Combinatorics of RSOS paths

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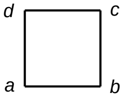
(partly with Patrick Jacob)

The (R)SOS models

- Variables: heights ℓ_i at the vertices of a square lattice
- SOS: $\ell_i \in \mathbb{Z}$
- Defining condition $|\ell_i - \ell_j| = 1$ for i, j nearest neighbors
- Interaction defined for the 4 sites of a paquette via w



$$w(a, b, c, d)$$

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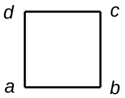

$$w(a,b,c,d)$$

- RSOS version: $\ell_i \in \{1, 2 \cdots, p-1\}$ and

$$\eta^{8V} = \frac{K(p-p')}{p}$$

[Andrews-Baxter-Forrester; Forrester-Baxter]

Scaling limit at criticality : minimal models

- Transition from regimes III to IV:

critical theory related to $M(p',p)$ with

$$c = 1 - 6\frac{(p-p')^2}{pp'}$$

unitary case: $p' = p - 1$

Minimal models: states vs paths

- Local state probabibliities: use CTM:

$$P_a \propto \text{1D configuration sum}$$

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- Focus here: display a weight preserving bijection between certain Dick paths (RSOS) to new Motzkin-type paths (generalized Bressoud)

Defining RSOS paths

and

relating paths to states

RSOS(p',p) paths (regime-III)

Configurations

- Configuration = sequence of values of the $\ell_i \in \{1, 2, \cdots, p-1\}$ $(0 \leq i \leq L)$
- with $|\ell_i - \ell_{i+1}| = 1$
- and the boundary conditions: ℓ_0, ℓ_{L-1} and ℓ_L fixed

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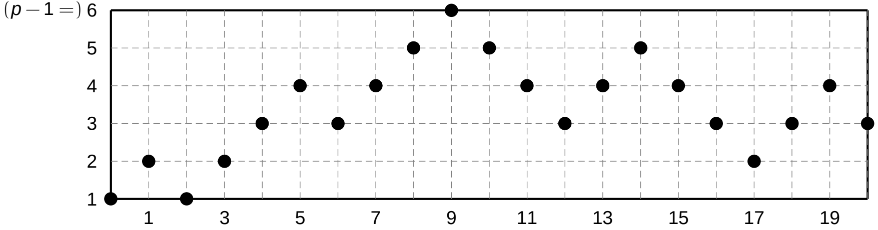
Paths

- A path is the contour of a configuration.
- Path = sequence of NE or SE edges
- choice $\ell_{L-1} = \ell_L + 1$: fixed last edge: SE

A typical RSOS$(p',7)$ configuration: $\ell_0 = 1$, $\ell_{19} = 4, \ell_{20} = 3$



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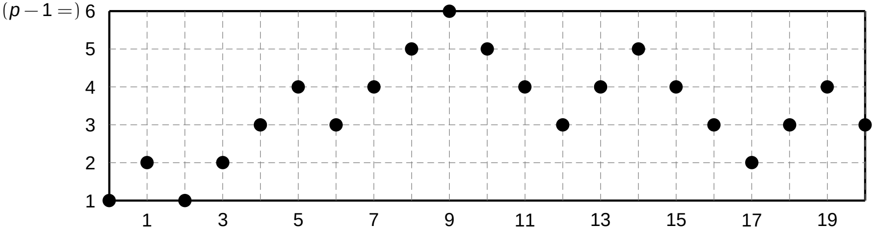
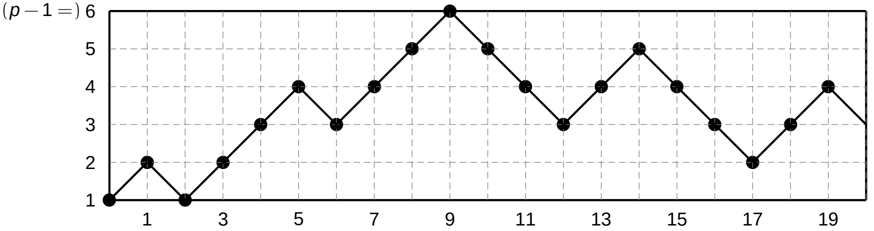
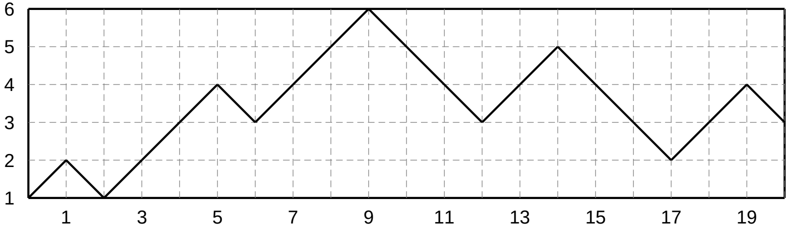


and the corresponding path (with $\ell_{20} = 3$)



A typical RSOS$(p', 7)$ path : $\ell_0 = 1$ and $\ell_{20} = 3$ and final SE



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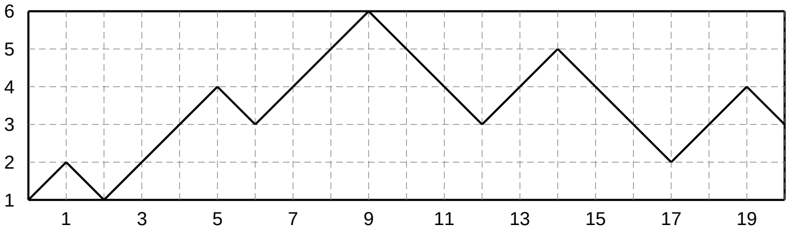
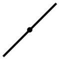
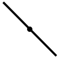


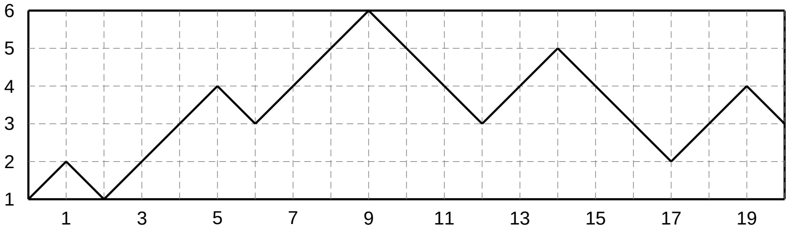
- But this corresponds to a state for which model ? (value of p'?)
- ...and to which module (r,s)?
- ...and what is its conformal dimension?

Weighting the path

The dependence of the path upon the parameter p' is via the weight:

$$\tilde{w} = \sum_{i=1}^{L-1} \tilde{w}_i$$

	Vertex		$\tilde{w}_i$		Vertex		$\tilde{w}_i$
h		i	$\frac{i}{2}$	$h+1$		i	$-i \left\lfloor h \frac{(p-p')}{p} \right\rfloor$
h		i	$\frac{i}{2}$	$h-1$		i	$i \left\lfloor h \frac{(p-p')}{p} \right\rfloor$



The expressions of $\tilde{w}_i / i$ for the extrema

	$p' = 2$		$p' = 3$		$p' = 6$	
h	max	min	max	min	max	min
6	-3	–	-2	–	0	–
5	-2	4	-2	3	0	0
4	-2	3	-1	2	0	0
3	-1	2	-1	2	0	0
2	0	2	0	1	0	0
1	–	1	–	1	–	0

The weight function is not positive

Weight vs conformal dimension

- Classes of paths are specified by ℓ_0 and ℓ_L
- Ground-state path = unique path with minimal weight, given ℓ_0, ℓ_L
- This path represents a highest-weight state
- Let its weight be $\tilde{w}_{\text{gs}}$
- The relative weight
$$\Delta\tilde{w} = \tilde{w} - \tilde{w}_{\text{gs}}$$
is the (relative) conformal dimension (function of p')

Generating functions for paths

- The GF is the q-enumeration of the paths

$$X_{\ell_0,\ell_L}^{(p',p)}(q) = \sum_{\substack{\text{paths with} \\ \ell_0 \text{ and } \ell_L \text{ fixed}}} q^{\Delta \tilde{w}}$$

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- For $L \to \infty$: when is this a character of $M(p',p)$?

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- For $L \to \infty$: when is this a character of $M(p',p)$?

  Need to restrict ℓ_L:

  the tail of the path must lie in one of the RSOS vaccua

A new weight function for the paths

[Foda-Lee-Pugai-Welsh]

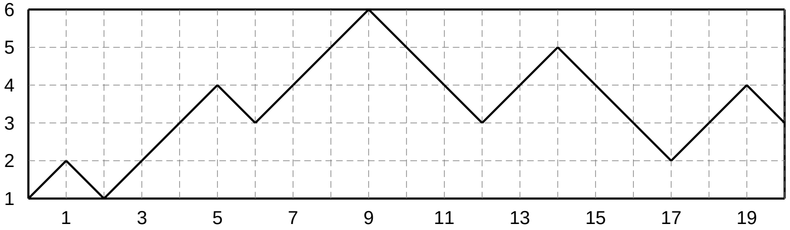
- Make the defining rectangle looks p'-dependent
- Color the $p'-1$ strips between the heights h and $h+1$ for which:

$$\left\lfloor \frac{hp'}{p} \right\rfloor = \left\lfloor \frac{(h+1)p'}{p} \right\rfloor - 1.$$

- Solutions:

$$h = h_t \equiv \left\lfloor \frac{tp}{p'} \right\rfloor \qquad \text{for} \qquad 1 \leq t \leq p'-1.$$

Our RSOS$(p', 7)$ path



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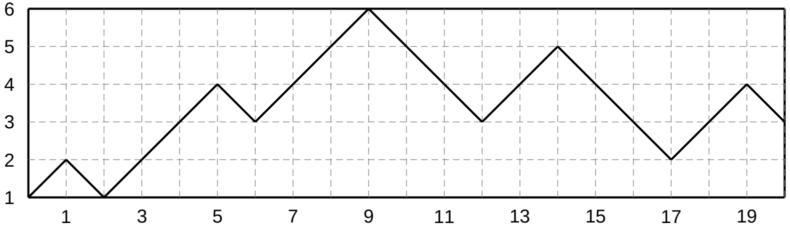
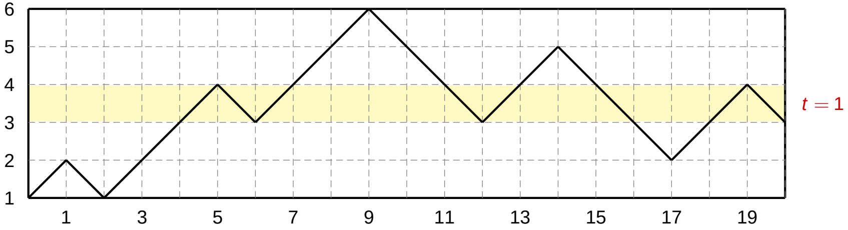
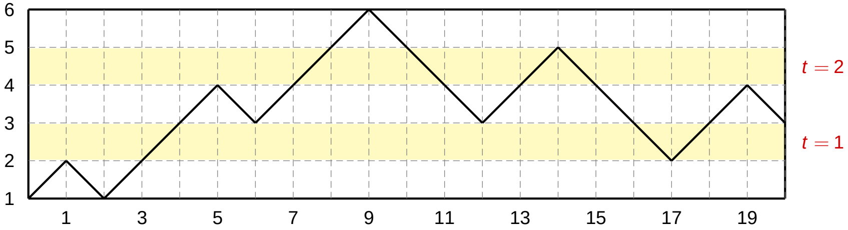


The same path for the RSOS$(2,7)$ model.



The same path for the RSOS$(3,7)$ model.



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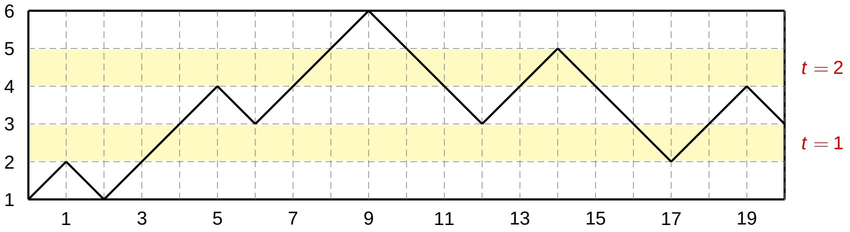
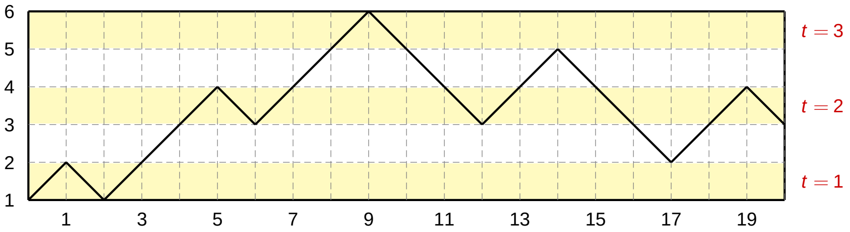
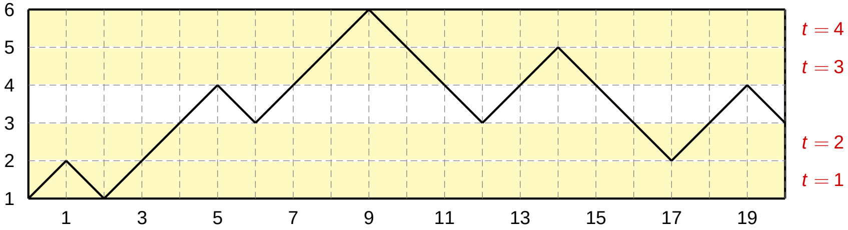


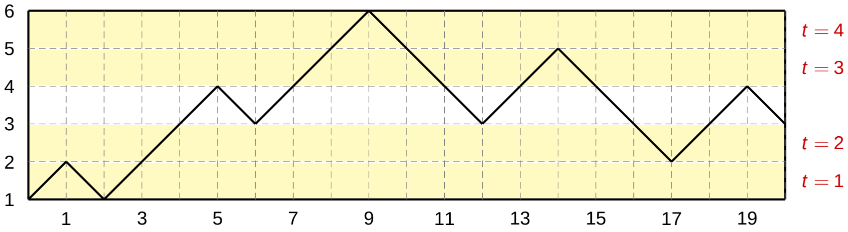
The same path for the RSOS$(4,7)$ model.



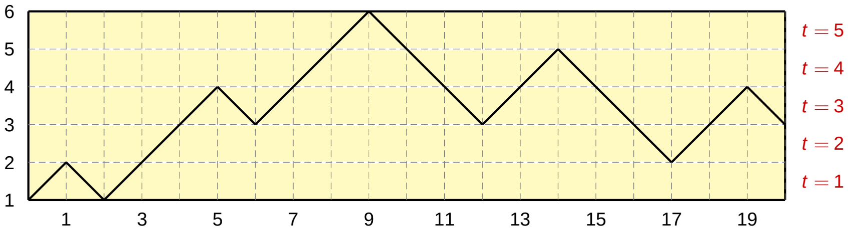
The same path for the RSOS$(5,7)$ model.



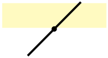
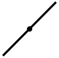
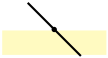
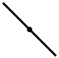
The same path for the RSOS$(5,7)$ model.



The same path for the RSOS$(6,7)$ model.



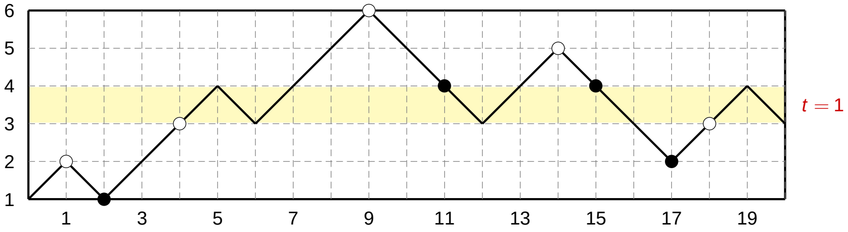
Scoring vertices

Vertex	Weight	Vertex	Weight
	0		u_i
	0		v_i
	u_i		0
	v_i		0

$$u_i = \frac{1}{2}(i - \ell_i + \ell_0)\ , \qquad v_i = \frac{1}{2}(i + \ell_i - \ell_0)$$

Our RSOS$(2,7)$ path with the "scoring vertices"

$$\circ \leftrightarrow \; u_i = \frac{1}{2}(i - \ell_i + \ell_0) \qquad\qquad \bullet \leftrightarrow \; v_i = \frac{1}{2}(i + \ell_i - \ell_0)$$



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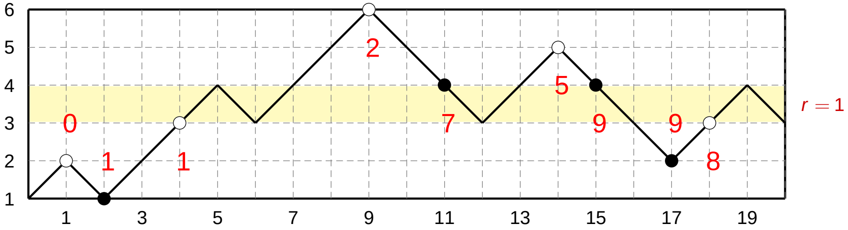
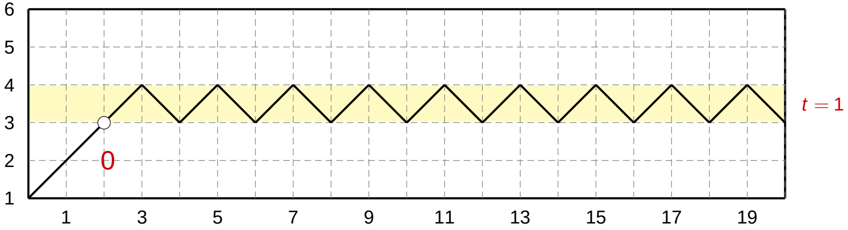


$$w = 1 + 1 + 2 + 7 + 5 + 8 + 9 + 8$$

Remark: this weighting is absolute

The ground-state path for the case $\ell_0 = 1$ and $\ell_L = 3$



The weight is absolute:

$$w_{\text{gs}} = 0 \qquad \Rightarrow \qquad w - w_{\text{gs}} = w$$

A constraint on ℓ_L

- Tails in colored bands have weight $w = 0$

  Or: colored bands correspond to the RSOS vacua

- Such tails are the proper ends for infinite paths

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- Previous question: When is

$$X^{(p',p)}_{\ell_0,\ell_L}(q) = \sum_{\substack{\text{paths with} \\ \ell_0 \text{ and } \ell_L \text{ fixed}}} q^{\Delta \tilde{w}}$$

  a character of $M(p',p)$ for $L \to \infty$?

  Answer: When

$$\ell_L = \left\lfloor \frac{tp}{p'} \right\rfloor \qquad \text{with} \qquad 1 \le t \le p' - 1$$

Module identification vs boundaries

- $$\ell_L = \left\lfloor \frac{tp}{p'} \right\rfloor \qquad \text{with} \qquad 1 \le t \le p' - 1$$

- There is no constraints on ℓ_0

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- How can we relate the Kac labels r, s where

$$1 \leq s \leq p-1 \qquad 1 \leq r \leq p'-1$$

to ℓ_0 and t?

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to ℓ_0 and t?

- Comparing the ranges suggests

$$s = \ell_0 \qquad \text{and} \qquad r = t$$

A bit of Virasoro representation theory

$M(p',p)$ irreducible modules:

- Highest-weight states of conformal dimensions

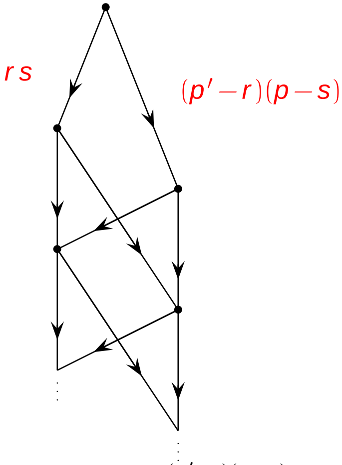
$$h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'} = h_{p'-r,p-s}$$

$$1 \le r \le p' - 1 \quad \text{and} \quad 1 \le s \le p - 1$$

- Highest-weight modules are completely degenerate

Embedding pattern of singular vectors

$$(r,s) \sim (p'-r, p-s)$$



$$\chi_{r,s}^{(p',p)}(q) = \frac{1}{(q)_\infty} - \frac{q^{rs}}{(q)_\infty} - \frac{q^{(p'-r)(p-s)}}{(q)_\infty} + \frac{q^{rs+(p'+r)(p-s)}}{(q)_\infty} + \cdots$$

Paths vs states

- Paths are blind to $h_{r,s}$:

$$w = h - h_{r,s}$$

with r, s fixed by ℓ_0 and ℓ_L (but yet to be fixed)

$\Rightarrow$ w cannot fix r, s

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RSOS= restriction of SOS

Restriction of the space of states: captured by the defining strip

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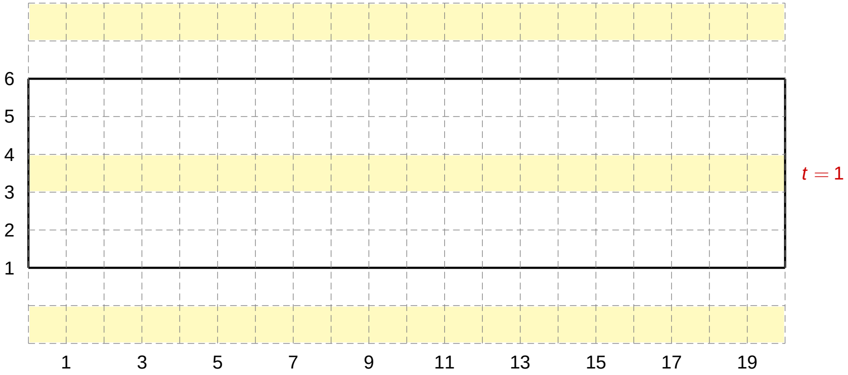
Restriction of the space of states: captured by the defining strip

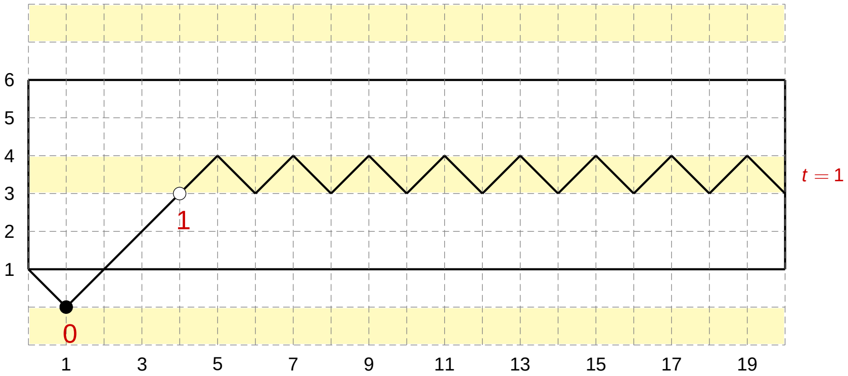
- Release the restrictions and identify the first two removed paths: candidates for the primitive SV

$$w_1 = rs \qquad w_2 = (p' - r)(p - s)$$

Identify singular vectors: extend the band structure



First singular vector: path below



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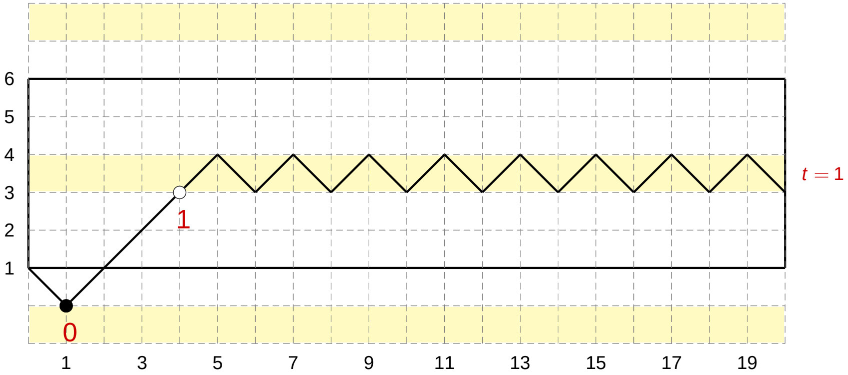
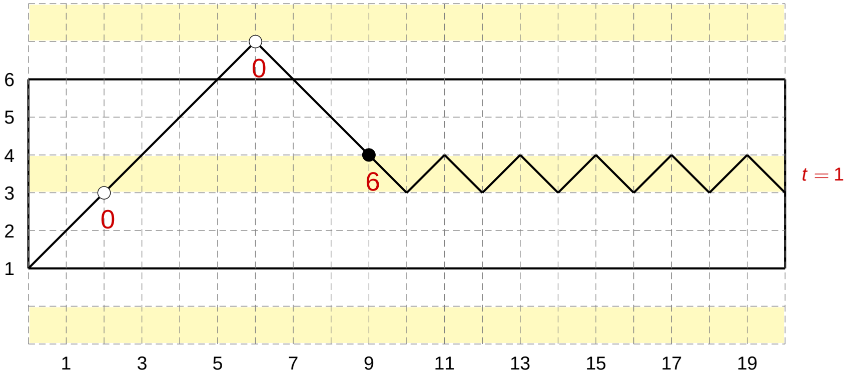


- The first excluded path from below has $w = 1$:
- Thus: the module with $\ell_0 = 1$ and $t = 1$ has a SV at level 1

Second singular vector: path above



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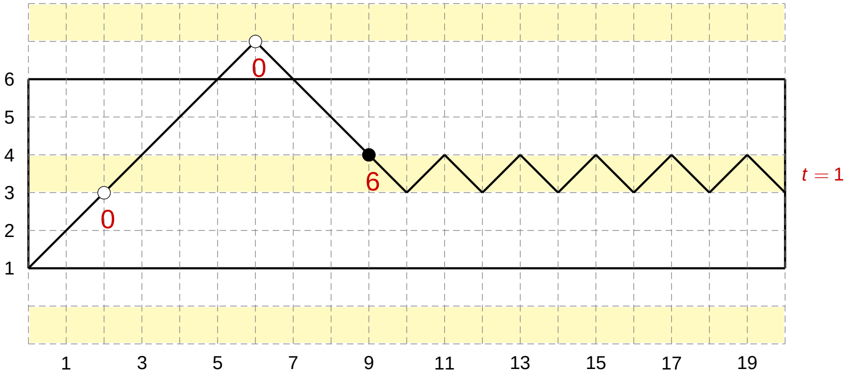

- The first excluded path from above has $w = 6$:
- Thus: the module with $\ell_0 = 1$ and $t = 1$ has a SV at level 6

Module identification vs boundaries

- In our example

$$sr = 1$$
$$(p' - r)(p - s) = (2 - r)(7 - s) = 6 \qquad \Rightarrow \qquad s = r = 1$$

Module identification vs boundaries

- In our example

$$sr = 1 \qquad\qquad \Rightarrow \qquad s = r = 1$$
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- More generally: SV analysis supports the identification

$$s = \ell_0 \qquad \text{and} \qquad r = t$$

Module identification vs boundaries

- In our example

$$\begin{aligned} sr &= 1 \\ (p'-r)(p-s) &= (2-r)(7-s) = 6 \end{aligned} \qquad \Rightarrow \qquad s = r = 1$$

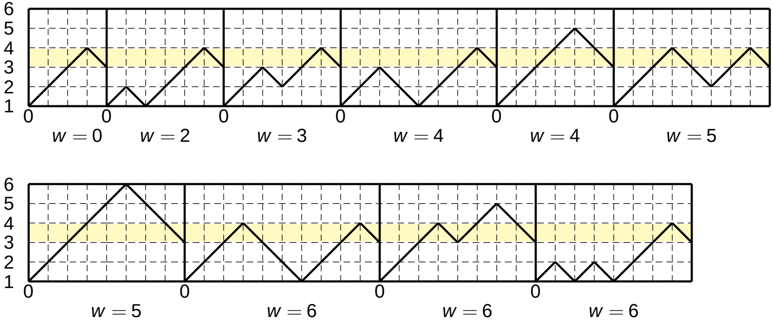
- More generally: SV analysis supports the identification

$$s = \ell_0 \qquad \text{and} \qquad r = t$$

- The Virasoro character is

$$\chi_{r,s}^{(p',p)}(q) = \lim_{L\to\infty} X_{s,\left\lfloor \frac{rp}{p'} \right\rfloor}^{(p',p)}(q)$$

The first few sates in the $M(2,7)$ vacuum module



These correspond to the first few terms in the character

$$\chi_{1,1}^{(2,7)}(q) = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + \cdots$$

RSOS paths, Partitions

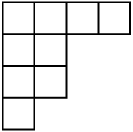
and

Bressoud paths

Partitions: hook differences

- To a partition $(\lambda_1, \lambda_2, \cdots)$, i.e., $\lambda_i \geq \lambda_{i+1}$
- corresponds a Young diagram, with λ_i boxes in the *i*-th row

$(4, 2, 2, 1):$



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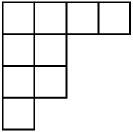


- For the box (i,j), the hook difference $H(i,j)$ is

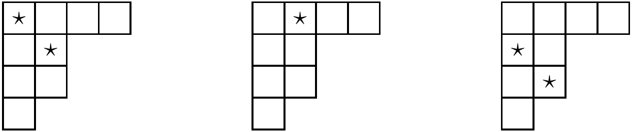
$$H(i,j) = \#\text{boxes in row } i - \#\text{boxes in column } j$$

0	1	3	3
$\bar{2}$	$\bar{1}$		
$\bar{2}$	$\bar{1}$		
$\bar{3}$			

$(\bar{a} \equiv -a)$

Partitions: diagonals

- Diagonal d: the set of boxes $(i, i-d)$.



$d = 0$ $d = 1$ $d = -1$

Partitions with prescribed hook differences (PHD)

[Andrews-Baxter-Bressoud-Burge-Forrester-Viennot]

Introduce 4 numbers

$$p, p', r, s$$

such that

$$1 \leq r \leq p' - 1 \quad \text{and} \quad 1 \leq s \leq p - 1 \quad \text{and} \quad p > p' \geq 2$$

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On the two diagonals

$$p' - r - 1 \qquad \text{and} \qquad 1 - r$$

impose the PHD

$$\begin{aligned} H(i, i - (p' - r - 1)) &\leq p - p' - s + r - 1 \\ H(i, i - (1 - r)) &\geq -s + r + 1 \end{aligned}$$

- Let
$$\mathsf{P}_{p,s}(p'-r,r;n) = \#\text{ of partitions of } n \text{ with PHD}$$

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$$\chi_{r,s}^{(p',p)}(q) = \sum_{n\geq 0} \mathsf{P}_{p,s}(p'-r,r;n)\, q^n.$$
- Or

RSOS paths $\leftrightarrow$ Partitions PHD

Partitions with prescribed successive ranks

- Special case where

$$p' = 2 \qquad \text{and} \qquad p = 2k + 1$$

so that (recall $1 \leq r \leq p' - 1$)

$$r = 1 \qquad \Rightarrow \qquad r - 1 = p' - r - 1 = 0$$

Partitions with prescribed successive ranks

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- The PHD reduce to

$$-s+2 \leq H(i,i) \leq 2k-1-s$$

- $H(i,i)$: successive ranks [Dyson, Andrews]

Restricted partitions

- Partitions with

$$-s+2 \leq H(i,i) \leq 2k-1-s$$

are in 1-1 correspondence with

- Restricted partitions: $(\lambda_1, \lambda_2, \cdots)$ s.t.

$$\lambda_i - \lambda_{i+k-1} \geq 2$$

and containing at most s parts equal to 1

$k = 2$: combinatorics of the sum-side of the RR identities

are in 1-1 correspondence with

Bressoud paths [Burge]

Integer lattice paths

- defined in the strip:

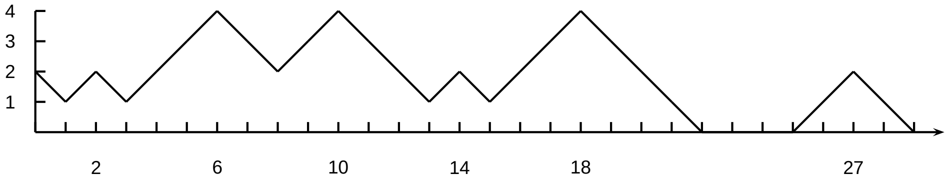
$$0 \leq x \leq \infty, \qquad 0 \leq y \leq k-1$$

  with initial point $(0, k-s)$

- composed of NE, SE and Horizontal edges (H iff $y = 0$)
- weight = x-position of the peaks

A Bressoud path for $k = 5$ and $s = 3$

$$0 \leq y \leq k - 1 = 4, \qquad y_0 = k - s = 2$$



Weight

$$w = 2 + 6 + 10 + 14 + 18 + 27$$

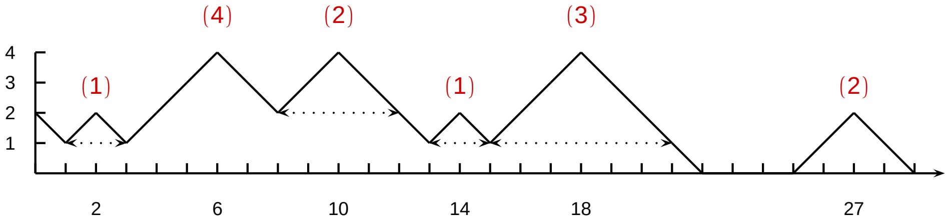
A Bressoud path : sequence of charged peaks

Isolated peak:

Charge = height

In a charge complex:

Charge = relative height



The charge ($\equiv$ particle) content of the path is:

$$m_1 = 2, m_2 = 2, m_3 = 1, m_4 = 1$$

$\{$Bressoud paths$\}$

as a fermi gas

Bressoud paths : generating function [Warnaar]

- For a fixed charge content (fixed $\{m_j\}$): determine the configuration of minimal weight (mwc)

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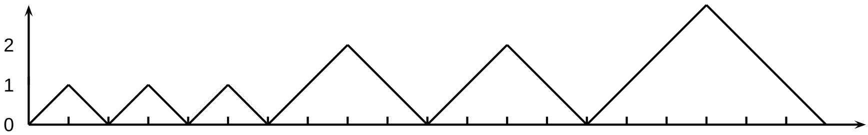

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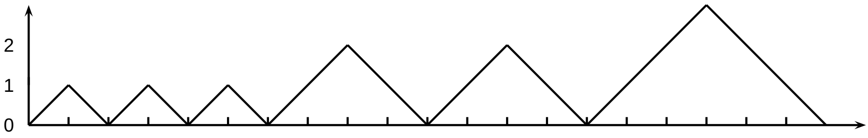

- Evaluate its weight: above $w_{\text{mwc}} = 1 + 3 + 5 + 8 + 12 + 17$

Bressoud paths : generating function [Warnaar]

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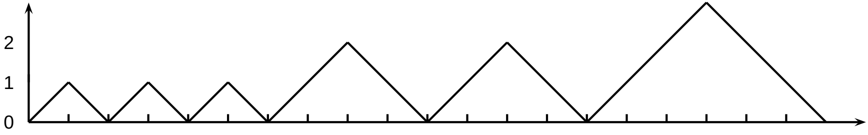
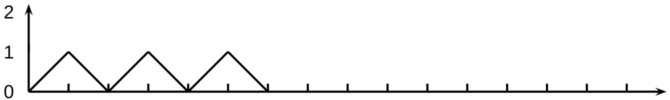


- Evaluate its weight: above $w_{\text{mwc}} = 1 + 3 + 5 + 8 + 12 + 17$

  In general

$$w_{\text{mwc}} = \sum_{i,j=1}^{k-1} \min(i,j)\, m_i\, m_j$$

- Move the particles (peaks) in all possible ways and q-count them
  Ex: consider $m_1 = 3$



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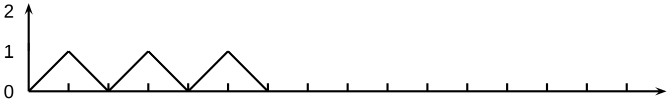
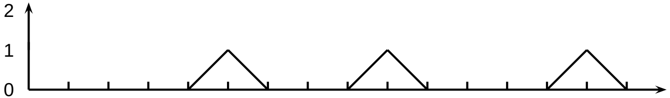


- Rule 1: Identical particles are impenetrable (hard-core repulsion):
  Ex: move the rightmost by 9, the next by 6 and the third by 4



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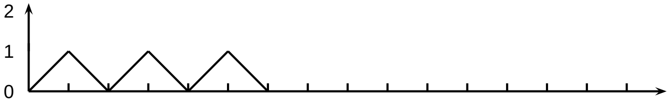

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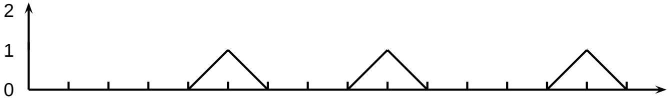
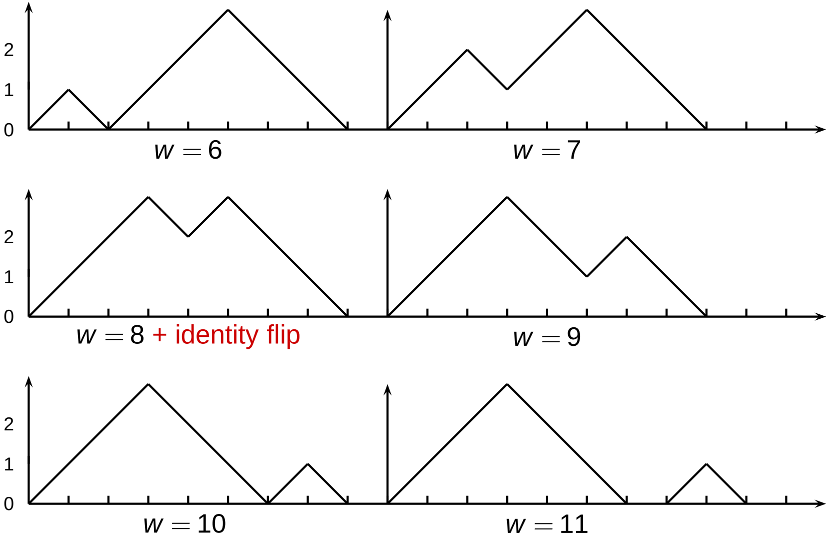


- Generating factor for these moves
  = the number of partitions with at most three parts:

$$\frac{1}{(1-q)(1-q^2)(1-q^3)} \equiv \frac{1}{(q)_3} \qquad \rightarrow \qquad \frac{1}{(q)_{m_1}}$$

- Rule 2: Particles of different charges can penetrate
  Consider the successive displacements of the peak 1 in 3:



- Every move of 1 unit increases the weight by 1 independently of the presence of higher charged particles

$$\text{i.e.} \quad \frac{1}{(q)_{m_1}} \quad \text{is generic}$$

- The same holds for the other particles:

$$\text{factor} \quad \frac{1}{(q)_{m_j}} \quad \text{for each type } 1 \leq j \leq k-1$$

- Generating functions for all paths with fixed charge content

$$G(\{m_j\}) = \frac{q^{w_{\text{mwc}}}}{(q)_{m_1} \ldots (q)_{m_{k-1}}}$$

with

$$w_{\text{mwc}} = \sum_{i,j=1}^{k-1} \min(i,j)\, m_i\, m_j$$

- Full generating function:

$$G = \sum_{m_1,\cdots,m_{k-1}} G(\{m_j\}) = \sum_{m_1,\cdots,m_{k-1}=0}^{\infty} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_1+\cdots+N_{k-1}}}{(q)_{m_1}\cdots(q)_{m_{k-1}}}$$

with N_j defined as

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- This is the fermionic character of the $M(2,2k+1)$ vacuum module (FNO)

- Bressoud paths have a clear particle interpretation

Particles in RSOS paths

RSOS$(2, 2k+1)$ vs Bressoud paths

- RSOS$(2, 2k+1)$ paths $\leftrightarrow$ Partitions PSR $\leftrightarrow$ Bressoud paths

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  Search for a direct bijection:

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- RSOS$(2, 2k+1)$ paths $\leftrightarrow$ Partitions PSR $\leftrightarrow$ Bressoud paths

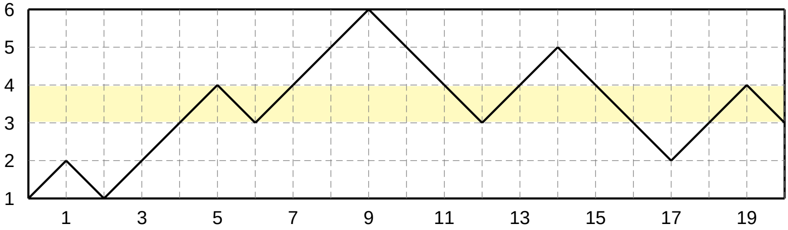
  Search for a direct bijection:

- RSOS$(2, 2k+1)$ paths $\leftrightarrow$ Bressoud paths

- Objective: identify particles in (generic) RSOS paths

Particles in RSOS$(2,p)$ paths?

E.g. in the RSOS$(2,7)$ path



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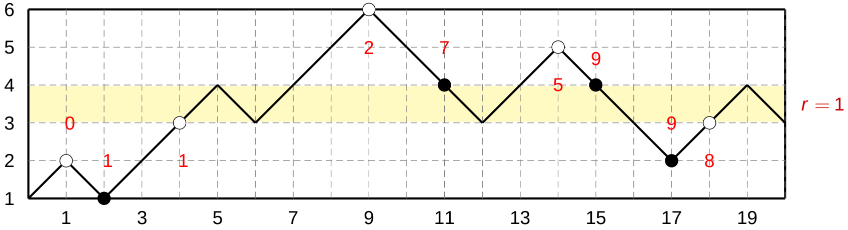

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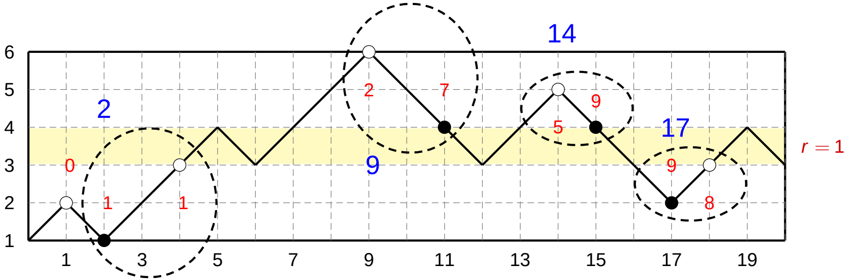

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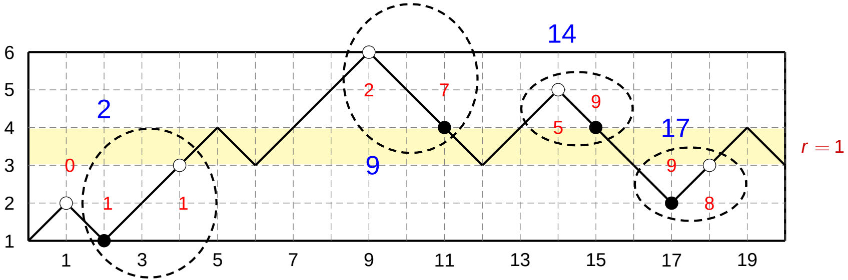
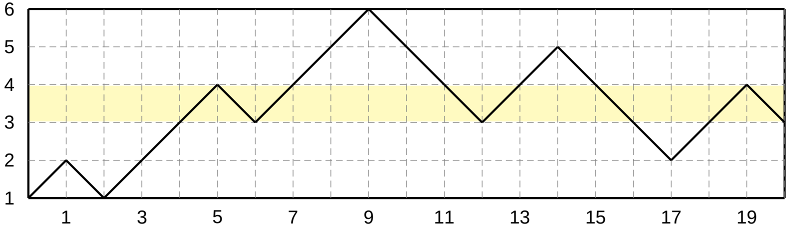


Observations:

- Peak above the yellow band: pair ○● with weight = position of ○
- Valley below the yellow band: pair ●○ with weight = position of ●

Transformation of the RSOS$(2, p)$ paths

These observations suggest to transform the RSOS$(2, 7)$ path



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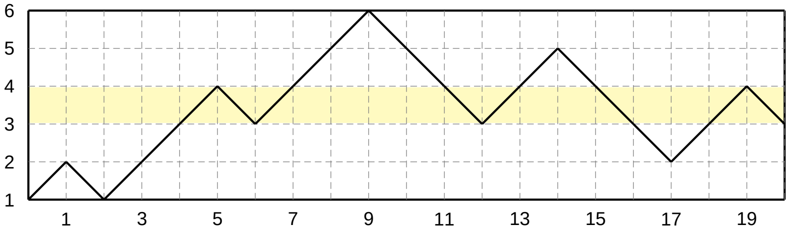
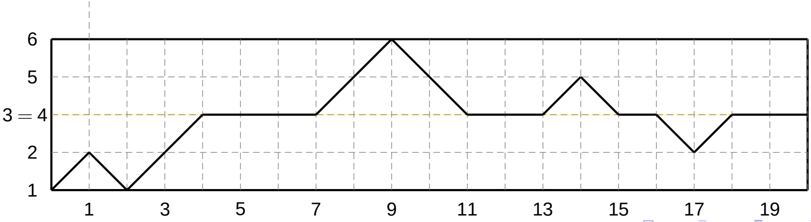
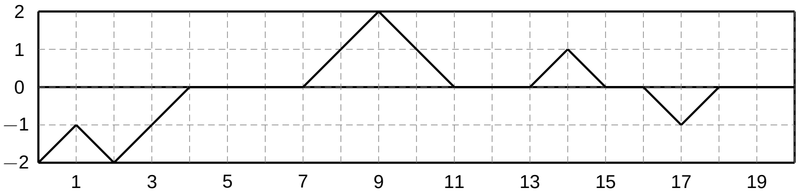


by flattening the colored band



redefine the vertical axis



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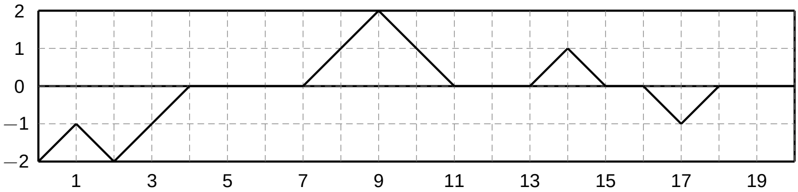
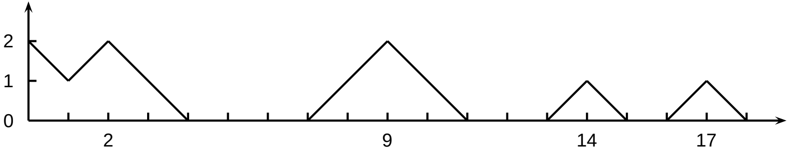


and fold the lower part onto the upper one



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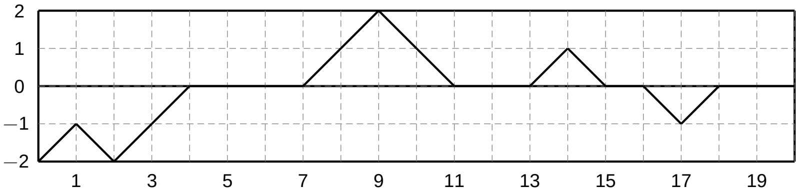

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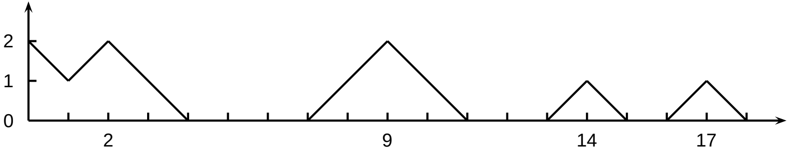
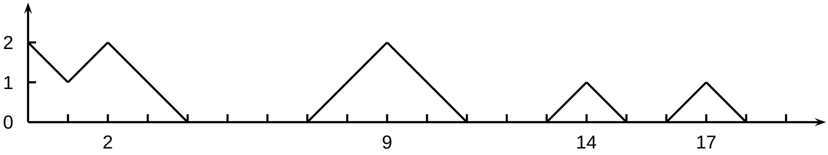


the result is a Bressoud path: weight = x position of the peaks:

$$w = 2 + 9 + 14 + 17$$

Is this 1-1?



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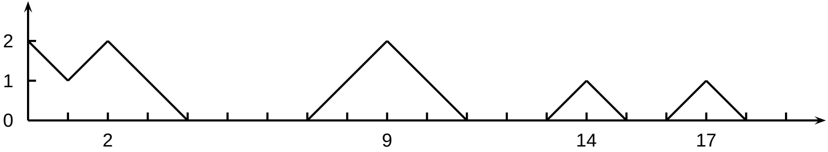
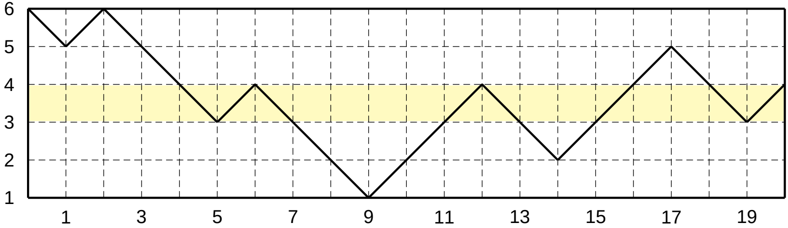


is also related to



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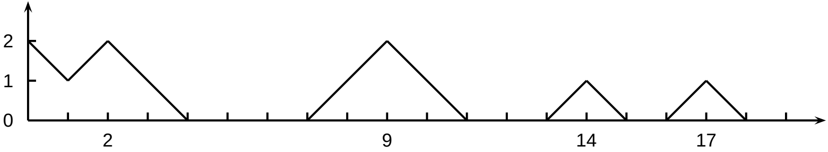

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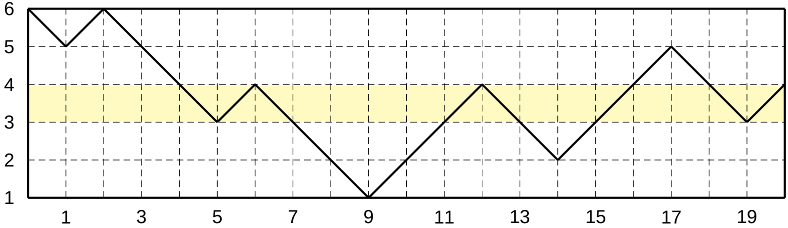


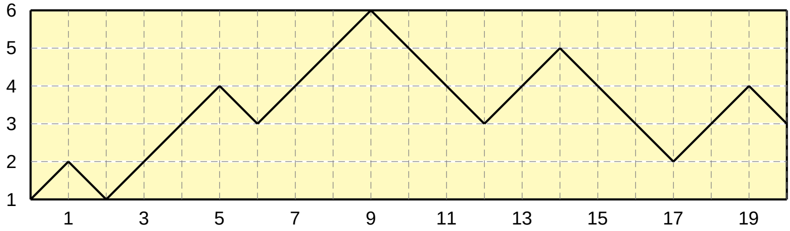
But this has a final NE edge: enforcing a final SE: 1-1 relation

From RSOS(p',p) to generalized Bressoud paths

- Flatten all colored bands

From RSOS(p',p) to generalized Bressoud paths

- Flatten all colored bands
- But restrictions are required: e.g., RSOS$(6,7)$:



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- Restriction to $p \geq 2p'-1$: isolated colored bands

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- Result: generalized Bressoud paths defined in

$$0 \leq y \leq p - p' - \left\lfloor \frac{p}{p'} \right\rfloor$$

- ...with H edges allowed at height

$$y(t) = \left\lfloor \frac{tp}{p'} \right\rfloor - \left\lfloor \frac{p}{p'} \right\rfloor - t + 1 \qquad (1 \leq t \leq p'-1)$$

  (with a condition relating the parity of successive H edges and the change of direction of the path)

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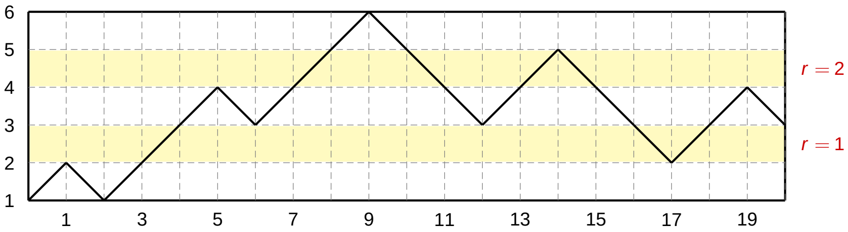
- ...and

$$w = \text{(half) } x \text{ position of the (half) peaks}$$

Our RSOS$(3,7)$ path



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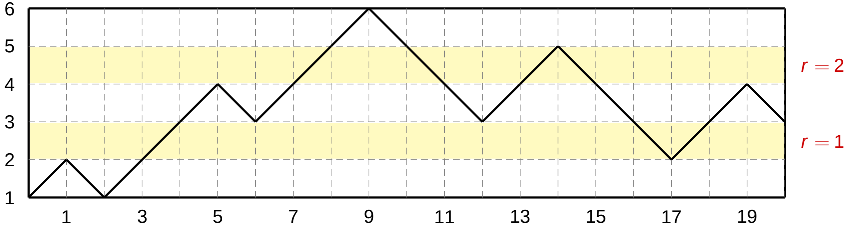
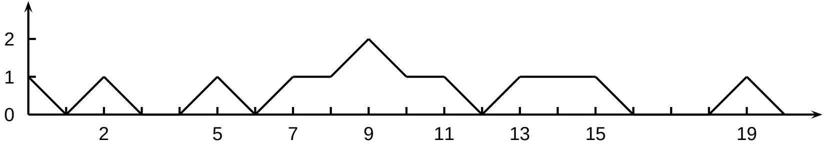
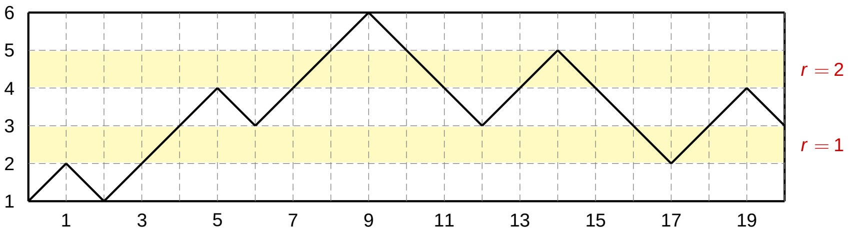


is transformed into



with H edges allowed at $y = 0, 1$ but not $y = 2$

Our RSOS$(3,7)$ path



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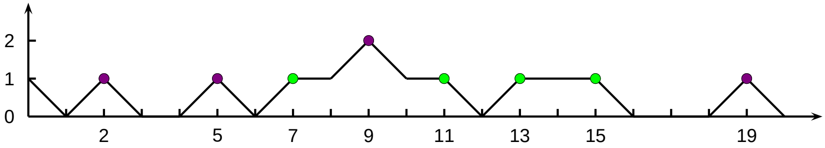
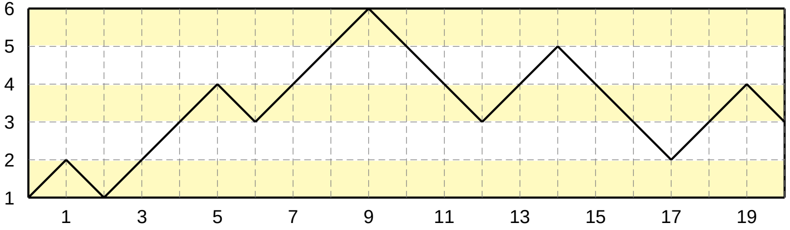
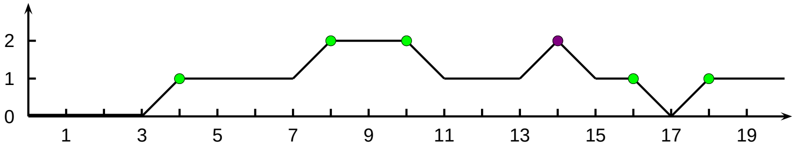


$$w = 2 + 5 + 9 + 19 + \frac{1}{2}(7 + 11 + 13 + 15)$$

Similary, our RSOS$(4,7)$ path



is transformed into:

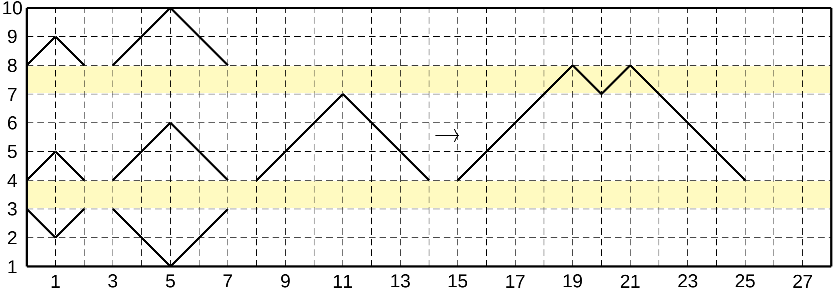


H edges at $y = 0, 1, 2$ and

$$w = 14 + \frac{1}{2}(4 + 8 + 10 + 16 + 18) - (w_{\mathrm{gs}} = 1)$$

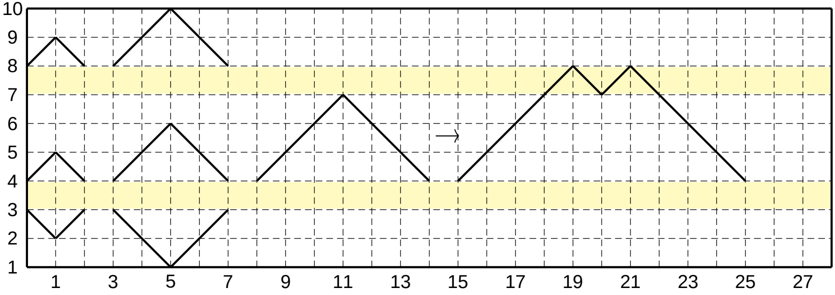
Fermi-gas analysis of the $\mathrm{B}(3,p)$ paths

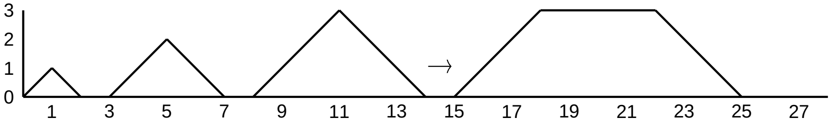
RSOS$(3,11)$ (case $p=3k+2$): 3 particles



Fermi-gas analysis of the $B(3,p)$ paths

RSOS$(3,11)$ (case $p = 3k+2$): 3 particles





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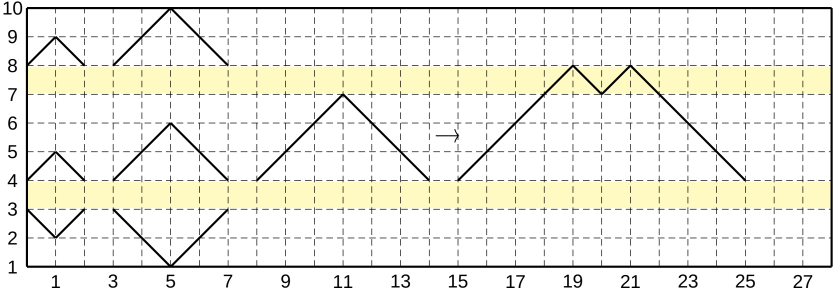

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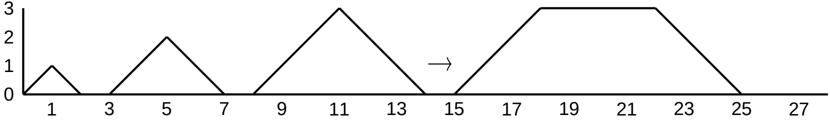
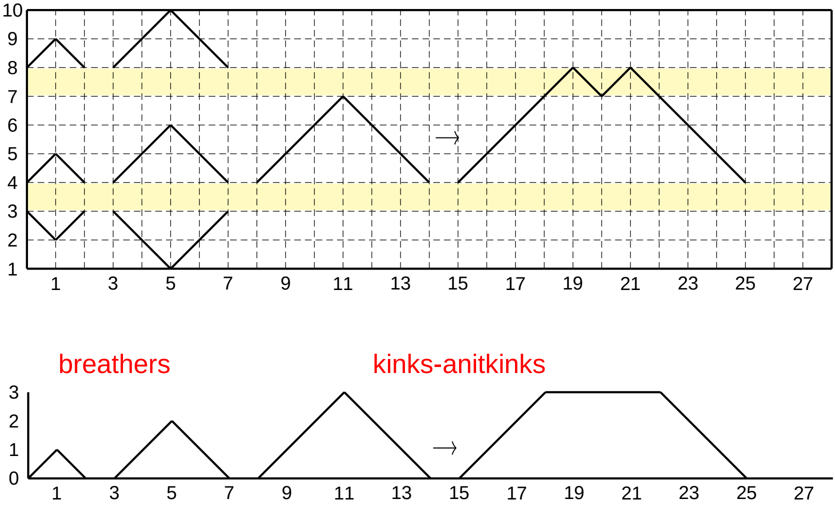


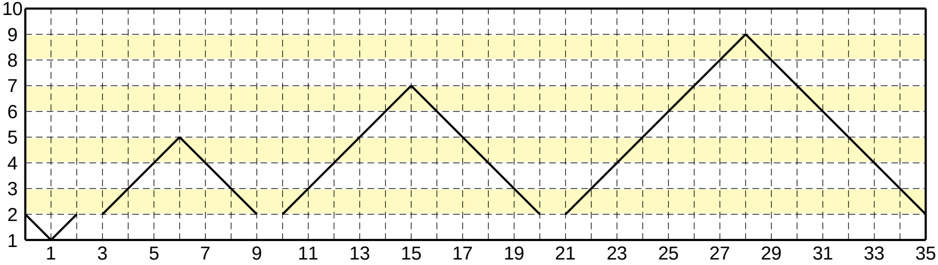
Fermi-gas analysis of the B$(3,p)$ paths

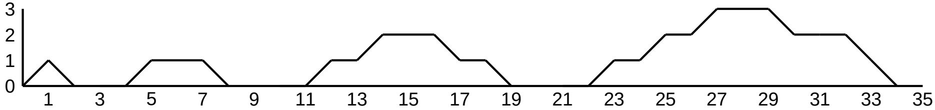
RSOS$(3,11)$ (case $p = 3k + 2$): 3 particles



Fermi-gas analysis of the $B(p', 2p'+1)$ paths

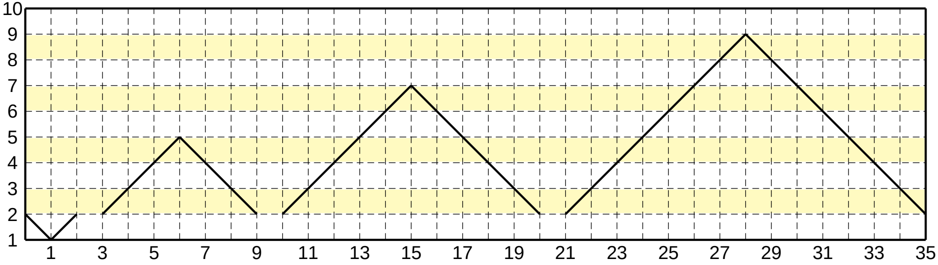
RSOS$(5, 11)$: 4 particles

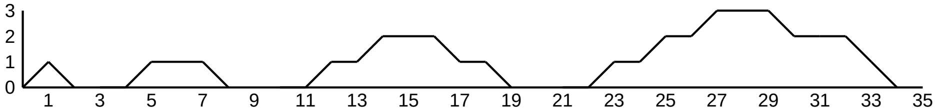




Fermi-gas analysis of the $\mathrm{B}(p', 2p'+1)$ paths

RSOS$(5, 11)$: 4 particles

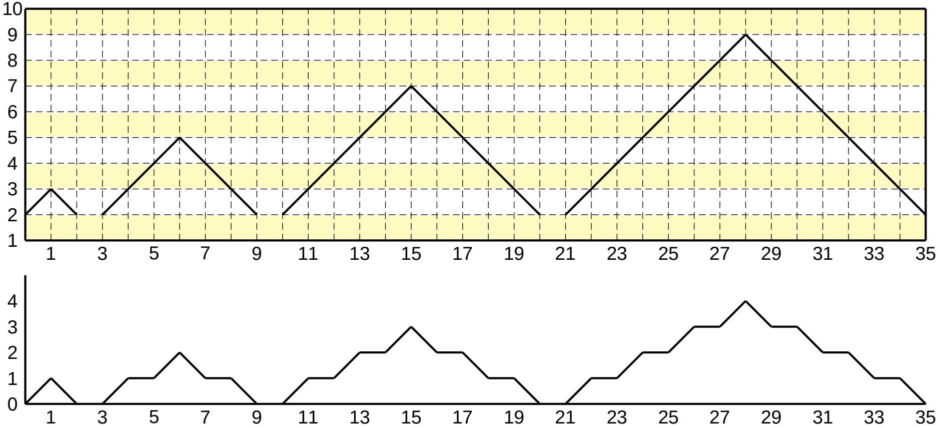




1 breather and kinks-antikinks of topological charge from 1 to 3

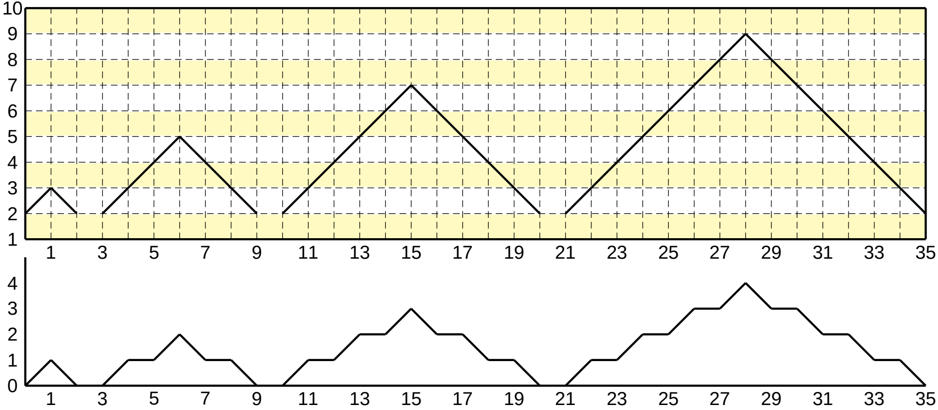
Fermi-gas analysis of the B$(p', 2p'-1)$ paths

RSOS$(6,11)$: 4 particles



Fermi-gas analysis of the B$(p', 2p'-1)$ paths

RSOS$(6,11)$: 4 particles



kinks-antikinks of topological charge from 1 to 4

no breathers

Particle content of RSOS paths

- Numbers of kinks = number of vacua -1

  kinks interpolate between yellow bands

$$\#\text{kinks} = (p' - 1) - 1$$

- Numbers of breathers = number bands below the first yellow one

$$\#\text{breathers} = \left\lfloor \frac{p}{p'} \right\rfloor - 1$$

  no breathers if $p < 2p'$

- Match the spectrum of the restricted sine-Gordon model with

$$\frac{\beta^2}{8\pi} = \frac{p'}{p}$$

A duality relation

- The finitized (polynomial e.g., $L < \infty$) form of the character allows for a duality relation

$$q \to \frac{1}{q}$$

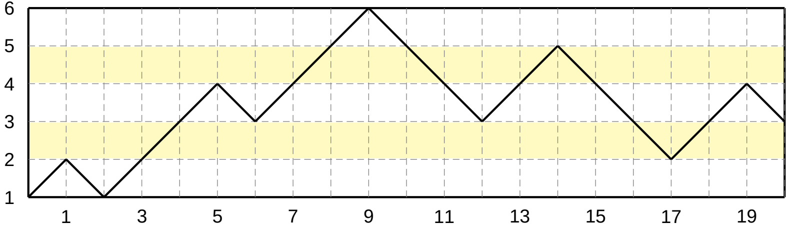
- Under this transformation

$$M(p',p) \to M(p-p',p)$$

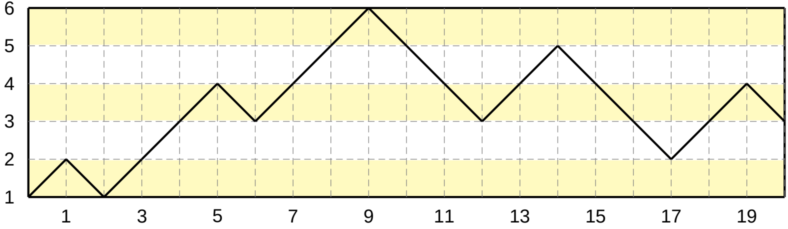
- Bands under duality: colored $\leftrightarrow$ white

Duality $M(p',p) \to M(p-p',p)$ in color

Compare RSOS$(3,7)$



vs RSOS$(4,7)$



Conclusion

- The transformation of RSOS(p',p) to B(p',p) paths is a key step for a direct fermi-gas analysis; it makes the particle interpretation transparent
- The particle interpretation match that of RSG which is a $\phi_{1,3}$-perturbation of $M(p',p)$ (= scaling limit of RSOS(p',p) in regiime III)
- More to be extracted from this?
- Can this be lifted to a CFT interpretation?

$M(k+2,2k+3)$ fermionic character

From the direct Fermi-gas analysis (k particles, no breathers)

$$\chi_{1,1}^{(k+2,2k+3)}(q) = \sum_{m_1,\cdots,m_k} \frac{q^{mBm+Cm}}{(q)_{p_0}} \prod_{i=1}^{k-1} \begin{bmatrix} m_i + p_j \\ m_j \end{bmatrix},$$

where

$$B_{i,j} = B_{j,i} \quad B_{i,j} = (2i-1)j \quad \text{if} \quad i \leq j \quad \text{and} \quad C_j = j$$

and

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \begin{cases} \frac{(q)_a}{(q)_{a-b}(q)_b} & \text{if} \quad 0 \leq b \leq a, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_j = 2m_{j+2} + 4m_{j+2} + \cdots + 2(k-j+1)m_k$$

so that

$$p_0 = \text{ number of half peaks}$$