Statistical Mechanical models + Cluster Algebras

Note Title 10/7/2009

R. KEDEM, Univ. Illinois

(1) T-systems and Q-systems for sl_n XXX

(2) Solutions in terms of path models

(3) Positivity conjectures in Cluster Algebras and non-commutative generalizations

Joint with P. Di Francesco
(see arXiv)

(1) T-systems and Q-systems in XXX-model

(V, V_i are finite-dimensional yangian-mod)

Boltzmann weights

$R_{V,W} =$ 

matrix of B.W.

Transfer matrix

$$t_V = Tr_V \left(R_{V_1, V} R_{V_2 V} \cdots \right)$$ Commuting family!

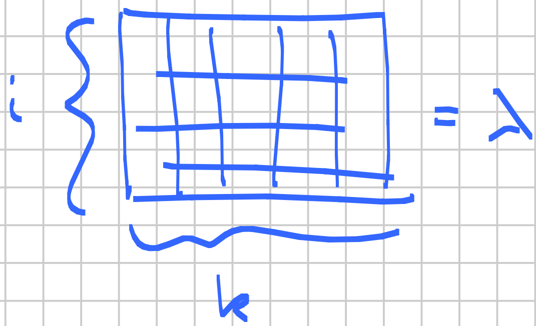
Integrable spin chain for any $\mathfrak{g}, \{V_i\}, V$.

$$\boxed{[t_V, t_{V'}] = 0}$$

Specialize - - -

$\mathfrak{g} = A_r$

$V =$ finite-dimensional evaluation module with $\mathfrak{g}$-h.w. of the form $\lambda = k\omega_i$



$V_{i,k}(j)$ = Kirillov-Resh. module

evaluation parameter

Then $\boxed{t_{i,j,k} = t_{V_{i,k}(j)}}$ the transfer matrix satisfies a discrete recursion (evolution) equation...

A_r T-system:

$$t_{i,j,k+1}\, t_{i,j,k-1} = t_{i,j+1,k}\, t_{i,j-1,k} - t_{i+1,j,k}\, t_{i-1,j,k}$$

$$t_{0,j,k} = t_{r+1,j,k} = 1$$

- a discrete evolution equation in k
- a discrete integrable system (Hirota)

associated with this: (drop j)

A_r Q-system:

$$q_{i,k+1}\, q_{i,k-1} = q_{i,k}^2 - q_{i+1,k}\, q_{i-1,k}$$

$$q_{0,k} = q_{r+1,k} = 1$$

Initial conditions:

$t_{i,j,0}$ = normalization constant

$q_{i,0} = 1$

Solutions to $\left\{\begin{matrix} T- \\ Q \end{matrix}\right\}$ system are $\left\{\begin{matrix} q- \\ \mathfrak{g} \end{matrix}\right\}$ characters of $\left\{\begin{matrix} Y(\mathfrak{g}) \\ \mathfrak{g} \end{matrix}\right\}$

Summary: We are interested in solutions of

Q

$$Q_{i,k+1} Q_{i,k-1} = Q_{i,k}^2 + Q_{i+1,k} Q_{i-1,k}$$

T

$$T_{i,j,k+1} T_{i,j,k-1} = T_{i,j+1,k} T_{i,j-1,k} + T_{i+1,j,k} T_{i-1,j,k}$$

With arbitrary initial conditions:

Example: Solutions of Q are fully specified once $\{Q_{i,0}, Q_{i,1}\}_{i=1,\dots,r}$ are given.

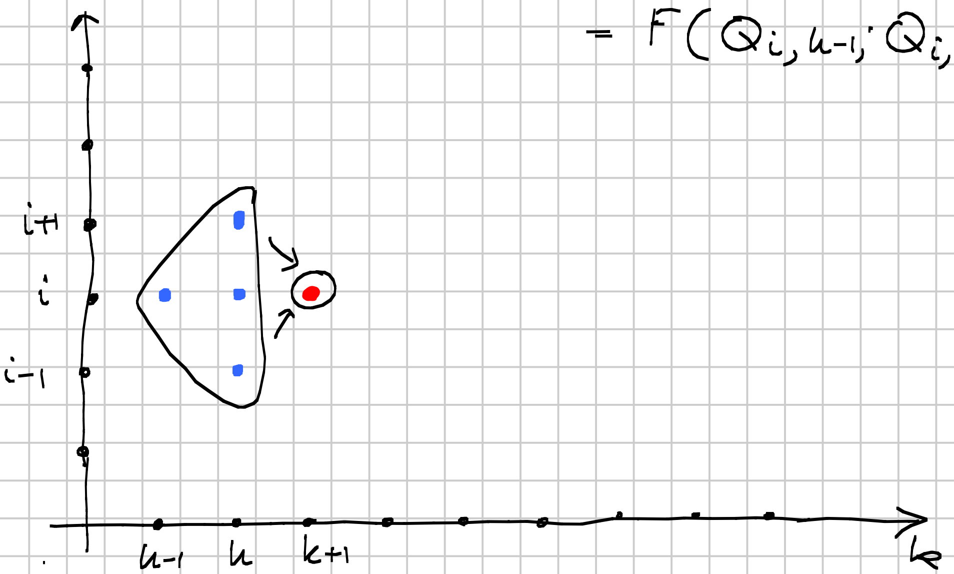
more generally we need to specify

$$\{Q_{i,m_i} ; Q_{i,m_{i+1}}\}_{i=1,\dots,r}$$

$$|m_i - m_{i+1}| \leq 1 \qquad (m = (m_1,\dots,m_r) \text{ is a Motzkin path})$$

Q-system: $\quad Q_{i,k+1} = Q_{i,k-1}^{-1}\left[Q_{i,k}^{2} + Q_{i-1,k}\, Q_{i+1,k} \right]$

$$= F(Q_{i,k-1};\, Q_{i,k};\, Q_{i\pm1,k})$$



So a valid set of variables contstituting an initial condition for the Q-system has

$$\{ Q_{i,m_i};\, Q_{i,m_i+1} \mid |m_i - m_{i+1}| \le 1 \}$$

That is, $(m_1, \dots, m_r)$ is a Motzkin path

Example: A_1

($q_{1,k}$ with $q_{1,0}=1$ are Chebyshev polynomials)

Q-system: $(Q_{1,k} \equiv Q_k)$

$$\boxed{Q_{k+1} Q_{k-1} = Q_k^2 + 1}$$

* Discrete integrable system with integral of motion

$$K = \frac{Q_{n+1}}{Q_n} + \frac{1}{Q_n Q_{n+1}} + \frac{Q_n}{Q_{n+1}}$$

is independent of n

* Q_n satisfy linear recursion relation with constant coefficients

$$\boxed{Q_{k+1} - K Q_k + Q_{k-1} = 0}$$

can be solved explicitly...

$$\sum_{n\geq 0} Q_0^{-1} Q_n t^n = \cfrac{1}{1 - t\,\cfrac{Q_1/Q_0}{1 - \cfrac{t/Q_0 Q_1}{1 - t\,Q_0/Q_1}}}$$

Note: Q_n is a positive Laurent polynomial in (Q_0, Q_1)

Fact: Q_n is a positive L.P. in (Q_k, Q_{k+1}) $\forall k, n$

(use $Q_n(Q_k, Q_{k+1}) = Q_{n-k}(Q_0, Q_1)\Big|_{\substack{Q_0 \to Q_k \\ Q_1 \to Q_{k+1}}}$)

More interesting: this is the partition function of paths on a weighted graph, with the coefficient of t^n having $2n$ steps...

Graph :

$$\left.\begin{aligned} y_1 &= Q_1/Q_0 \\ y_2 &= \frac{1}{Q_0 Q_1} \\ y_3 &= Q_0/Q_1 \end{aligned}\right\} \text{weights}$$

the p.f. of paths from 1 to 1 with $2n$ steps is $Q_0^{-1} Q_1$

Transfer matrix of this graph:

$$T = \begin{pmatrix} 0 & ty_1 & 0 & 0 \\ 1 & 0 & ty_2 & 0 \\ 0 & 1 & 0 & ty_3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$f(t) = (1-T)^{-1}_{11}$ is obtained by Gaussian elimination = the finite continued fraction which gives $\sum_{n\geq 0} Q_0^{-1} Q_n t^n$

T- system generalization

$T_{1,j,k} \equiv T_{j,k}$

$$\boxed{T_{j,k+1}\, T_{j,k-1} = T_{j+1,k}\, T_{j-1,k} + 1}$$

A_1 T-sys.

Integrals of motion and linear recursions

$$T_{j-1,k+1} - c(j-k)\, T_{j,k} + T_{j+1,k-1} = 0$$

$$T_{j+1,k+1} - d(j+k)\, T_{j,k} + T_{j-1,k-1} = 0$$

The solution of the A_1 T-system is the partition function

$$T_{j,k}\, T_{j+k,0}^{-1} = Z^{1,1}_{j-k,j+k}$$

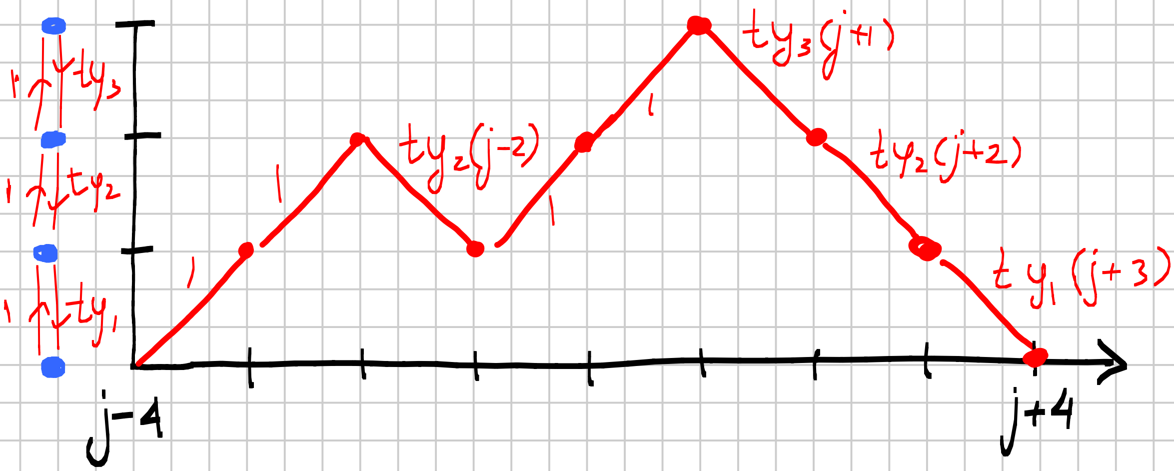
where we define $Z^{1,1}_{j-k,j+k}$ to be the p.f. of paths of length $2k$ with "time dependent" weights $y_i(t)$ starting at $t=j-k$ ending at $t=j+k$

$$y_1(j) = \frac{T_{j,1}}{T_{j+1,0}}$$

$$y_2(j) = \frac{1}{T_{j,0}\,T_{j+1,1}}$$

$$y_3(j) = \frac{T_{j+1,0}}{T_{j,1}}$$

again, $T_{j,k}$ is a positive Laurent poly. of the initial data.



$$wt = t^4\, y_2(j-2)\, y_1(j+1)\, y_2(j+2)\, y_1(j+3)$$

For A_r, we also have

Integrability, linear recursion relations, paths on graphs.

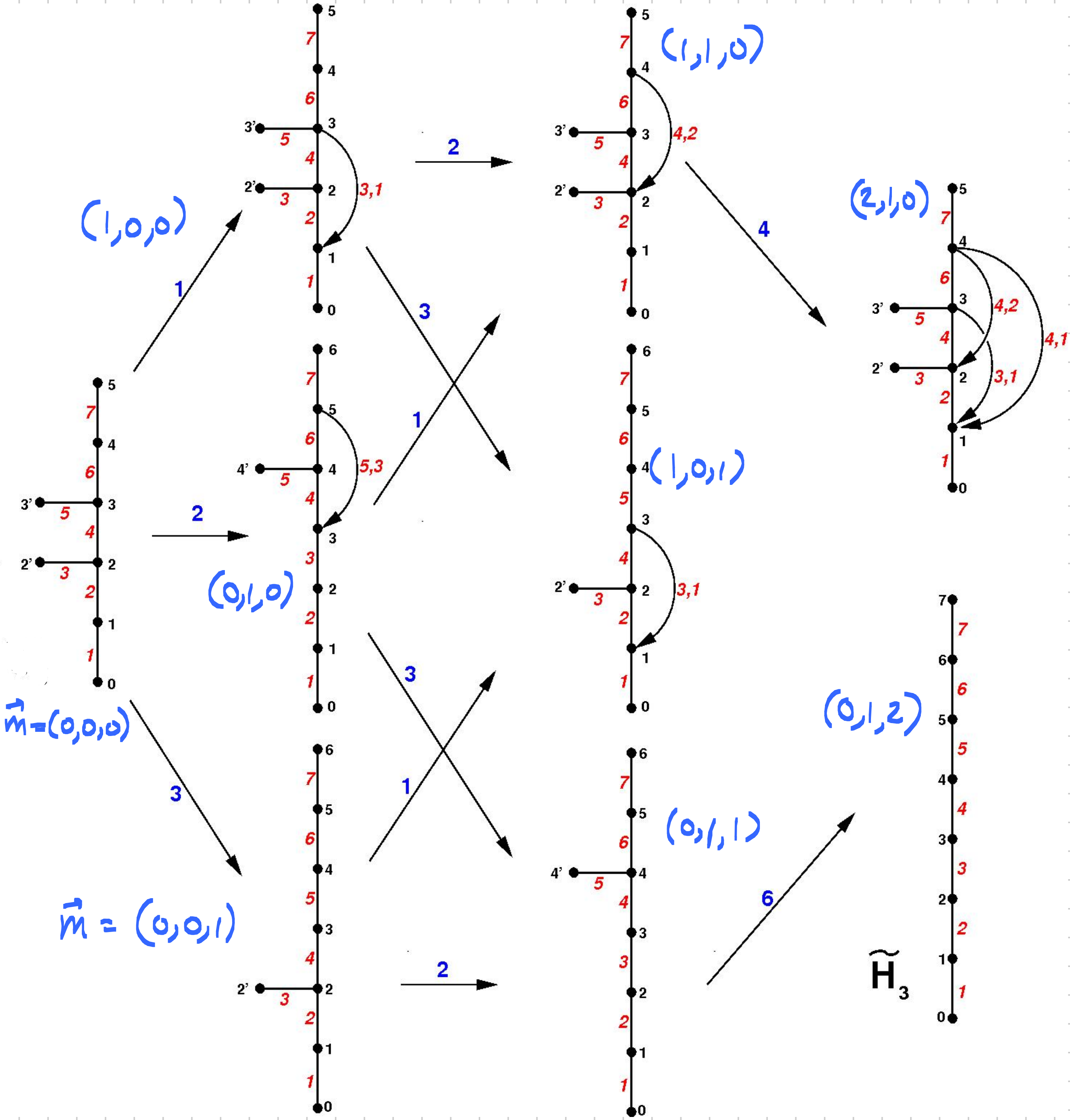
2 added ingredients:

(1) Graphs depend on choice of <u>initial conditions</u>

(2) Only $Q_{1,k}$ and $T_{1,j,k}$ are path p.f.

$Q_{\alpha,k}$ ($T_{\alpha,j,k}$) are p.f. for (strongly) non-intersecting paths on the same graphs.

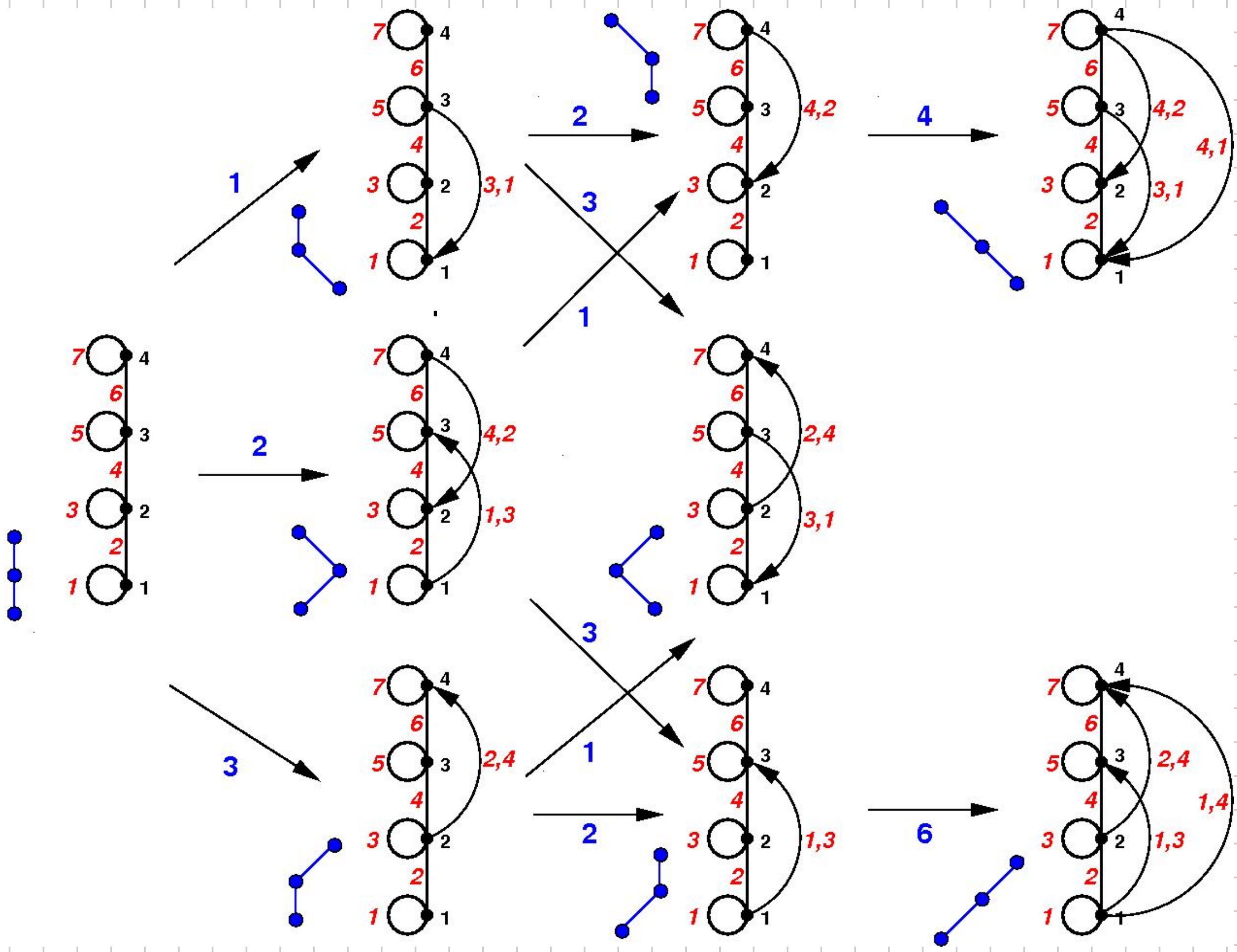
Example of graphs A_3 :



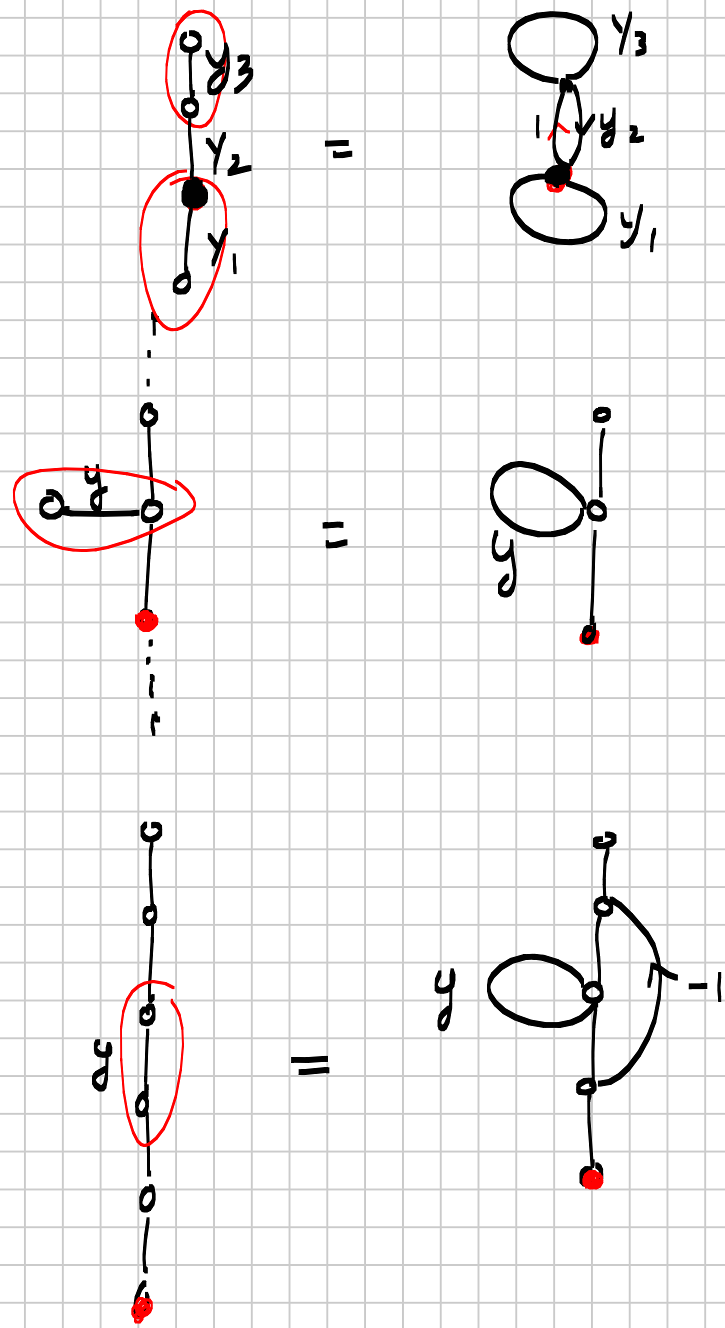
Relation to total positivity

$$(1-T)^{-1}_{11} \approx (1-\tilde{T})^{-1}_{11}$$

where $\tilde{T}$ is the transfer matrix of a compactified graph with $r+1$ vertices:



Examples



Totally positive matrices of the form

$$A = FDE$$

where $F = f_{i_1} f_{i_2} \cdots f_{i_r}$, $E = e_{j_1} \cdots e_{j_r}$

are lower and upper Δ elementary matrices and D is a diagonal matrix.

are encoded by networks.

Take: $f_i = \mathbb{1} + E_{i+1,i}$

$$e_i = \mathbb{1} + \frac{y_{2i}}{y_{2i-1}} E_{i,i+1}$$

$$d_i - \mathbb{1} = (y_{2i-1} - 1) E_{ii}$$

Thm: $(\mathbb{1} - \tilde{T})^{-1}_{11} = (\mathbb{1} - DEF)^{-1}_{11}$

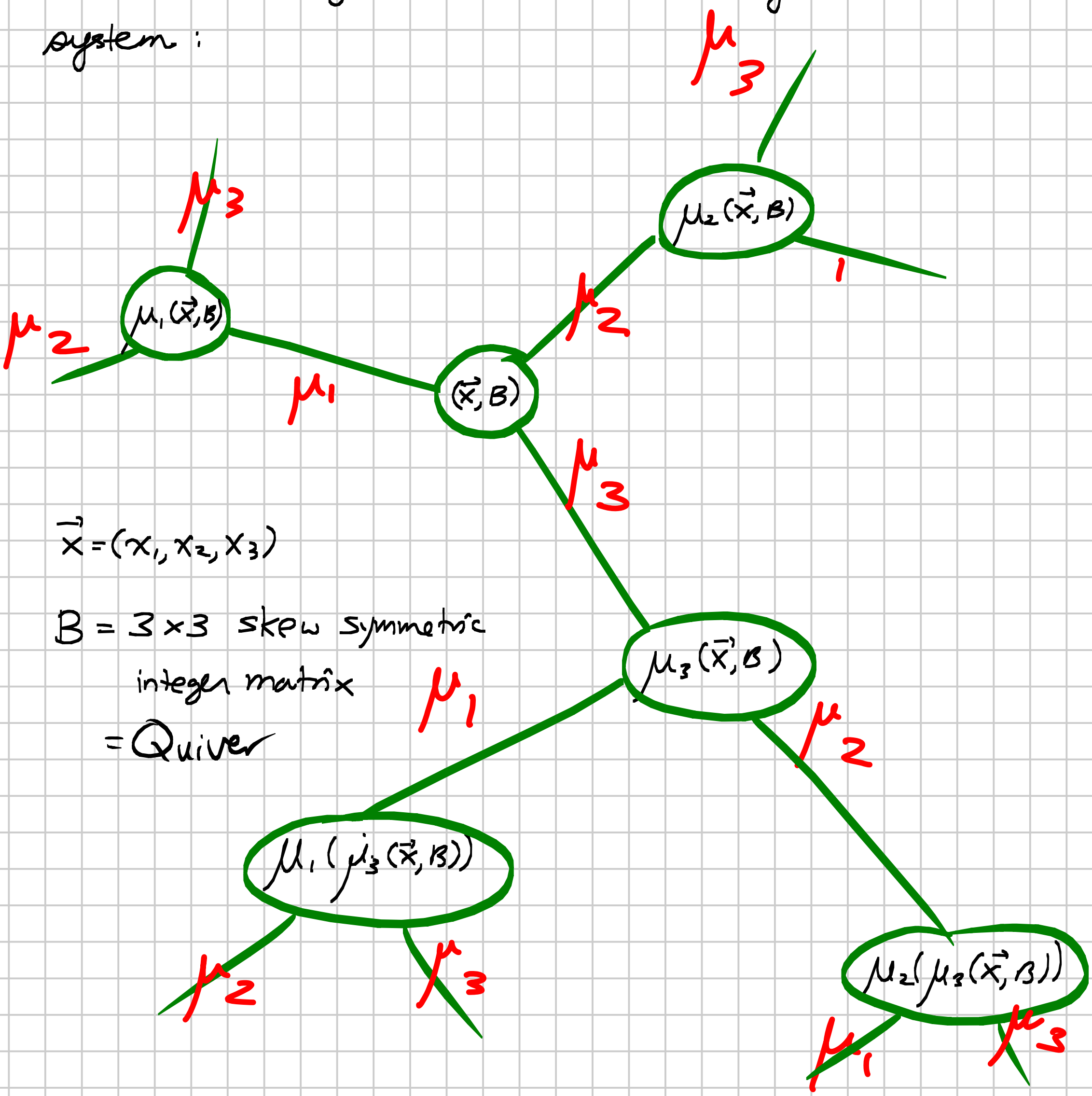
where $\begin{pmatrix} (i_1, \ldots, i_r) \\ (j_1, \ldots, j_r) \end{pmatrix}$ are permutations of $(1, \ldots, r)$

encoded by $\vec{m}$ ...

Cluster positivity

T- systems and Q- systems are also examples of cluster algebras

A cluster algebra is a discrete dynamical system :



$\vec{x} = (x_1, x_2, x_3)$

$B = 3 \times 3$ skew symmetric integer matrix = Quiver

$$\mu_k : \begin{cases} x_k \mapsto x_k^{-1}\left(\prod_{j \leftarrow k} x_j + \prod_{j \to k} x_j \right) \\ x_{i \neq k} \mapsto x_i \end{cases}$$

<u>Property</u>: Any cluster variable at any node (ie, $\mu_{i_1} \cdots \mu_{i_N}(\vec{x})$) is a Laurent polynomial in the variables of any node of the cluster graph.

<u>To be proven</u>: For any cluster algebra, it is a <u>positive</u> Laurent polynomial.

<u>Thm</u>:
Q-systems and T-systems are Cluster algebra exchange relations

So what we are doing is proving positivity.

<u>Example</u>: A_1 Q-system is a rank 2 cluster algebra:

$$
\begin{array}{ccccccccccc}
+B & & -B=\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} & & +B & & -B & & +B & \\
-\begin{pmatrix} Q_{-2} \\ Q_{-1} \end{pmatrix} & \overset{1}{\text{———}} & \begin{pmatrix} Q_0 \\ Q_{-1} \end{pmatrix} & \overset{2}{\text{———}} & \begin{pmatrix} Q_0 \\ Q_1 \end{pmatrix} & \overset{1}{\text{———}} & \begin{pmatrix} Q_2 \\ Q_1 \end{pmatrix} & \overset{2}{\text{———}} & \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} & \overset{1}{\text{———}} &
\end{array}
$$

$$\mu: \quad Q_{k+1} = Q_{k-1}^{-1}\left(Q_k^2+1\right)$$

T-system as a non-commutative Q-system:

If we define $\mathcal{D}$:

$$\mathcal{D}^{-1} T_{i,j,k} \mathcal{D} = T_{i,j+1,k}$$

Then the T-system is equivalent to

$$T_{i,j,k+1} T_{i,j,k-1} = \mathcal{D}^{-1} T_{i,j,k} \mathcal{D}^{2} T_{i,j,k} \mathcal{D}^{-1} + T_{i+1,j,k} T_{i-1,j,k}$$

a non-commutative Q-system:

$$\boxed{Q_{i,k+1} Q_{i,k-1} = \mathcal{D}^{-1} Q_{i,k} \mathcal{D}^{2} Q_{i,k} \mathcal{D}^{-1} + Q_{i+1,k} Q_{i-1,k}}$$

(T-system is obtained as a matrix element)

$\Rightarrow$ Solutions are p.f. of paths on graphs with non-commutative weights.

Conclusion:

For integrable discrete evolutions, the path formulation works for both commutative + non-commutative systems.

Non commutative generalizations

Rank 2: Kontsevich introduced the following conjecture: let x, y be n.c. variables, define a symplectomorphism ($C = xyx^{-1}y^{-1}$ conserved)

$$T_a : \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} xyx^{-1} \\ (1+y^a)x^{-1} \end{pmatrix} \quad (a \in \mathbb{N})$$

(wall crossing)

Then $(T_a T_b)^n \begin{pmatrix} x \\ y \end{pmatrix}$ is positive Laurent

Non commutative rank 2 cluster algebra

Example:

$a = b = 2$: $Q_0 = C^{-1}x$, $Q_1 = y$, $T_2 : (CQ_k, Q_{k+1}) \to (CQ_{k+1}, Q_{k+2})$

where $Q_{k+1}\, C\, Q_{k-1} = 1 + Q_k^2$ PROVED!