Two-parameter Deformation of Multivariate Hook Product Formulae

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Two-dimensional Lattice Models
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Hook Product Formulae

- Frame–Robinson–Thrall

$$f^\lambda = \frac{n!}{\prod_{v \in D(\lambda)} h_\lambda(v)}$$

- Stanley (univariate z)

$$\sum_{\substack{\pi\,:\,\text{reverse plane partition} \\ \text{of shape } \lambda}} z^{|\pi|} = \frac{1}{\prod_{v \in D(\lambda)} (1 - z^{h_\lambda(v)})}$$

- Gansner (multivariate $\boldsymbol{z} = (\cdots, z_{-1}, z_0, z_1, \cdots)$)

$$\sum_{\substack{\pi\,:\,\text{reverse plane partition} \\ \text{of shape } \lambda}} \boldsymbol{z}^{\pi} = \frac{1}{\prod_{v \in D(\lambda)} (1 - \boldsymbol{z}[H_{D(\lambda)}(v)])}$$

Goal : (q,t)-deformations of multivariate hook product formulae

$$\frac{1}{1-x} \longrightarrow \frac{(tx;q)_\infty}{(x;q)_\infty},$$

where $(a;q)_\infty = \prod_{i \geq 0}(1-aq^i)$.

Our formulae look like

$$\sum_{\sigma \in \mathcal{A}(P)} W_P(\sigma;q,t)\boldsymbol{z}^\sigma = \prod_{v \in P} \frac{(t\boldsymbol{z}[H_P(v)];q)_\infty}{(\boldsymbol{z}[H_P(v)];q)_\infty}.$$

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Plan

1. **Symmetric function approach to Gansner's formula**

   (an approach by Okounkov–Reshetikhin)

2. **(q,t)-deformation of Gansner's formula**

   (for ordinary or shifted reverse plane partitions)

3. **(q,t)-deformation of Peterson–Proctor's formula**

   (for P-partitions on d-complete poset P)

Symmetric Function Approach to Gansner's Formula

Diagrams and Shifted Diagrams

For a partition λ, we denote its diagram by $D(\lambda)$:

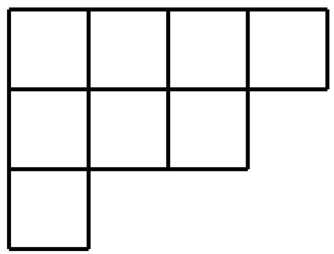
$$D(\lambda) = \{(i,j) \in \mathbb{P}^2 : 1 \leq j \leq \lambda_i\}.$$

For a strict partition μ, we denote its shifted diagram by $S(\mu)$:

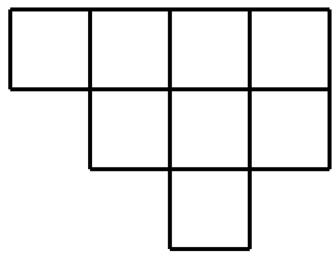
$$S(\mu) = \{(i,j) \in \mathbb{P}^2 : i \leq j \leq \mu_i + i - 1\}.$$

Example :

$D((4,3,1))$



$S((4,3,1))$



Reverse Plane Partitions

A (weak) reverse plane partition of shape λ is an array of non-negative integers

$$\pi = \begin{matrix} \pi_{1,1} & \pi_{1,2} & \cdots\cdots & \pi_{1,\lambda_1} \\ \pi_{2,1} & \pi_{2,2} & \cdots & \pi_{2,\lambda_2} \\ \vdots & \vdots & & \\ \pi_{r,1} & \pi_{r,2} & \cdots & \pi_{r,\lambda_r} \end{matrix}$$

(i.e., a map $D(\lambda) \longrightarrow \mathbb{N}$) satisfying

$$\pi_{i,j} \le \pi_{i,j+1}, \quad \pi_{i,j} \le \pi_{i+1,j}.$$

Let $\mathcal{A}(D(\lambda))$ be the set of reverse plane partitions of shape λ :

$$\mathcal{A}(D(\lambda)) = \{\pi : \text{reverse plane partition of shape } \lambda\}.$$

A shifted (weak) reverse plane partition of shifted shape μ is an array of non-negative integers

$$\sigma = \begin{array}{ccccccc} \sigma_{1,1} & \sigma_{1,2} & \sigma_{1,3} & & \cdots\cdots & & \sigma_{1,\mu_1} \\ & \sigma_{2,2} & \sigma_{2,3} & & \cdots & \sigma_{2,\mu_2+1} & \\ & & \ddots & & & & \\ & & & \sigma_{r,r} & \cdots & \sigma_{r,\mu_r+r-1} & \end{array}$$

(i.e., a map $S(\mu) \longrightarrow \mathbb{N}$) satisfying

$$\sigma_{i,j} \leq \sigma_{i,j+1}, \quad \sigma_{i,j} \leq \sigma_{i+1,j}.$$

Let $\mathcal{A}(S(\mu))$ be the set of shifted reverse plane partitions of shape μ :

$$\mathcal{A}(S(\mu)) = \{\sigma : \text{shifted reverse plane partition of shape } \mu\}.$$

Trace Generating Function

Given an ordinary or shifted reverse plane partition $\pi = (\pi_{i,j})$, we define its k-th trace $t_k(\pi)$ by

$$t_k(\pi) = \sum_i \pi_{i,i+k}.$$

We write

$$\boldsymbol{z}^\pi = \prod_k z_k^{t_k(\pi)} = \prod_{i,j} z_{j-i}^{\pi_{i,j}},$$

and consider trace generating functions with respect to this weight.

Example : For $\pi = \begin{matrix} 0 & 1 & 3 & 3 \\ 1 & 1 & 3 & \\ 2 & 4 & & \end{matrix}$, we have

$$\boldsymbol{z}^\pi = {z_{-2}}^2 {z_{-1}}^{1+4} {z_0}^{0+1} {z_1}^{1+3} {z_2}^3 {z_3}^3.$$

Hook and Shifted Hook

For a partition λ, the hook at (i, j) in $D(\lambda)$ is defined by

$$\begin{aligned} H_{D(\lambda)}(i,j) = \{(i,j)\} &\cup \{(i,l) \in D(\lambda) : l > j\} \\ &\cup \{(k,j) \in D(\lambda) : k > i\}. \end{aligned}$$

For a strict partition μ, the shifted hook at (i, j) in $S(\mu)$ is defined by

$$\begin{aligned} H_{S(\mu)}(i,j) = \{(i,j)\} &\cup \{(i,l) \in S(\mu) : l > j\} \\ &\cup \{(k,j) \in S(\mu) : k > i\} \\ &\cup \{(j+1,l) \in S(\mu) : l > j\}. \end{aligned}$$

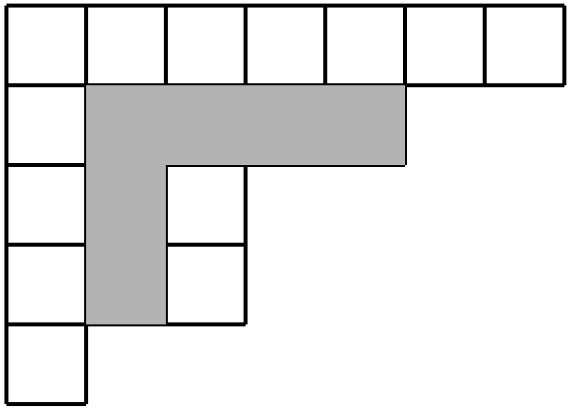
We write

$$\boldsymbol{z}[H] = \prod_{(i,j) \in H} z_{j-i}$$

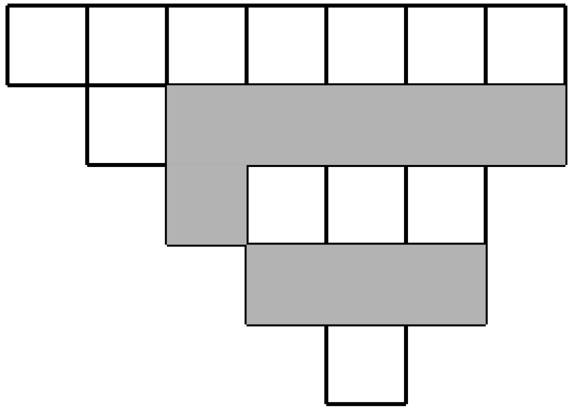
for a finite subset $H \subset \mathbb{P}^2$.

Example :

The hook at $(2,2)$
in $D((7,5,3,3,1))$



The shifted hook at $(2,3)$
in $S((7,6,4,3,1))$



Gansner's Hook Product Formula

(a) For a partition λ, the trace generating function of $\mathcal{A}(D(\lambda))$ is given by

$$\sum_{\pi \in \mathcal{A}(D(\lambda))} \boldsymbol{z}^{\pi} = \prod_{v \in D(\lambda)} \frac{1}{1 - \boldsymbol{z}[H_{D(\lambda)}(v)]}.$$

(b) For a strict partition μ, the trace generating function of $\mathcal{A}(S(\mu))$ is given by

$$\sum_{\sigma \in \mathcal{A}(S(\mu))} \boldsymbol{z}^{\sigma} = \prod_{v \in S(\mu)} \frac{1}{1 - \boldsymbol{z}[H_{S(\mu)}(v)]}.$$

Idea of Proof of Gansner's formula

Consider generating functions

$$R_{S(\mu),\tau}(\boldsymbol{z}) = \sum_{\sigma \in \mathcal{A}(S(\mu),\tau)} z^{\sigma}$$

of shifted reverse plane partitions of shifted shape μ with profile τ, and express them in terms of Schur functions by using operator calculus on the ring of symmetric functions.

Then we have

$$\sum_{\pi \in \mathcal{A}(D(\lambda))} \boldsymbol{z}^{\pi} = \sum_{\tau} R_{S(\mu),\tau}(\boldsymbol{x}) R_{S(\nu),\tau}(\boldsymbol{y}),$$

$$\sum_{\sigma \in \mathcal{A}(S(\mu))} \boldsymbol{z}^{\sigma} = \sum_{\tau} R_{S(\mu),\tau}(\boldsymbol{z}).$$

Hence Gansner's formulae follow from Cauchy and Schur–Littlewood identities.

Diagonals and Profile

For an array of non-negative integers σ of shifted shape μ, we define its k-th diagonal $\sigma[k]$ by putting

$$\sigma[k] = (\cdots, \sigma_{2,k+2}, \sigma_{1,k+1}) \quad (k = 0, 1, 2, \cdots).$$

We call $\sigma[0]$ the profile and put

$$\mathcal{A}(S(\mu), \tau) = \{\sigma \in \mathcal{A}(S(\mu)) : \sigma[0] = \tau\}.$$

Example : For $\sigma = \begin{matrix} 0 & 0 & 1 & 2 & 3 & 3 \\ & 1 & 2 & 3 & 3 & 3 \\ & & 2 & 4 & & \end{matrix}$, we have

$$\begin{aligned} &\sigma[0] = (2, 1, 0), \ \sigma[1] = (4, 2, 0), \ \sigma[2] = (3, 1), \\ &\sigma[3] = (3, 2), \quad \ \ \sigma[4] = (3, 3), \quad \ \ \sigma[5] = (3). \end{aligned}$$

A key is the following observation.

Lemma The following are equivalent:

(i) σ is a shifted reverse plane partition.

(ii) Each $\sigma[k]$ is a partition and

$$\begin{cases} \sigma[k-1] \succ \sigma[k] & \text{if } k \text{ is a part of } \mu, \\ \sigma[k-1] \prec \sigma[k] & \text{otherwise.} \end{cases}$$

where we write $\alpha \succ \beta$ if

$$\alpha_1 \geq \beta_1 \geq \alpha_2 \geq \beta_2 \geq \cdots ,$$

i.e., the skew diagram α/β is a horizontal strip.

Let h_k and $h_k^\perp$ be the multiplication and skewing operators on the ring of symmetric functions Λ associated to the complete symmetric function h_k. Consider the generating functions

$$H^+(u) = \sum_{k\geq 0} h_k u^k, \quad H^-(u) = \sum_{k\geq 0} h_k^\perp u^k.$$

and the operator $D(z) : \Lambda \longrightarrow \Lambda$ defined by

$$D(z)s_\lambda = z^{|\lambda|}s_\lambda.$$

First we apply the Pieri rule

$$H^+(t)s_\lambda = \sum_{\kappa\succ\lambda} t^{|\kappa|-|\lambda|}s_\kappa, \quad H^-(t)s_\lambda = \sum_{\kappa\prec\lambda} t^{|\lambda|-|\kappa|}s_\kappa,$$

and Lemma above to obtain

Lemma If we define $\varepsilon_1, \cdots, \varepsilon_N$ $(N \geq \mu_1)$ by

$$\varepsilon_k = \begin{cases} + & \text{if } k \text{ is a part of } \mu, \\ - & \text{otherwise,} \end{cases}$$

then we have

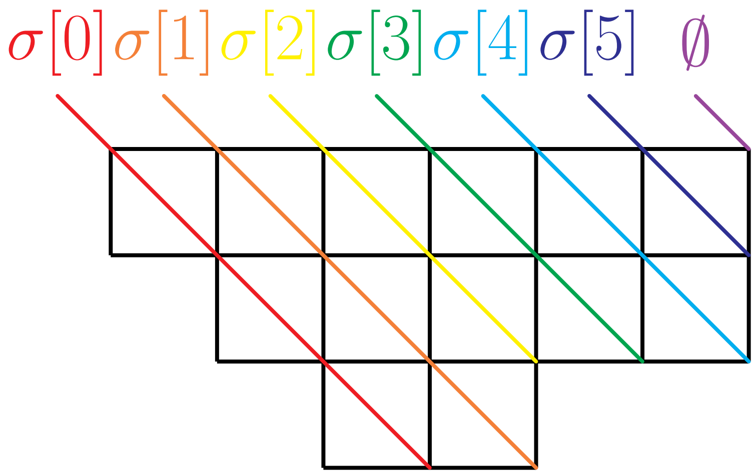
$$D(z_0)H^{\varepsilon_1}(1)D(z_1)H^{\varepsilon_2}(1)D(z_2)H^{\varepsilon_2}(1)\cdots H^{\varepsilon_{N-1}}(1)D(z_{N-1})H^{\varepsilon_N}(1)1 \\ = \sum_{\tau} R_{S(\mu),\tau}(\boldsymbol{z}) s_\tau,$$

where $R_{S(\mu),\tau}(\boldsymbol{z})$ is the generating function of shifted reverse plane partitions of shifted shape μ with profile τ:

$$R_{S(\mu),\tau}(\boldsymbol{z}) = \sum_{\sigma \in \mathcal{A}(S(\mu),\tau)} \boldsymbol{z}^{\sigma}.$$

Example : If $\mu = (6,5,2)$ and $N = 6$, then $\varepsilon = (-,+,-,-,+,+)$ and we compute

$$D(z_0)H^-(1)D(z_1)H^+(1)D(z_2)H^-(1)D(z_3)H^-(1)$$
$$D(z_4)H^+(1)D(z_5)H^+(1)1.$$



$$\sigma[0] \prec \sigma[1] \succ \sigma[2] \prec \sigma[3] \prec \sigma[4] \succ \sigma[5] \succ \emptyset.$$

Commutation Relations

By using the commutation relations

$$D(z)H^{+}(u) = H^{+}(zu)D(z),$$
$$D(z)H^{-}(u) = H^{-}(z^{-1}u)D(z),$$
$$D(z)D(z') = D(zz'),$$

we obtain

$$\begin{aligned} D(z_0)H^{\varepsilon_1}(1)D(z_1)H^{\varepsilon_2}(1)D(z_2)H^{\varepsilon_2}(1)\cdots H^{\varepsilon_{N-1}}(1)D(z_{N-1})H^{\varepsilon_N}(1) \\ = H^{\varepsilon_1}(\tilde{z}_1^{\varepsilon_1})H^{\varepsilon_2}(\tilde{z}_2^{\varepsilon_2})\cdots H^{\varepsilon_N}(\tilde{z}_N^{\varepsilon_N})D(\tilde{z}_N), \end{aligned}$$

where we put

$$\tilde{z}_k = z_0 z_1 \cdots z_{k-1}.$$

Further, by using the commutation relation

$$H^{-}(u)H^{+}(v) = \frac{1}{1-uv}H^{+}(v)H^{-}(u),$$

we can derive

$$H^{\varepsilon_1}(\tilde{z}_1^{\varepsilon_1})H^{\varepsilon_2}(\tilde{z}_2^{\varepsilon_2}) \cdots H^{\varepsilon_N}(\tilde{z}_N^{\varepsilon_N})$$
$$= \prod_{\mu_k^c < \mu_l} \frac{1}{1 - \tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}} \prod_{k=1}^{r} H^{+}(\tilde{z}_{\mu_k}) \prod_{l=1}^{N-r} H^{-}(\tilde{z}_{\mu_l^c}).$$

where μ^c is the strict partition formed by the complement of μ in $\{1, 2, \cdots, N\}$:

$$\{\mu_1, \cdots, \mu_r\} \sqcup \{\mu_1^c, \cdots, \mu_{N-r}^c\} = \{1, 2, \cdots, N\}.$$

Generating Functions in terms of Schur Functions

Finally, by using the Cauchy identity

$$\prod_{k=1}^{r} H^{+}(\tilde{z}_{\mu_k})1 = \sum_{\tau} s_{\tau}(\tilde{z}_{\mu_1}, \cdots, \tilde{z}_{\mu_r}) s_{\tau},$$

we have

Proposition The generating function of shifted reverse plane partitions of shifted shape μ with profile τ is given by

$$\sum_{\sigma \in \mathcal{A}(S(\mu);\tau)} z^{\sigma} = \prod_{\mu_k^c < \mu_l} \frac{1}{1 - \tilde{z}_{\mu_k^c}^{-1} \tilde{z}_{\mu_l}} \cdot s_{\tau}(\tilde{z}_{\mu_1}, \cdots, \tilde{z}_{\mu_r}),$$

where $\{\mu_1, \cdots, \mu_r\} \sqcup \{\mu_1^c, \cdots, \mu_{N-r}^c\} = \{1, 2, \cdots, N\}$, and $\tilde{z}_k = z_0 z_1 \cdots z_{k-1}$.

Proof of Gansner's Formula (a) for Shapes

A reverse plane partition $\pi \in \mathcal{A}(D(\lambda))$ is obtained by gluing two shifted reverse plane partitions $\sigma \in \mathcal{A}(S(\mu))$ and $\rho \in \mathcal{A}(S(\nu))$ with the same profile $\tau = \sigma[0] = \rho[0]$, where two strict partitions μ and ν are defined by

$$\mu_i = \lambda_i - i + 1, \quad \nu_i = {}^t\lambda_i - i + 1 \quad (1 \le i \le p(\lambda)).$$

Example If $\lambda = (4,3,1)$, then $\mu = (4,2)$, $\nu = (3,1)$ and

$$\begin{matrix} 0 & 0 & 1 & 3 \\ 1 & 2 & 2 & \\ 3 & & & \end{matrix} \longleftrightarrow \left(\begin{matrix} 0 & 0 & 1 & 3 \\ & 2 & 2 & \end{matrix} , \begin{matrix} 0 & 1 & 3 \\ & 2 & \end{matrix} \right).$$

Hence Gansner's formula follows from the Cauchy identity

$$\sum_{\tau} s_{\tau}(X) s_{\tau}(Y) = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$

Proof of Gansner's Formula (b) for Shifted Shapes

We have

$$\sum_{\sigma \in \mathcal{A}(S(\mu))} \boldsymbol{z}^{\sigma} = \sum_{\tau} R_{S(\mu),\tau}(\boldsymbol{z}),$$

so Gansner's formula follows from the Schur–Littlewood identity

$$\sum_{\tau} s_{\tau}(X) = \prod_{i} \frac{1}{1 - x_i} \prod_{i<j} \frac{1}{1 - x_i x_j}.$$

(q,t)-Deformation of Gansner's Formula

Generalization by Macdonald Symmetric Functions

We can play the same game for Macdonald functions instead of Schur functions to obtain weighted trace generating functions for reverse plane partitions. (See also works by Foda–Wheeler-Zuparic, Vuletić.)

We denote by $P_\lambda = P_\lambda(X; q, t)$ the Macdonald symmetric function characterized by

- $P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda,\mu} m_\mu$.
- If $\lambda \neq \mu$, then $\langle P_\lambda, P_\mu \rangle = 0$.

Let $Q_\lambda = Q_\lambda(X; q, t)$ be the dual basis defined by

$$\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda,\mu}.$$

Note that, if we put $q = t$, then

$$P_\lambda(X; q, q) = Q_\lambda(X; q, q) = s_\lambda(X).$$

We write

$$g_k = g_k(X;q,t) = Q_{(k)}(X;q,t).$$

Note that, if we put $q = t$, then $g_k(X;q,q) = h_k(X)$.

Let $g_k^+ : \Lambda \longrightarrow \Lambda$ be the multiplication operator by g_k and let $g_k^- : \Lambda \longrightarrow \Lambda$ be the skewing operator by g_k, i.e., the adjoint operator of g_k^+ :

$$g_k^+(h) = hg_k \quad (h \in \Lambda),$$
$$\langle g_k^-(h), f\rangle = \langle h, g_k f\rangle \quad (f, h \in \Lambda).$$

Consider generating functions

$$G^+(u) = \sum_{k\geq 0} g_k^+ u^k, \quad G^-(u) = \sum_{k\geq 0} g_k^- u^k,$$

and the operator $D(z) : \Lambda \longrightarrow \Lambda$ defined by

$$D(z)P_\lambda = z^{|\lambda|}P_\lambda.$$

The Pieri rule for Macdonald functions can be stated as follows:

$$G^{+}(u)P_{\beta}= \sum_{\alpha \succ \beta} \varphi^{+}_{\alpha,\beta}(q,t)u^{|\alpha|-|\beta|}P_{\alpha},$$

$$G^{-}(u)P_{\alpha}= \sum_{\beta \prec \alpha} \varphi^{-}_{\beta,\alpha}(q,t)u^{|\alpha|-|\beta|}P_{\beta},$$

where

$$\varphi^{+}_{\alpha,\beta}(q,t) = \prod_{i\le j} \frac{f_{q,t}(\alpha_i - \beta_j; j-i) f_{q,t}(\beta_i - \alpha_{j+1}; j-i)}{f_{q,t}(\alpha_i - \alpha_j; j-i) f_{q,t}(\beta_i - \beta_{j+1}; j-i)},$$

$$\varphi^{-}_{\beta,\alpha}(q,t) = \prod_{i\le j} \frac{f_{q,t}(\alpha_i - \beta_j; j-i) f_{q,t}(\beta_i - \alpha_{j+1}; j-i)}{f_{q,t}(\alpha_i - \alpha_{j+1}; j-i) f_{q,t}(\beta_i - \beta_j; j-i)},$$

and

$$f_{q,t}(n;m) = \prod_{i=0}^{n-1} \frac{1-q^i t^{m+1}}{1-q^{i+1} t^m}.$$

Proposition The weighted generating function of shifted reverse plane partitions of shape μ with profile τ is given by

$$\sum_{\sigma \in \mathcal{A}(S(\mu);\tau)} V_{S(\mu)}(\sigma;q,t)\boldsymbol{z}^{\sigma}$$

$$= \prod_{\mu_k^c < \mu_l} \frac{(t\tilde{z}_{\mu_k^c}^{-1}\tilde{z}_{\mu_l};q)_\infty}{(\tilde{z}_{\mu_k^c}^{-1}\tilde{z}_{\mu_l};q)_\infty} \cdot Q_\tau(\tilde{z}_{\mu_1},\cdots,\tilde{z}_{\mu_r};q,t),$$

where $\{\mu_1,\cdots,\mu_r\} \sqcup \{\mu_1^c,\cdots,\mu_{N-r}^c\} = \{1,2,\cdots,N\}$, and $\tilde{z}_k = z_0 z_1 \cdots z_{k-1}$. And the weight $V_{S(\mu)}(\sigma;q,t)$ is given by

$$V_{S(\mu)}(\sigma;q,t) = \prod_{k=1}^{N} \varphi_{\sigma[k-1],\sigma[k]}^{\varepsilon_k}(q,t),$$

where $\varepsilon_k = +$ if k is a part of μ and $\varepsilon_k = -$ otherwise.

This weight function can be written explicitly as

$$
\begin{aligned}
&V_{S(\mu)}(\sigma; q, t) \\
&= \prod_{\substack{(i,j)\in S(\mu) \\ i<j}} \prod_{m\geq 0} \frac{f_{q,t}(\sigma_{i,j}-\sigma_{i-m,j-m-1}; m) f_{q,t}(\sigma_{i,j}-\sigma_{i-m-1,j-m}; m)}{f_{q,t}(\sigma_{i,j}-\sigma_{i-m,j-m}; m) f_{q,t}(\sigma_{i,j}-\sigma_{i-m-1,j-m-1}, m)} \\
&\quad \times \prod_{(i,i)\in S(\mu)} \prod_{m\geq 0} \frac{f_{q,t}(\sigma_{i,i}-\sigma_{i-m-1,i-m}; m)}{f_{q,t}(\sigma_{i,i}-\sigma_{i-m,i-m}; m)}.
\end{aligned}
$$

Theorem A (for shapes)

Let λ be a partition. For a reverse plane partition $\pi \in \mathcal{A}(D(\lambda))$, we define

$$W_{D(\lambda)}(\pi; q, t)$$
$$= \prod_{(i,j) \in D(\lambda)} \prod_{m \geq 0} \frac{f_{q,t}(\pi_{i,j} - \pi_{i-m,j-m-1}; m) f_{q,t}(\pi_{i,j} - \pi_{i-m-1,j-m}; m)}{f_{q,t}(\pi_{i,j} - \pi_{i-m,j-m}; m) f_{q,t}(\pi_{i,j} - \pi_{i-m-1,j-m-1}; m)},$$

where $\pi_{k,l} = 0$ if $k < 0$ or $l < 0$. Then we have

$$\sum_{\pi \in \mathcal{A}(D(\lambda))} W_{D(\lambda)}(\pi; q, t) \boldsymbol{z}^{\pi} = \prod_{v \in D(\lambda)} \frac{(t\boldsymbol{z}[H_{D(\lambda)}(v)]; q)_{\infty}}{(\boldsymbol{z}[H_{D(\lambda)}(v)]; q)_{\infty}}.$$

Plane partitions of rectangular shape (c^r) are obtained by 180° rotation from reverse plane partitions of the same shape. Hence we obtain Vuletić's generalization of MacMahon formula.

Example : If $\lambda = (3,3)$, then the weight is given by

$$
\begin{aligned}
&W_{D(3,3)}\left(\begin{matrix} a & b & c \\ d & e & f \end{matrix};q,t\right) \\
&\quad = f_{q,t}(a-0;0) \times f_{q,t}(b-a;0) \times f_{q,t}(c-b;0) \times f_{q,t}(d-a;0) \\
&\qquad \times \frac{f_{q,t}(e-b;0)f_{q,t}(e-d;0)f_{q,t}(e-0;1)}{f_{q,t}(e-a;0)f_{q,t}(e-a;1)} \\
&\qquad \times \frac{f_{q,t}(f-c;0)f_{q,t}(f-e;0)f_{q,t}(f-a;1)}{f_{q,t}(f-b;0)f_{q,t}(f-b;1)}.
\end{aligned}
$$

Theorem B (for shifted shapes)
Let μ be a strict partition. For a shifted reverse plane partition $\sigma \in \mathcal{A}(S(\mu))$, we define

$$W_{S(\mu)}(\sigma; q, t)$$
$$= \prod_{\substack{(i,j)\in S(\mu)\\ i<j}} \prod_{m\geq 0} \frac{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m-1}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m}; m)}{f_{q,t}(\sigma_{i,j} - \sigma_{i-m,j-m}; m) f_{q,t}(\sigma_{i,j} - \sigma_{i-m-1,j-m-1}, m)}$$
$$\times \prod_{(i,i)\in S(\mu)} \prod_{m\geq 0} \frac{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-1,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-1}; 2m+1)}{f_{q,t}(\sigma_{i,i} - \sigma_{i-2m,i-2m}; 2m) f_{q,t}(\sigma_{i,i} - \sigma_{i-2m-2,i-2m-2}; 2m+1)},$$

where $\sigma_{k,l} = 0$ if $k < 0$. Then we have

$$\sum_{\sigma\in\mathcal{A}(S(\mu))} W_{S(\mu)}(\sigma; q, t)\boldsymbol{z}^{\sigma} = \prod_{v\in S(\mu)} \frac{(t\boldsymbol{z}[H_{S(\mu)}(v)]; q)_\infty}{(\boldsymbol{z}[H_{S(\mu)}(v)]; q)_\infty}.$$

Example : If $\mu = (3,2,1)$, then the weight is given by

$$W_{S(3,2,1)}\left(\begin{matrix} a & b & c \\ & d & e \\ & & f \end{matrix}\; ; q, t\right)$$

$$= f_{q,t}(a-0;0) \times f_{q,t}(b-a;0) \times f_{q,t}(c-b;0) \times f_{q,t}(d-b;0)$$

$$\times \frac{f_{q,t}(e-c;0) f_{q,t}(e-d;0) f_{q,t}(e-a;1)}{f_{q,t}(e-b;0) f_{q,t}(e-b;1)}$$

$$\times \frac{f_{q,t}(f-e;0) f_{q,t}(f-b;1) f_{q,t}(f-0;2)}{f_{q,t}(f-a;1) f_{q,t}(f-a;2)}.$$

Proof of Theorems A and B : Same as the proof of Gansner's formula. Note that the weights are related as

$$W_{D(\lambda)}(\pi;q,t) = \frac{1}{b_\tau(q,t)} V_{S(\mu)}(\sigma;q,t) V_{S(\nu)}(\rho;q,t),$$

$$W_{S(\mu)}(\sigma;q,t) = \frac{b^{\mathsf{el}}_\tau(q,t)}{b_\tau(q,t)} V_{S(\mu)}(\sigma;q,t),$$

where

$$b_\tau(q,t) = \prod_{i \le j} \frac{f_{q,t}(\tau_i - \tau_{j+1}; j-i)}{f_{q,t}(\tau_i - \tau_j; j-i)} = \langle P_\tau, P_\tau \rangle,$$

$$b^{\mathsf{el}}_\tau(q,t) = \prod_{\substack{i \le j \\ j-i \text{ is even}}} \frac{f_{q,t}(\tau_i - \tau_{j+1}; j-i)}{f_{q,t}(\tau_i - \tau_j; j-i)}.$$

Hence Theorems A and B follow from Cauchy-type and Schur–Littlewood–type identities respectively.

(q, t)-Deformation of Peterson–Proctor's Hook Product Formula for d-Complete Posets

P-Partitions

Let P be a poset. A P-partition is a map $\sigma : P \to \mathbb{N}$ satisfying

$$x \leq y \text{ in } P \quad \Longrightarrow \quad \sigma(x) \geq \sigma(y) \text{ in } \mathbb{N}.$$

Let $\mathcal{A}(P)$ be the set of P-partitions:

$$\mathcal{A}(P) = \{\sigma : P \to \mathbb{N} : P\text{-partition}\}.$$

The diagram $D(\lambda)$ and the shifted diagram $S(\mu)$ are posets w.r.t

$$(i,j) \geq (k,l) \quad \Longleftrightarrow \quad i \leq k, \text{ and } j \leq l.$$

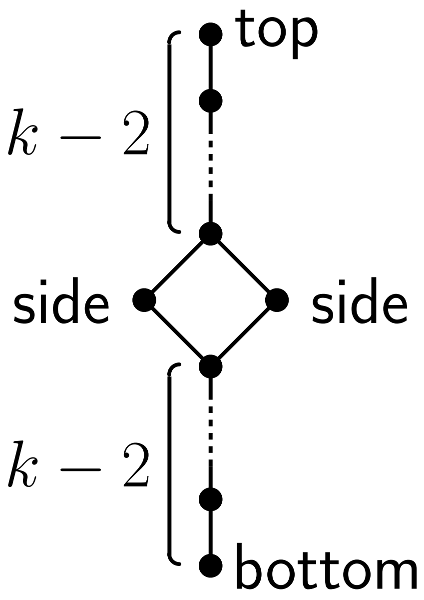
Then

$D(\lambda)$-partition $=$ reverse plane partition of shape λ,
$S(\mu)$-partition $=$ shifted reverse plane partition of shifted shape μ.

Gansner's hook product formula is generalized to the generating function of P-partitions for d-complete posets P (Peterson–Proctor).

d-Complete Posets

- The double-tailed diamond poset $d_k(1)$ is the poset depicted below:



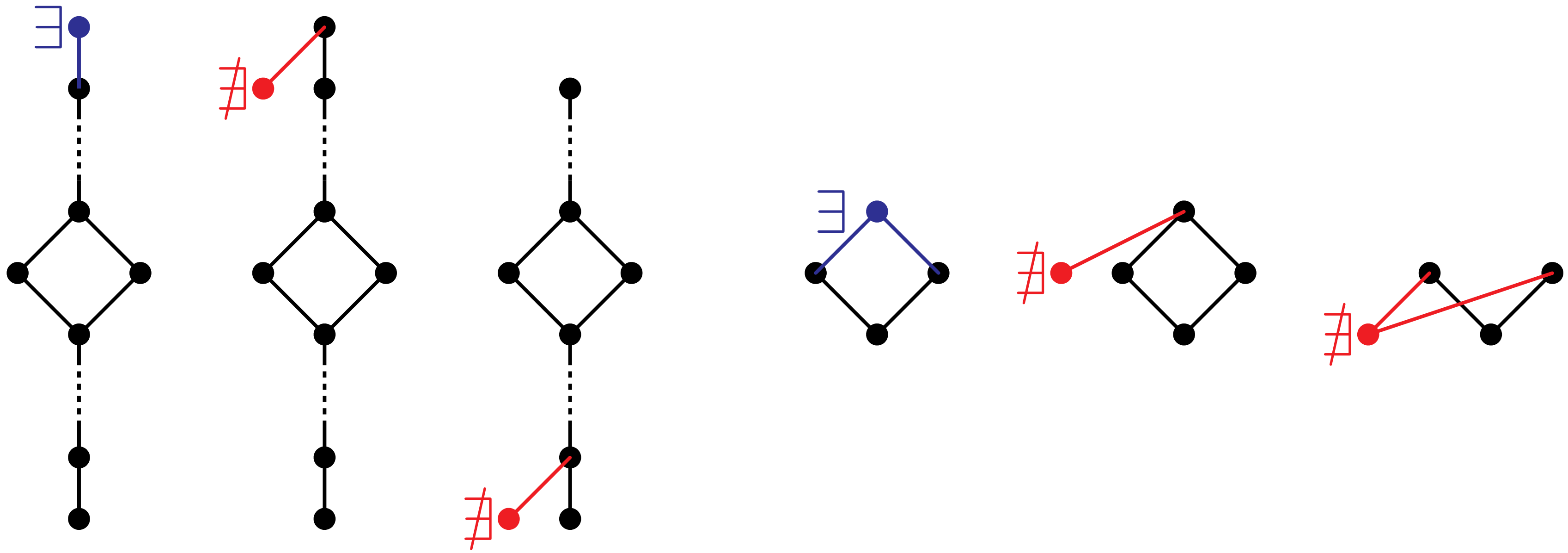
- A d_k-interval is an interval isomorphic to $d_k(1)$.
- A d_k^--interval ($k \geq 4$) is an interval isomorphic to $d_k(1) - \{\text{top}\}$.
- A d_3^--interval consists of three elements x, y and w such that w is covered by x and y.

Definition A finite poset P is d-complete if it satisfies the following three conditions for every k:

(D1) If I is a d_k^--interval, then there exists an element v such that v covers the maximal elements of I and $I \cup \{v\}$ is a d_k-interval.

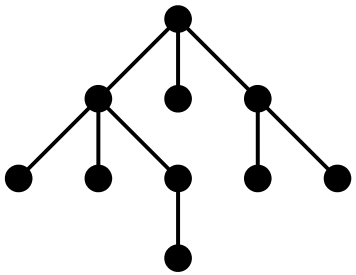
(D2) If $I = [w, v]$ is a d_k-interval and v covers u in P, then $u \in I$.

(D3) There are no d_k^--intervals which differ only in the minimal elements.

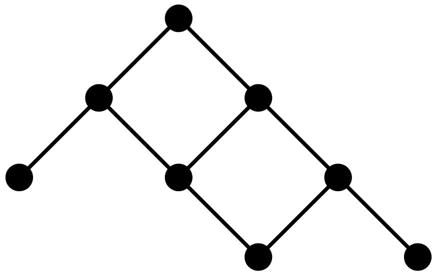


Example :

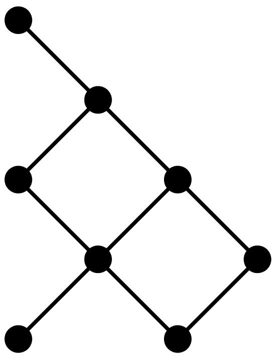
- rooted tree



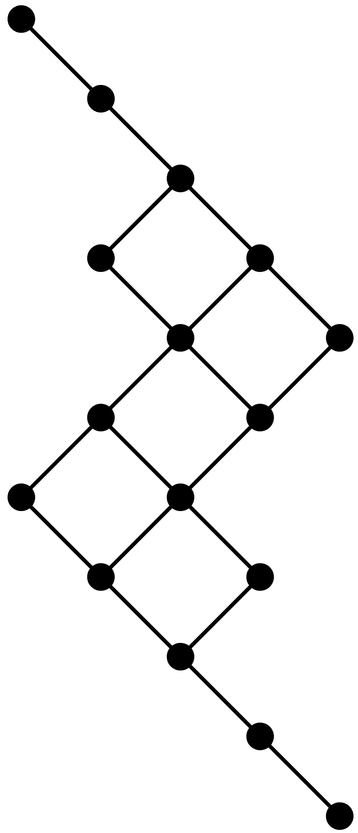
- shape



- shifted shape



- swivel



Fact If P is a connected d-complete poset, then

(a) P has a unique maximal element.

(b) P is ranked, i.e., there exists a rank function $r : P \to \mathbb{N}$ such that $r(x) = r(y) + 1$ if x covers y.

Fact

(a) Any connected d-complete poset is uniquely decomposed into a slant sum of one-element posets and slant-irreducible d-complete posets.

(b) Slant-irreducible d-complete posets are classified into 15 families :

shapes, shifted shapes, birds, insets, tailed insets, banners, nooks, swivels, tailed swivels, tagged swivels, swivel shifts, pumps, tailed pumps, near bats, bat.

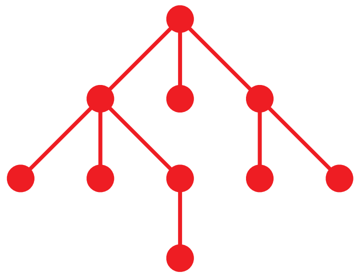
Top Tree

For a connected d-complete poset P, we define its top tree by putting

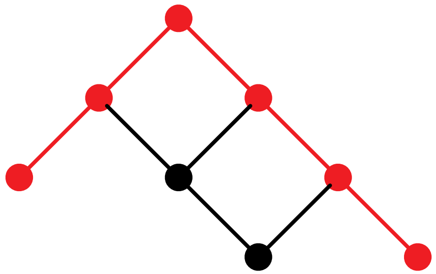
$$T = \{x \in P : \text{ every } y \geq x \text{ is covered by at most one other element } \}$$

Example : Top trees

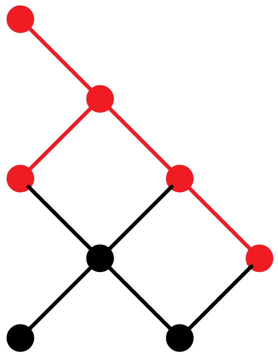
- rooted tree



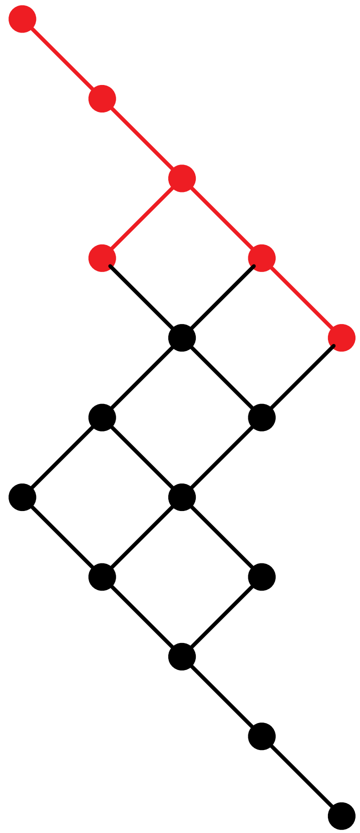
- shape



- shifted shape



- swivel



Top Tree and d-Complete Coloring

For a connected d-complete poset P, we define its top tree by putting

$$T = \{x \in P : \text{ every } y \geq x \text{ is covered by at most one other element } \}$$

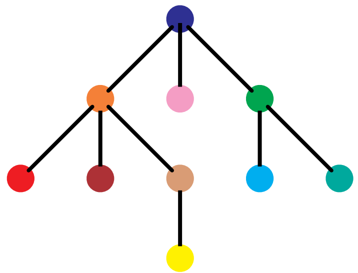
Fact Let I be a set of colors such that $\#I = \#T$. Then a bijection $c : T \to I$ can be uniquely extended to a map $c : P \to I$ satisfying the following four conditions:

- If x and y are incomparable, then $c(x) \neq c(y)$.
- If an interval $[w, v]$ is a chain, then the colors $c(x)$ ($x \in [w, v]$) are distinct.
- If $[w, v]$ is a d_k-interval then $c(w) = c(v)$.

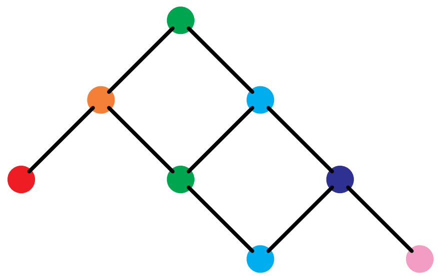
Such a map $c : P \to I$ is called a d-complete coloring.

Example : d-Complete colorings

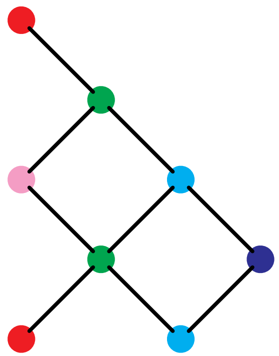
- rooted tree



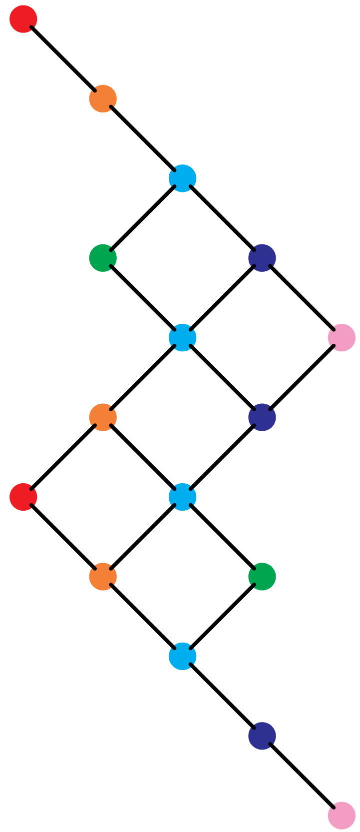
- shape



- shifted shape



- swivel



Monomials associated to Hooks

Let P be a connected d-complete poset and T its top tree. Let z_v ($v \in T$) be indeterminate. Let $c : P \to T$ be the d-complete coloring.

For each $v \in P$, we define monomials $\boldsymbol{z}[H_P(v)]$ by induction as follows:

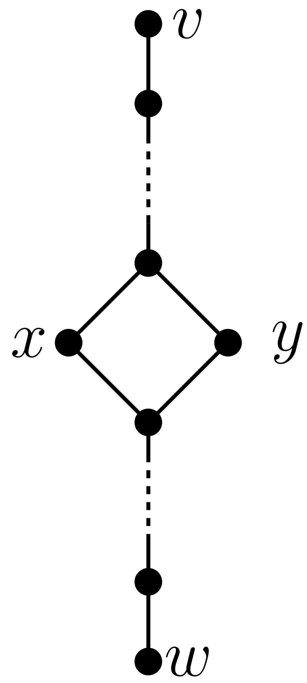
(a) If v is not the top of any d_k-interval, then we define

$$\boldsymbol{z}[H_P(v)] = \prod_{w \le v} z_{c(w)}.$$

(b) If v is the top of a d_k-interval $[w, v]$, then we define

$$\boldsymbol{z}[H_P(v)] = \frac{\boldsymbol{z}[H_P(x)] \cdot \boldsymbol{z}[H_P(y)]}{\boldsymbol{z}[H_P(w)]},$$

where x and y are the sides of $[w, v]$.



Conjecture

Let P be a connected d-complete poset with maximum element v_0 and top tree T. Let $r : P \to \mathbb{N}$ be the rank function and $c : P \to T$ the d-complete coloring. Given a P-partition $\sigma \in \mathcal{A}(P)$, we define

$$W_P(\sigma; q, t) = \frac{\displaystyle\prod_{\substack{x,y \in P \\ x<y,\, c(x) \sim c(y)}} f_{q,t}(\sigma(x) - \sigma(y); d(x,y)) \prod_{\substack{x \in P \\ c(x) = v_0}} f_{q,t}(\sigma(x); e(x, v_0))}{\displaystyle\prod_{\substack{x,y \in P \\ x<y,\, c(x) = c(y)}} f_{q,t}(\sigma(x) - \sigma(y); e(x,y)) f_{q,t}(\sigma(x) - \sigma(y); e(x,y) - 1)},$$

where $c(x) \sim c(y)$ means that $c(x)$ and $c(y)$ are adjacent in T, and

$$d(x,y) = (r(y) - r(x) - 1)/2, \quad e(x,y) = (r(y) - r(x))/2.$$

Recall $f_{q,t}(n; m) = \prod_{i=0}^{n-1} (1 - q^i t^{m+1})/(1 - q^{i+1} t^m)$.

And we write

$$\boldsymbol{z}^{\sigma} = \prod_{v \in P} z_{c(v)}^{\sigma(v)}.$$

Conjecture

$$\sum_{\sigma \in \mathcal{A}(P)} W_P(\sigma; q, t) \boldsymbol{z}^{\sigma} = \prod_{v \in P} \frac{(t\boldsymbol{z}[H_P(v)]; q)_\infty}{(\boldsymbol{z}[H_P(v)]; q)_\infty}.$$

Known cases

- $q = t$ case (Peterson–Proctor's hook product formula).
- Rooted trees (use the binomial theorem and induction).
- Shapes (Theorem A).
- Shifted shapes (a modification of Theorem B by Warnaar's formula).