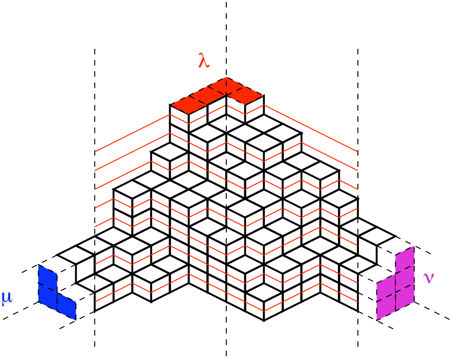
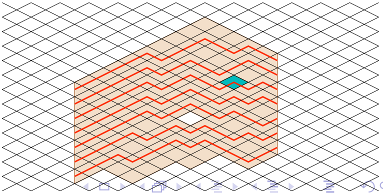
A matrix model for counting plane partitions and tilings

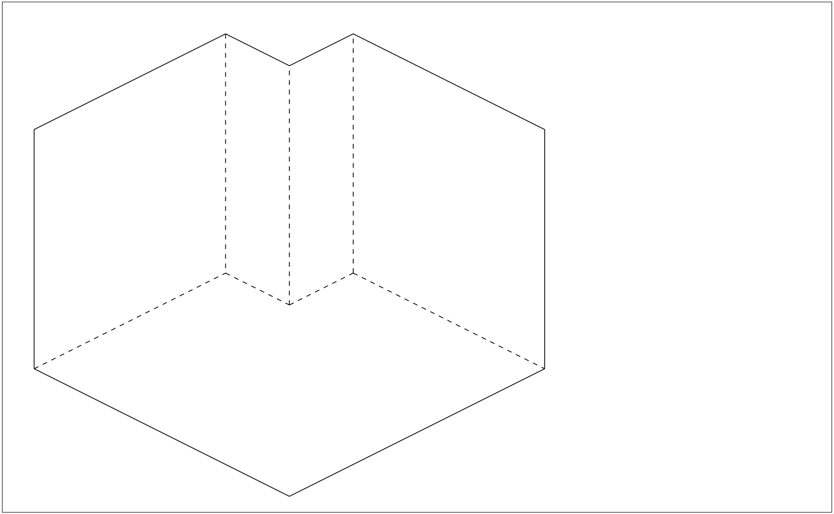
Bertrand Eynard, IPHT CEA-SACLAY

STATCOMB, Dimer models and random tilings, oct. 2009

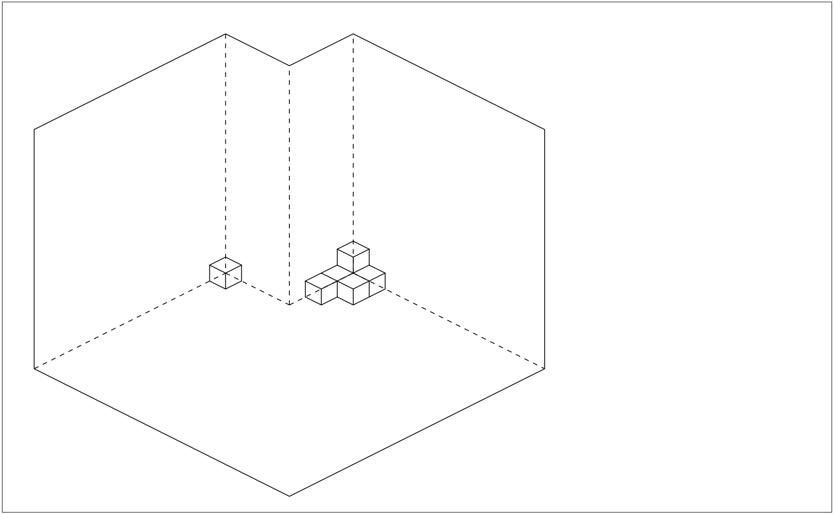




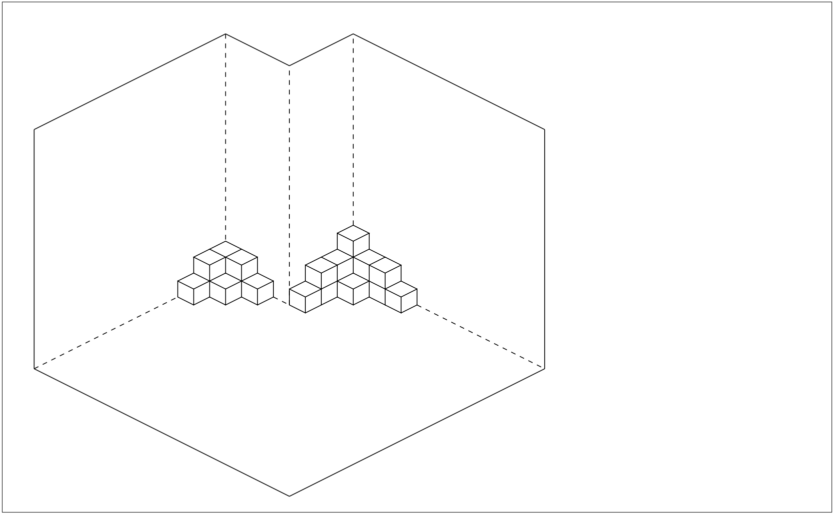
Introduction



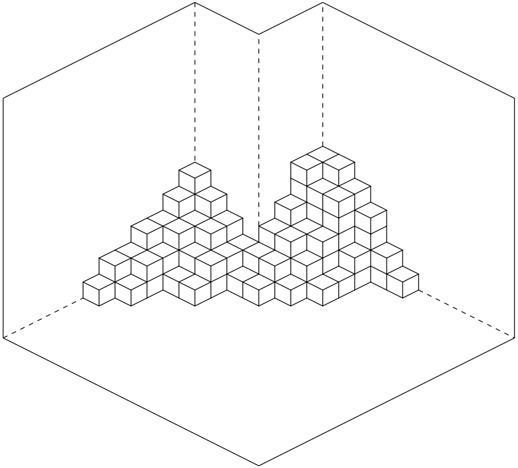
Introduction



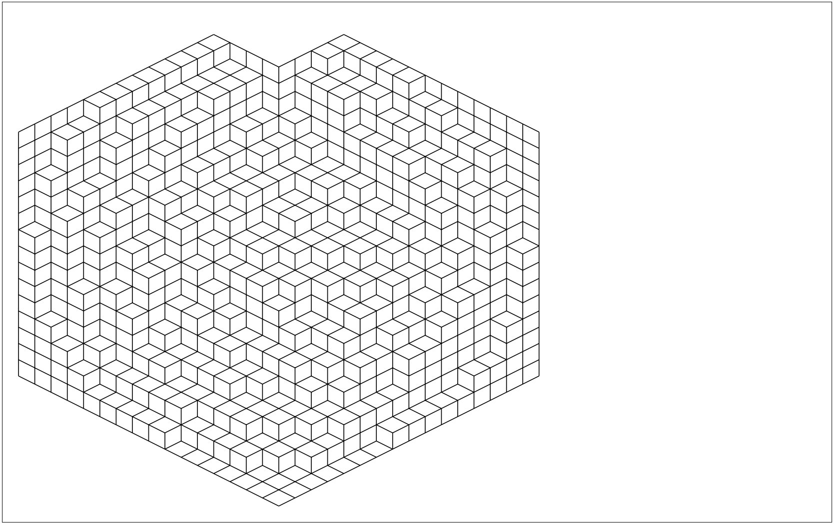
Introduction



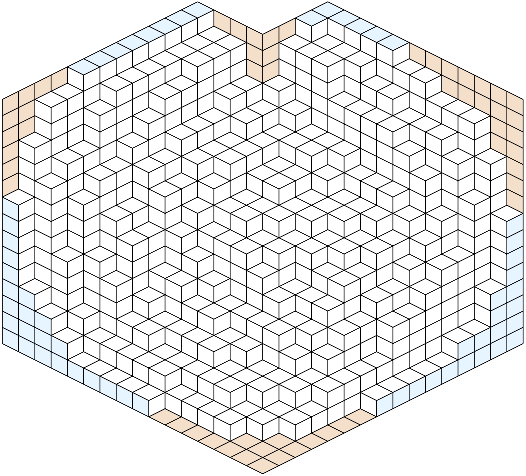
Introduction



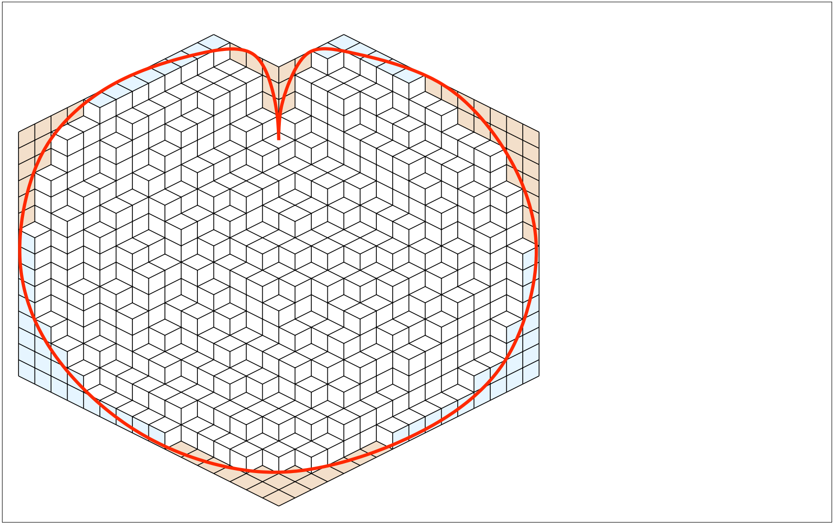
Introduction



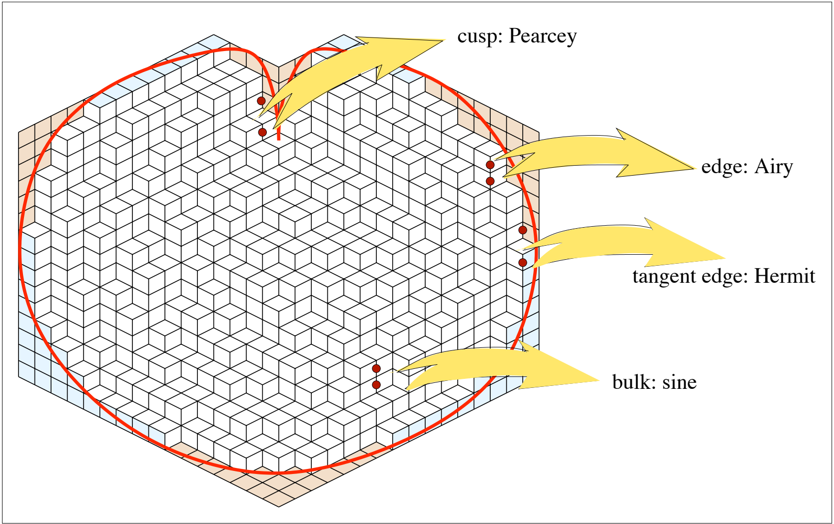
Introduction



Introduction

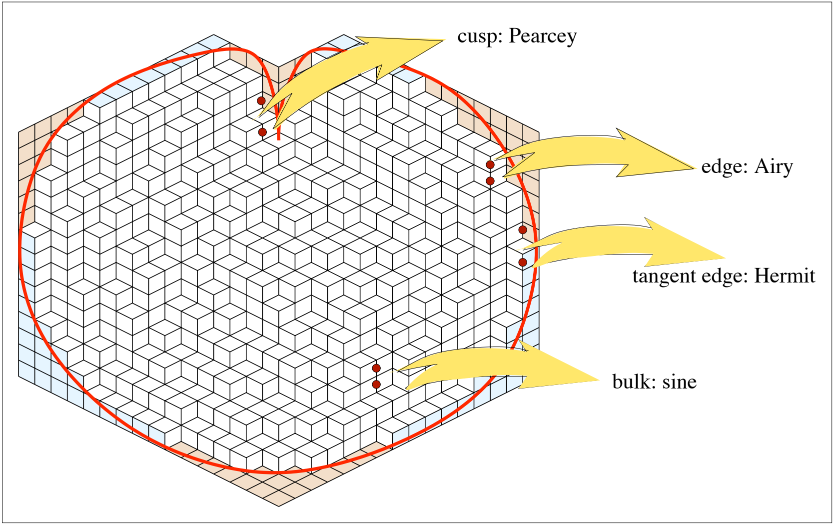


Introduction



In all those limits: statistics of cubes $\sim$ random matrix eigenvalues statistics.

Introduction



Question: is there a matrix model whose eigenvalues statistics = statistics of cubes ? before any limit ?

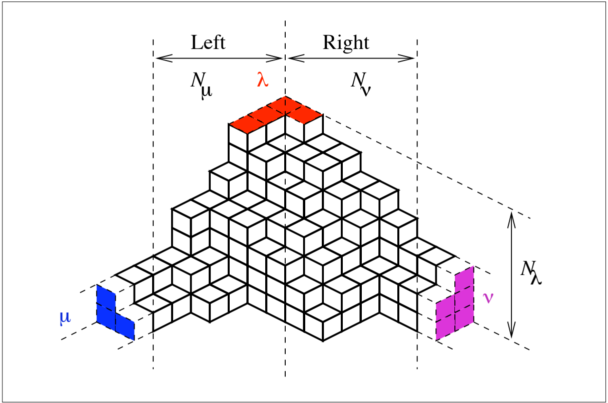
Outline

Outline:

- Plane partitions, tilings and TASEP
- Rewriting as a matrix integral
- Tools available for matrix models
  Orthogonal polynomials, determinantal formulae, integrability, loop equations.
- topological expansion of the matrix model
  Large size asymptotics, liquid region.
- Examples
  Tiling the hexagon, the cardioid, TSSCPPs.
- Conclusion

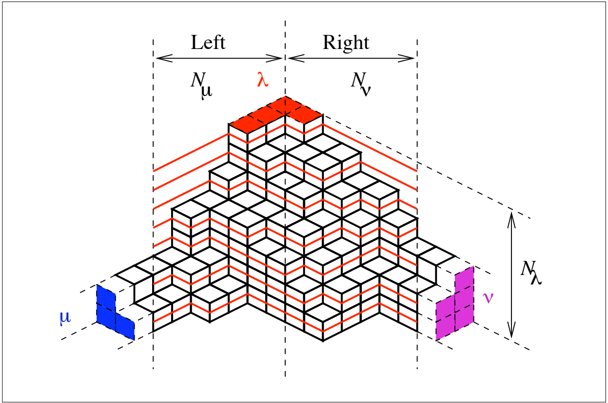
Plane partitions

- Plane partition, with 3 given boundaries λ, μ, ν:



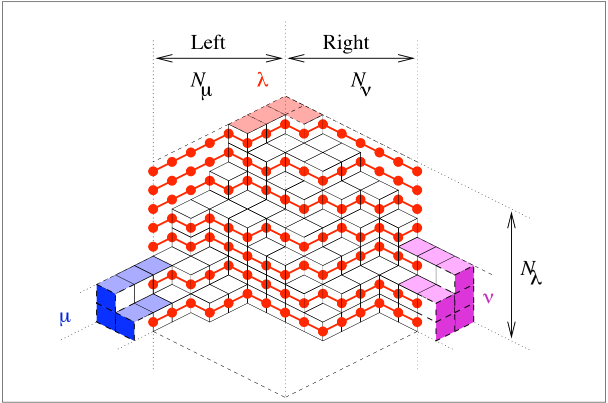
Plane partitions

- Plane partition, with 3 given boundaries λ, μ, ν:



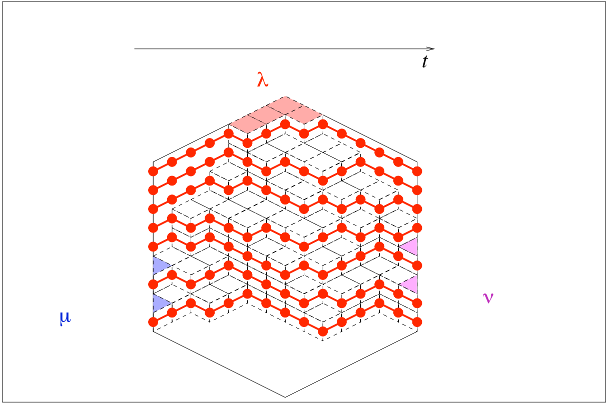
Plane partitions

- Plane partition, with 3 given boundaries λ, μ, ν:



Plane partitions

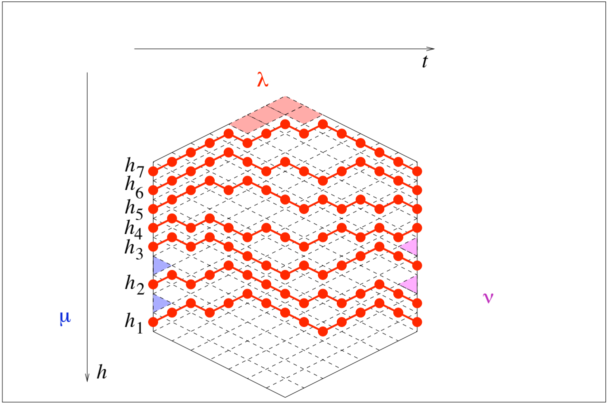
- Plane partition, with 3 given boundaries λ, μ, ν:



= N self avoiding particles moving in a given region of the Rhombus lattice.

Plane partitions

- Plane partition, with 3 given boundaries λ, μ, ν:



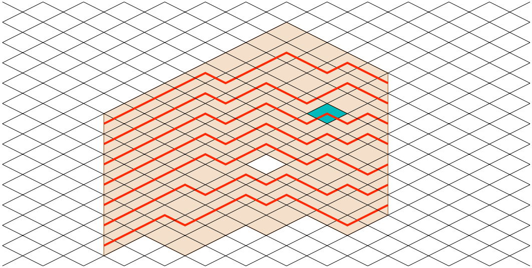
= N self avoiding particles moving in a given region of the Rhombus lattice.

$h_i(t)$, $i = 1, \ldots, N$, $h_i(t) - \frac{t}{2} \in \mathbb{Z}$, $\quad h_i(t+1) = h_i(t) \pm \frac{1}{2}$,

$h_1(t) > h_2(t) > h_3(t) > \cdots > h_N(t)$.

Generalization

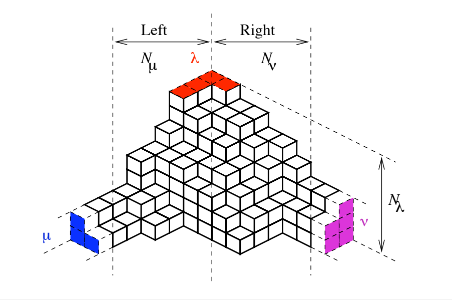
N self avoiding particles moving in a given arbitrary domain $\mathcal{D}$ of the Rhombus lattice.



$h_i(t)$, $i = 1, \ldots, N$, $h_i(t) - \frac{t}{2} \in \mathbb{Z}$,
$h_1(t) > h_2(t) > h_3(t) > \cdots > h_N(t)$,
$h_i(t+1) = h_i(t) + \frac{1}{2}$ with proba $\alpha(t + \frac{1}{2})$
$h_i(t+1) = h_i(t) - \frac{1}{2}$ with proba $\beta(t + \frac{1}{2})$

Possibility of having forbidden places, obliged places, non flat landscape, jumps other than $\pm\frac{1}{2}$,...

Partition function



Plane partitions:

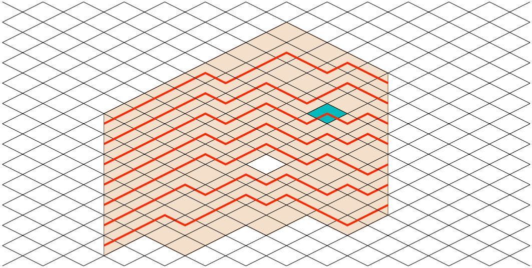
$$Z_{N_\lambda,N_\mu,N_\nu}(\lambda,\mu,\nu) = \sum_{\pi, \partial\pi=(\lambda,\mu,\nu)} q^{|\pi|}$$

Example, Mac-Mahon formula $N_\lambda = N_\mu = N_\nu = \infty$, $\lambda,\mu,\nu = \emptyset$:

$$Z = \sum_\pi q^{|\pi|} = \prod_{k=1}^{\infty} (1-q^k)^{-k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + \dots$$

Partition function, TASEP

Generalization self-avoiding process in a domain $\mathcal{D}$:



$$Z = \sum_{h_1(t)>\dots>h_N(t)} \prod_{t=t_{\min}}^{t_{\max}-1} \prod_{i=1}^{N} \mathrm{e}^{-V_t(h_i(t))}\, q^{h_i(t)}$$

$$\prod_{t'} \prod_{i} \left(\alpha(t')\, \delta_{h_i(t'+\frac{1}{2}),h_i(t'-\frac{1}{2})+\frac{1}{2}} + \beta(t')\, \delta_{h_i(t'+\frac{1}{2}),h_i(t'-\frac{1}{2})-\frac{1}{2}} \right)$$

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\dots>h_N(t)} \prod_i \text{paths} = \sum_{h_i(t)} \det(\text{paths})$.
- Fourrier transform δ-functions:
- Harish Chandra-Itzykson-Zuber: $H = \text{diag}(h_i)$, $R = \text{diag}(r_i)$
- Matrices: $M(t) = U\, H(t)\, U^\dagger\,,\ \ \Delta(H(t))^2\, dH(t)\, dU = dM(t)$,

and $\tilde{M}(t') = U^\dagger\ R(t')\, U\,, \quad \Delta(R(t'))^2\ dR(t')\, dU = d\tilde{M}(t')$.

$h_i(t) =$eigenvalues of $M(t)$, and $r_i(t) =$eigenvalues of $\tilde{M}(t')$.

$\rightarrow$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\dots>h_N(t)} \prod_i \mathrm{paths} = \sum_{h_i(t)} \det(\mathrm{paths})$.
- Fourrier transform δ-functions:
- Harish Chandra-Itzykson-Zuber: $H = \mathrm{diag}(h_i)$, $R = \mathrm{diag}(r_i)$
- Matrices: $M(t) = U\, H(t)\, U^\dagger\,,\ \ \Delta(H(t))^2\, dH(t)\, dU = dM(t)$,

and $\tilde{M}(t') = U^\dagger\ R(t')\, U\,,\quad \Delta(R(t'))^2\, dR(t')\, dU = d\tilde{M}(t')$.

$h_i(t)$ =eigenvalues of $M(t)$, and $r_i(t)$ =eigenvalues of $\tilde{M}(t')$.

$\rightarrow$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\cdots>h_N(t)} \prod_i \text{paths} = \sum_{h_i(t)} \det(\text{paths})$.
- Fourrier transform δ-functions:

$\delta(h(t+1) - h(t) \pm \frac{1}{2}) = \int_{-i\infty}^{i\infty} dr \;\; \mathrm{e}^{r(h(t+1)-h(t))} \; \mathrm{e}^{\pm r/2}$.

- Harish Chandra-Itzykson-Zuber: $H = \mathrm{diag}(h_i)$, $R = \mathrm{diag}(r_i)$
- Matrices: $M(t) = U\, H(t)\, U^\dagger$, $\Delta(H(t))^2\, dH(t)\, dU = dM(t)$,

and $\tilde{M}(t') = U^\dagger\; R(t')\, U$, $\quad \Delta(R(t'))^2\; dR(t')\, dU = d\tilde{M}(t')$.

$h_i(t)$ =eigenvalues of $M(t)$, and $r_i(t)$ =eigenvalues of $\tilde{M}(t')$.

$\rightarrow$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\cdots>h_N(t)} \prod_i \text{paths} = \sum_{h_i(t)} \det(\text{paths})$.
- Fourrier transform δ-functions:

$$\alpha\,\delta(h(t+1)-h(t)+\tfrac{1}{2})+\beta\,\delta(h(t+1)-h(t)-\tfrac{1}{2}) =$$
$$\int_{-i\infty}^{i\infty} dr\ \ e^{r(h(t+1)-h(t))}\,(\alpha\,e^{r/2}+\beta\,e^{-r/2}).$$

- Harish Chandra-Itzykson-Zuber: $H=\mathrm{diag}(h_i)$, $R=\mathrm{diag}(r_i)$
- Matrices: $M(t)=U\,H(t)\,U^\dagger\,,\ \ \Delta(H(t))^2\,dH(t)\,dU = dM(t)$, and $\tilde M(t')=U^\dagger\ R(t')\,U\,,\quad \Delta(R(t'))^2\,dR(t')\,dU = d\tilde M(t')$.

$h_i(t)=$eigenvalues of $M(t)$, and $r_i(t)=$eigenvalues of $\tilde M(t')$.

$\to$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\dots>h_N(t)} \prod_i \text{paths} = \sum_{h_i(t)} \det(\text{paths})$.
- Fourrier transform δ-functions:

$$\alpha\,\delta(h(t+1)-h(t)+\tfrac{1}{2}) + \beta\,\delta(h(t+1)-h(t)-\tfrac{1}{2}) = \int_{-i\infty}^{i\infty} dr\;\, \mathrm{e}^{r(h(t+1)-h(t))}\,(\alpha\,\mathrm{e}^{r/2}+\beta\,\mathrm{e}^{-r/2}).$$

- Harish Chandra-Itzykson-Zuber: $H=\mathrm{diag}(h_i)$, $R=\mathrm{diag}(r_i)$

$$\det(\mathrm{e}^{r_i\,h_j}) = \Delta(H)\Delta(R)\;\int_{U(N)} dU\;\, \mathrm{e}^{\mathrm{Tr}\,R\,U\,H\,U^\dagger}.$$

- Matrices: $M(t)=U\,H(t)\,U^\dagger\,,\;\; \Delta(H(t))^2\,dH(t)\,dU = dM(t)$, and $\tilde{M}(t')=U^\dagger\;R(t')\,U\,,\quad \Delta(R(t'))^2\,dR(t')\,dU = d\tilde{M}(t')$.

$h_i(t)=$eigenvalues of $M(t)$, and $r_i(t)=$eigenvalues of $\tilde{M}(t')$.

$\to$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\dots>h_N(t)} \prod_i \text{paths} = \sum_{h_i(t)} \det(\text{paths})$.
- Fourrier transform δ-functions:

$$\alpha\,\delta(h(t+1)-h(t)+\tfrac{1}{2})+\beta\,\delta(h(t+1)-h(t)-\tfrac{1}{2}) = \int_{-i\infty}^{i\infty} dr\ \mathrm{e}^{r(h(t+1)-h(t))}\,(\alpha\,\mathrm{e}^{r/2}+\beta\,\mathrm{e}^{-r/2}).$$

- Harish Chandra-Itzykson-Zuber: $H=\mathrm{diag}(h_i)$, $R=\mathrm{diag}(r_i)$

$$\det(\mathrm{e}^{r_i(t+\frac{1}{2})h_j(t+1)}) = \Delta(H(t+1))\Delta(R(t+\tfrac{1}{2}))\ \int_{U(N)} dU\ \mathrm{e}^{\mathrm{Tr}\,R(t+\frac{1}{2})\,U\,H(t+1)\,U^\dagger},$$

$$\det(\mathrm{e}^{-r_i(t+\frac{1}{2})h_j(t)}) = \Delta(H(t))\Delta(R(t+\tfrac{1}{2}))\ \int_{U(N)} dU\ \mathrm{e}^{-\,\mathrm{Tr}\,R(t+\frac{1}{2})\,U\,H(t)\,U^\dagger}.$$

- Matrices: $M(t)=U\,H(t)\,U^\dagger$, $\Delta(H(t))^2\,dH(t)\,dU=dM(t)$, and $\tilde M(t')=U^\dagger\,R(t')\,U$, $\Delta(R(t'))^2\,dR(t')\,dU=d\tilde M(t')$.

$h_i(t)$ =eigenvalues of $M(t)$, and $r_i(t)$ =eigenvalues of $\tilde M(t')$.

$\rightarrow$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\dots>h_N(t)} \prod_i \mathrm{paths} = \sum_{h_i(t)} \det(\mathrm{paths})$.
- Fourrier transform δ-functions:

$\alpha\,\delta(h(t+1)-h(t)+\frac{1}{2}) + \beta\,\delta(h(t+1)-h(t)-\frac{1}{2}) =$
$\int_{-i\infty}^{i\infty} dr\;\; \mathrm{e}^{r(h(t+1)-h(t))}\;(\alpha\,\mathrm{e}^{r/2}+\beta\,\mathrm{e}^{-r/2})$.

- Harish Chandra-Itzykson-Zuber: $H=\mathrm{diag}(h_i)$, $R=\mathrm{diag}(r_i)$

$\det(\mathrm{e}^{r_i(t+\frac{1}{2})h_j(t+1)}) =$
$\Delta(H(t+1))\Delta(R(t+\frac{1}{2}))\;\int_{U(N)} dU\;\; \mathrm{e}^{\mathrm{Tr}\,R(t+\frac{1}{2})\,U\,H(t+1)\,U^\dagger}$,

$\det(\mathrm{e}^{-r_i(t+\frac{1}{2})h_j(t)}) =$
$\Delta(H(t))\Delta(R(t+\frac{1}{2}))\;\int_{U(N)} dU\;\; \mathrm{e}^{-\,\mathrm{Tr}\,R(t+\frac{1}{2})\,U\,H(t)\,U^\dagger}$.

- Matrices: $M(t)=U\;H(t)\,U^\dagger\,,\;\;\Delta(H(t))^2\;dH(t)\,dU=dM(t)$,

and $\tilde M(t')=U^\dagger\;R(t')\,U\,,\quad \Delta(R(t'))^2\;dR(t')\,dU=d\tilde M(t')$.

$h_i(t)$ =eigenvalues of $M(t)$, and $r_i(t)$ =eigenvalues of $\tilde M(t')$.

$\rightarrow$ Matrix model

Transformation into a matrix integral

Idea:

- Gessel-Viennot: $\sum_{h_1(t)>\dots>h_N(t)} \prod_i \mathrm{paths} = \sum_{h_i(t)}$ det(paths).
- Fourrier transform δ-functions:

$\alpha\,\delta(h(t+1)-h(t)+\frac{1}{2})+\beta\,\delta(h(t+1)-h(t)-\frac{1}{2}) =$
$\int_{-i\infty}^{i\infty} dr\;\; \mathrm{e}^{r(h(t+1)-h(t))}\;(\alpha\,\mathrm{e}^{r/2}+\beta\,\mathrm{e}^{-r/2})$.

- Harish Chandra-Itzykson-Zuber: $H=\mathrm{diag}(h_i)$, $R=\mathrm{diag}(r_i)$

$\det(\mathrm{e}^{r_i(t+\frac{1}{2})h_j(t+1)}) =$
$\Delta(H(t+1))\Delta(R(t+\frac{1}{2}))\;\int_{U(N)} dU\;\; \mathrm{e}^{\mathrm{Tr}\, R(t+\frac{1}{2})\,U\,H(t+1)\,U^\dagger}$,
$\det(\mathrm{e}^{-r_i(t+\frac{1}{2})h_j(t)}) =$
$\Delta(H(t))\Delta(R(t+\frac{1}{2}))\;\int_{U(N)} dU\;\; \mathrm{e}^{-\,\mathrm{Tr}\, R(t+\frac{1}{2})\,U\,H(t)\,U^\dagger}$.

- Matrices: $M(t)=U\;H(t)\,U^\dagger\,,\;\;\Delta(H(t))^2\;dH(t)\,dU = dM(t)$,

and $\tilde{M}(t')=U^\dagger\;R(t')\,U\,,\qquad \Delta(R(t'))^2\;dR(t')\,dU = d\tilde{M}(t')$.
$h_i(t)=$eigenvalues of $M(t)$, and $r_i(t)=$eigenvalues of $\tilde{M}(t')$.

$\rightarrow$ Matrix model

Matrix integral

We end up with

$$Z = \frac{\Delta(H_{t_{\max}})}{\Delta(H_{t_{\min}})} \int \prod_{t=t_{\min}}^{t_{\max}-1} dM(t)\, \mathrm{e}^{-\mathrm{Tr}\, V_t(M(t))}\, q^{\mathrm{Tr}\, M(t)}$$

$$\int \prod_{t'=t_{\min}+\frac{1}{2}}^{t_{\max}-\frac{1}{2}} d\tilde{M}(t')\, \mathrm{e}^{-\mathrm{Tr}\, \tilde{\mathcal{V}}_{t'}(\tilde{M}(t'))}\, \mathrm{e}^{\mathrm{Tr}\, \tilde{M}(t')\, (M(t'+\frac{1}{2})-M(t'-\frac{1}{2}))}$$

The potentials $V_t(h)$ encode the domain, and landscape weight.

The potentials $\tilde{\mathcal{V}}_{t'}$ encode the jumps:
$\mathrm{e}^{-\tilde{\mathcal{V}}_{t'}(x)} = (\alpha(t')\mathrm{e}^{-\frac{x}{2}} + \beta(t')\mathrm{e}^{\frac{x}{2}})$.

The eigenvalues of $M(t)$ are $h_i(t)$ =position of the i^{th} particle at time t.

Summary matrix model

Theorem

The "lozenge tiling/plane partitions/particle process" generating function Z, is a matrix integral.

$\rightarrow$ Chain of matrices, with $2(t_{\max} - t_{\min}) + 1$ matrices.

Summary:

• matrices $M(t)$, $t \in \mathbb{Z}$: eigenvalues $h_i(t)$ = particles, potential $\mathrm{e}^{-V_t(h)}$ characterizes the domain+landscape.

• matrices $\tilde{M}(t')$, $t' \in \mathbb{Z} + \frac{1}{2}$, eigenvalues $r_i(t') =$ Lagrange multipliers for jumps,
potential $\mathrm{e}^{-\tilde{\mathcal{V}}_{t'}(r)} = \alpha(t')\mathrm{e}^{-r/2} + \beta(t')\mathrm{e}^{r/2}$.

• Angular parts= Fourier transform of Gessel-Viennot $\rightarrow$ HCIZ.

Consider a general chain of matrices:

$$Z = \int dM_1 \dots dM_k \, \mathrm{e}^{-Q \,\mathrm{Tr}\, \sum_{i=1}^{k} V_i(M_i) - c_i M_i M_{i+1}}$$

• method of biorthogonal polynomials $\rightarrow$ determinantal formuale. Correlation functions of eigenvalues are determinant of some Christoffel-Darboux kernel [E., Mehta 1997].

• Integrability $\rightarrow Z =$tau-function, Hirota equation, various pde's.

• method of loop equations $\rightarrow$ topological expansion $\ln Z = \sum_{g=0}^{\infty} Q^{2-2g} F_g$.

• Matrix laws universal limits $=$ Bergman, Sine, Tracy-Widom $(= (1,2))$, Pearcey, Hermit, (p, q) conformal laws,...

Generalities: Loop equations

Consider a general chain of matrices

$$Z = \int dM_1 \dots dM_k \ \mathrm{e}^{-Q \,\mathrm{Tr}\, \sum_{i=1}^{k} V_i(M_i) - c_i M_i M_{i+1}}$$

Assume it has a topological expansion:
$\ln Z = \sum_{g=0}^{\infty} Q^{2-2g} F_g$.

$$W_n(x_1, \dots, x_n) = \left\langle \mathrm{Tr}\, \frac{1}{x_1 - M_1} \dots \mathrm{Tr}\, \frac{1}{x_n - M_1} \right\rangle_c = \sum_g Q^{2-2g-n} W_n^{(g)}$$

then, by solving loop equations (=integration by parts $\Rightarrow$ equations relating correlation functions) we get:

Theorem (E.-Prats Ferrer 2008)

For every chain of matrices having a topological expansion, the $W_n^{(g)}$'s and F_g's satisfy the "topological recursion".

Topological recursion

Define the recursion kernel:

$$H(x_0, x) = \frac{\int_x^{\bar{x}} W_2^{(0)}(x_0, x')\, dx'}{2(W_1^{(0)}(x) - W_1^{(0)}(\bar{x}))}$$

Then the topological recursion [E.-Orantin 2007] is:

$$W_{n+1}^{(g)}(x_0, \overbrace{x_1, \dots, x_n}^{J}) = \sum_i \mathrm{Res}_{x \to a_i} H(x_0, x) \Big[W_{n+2}^{(g-1)}(x, x, J) + \sum_{h=0}^{g} \sum_{I \subset J}^{\prime} W_{1+\#I}^{(h)}(x, I) W_{1+n-\#I}^{(g-h)}(x, J \setminus I) \Big]$$

$\rightarrow$ if one knows $W_1^{(0)}(x)$ (called spectral curve $\mathcal{S}$) and $W_2^{(0)}$ (called Bergman kernel of $\mathcal{S}$), then this recursion easily computes every $W_n^{(g)}$.

Symplectic invariants

The $F_g = W_0^{(g)}$'s are computed (for $g \geq 2$) by:

$$F_g = W_0^{(g)} = \frac{1}{2-2g} \sum_i \mathrm{Res}_{x \to a_i} \Phi(x)\ W_1^{(g)}\ dx$$

where $d\Phi/dx = W_1^{(0)}(x)$.

+ more sophisticated expressions for F_0 and F_1, see [E. Orantin 2007].

Remark: F_g's and $W_n^{(g)}$'s can be computed for any function $W_1^{(0)}(x)$, related to any matrix model or not.

The $F_g(\mathcal{S})$'s and $W_n^{(g)}$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\text{Airy kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}(1 - \sqrt{1-12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin(\sqrt{x})$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $\mathrm{e}^x = y\mathrm{e}^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(\mathrm{e}^x, \mathrm{e}^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\mathrm{Airy\,kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}(1 - \sqrt{1-12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin(\sqrt{x})$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $\mathrm{e}^x = y\mathrm{e}^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(\mathrm{e}^x, \mathrm{e}^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\mathrm{Airy\ kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}(1 - \sqrt{1 - 12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin(\sqrt{x})$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $\mathrm{e}^x = y\mathrm{e}^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(\mathrm{e}^x, \mathrm{e}^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\text{Airy kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}(1 - \sqrt{1-12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin(\sqrt{x}\,)$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $e^x = ye^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(e^x, e^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\mathrm{Airy\ kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}\,(1 - \sqrt{1-12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin\,(\sqrt{x}\,)$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $\mathrm{e}^x = y\mathrm{e}^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(\mathrm{e}^x, \mathrm{e}^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\text{Airy kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}\,(1 - \sqrt{1 - 12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin(\sqrt{x})$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $\mathrm{e}^x = y\mathrm{e}^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(\mathrm{e}^x, \mathrm{e}^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

The $F_g(\mathcal{S})$'s are functionals of a spectral curve $\mathcal{S} = \{W_1^{(0)}(x)\}$.

Examples:

- $W_1^{(0)}(x) = \sqrt{x}$, then $W_n = \sum_g W_n^{(g)} = \det_{n\times n}(\text{Airy kernel})$.
- $W_1^{(0)}(x) = t\,x\,\sqrt{x^2 - a^2}$, $a^2 = \frac{2}{3}(1 - \sqrt{1-12t})$, then $F_g =$ enumerating quadrangulations of genus g.
- $W_1^{(0)}(x) = \sin(\sqrt{x})$, then $F_g = \mathrm{Vol}(\mathcal{M}_g)$= Weil-Petersson.
- $y = W_1^{(0)}(x)$, solution of $\mathrm{e}^x = y\mathrm{e}^{-y}$, then $W_n^{(g)}$ =gen. function of Hurwitz numbers of genus g.
- $y = W_1^{(0)}(x)$, solution of $0 = P(\mathrm{e}^x, \mathrm{e}^y)$ =mirror curve of some toric CY 3-fold $\mathfrak{X}$, then $F_g = \mathrm{Gromov-Witten}_g(\mathfrak{X})$ (=[BKMP 2007] conjecture).
- + plenty of other examples...

Symplectic invariants

Some general properties of the invariants $F_g(\mathcal{S})$:
$(\mathcal{S} = \{y(x)\})$

• homogeneity ($g \geq 2$): $F_g(\lambda\mathcal{S}) = \lambda^{2-2g}\, F_g(\mathcal{S})$.

• symplectic invariance $F_g(\{y(x) + \mathrm{Ratl}(x)\}) = F_g(\{y(x)\})$, $F_g(\{-x(y)\}) = F_g(\{y(x)\})$, $F_g(\{\lambda y(x/\lambda)\}) = F_g(\{y(x)\})$.

• Special geometry formulae for the derivatives
$\partial_t F_g = \oint_{t^*} W_1^{(g)}$, where t^* =dual cycle to the form $\partial_t ydx$.

• Commute with limits: $\lim F_g(\mathcal{S})" = "\ F_g(\lim \mathcal{S})$.

• Integrability: $\tau = \mathrm{e}^{\sum_g F_g}\,\Theta$ =Tau-function, satisfies Hirota.
Determinantal formulae: $W_n = \sum_g W_n^{(g)}$ =determinants.

• modular properties, ... etc

Summary loop equations

Summary of the loop equation method:

• If one knows the "spectral curve"
$\mathcal{S} = W_1^{(0)}(x)$ " = " $\lim \frac{1}{Q} \left\langle \mathrm{tr} \left(\frac{1}{x - M_1} \right) \right\rangle$

• then: $W_2^{(0)}$ =Bergman kernel of $\mathcal{S}$ ($\rightarrow$ heat equation on $\mathcal{S}$).

• then: $\ln Z = \sum_g Q^{2-2g} F_g(\mathcal{S})$, where $F_g(\mathcal{S})$ =symplectic invariants of $\mathcal{S}$.

• and the $W_n^{(g)}$'s satisfy the topological recursion:

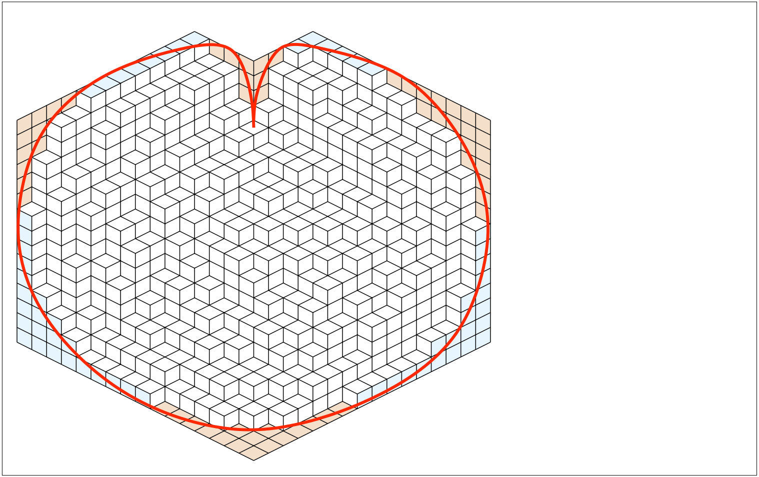
$$W_n(x_1, \dots, x_n) = \left\langle \mathrm{Tr} \frac{1}{x_1 - M_1} \dots \mathrm{Tr} \frac{1}{x_n - M_1} \right\rangle_c = \sum_g Q^{2-2g-n} W_n^{(g)}$$

$\rightarrow$ so, once $W_1^{(0}(x)$ is known, corrections to all orders can be easily computed.

Spectral curve for plane-partitions and TASEP

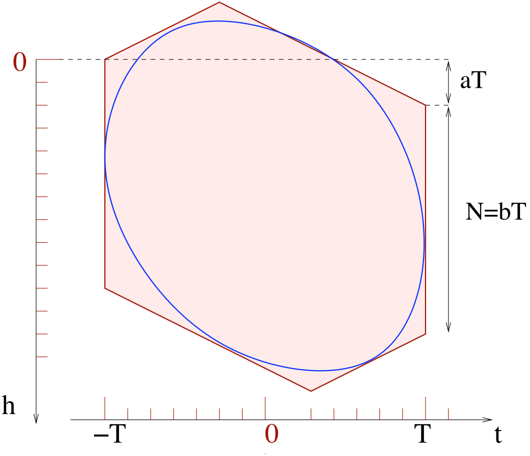
Result: Matrix model's spectral curve

$W_1^{(0)}(x) \leftrightarrow$ limit shape of [Kenyon-Okounkov-Sheffield]



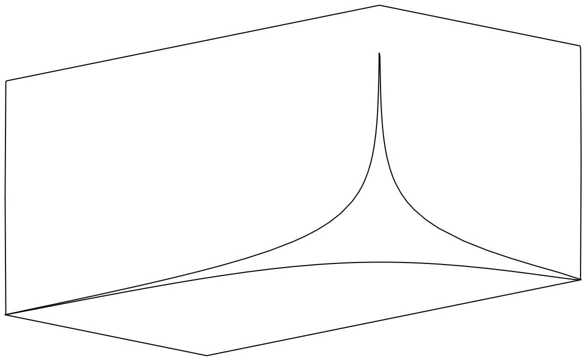
Examples

Hexagon' spectral curve:



Examples

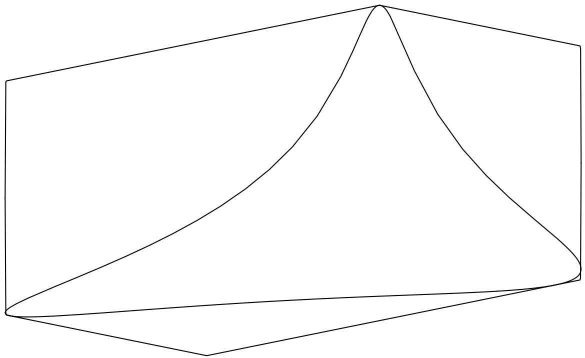
Hexagon' spectral curve:



$b = 2, a = 0.3, \qquad q = 0.001$

Examples

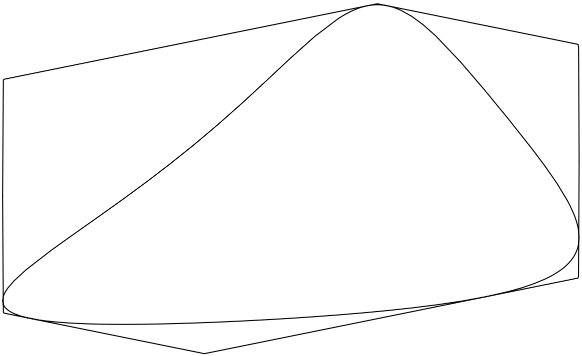
Hexagon' spectral curve:



$b = 2, a = 0.3, \qquad q = 0.1$

Examples

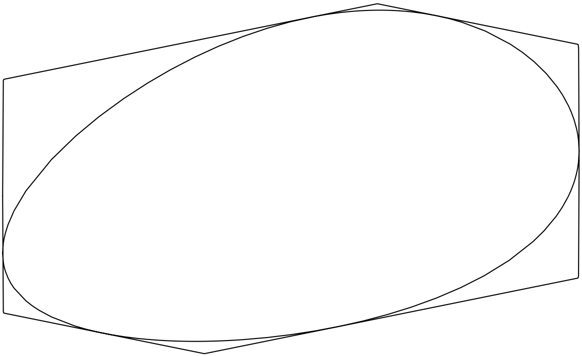
Hexagon' spectral curve:



$b = 2, a = 0.3, \qquad q = 0.3$

Examples

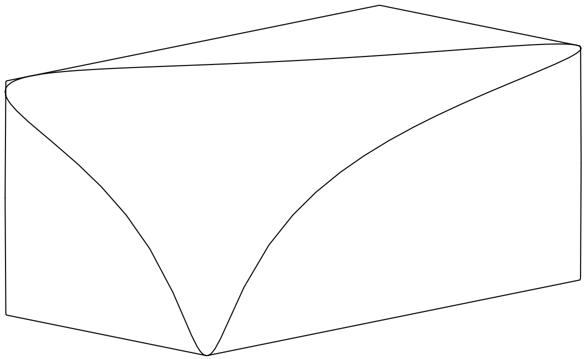
Hexagon' spectral curve:



$b = 2, a = 0.3, \qquad q = 0.8$

Examples

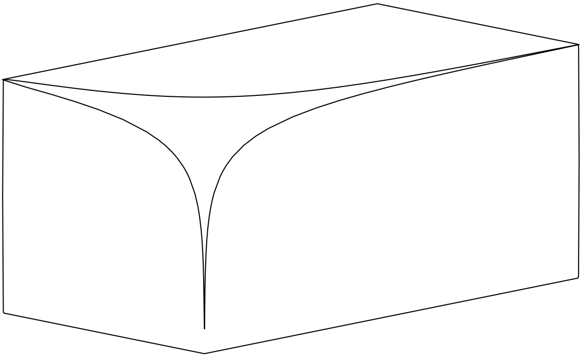
Hexagon' spectral curve:



$b = 2, a = 0.3, \qquad q = 10$

Examples

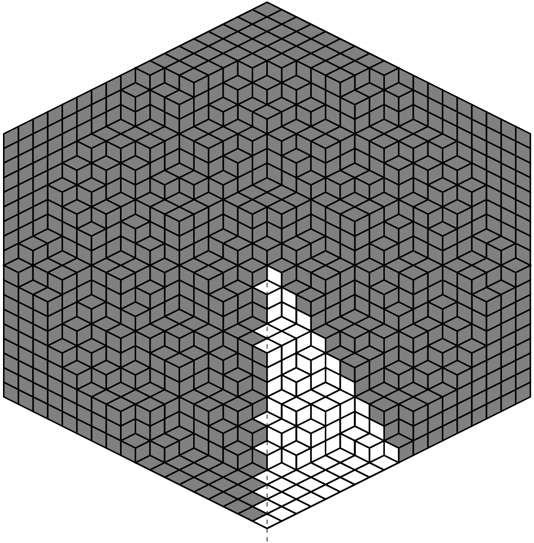
Hexagon' spectral curve:



$b = 2, a = 0.3, \qquad q = 1000$

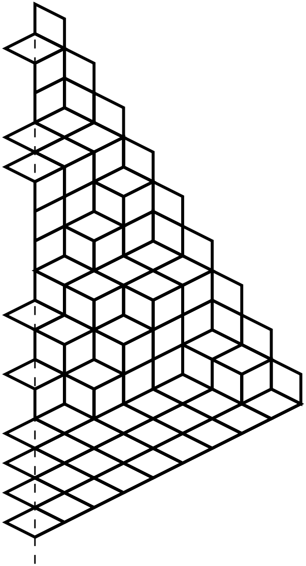
Example: TSSCPP's

Totally Symmetric Self Complementary Plane Partitions



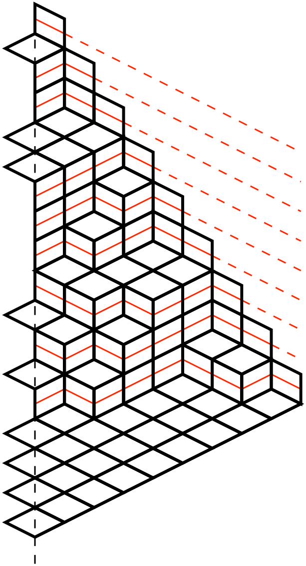
Example: TSSCPP's

Totally Symmetric Self Complementary Plane Partitions



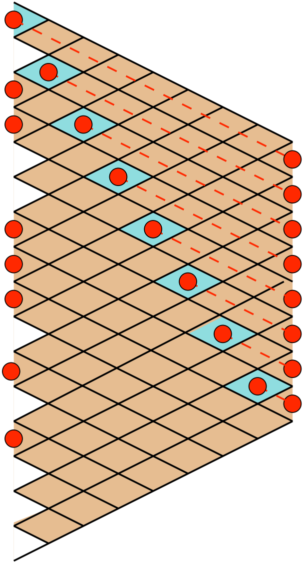
Example: TSSCPP's

Totally Symmetric Self Complementary Plane Partitions

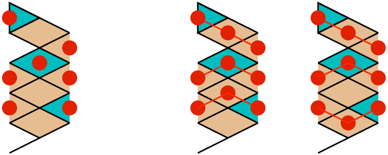


Example: TSSCPP's

Totally Symmetric Self Complementary Plane Partitions



Example: TSSCPP's



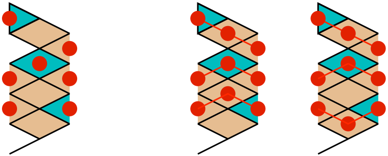
Example: $N = 3$

$$Z_3(1^2) = \frac{1}{144}\int dM_1\, dM_2\, d\tilde{M}_{3/2}\, d\tilde{M}_{5/2}\; e^{-\operatorname{tr}(V_1(M_1))}\, q^{\operatorname{tr}(M_1+M_2+M_3)}$$
$$e^{-\operatorname{tr}\left(\tilde{\mathcal{V}}_{3/2}(\tilde{M}_{3/2})+\tilde{\mathcal{V}}_{5/2}(\tilde{M}_{5/2})\right)}\; e^{\operatorname{tr}\left(\tilde{M}_{3/2}(M_2-M_1)+\tilde{M}_{5/2}(M_3-M_2)\right)}$$
$$\operatorname{tr}\,(P_2(M_2))$$

$$M_3 = \operatorname{diag}(2,3,4)\,,\qquad P_2(x) = \frac{1}{2}(x-\frac{3}{2})(x-\frac{7}{2})(x-\frac{9}{2})$$

$$\mathrm{e}^{-V_1(x)} = (x-2)(x-5)$$

Example: TSSCPP's



Example: $N = 3$ $\rightarrow$ ASM ?

$$Z_3(1^2) = \frac{1}{144} \int dM_1 dM_2 d\tilde{M}_{3/2} d\tilde{M}_{5/2}\; e^{-\operatorname{tr}(V_1(M_1))}\, q^{\operatorname{tr}(M_1+M_2+M_3)}$$
$$e^{-\operatorname{tr}\left(\tilde{\mathcal{V}}_{3/2}(\tilde{M}_{3/2})+\tilde{\mathcal{V}}_{5/2}(\tilde{M}_{5/2})\right)}\; e^{\operatorname{tr}\left(\tilde{M}_{3/2}(M_2-M_1)+\tilde{M}_{5/2}(M_3-M_2)\right)}$$
$$\operatorname{tr}\,(P_2(M_2))$$

$$M_3 = \operatorname{diag}(2,3,4)\,, \quad P_2(x) = \frac{1}{2}(x-\frac{3}{2})(x-\frac{7}{2})(x-\frac{9}{2})$$

$$e^{-V_1(x)} = (x-2)(x-5)$$

Conclusion

- General method: tiling problem → matrix model

- saying that limit laws of plane partitions = matrix models limit laws, is a truism.

- possibility to use the huge matrix models toolbox: orthogonal polynomials, integrability, loop equations, ...

- loop equations → possibility to use the "topological recursion" to find the asymptotic expansion (large size, or small ln q), to ALL ORDERS.

- Possible application: Gromov-Witten invariants of toric CY 3-folds, [BKMP 2007] conjecture ("remodelling the B-model"): *The Gromov-Witten invariants do satisfy the topological recursion ?*