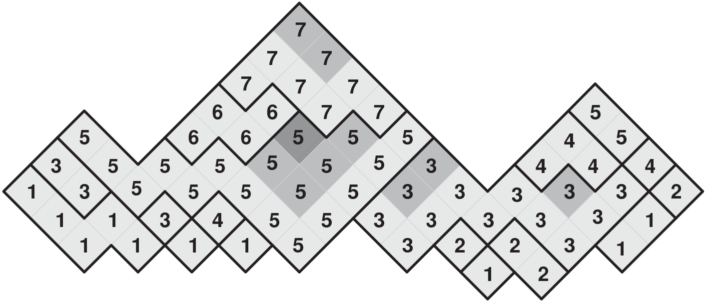
MacDonald (95)

$$\sum_{\lambda} Q_\lambda(x;t) P_\lambda(y;t) = \prod_{i,j} \frac{1 - t x_i y_j}{1 - x_i y_j}.$$

$\Rightarrow$

$$\sum_{\Pi \in \mathcal{P}(r,c)} A_\Pi(t) q^{|\Pi} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 - t q^{i+j-1}}{1 - q^{i+j-1}}.$$

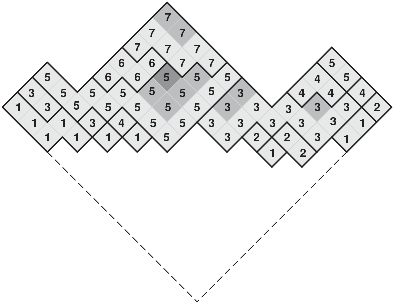
Interlacing sequences



$$A = (0,0,0,1,1,0,0,0,0,0,0,1,1,1,1,1,1,1,0,0,0,0,1,1,1,1)$$
$$T_\Pi = (1-t)^{19}(1-t^2)^4(1-t^3)$$

$$\sum_{\Pi} T_\Pi q^{|\Pi|} = \prod_{\substack{i<j \\ A[i]=0,\ A[j]=1}} \frac{1-tq^{j-i}}{1-q^{j-i}}$$

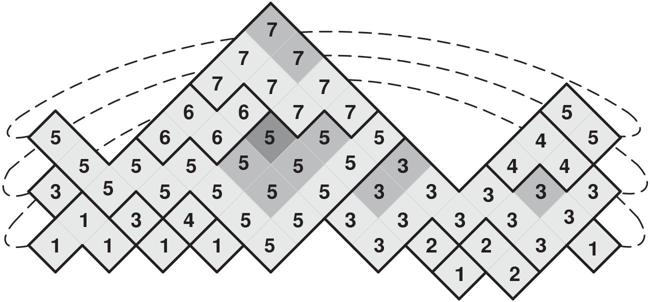
Skew (or reverse) plane partitions



$$\prod_{\substack{i<j \\ A[i]=0,\ A[j]=1}} \frac{1-tq^{j-i}}{1-q^{j-i}} = \prod_{x\in\lambda} \frac{1-tq^{h_x}}{1-q^{h_x}}.$$

$t=0$ Gansner (76), Mac Donald case : Okada (09)

Cylindric partitions



Cylindric plane partitions of a given profile $(A_1, \ldots, A_T)$

$$\prod_{n=1}^{\infty} \frac{1}{1-q^{nT}} \prod_{\substack{1 \le i,j \le T \\ A_i=1,\, A_j=0}} \frac{1 - tq^{(i-j)(T)+(n-1)T}}{1 - q^{(i-j)(T)+(n-1)T}}$$

$t = 0$ Gessel and Krattenthaler (97), Borodin (03)

More ?

- d-complete posets (Conjecture Okada 09)
- Link between cylindric partitions ($t = 0$) and the representation of $\hat{sl}_n$ (Tingley 07)

Thanks