Domino tilings, lattice paths and plane overpartitions

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Aztec diamond of order $n$: 4 staircase of height $n$ glued together.
Domino Tilings

Tile the aztec diamond of order $n$ with $n(n + 1)$ dominos.

$2^{\left\lfloor \frac{n+1}{2} \right\rfloor}$ tilings of the aztec diamond of order $n$ (Elkies et al 92)
Flip
Flip
Flip
Flip

[Diagram showing various shapes and arrows indicating the flipping action]
Flip

[Diagram showing various configurations and transformations related to flipping]
Flip

[Diagram of various shapes and transformations]
Flip

Rank: minimal number of flips from the horizontal tiling
Generating function

Tiling $T$. Number of vertical dominos : $v(T)$. Rank : $r(T)$.

$$A_n(x, q) = \sum_{T \text{ tiling of order } n} x^{v(T)} q^{r(T)} = \prod_{k=0}^{n-1} \left(1 + xq^{2k+1}\right)^{n-k}.$$  

(Elkies et al, Stanley, Benchetrit)
Tilings and lattice paths
Tilings and lattice paths
Tilings and lattice paths

Rule
Tilings and lattice paths

Rule
Tilings and lattice paths

Rule
Tilings and lattice paths

Rule
Generating function

• Vertical dominos = North-East and South-East steps
• Rank = height of the paths + constant

Non intersecting paths: Lindström, Gessel-Viennot (70-80s)

\[ A_n(x, q) = \text{determinant } ((x, q)\text{-Schröder numbers}) \]

Combinatorics of lattice paths \(\Rightarrow\)

\[ A_n(x, q) = (1 + xq)^m A_{n-1}(xq^2, q), \quad A_0(x, q) = 1. \]

\[ A_n(x, q) = \prod_{k=0}^{n-1} (1 + xq^{2k+1})^{n-k}. \]
Artic circle

(Johansson 05)
Lattice paths and monotone triangles
Lattice paths and monotone triangles
Lattice paths and monotone triangles
Monotone triangles

Monotone triangles with weights 2 on the non-diagonal rim hooks

```
\begin{array}{cccccc}
\bar{3} & & & & \\
3 & & \bar{4} & & \\
2 & 3 & 5 & & \\
2 & 3 & 4 & 5 & \\
1 & 2 & 3 & 4 & 5 \\
\end{array}
```

Alternating sign matrices with weight 2 on each -1.

```
0 0 1 0 0 \\
0 0 0 1 0 \\
0 1 0 \text{-1} 1 \\
0 0 0 1 0 \\
1 0 0 0 0 \\
```

Domino Tilings and plane overpartitions
Plane overpartitions
Tilings and flips

Flips and lattice steps
Plane overpartitions

An overpartition is a partition where the last occurrence of a part can be overlined.
\((6, 5, 5, 5, 3, 3, 3, 1)\)

A plane overpartition is a two-dimensional array such that each row is an overpartition and each column is a superpartition.

\[
\begin{array}{cccc}
5 & 5 & 5 & 3 \\
5 & 3 & 2 & 2 \\
5 & 3 \\
5
\end{array}
\]

Generating function:
\[
\sum_{\Pi} q^{\left|\Pi\right|} = \prod_{i \geq 1} \left(\frac{1 + q^i}{1 - q^i}\right)^i.
\]
Lattice paths and plane overpartitions

Plane overpartitions of shape $\lambda$

$$q^{\sum_i i\lambda_i} \prod_{x \in \lambda} \frac{1 + aq^{c_x}}{1 - q^{h_x}}$$

Krattenthaler (96), $a = -q^n$ Stanley content formula

Reverse plane overpartitions included in the shape $\lambda$

$$\prod_{x \in \lambda} \frac{1 + q^{h_x}}{1 - q^{h_x}}$$
Related objects

Plane overpartitions are in bijection with super semi-standard young tableaux.

Representation of Lie Superalgebras
Berele and Remmel (85), Krattenthaler (96)
Related objects

Plane overpartitions are in bijection with diagonally strict partitions where each rim hook counts 2

Vuletic (07), Foda and Wheeler (07, 08)

\[
\begin{array}{cccccccc}
5 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
4 & 4 & 3 & 2 & 1 & 1 & \bar{1} & \\
3 & 3 & 2 & 2 & \bar{1} & & & \\
\bar{3} & 2 & 2 & & & & & \\
2 & & & & & & & \\
\end{array}
\quad \leftrightarrow \quad
\begin{array}{cccccccc}
5 & 4 & 3 & 3 & 2 & 2 & 1 & 1 \\
4 & 4 & 3 & 2 & 1 & 1 & 1 & \\
3 & 3 & 2 & 2 & 1 & & & \\
3 & 2 & 2 & & & & & \\
2 & & & & & & & \\
\end{array}
\]
Limit shape

Diagonally strict polane partitions weighted by $2^{k(\Pi)} q^{|\Pi|}$
Ronkin function of the polynomial $P(z, w) = z + w + zw$

Vuletic (07)
RSK type algorithms

Generating function of plane overpartitions with at most \( r \) rows and \( c \) columns

\[
\prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 + q^{i+j-1}}{1 - q^{i+j-1}}.
\]

Generating function of plane overpartitions with entries at most \( n \)

\[
\prod_{i=1}^{n} \frac{\prod_{j=1}^{n} (1 + aq^{i+j})}{\prod_{j=0}^{i-1} (1 - q^{i+j})(1 - aq^{i+j})}
\]

Generating function of plane overpartitions with at most \( r \) rows and \( c \) columns and entries at most \( n \)? NICE?
Plane partitions

Interlacing sequences

Generating function

\[ \sum_{\Pi} q^{\vert\Pi\vert} = \prod_{i=1}^{\infty} \left( \frac{1}{1 - q^i} \right)^i. \]
Plane partitions

Plane partitions ↔ Non intersecting paths

\[
\sum_{\Lambda \in \mathcal{P}(a,b,c)} q^{\lvert \Lambda \rvert} = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{1 - q^{i+j+k-2}}{1 - q^{i+j+k-1}}
\]
Plane overpartitions are not a generalization of plane partitions.

\[
\sum_{|\Pi|} a^{o(\Pi)} q^{|\Pi|} = \prod_{i=1}^{\infty} \frac{(1 + aq^i)^{i-1}}{((1 - q^i)(1 - aq^i))^{[(i+1)/2]}}.
\]
Plane (over)partitions

\[ A_\Pi(t) = (1 - t)^{10} (1 - t^2)^2 (1 - t^3) \]

\[
\sum_{\Pi \in \mathcal{P}(r,c)} A_\Pi(t) q^{|\Pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 - tq^{i+j-1}}{1 - q^{i+j-1}}.
\]

Vuletic (07) + Mac Donald case

\[ t = 0: \text{plane partitions, } t = -1: \text{plane overpartitions} \]
Hall-Littlewood functions

Column strict plane partitions ↔ Plane partition

Knuth (70)

\[
\begin{pmatrix}
4444 & 4433 \\
2221 & 3322 \\
111 & 111
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
4444 \\
443 \\
111
\end{pmatrix}
\]

MacDonald (95)

\[
\sum_{\lambda} Q_{\lambda}(x; t) P_{\lambda}(y; t) = \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j}.
\]

\[
\sum_{\Pi \in \mathcal{P}(r,c)} A_{\Pi}(t) q^{|\Pi|} = \prod_{i=1}^{r} \prod_{j=1}^{c} \frac{1 - tq^{i+j-1}}{1 - q^{i+j-1}}.
\]
Interlacing sequences

\[ A = (0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1) \]

\[ T_\Pi = (1 - t)^{19}(1 - t^2)^4(1 - t^3) \]

\[
\sum_\Pi T_\Pi q^{\vert \Pi \vert} = \prod_{i<j} \frac{1 - t q^{j-i}}{1 - q^{j-i}}
\]

\(A[i]=0, \ A[j]=1\)
Skew (or reverse) plane partitions

\[ \prod_{i<j} \frac{1 - tq^{j-i}}{1 - q^{j-i}} = \prod_{x \in \lambda} \frac{1 - tq^{h_x}}{1 - q^{h_x}}. \]

\( t = 0 \) Gansner (76), Mac Donald case : Okada (09)
Cylindric partitions of a given profile \((A_1, \ldots, A_T)\)

\[
\prod_{n=1}^{\infty} \frac{1}{1 - q^n T} \prod_{1 \leq i, j \leq T \atop A_i = 1, A_j = 0} \frac{1 - tq^{(i-j)(T)+(n-1)T}}{1 - q^{(i-j)(T)+(n-1)T}}
\]

\(t = 0\) Gessel and Krattenthaler (97), Borodin (03)
More?

- $d$-complete posets (Conjecture Okada 09)
- Link between cylindric partitions ($t = 0$) and the representation of $\hat{sl}_n$ (Tingley 07)

Thanks