STATCOMB 2009, IHP Paris, October 2009

Arctic curves for the domain-wall six-vertex model

A.G. Pronko, PDMI Steklov, Saint Petersbourg

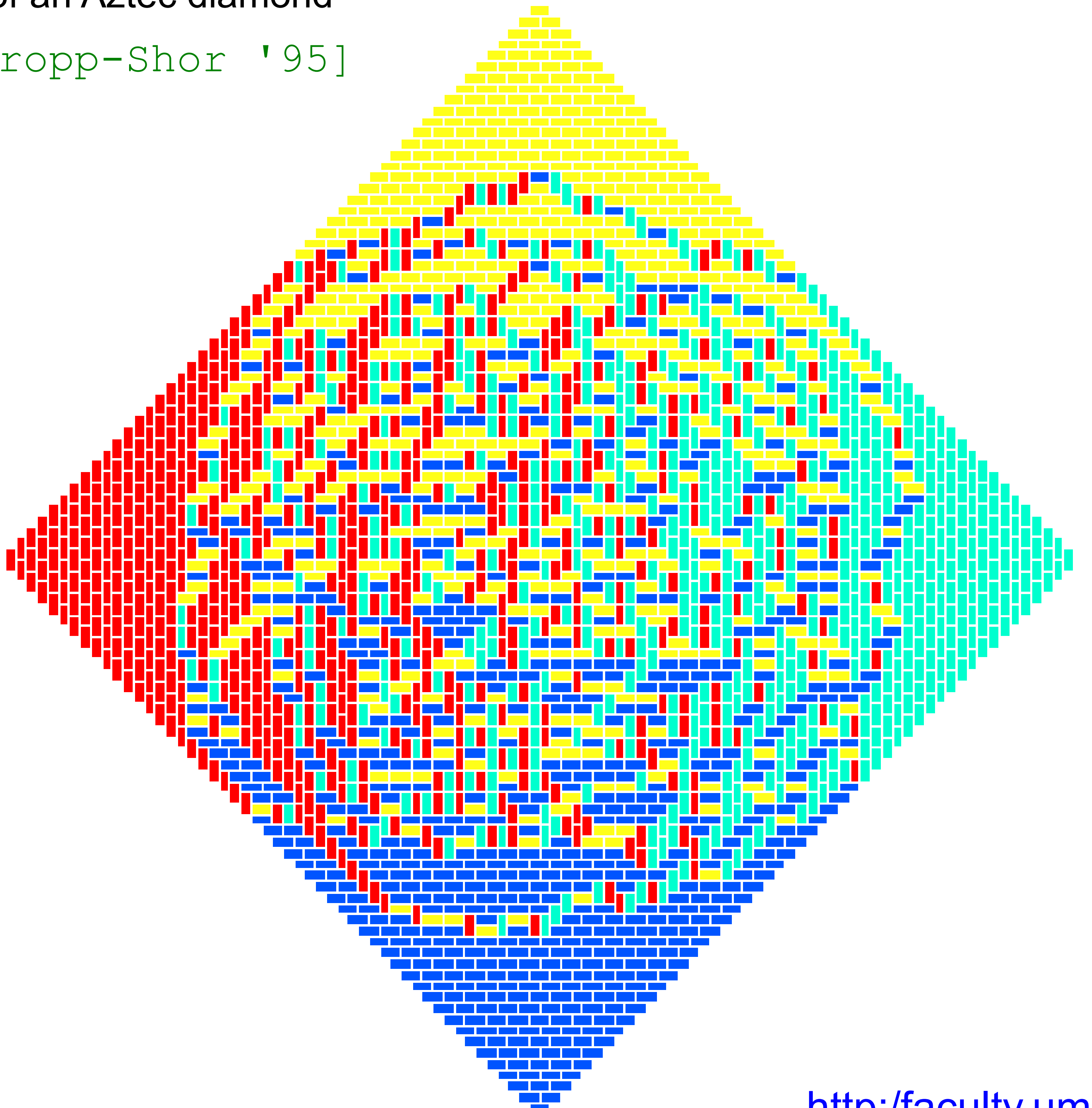
F.C. INFN, Florence

- Emptiness Formation Probability in the domain wall six-vertex model, `arXiv:0712.1524` *Nucl. Phys. B 798 (2008) 340*
- The Arctic Circle revisited, `arXiv:0704.0362` *Contemp. Math. 458 (2008) 361*
- The limit shape of large Alternating Sign Matrices, `arXiv:0803.2697` subm. to *SIAM J. Discr. Math.*
- The Arctic curve of the domain-wall six-vertex model, `arXiv:0907.1264` subm. to *Comm. Math. Phys.*

Domino tiling of an Aztec diamond

[Jockush-Propp-Shor '95]

$N = 64$



http:/faculty.uml.edu/jpropp

The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \varepsilon > 0, \quad \exists N$ such that "almost all" (i.e. with probability $P > 1 - \varepsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance εN from the circle of radius $N/\sqrt{2}$.

The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \varepsilon > 0, \quad \exists N$ such that "almost all" (i.e. with probability $P > 1 - \varepsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance εN from the circle of radius $N/\sqrt{2}$.

Fluctuations:

- boundary fluctuations $N^{1/3}$ [Johansson'00]
- fluctuations of boundary intersection with main diagonal obey Tracy-Widom distribution [Johansson'02]
- after suitable rescaling, boundary has limit as a random function, governed by an Airy stochastic process [Johansson'05]

The Arctic Circle Theorem

[Jockush-Propp-Shor '95]

$\forall \varepsilon > 0, \quad \exists N$ such that "almost all" (i.e. with probability $P > 1 - \varepsilon$) randomly picked domino tilings of $AD(N)$ have a temperate region whose boundary stays uniformly within distance εN from the circle of radius $N/\sqrt{2}$.

Example of more general phenomena: phase separation, limit shapes, frozen boundaries/arctic curves, e.g:

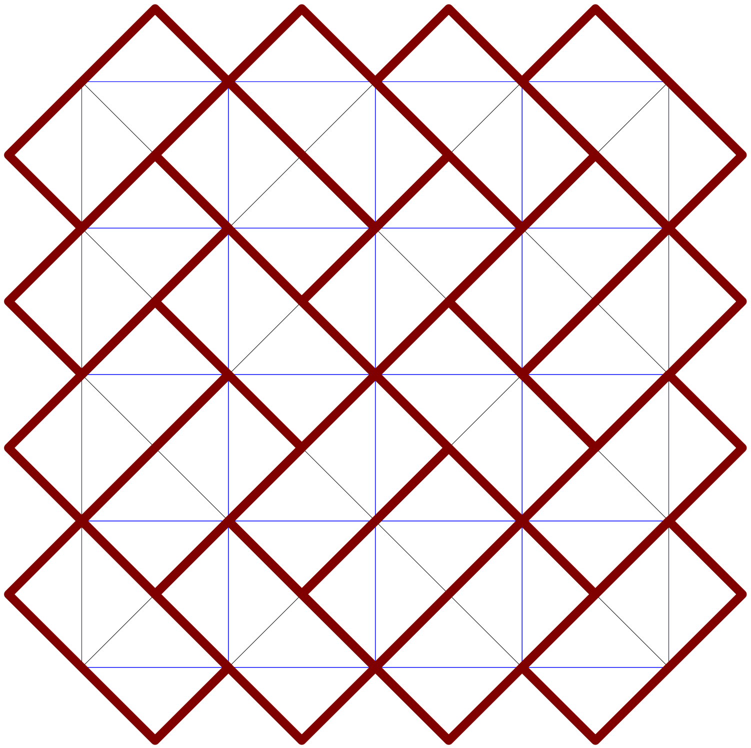
- Young diagrams [Kerov-Vershik '77] [Logan-Shepp '77]
- Boxed plane partitions [Cohn-Larsen-Propp '98]
- Corner melting of a crystal [Ferrari-Spohn '02]
- Plane partitions [Cerf-Kenyon'01][Okounkov-Reshetikhin'01]
- Skewed plane partitions [Okounkov-Reshetikhin '05]

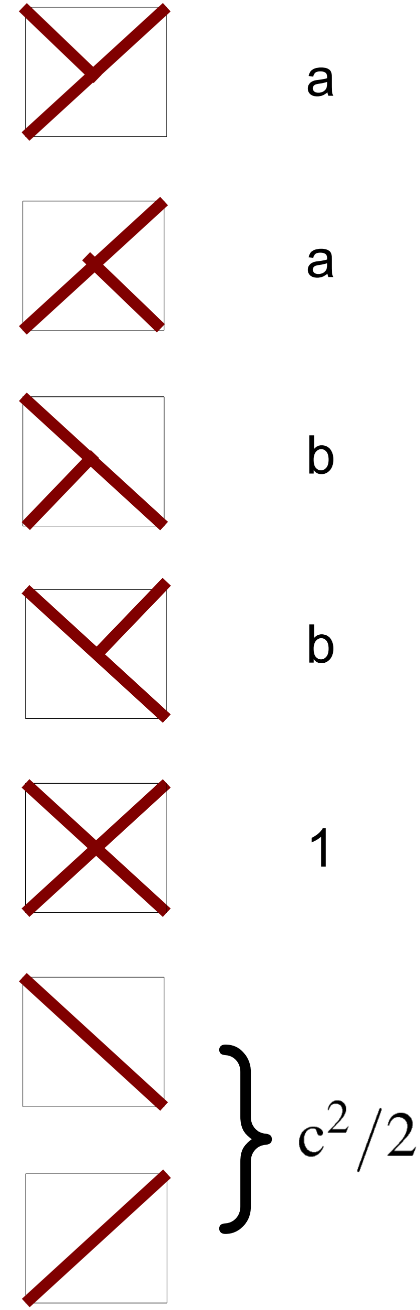
Dimer models and algebraic geometry

[Kenyon, Sheffield, Okounkov, '03-'05]

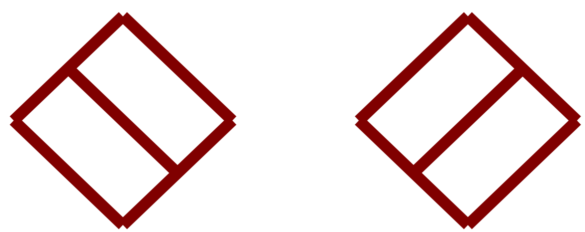
The DW 6VM as a model of interacting dimers

[Elkies-Kuperberg-Larsen-Propp'92]





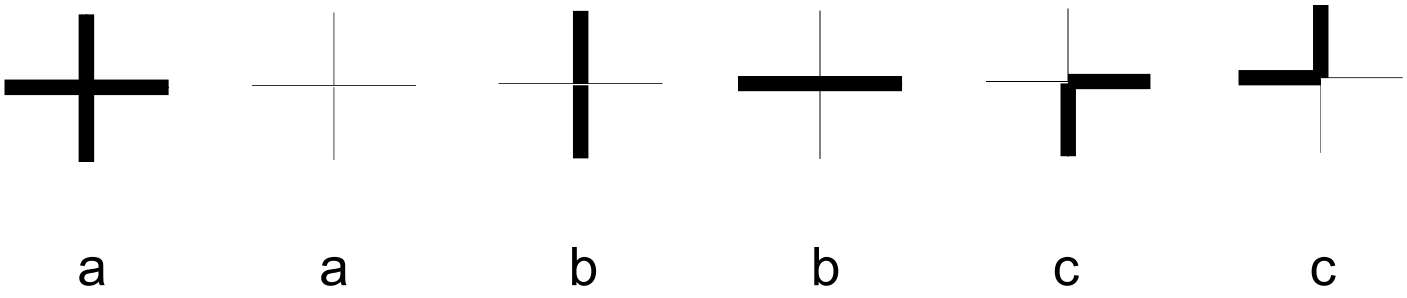
DW 6VM partition function can be seen as a weighted enumeration of the Domino Tilings of Aztec Diamond; in particular a weight $c^2/2$ is assigned to configurations:

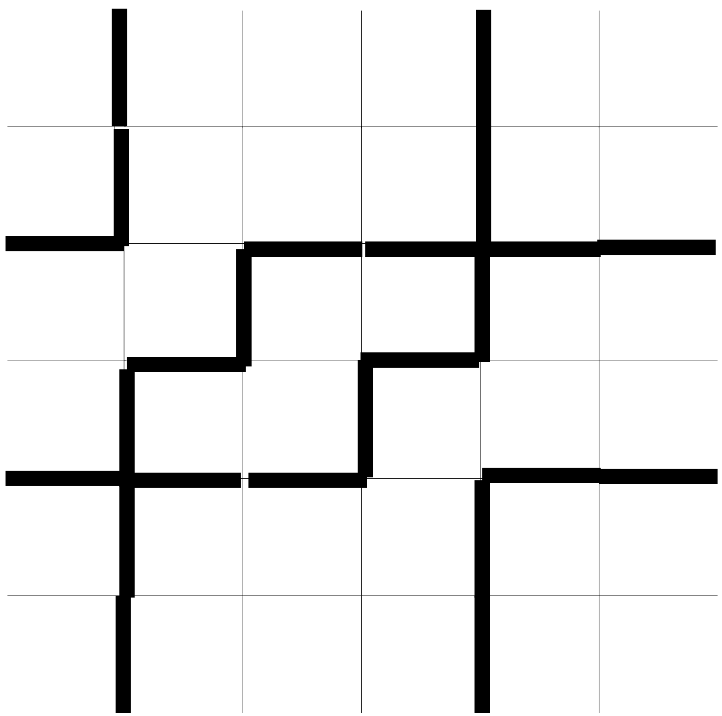


DW 6VM can be seen as a model of interacting dimers on Aztec Diamond.

The six-vertex model

[Lieb '67] [Sutherland'67]



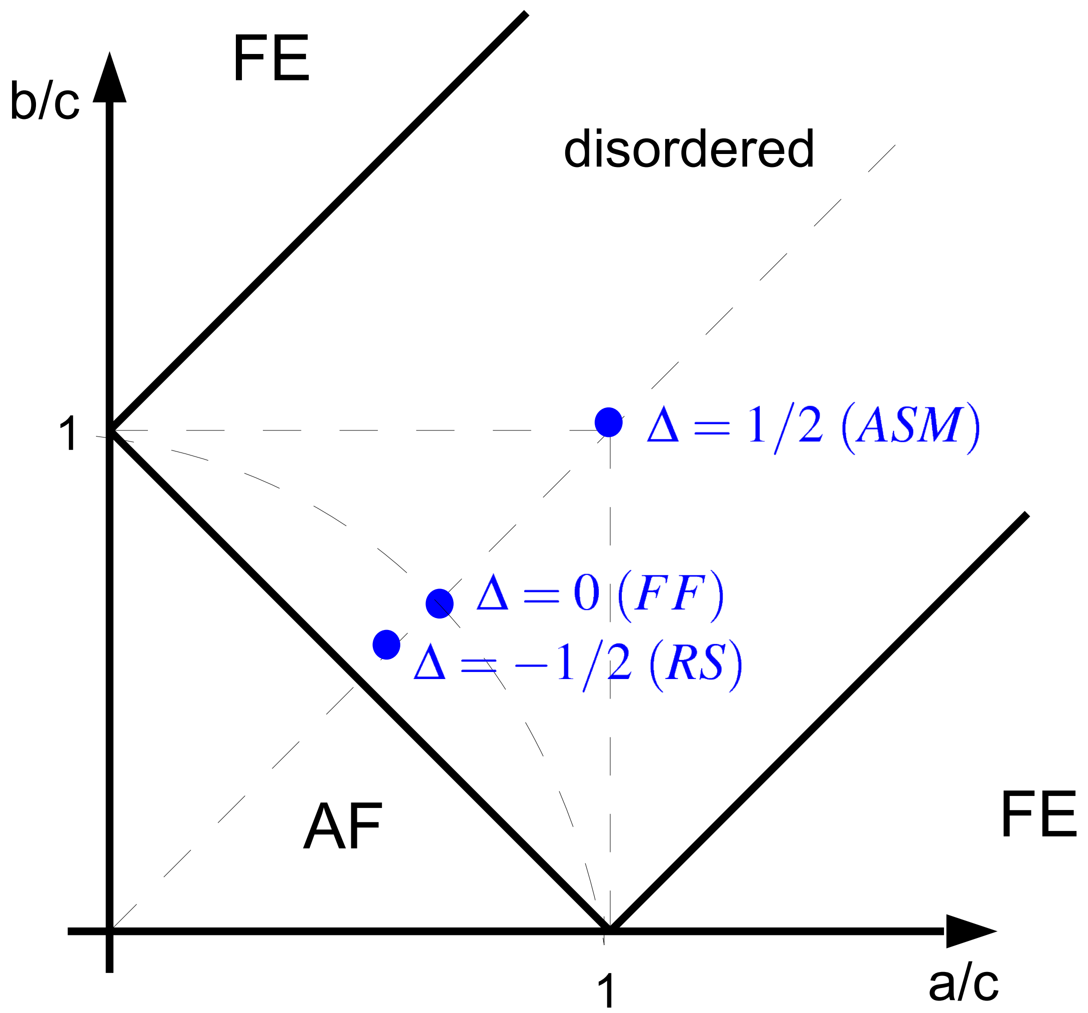


$$
\begin{aligned}
a &= \sin(\lambda + \eta) \\
b &= \sin(\lambda - \eta) \\
c &= \sin 2\eta
\end{aligned}
$$

$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

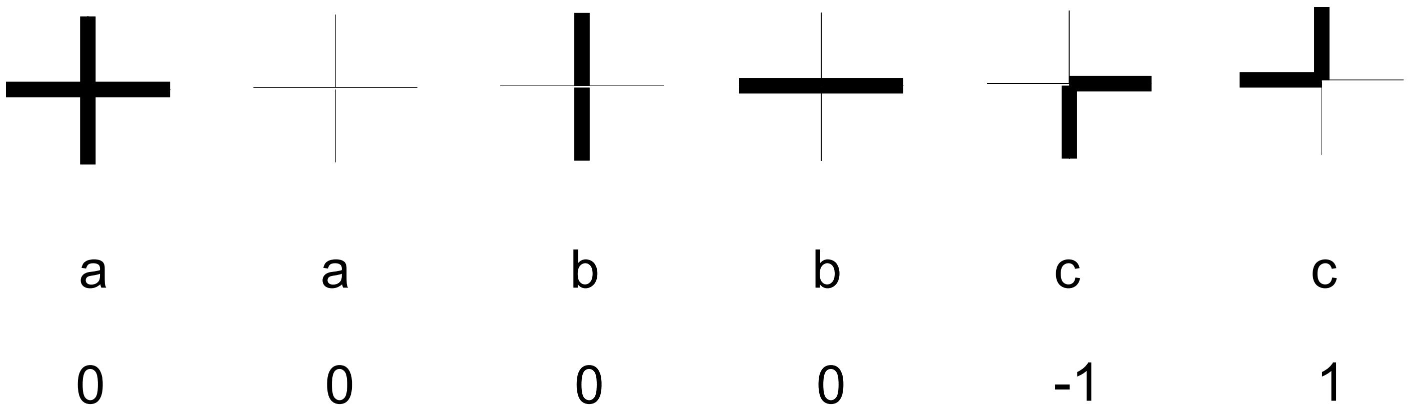
$$t = \frac{b}{a}$$

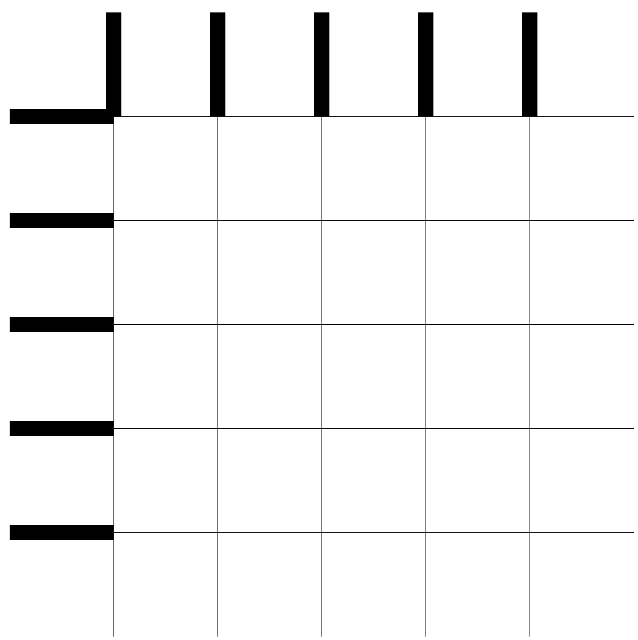
$Z_N = \sum a^{n_1} b^{n_2} c^{n_3}$ $\quad$ $n_1 + n_2 + n_3 = N^2$ $\quad$ Periodic BC



The Domain Wall six-vertex model

[Korepin '82]

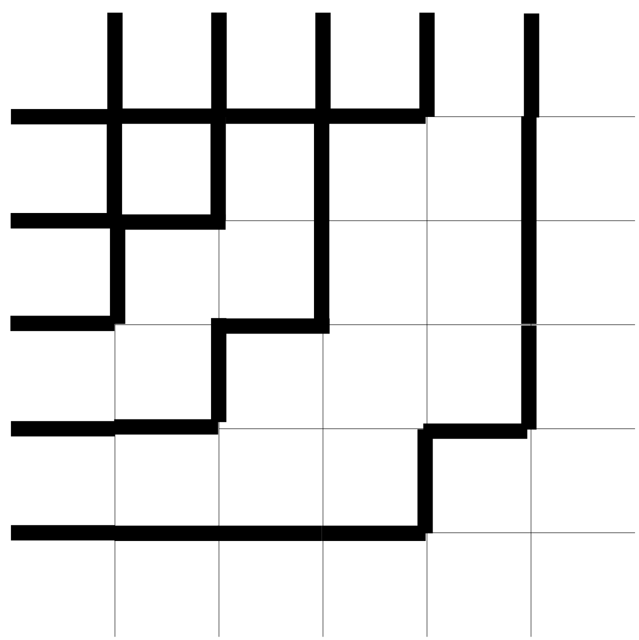




The Domain Wall six-vertex model

[Korepin '82]

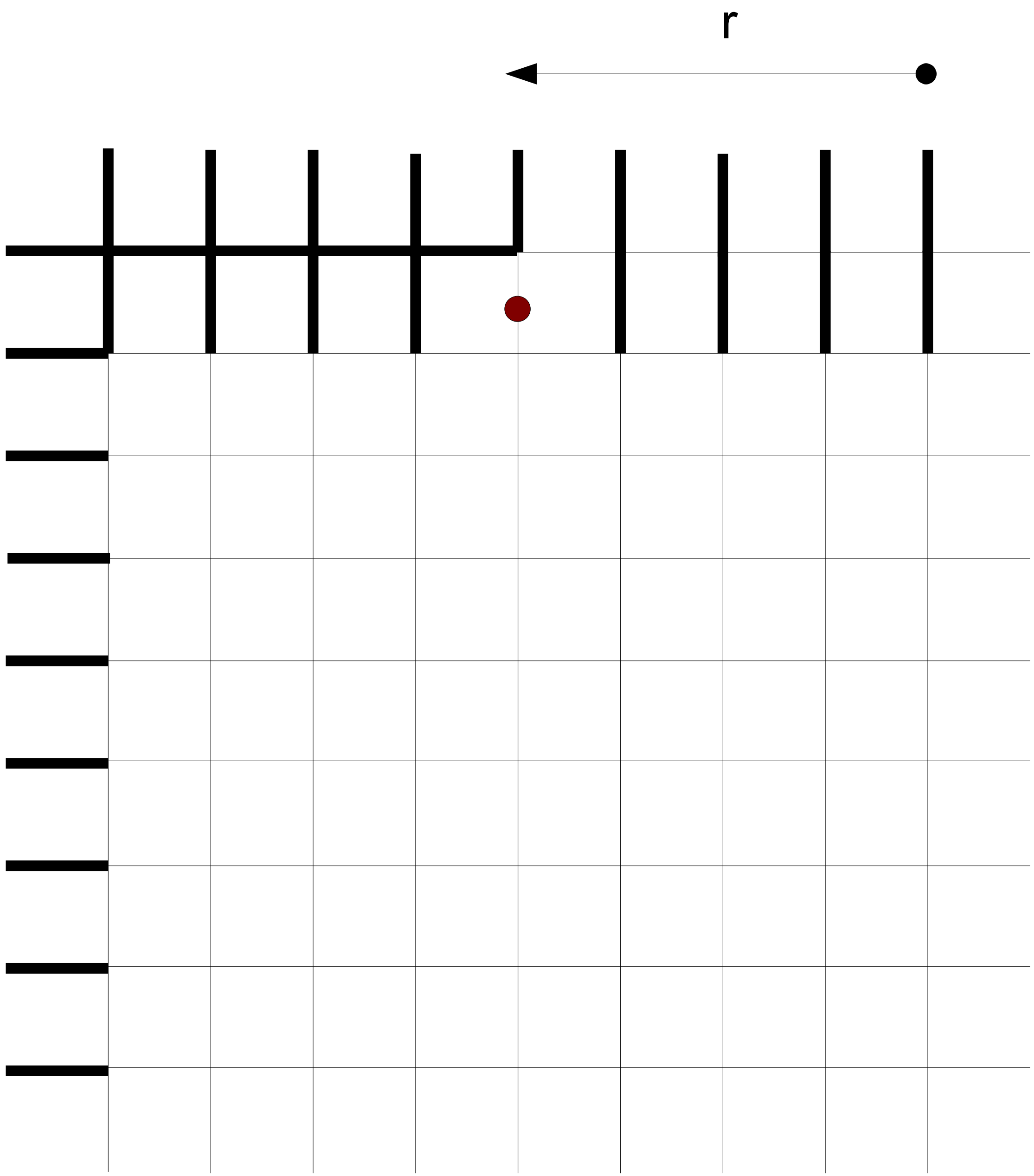
a	a	b	b	c	c
0	0	0	0	-1	1



Domain Wall six vertex model: known results

- Izergin'87: I-K determinant representation and Hankel determinant representation for Z_N ;
- Bogoliubov-Pronko-Zvonarev'02: one point boundary correlation function;

$H_N(r)$



Domain Wall six vertex model: known results

- Izergin'87: I-K determinant representation and Hankel determinant representation for Z_N ;
- Bogoliubov-Pronko-Zvonarev'02: one point boundary correlation function;
- Colomo-Pronko'05: two point boundary correlation function.

All above results have rather implicit form, in terms of determinants.

Domain Wall six vertex model: known results

- Izergin'87: I-K determinant representation and Hankel determinant representation for Z_N ;

- Bogoliubov-Pronko-Zvonarev'02: one point boundary correlation function;

- Colomo-Pronko'05: two point boundary correlation function.

All above results have rather implicit form, in terms of determinants.

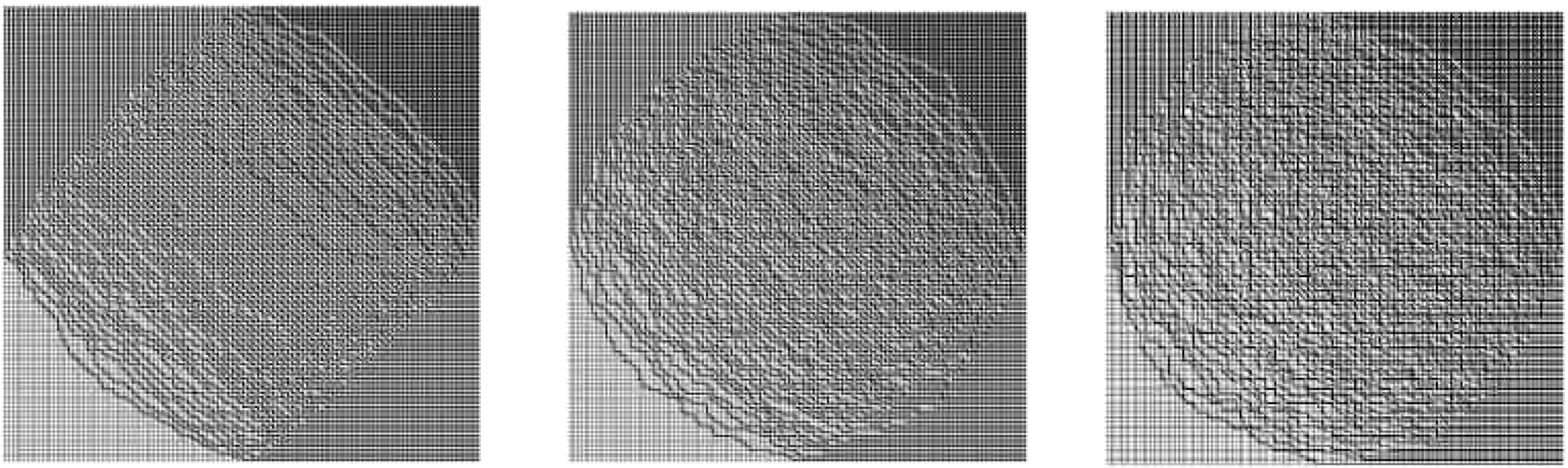
- Korepin Zinn-Justin'00, Zinn-Justin'01, Bleher-Fokin'05-'09: Large N behaviour of Z_N:

Bulk free energy: DWBC $\neq$ PBC

In addition, there are many other results, of more explicit form, for the three specific cases of $\Delta = 0,\ 1/2,\ -1/2$.

Domain Wall six vertex model: numerical results

[Eloranta'99] [Zvonarev-Syluasen'04] [Allison-Reshetikhin'05]



$\Delta = -3$ $\quad$ $\Delta = -0.92$ $\quad$ $\Delta = 0$ (free fermions)

$N = 225$

[Allison-Reshetikhin'05]

The problem

Extend the Arctic Circle Theorem [DWBC 6VM at $\Delta = 0$] to generic values of Δ (including e.g. $\Delta = \frac{1}{2}$: limit shape of ASMs).

- Compute a suitable bulk correlation function

$$F_N(r,s) \qquad\qquad 1 \leq r,s \leq N$$

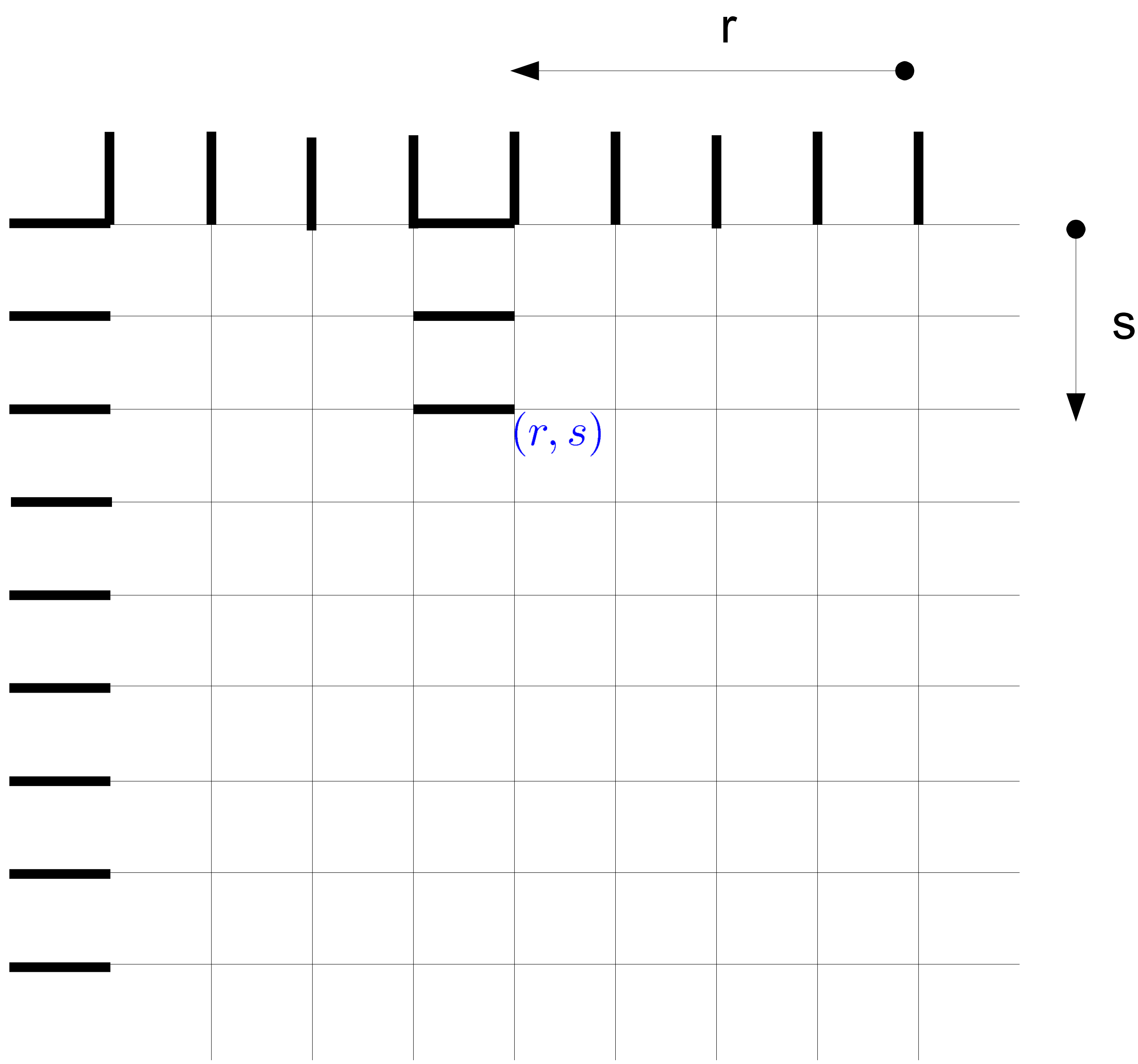
- Evaluate it in the "scaling" limit:

$$N,r,s \longrightarrow \infty \qquad\qquad \frac{r}{N} = x \qquad \frac{s}{N} = y$$

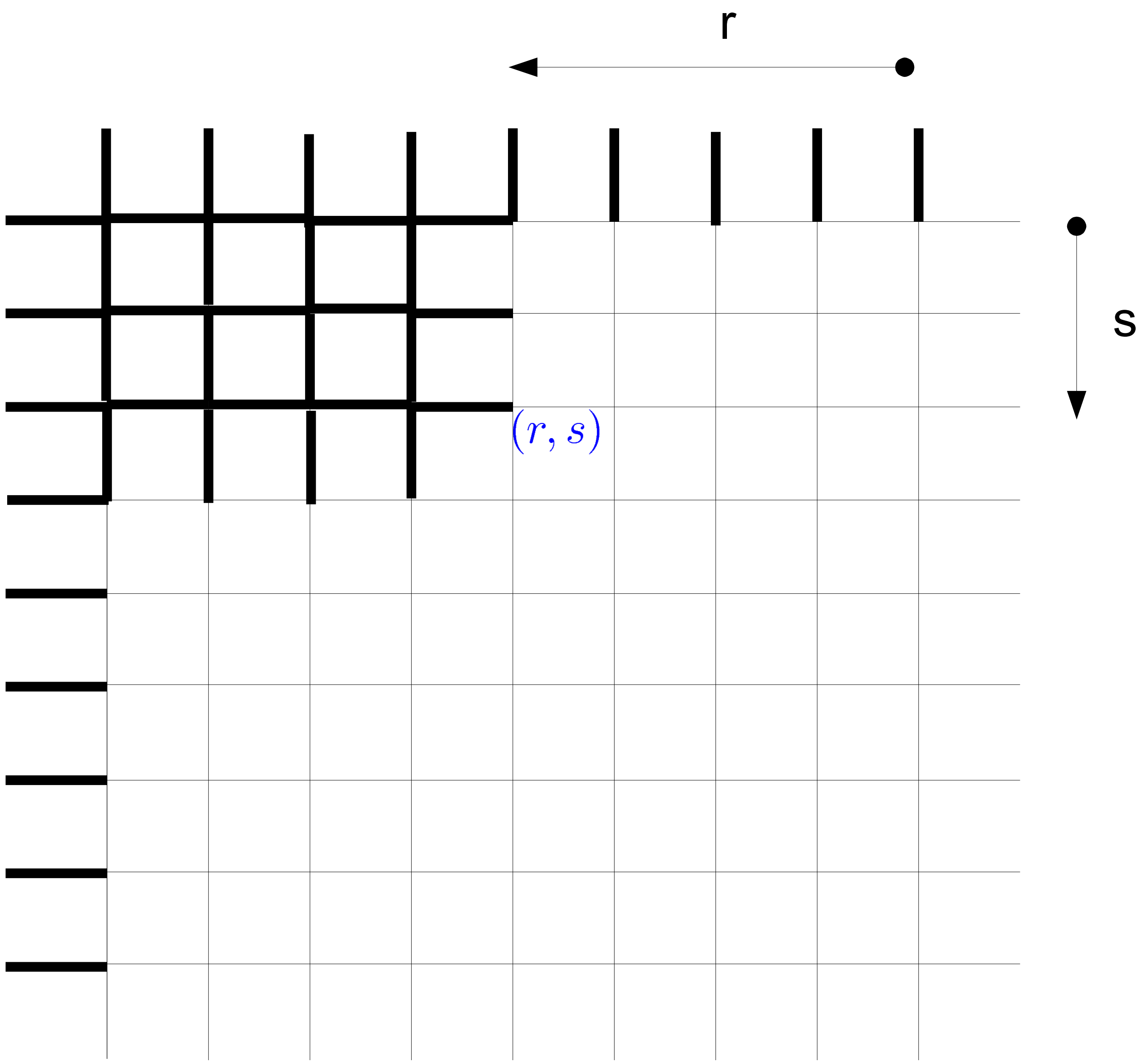
i.e.: evaluate asymptotic behaviour of

$$F(x,y) := \lim_{N\to\infty} F_N(xN, yN) \qquad\qquad x,y \in [0,1]$$

$F_N(r,s)$ Emptiness Formation Probability (EFP)



$F_N(r,s)$ Emptiness Formation Probability (EFP)



Multiple Integral Representation for EFP

Define the generating function for the 1-point boundary correlator:

$$h_N(z) := \sum_{r=1}^{N} H_N(r) z^{r-1}, \qquad h_N(1) = 1 .$$

Now define, for $s = 1, \ldots, N$:

$$h_N^{(s)}(z_1, \ldots, z_s) := \frac{1}{\Delta_s(z_1, \ldots, z_s)} \det_{1 \le j,k \le s} \left[h_{N-s+k}(z_j)(z_j - 1)^{k-1} z_j^{s-k} \right]$$

- The functions $h_N^{(s)}(z_1, \ldots, z_s)$ are totally symmetric polynomials of order $N-1$ in $z_1, \ldots, z_s$.
- They encode the full functional dependence of the partially inhomogeneous partition function from its spectral parameters.

Two important properties of $h_N^{(s)}(z_1, \ldots, z_s)$:

$$h_N^{(s)}(z_1, \ldots, z_{s-1}, 0) = h_N(0) h_{N-1}^{(s-1)}(z_1, \ldots, z_{s-1}) ,$$

$$h_N^{(s)}(z_1, \ldots, z_{s-1}, 1) = h_N^{(s-1)}(z_1, \ldots, z_{s-1}) .$$

NB: An explicit expression of $h_N(z)$ is known for $\Delta = 0, 1/2, -1/2$.

The following Multiple Integral Representation is valid for EFP ($r, s = 1, 2, \ldots, N$):

$$F_N(r,s) = \left(-\frac{1}{2\pi i}\right)^s \oint_{C_0} \cdots \oint_{C_0} d^s z \, h_N^{(s)}(z_1, \ldots, z_s) \prod_{j=1}^{s} \frac{1}{z_j^r (z_j - 1)^s} \times \prod_{1 \le j < k \le s} \frac{(\tilde{z}_j - 1)(z_k - 1)(z_j - z_k)}{\tilde{z}_j z_k - 1} \, .$$

where

$$\tilde{z}_j := \frac{t^2 z_j}{2\Delta t z_j - 1} \, , \qquad j = 1, \ldots, s \, .$$

The contours C_0 are simple anticlockwise contours, enclosing $z = 0$ and no other singularity of the integrand.

Ingredients:

- Quantum Inverse Scattering Method to obtain a determinant representation on the lines of Izergin-Korepin formula;
- Orthogonal Polynomial and Random Matrices technologies to rewrite it as a multiple integral.

Free Fermion point

In this case:

$$\Delta = 0 \qquad\qquad t = 1 \qquad\qquad \tilde{z}_j = -z_j$$

Moreover in this case function $h_N(z)$ is exactly known:

$$h_N(z) = \left(\frac{1+z}{2}\right)^{N-1}$$

MIR for EFP reduces simply to

$$F_N(r,s) = \frac{(-1)^{s(s+1)/2}}{s!2^{s(N-s)}(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{1\le j<k\le s} (z_j - z_k)^2 \prod_{j=1}^{s} \frac{(z_j+1)^{N-s}}{(z_j-1)^s z_j^r}\,.$$

Note the squared Vandermonde determinant.

Saddle point equation and Random Matrices $(\Delta = 0)$

We can view MIR as a Random Matrix Model with logarithmic potential (Triple Penner Model):

$$F_N(r,s) = \frac{(-1)^{s(s+1)/2}}{s!2^{s^2(1/y-1)}(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \exp\Bigg\{ \sum_{\substack{j,k=1 \\ j\neq k}}^{s} \ln|z_j - z_k| + s \sum_{j=1}^{s} \left[\left(\frac{1}{y} - 1 \right) \ln(z_j + 1) - \ln(z_j - 1) - \frac{x}{y} \ln(z_j) \right] \Bigg\}.$$

Saddle Point Equation (SPE) reads:

$$\frac{1}{z_j - 1} + \frac{x/y}{z_j} - \frac{(1/y-1)}{z_j+1} = \frac{2}{s} \sum_{k=1}^{s} \frac{1}{z_j - z_k}, \qquad j = 1, 2, \ldots, s$$

There is some standard approach developed for Random Matrix models, to solve such saddle-point eq. In the present case it turns out to be rather involved, and cannot be generalized to the case of generic $\Delta \neq 0$.

Even in the $\Delta = 0$, this is rather complicate. But we do not need the full solution!

A simple exercise: $s = 1$

$$F_N(r,1) = -\frac{1}{2^{N-1}} \oint_{C_0} \frac{(z+1)^{N-1}}{(z-1)z^r}\, dz$$

Large N behaviour: $x = \frac{r}{N}$ fixed.

Solution of saddle point equation is: $z_{sp} = \frac{x}{1-x}$

A simple exercise: $s = 1$

$$F_N(r,1) = -\frac{1}{2^{N-1}} \oint_{C_0} \frac{(z+1)^{N-1}}{(z-1)z^r}\, dz$$

Large N behaviour: $x = \dfrac{r}{N}$ fixed.

Solution of saddle point equation is: $z_{sp} = \dfrac{x}{1-x}$

- When $z_{sp} < 1$ $\quad (0 < x < \frac{1}{2})$ we get: $\quad F_N(x) \sim \mathrm{e}^{-Nf(x)}$
- When $z_{sp} > 1$ $\quad (\frac{1}{2} < x < 1)$ we get: $\quad F_N(x) \sim \underbrace{-\mathrm{Res}_{z=1}}_{=1!} + \mathrm{e}^{-Nf(x)}$

As $N \to \infty$ we get a step function behavior.

The step occurs when x is such that $z_{sp} = 1$: $\quad \Longrightarrow \quad \Theta(x - 1/2)$

This mechanism holds for any finite value of s.

A nice identity

The following identity holds:

$$\frac{(-1)^{s(s+1)/2}}{s!(t^2+1)^{s(N-s)}(2\pi i)^s} \oint_{C_1} \cdots \oint_{C_1} d^s z \prod_{1\le j<k\le s} (z_j - z_k)^2 \prod_{j=1}^{s} \frac{(z_j+1)^{N-s}}{(z_j-1)^s \, z_j^r} = 1 \,,$$

Note the different contour C_1: clockwise, encircling $z = 1$, and no other singularity of the integrand.

Single Penner Model

[Penner'88][Ambjorn-Kristjansen-Makeenko'94]

$$Z_N \propto \int \mathcal{D}M \mathrm{e}^{-N\mathrm{Tr}[V(M)]} \qquad M = M^+$$

$$\propto \int d^N z\, \Delta_N^2(z)\, \mathrm{e}^{-N\sum_{j=1}^N V(z_j)} \qquad V(M) = q \log M + aM$$

When $q = 1$, the coefficient of $\ln M$ is exactly equal to the order of the Vandermonde. In this case, possibility of `total' condensation of roots of SPEs into the logarithmic well.

Strictly speaking total condensation is impossible (it does not satisfy SPEs). It is to be intended in the sense of condensation of `almost all' roots, but a vanishing fraction.

In the case of `total condensation', among this vanishing fraction of uncondensed roots, there must necessarily be a pair of coinciding real roots.

Summarizing:

- EFP has a step function behaviour in the scaling limit;
- EFP behaviour is governed by the position of SPE roots with respect to the pole at $z = 1$;
- the cumulative residue at such pole is exactly 1;
- Penner model allows for partial/total condensation of eigenvalues in the logarithmic potential well.
- The coefficient of our logarithmic potential well at $z = 1$ is exactly s: possibility of total condensation.

Summarizing:

- EFP has a step function behaviour in the scaling limit;
- EFP behaviour is governed by the position of SPE roots with respect to the pole at $z = 1$;
- the cumulative residue at such pole is exactly 1;
- Penner model allows for partial/total condensation of eigenvalues in the logarithmic potential well.
- The coefficient of our logarithmic potential well at $z = 1$ is exactly s: possibility of total condensation.

Condensation of `almost all' SPE roots at $z = 1$ $\Longleftrightarrow$ Arctic Curves

NB: This last statement is in fact a theorem in the Free Fermion case $(\Delta = 0)$

[Colomo-Pronko'07, Bleher-McLaughlin *(to appear)*]

The Arctic curve ($\Delta = 0$)

SPE reads:

$$\frac{1}{z_j - 1} + \frac{x/y}{z_j} - \frac{(1/y - 1)}{z_j + 1} = \frac{2}{s} \sum_{k=1}^{s} \frac{1}{z_j - z_k} .$$

If we assume condensation, in the large s limit $\rho(w) = \delta(w-1)$, and LHS in SPE becomes:

$$\frac{2}{z_j - 1}$$

And the `reduced' SPE thus reads simply

$$-\frac{1}{z_j - 1} + \frac{x/y}{z_j} - \frac{(1/y - 1)}{z_j + 1} = 0 ,$$

and determines the position of the `very few' possibly uncondensed roots.

We require two coinciding roots:

$$\begin{cases} (x-1)z^2 + (1-2y)z - x = 0 \\ 2(x-1)z + (1-2y) = 0 \end{cases}$$

The solution of the above system (linear in x, y) is

$$x = \frac{1}{z^2 + 1}, \qquad y = \frac{(z-1)^2}{2(z^2+1)} \qquad z \in [1, +\infty)$$

Which is exactly the parametric form of the (top left quarter of the) Arctic Circle!
Indeed, eliminating z :

$$(x - \tfrac{1}{2})^2 + (y - \tfrac{1}{2})^2 = \tfrac{1}{4}$$

Generic values of $\Delta \neq 0$

1) Our nice identity still holds:

$$\left(-\frac{1}{2\pi i}\right)^s \oint_{C_1} \cdots \oint_{C_1} d^s z\, h_{N,s}(z_1,\ldots,z_s) \prod_{j=1}^{s} \frac{1}{z_j^r (z_j-1)^s} \prod_{1\le j<k\le s} \frac{(\tilde{z}_j-1)(z_k-1)(z_j-z_k)}{\tilde{z}_j z_k - 1} = 1$$

2) again the poles at $z_j = 1$ ($j = 1, \ldots, s$) have power s just as the order of the Vandermonde determinant.

Generic values of $\Delta \neq 0$

1) Our nice identity still holds:

$$\left(-\frac{1}{2\pi i}\right)^s \oint_{C_1} \cdots \oint_{C_1} d^s z\, h_{N,s}(z_1,\ldots,z_s) \prod_{j=1}^{s} \frac{1}{z_j^r (z_j-1)^s} \prod_{1\le j<k\le s} \frac{(\tilde{z}_j-1)(z_k-1)(z_j-z_k)}{\tilde{z}_j z_k - 1} = 1$$

2) again the poles at $z_j = 1$ ($j = 1,\ldots,s$) have power s just as the order of the Vandermonde determinant.

Main assumption

Arctic Curve occurs in correspondence to the following configuration of SPE solutions:

- "almost all" SPE solutions condense to the value $z = 1$;
- a vanishing fraction of SPE solutions survive condensation and lies somewhere in the complex plane; among them there is a pair of coinciding real roots, lying in $[1, +\infty[$.

Generic values of $\Delta \neq 0$

The saddle Point Equation now reads:

$$-\frac{s}{z_j-1}-\frac{r}{z_j}+s\frac{t^2-2\Delta t}{t^2 z_j-2\Delta t z_j+1}+\partial_{z_j}\ln h_N^{(s)}(z_1,\ldots,z_s)$$
$$-\frac{t^2-2\Delta t+1}{(t^2 z_j-2\Delta t z_j+1)^2}\partial_{u_j}\ln h_s^{(s)}(u_1,\ldots,u_s)+2\sum_{\substack{k=1\\k\neq j}}^{s}\frac{1}{z_j-z_k}$$
$$-\sum_{\substack{k=1\\k\neq j}}^{s}\frac{t^2 z_k-2\Delta t}{t^2 z_j z_k-2\Delta t z_j+1}-\sum_{\substack{k=1\\k\neq j}}^{s}\frac{t^2 z_k}{t^2 z_j z_k-2\Delta t z_k+1}=0,$$

$$\left(u_j=\frac{1-z_j}{(t^2-2\Delta t)z_j+1}\right)$$

The procedure of condensation leads to the following equation for the vanishing fraction of uncondensed roots

$$\frac{y}{z-1}-\frac{x}{z}-\frac{yt^2}{t^2 z-2\Delta t+1}+\lim_{N\to\infty}\frac{1}{N}\partial_z\ln h_N(z)=0.$$

Generic values of Δ (disordered regime $|\Delta| < 1$)

So, the reduced SPE, for the vanishing fraction of uncondensed roots, is:

$$\frac{y}{z-1} - \frac{x}{z} - \frac{yt^2}{t^2 z - 2\Delta t + 1} + \lim_{N\to\infty} \frac{1}{N} \partial_z \ln h_N(z) = 0\,.$$

We need now the large N behaviour of $h_N(z)$, for generic Δ.

Generic values of Δ (disordered regime $|\Delta| < 1$)

So, the reduced SPE, for the vanishing fraction of uncondensed roots, is:

$$\frac{y}{z-1} - \frac{x}{z} - \frac{yt^2}{t^2 z - 2\Delta t + 1} + \lim_{N\to\infty} \frac{1}{N} \partial_z \ln h_N(z) = 0\,.$$

We need now the large N behaviour of $h_N(z)$, for generic Δ.

For $|\Delta| < 1$, we have:

$$h_N(z) \underset{N\to\infty}{\sim} \left[\frac{\sin\gamma(\lambda-\eta)}{\gamma\sin(\lambda-\eta)}\right]^N \left[\frac{\sin(\zeta+\lambda-\eta)\sin(\gamma\zeta)}{\sin\gamma(\zeta+\lambda-\eta)\sin\zeta}\right]^N \mathrm{e}^{o(N)}$$

where

$$z(\zeta) = \frac{\sin(\lambda+\eta)}{\sin(\lambda-\eta)} \frac{\sin(\zeta+\lambda-\eta)}{\sin(\zeta+\lambda+\eta)}, \qquad \text{and} \qquad \gamma := \frac{\pi}{\pi - 2\eta}\,.$$

$$\Delta = \cos 2\eta \qquad\qquad t = \frac{\sin(\lambda-\eta)}{\sin(\lambda+\eta)}$$

NB: $z \in [1, +\infty)$ corresponds to $\zeta \in [0, \pi - \lambda - \eta)$

The equation for uncondensed roots now read:

$$x\Phi(\zeta+\lambda-\eta,2\eta)-y\Phi(\zeta,2\eta)+\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))=0$$

where

$$\Phi(\mu,\nu)=\frac{\sin(\nu)}{\sin(\mu)\sin(\mu+\nu)}\,.$$

Its derivative is:

$$x\Psi(\zeta+\lambda-\eta,2\eta)-y\Psi(\zeta,2\eta)+\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))=0\,,$$

where

$$\Psi(\mu,\nu)=\frac{\sin\nu\sin(2\mu+\nu)}{\sin^2\mu\sin^2(\mu+\nu)}\,.$$

Solve the above system, linear in x, y:

We get:

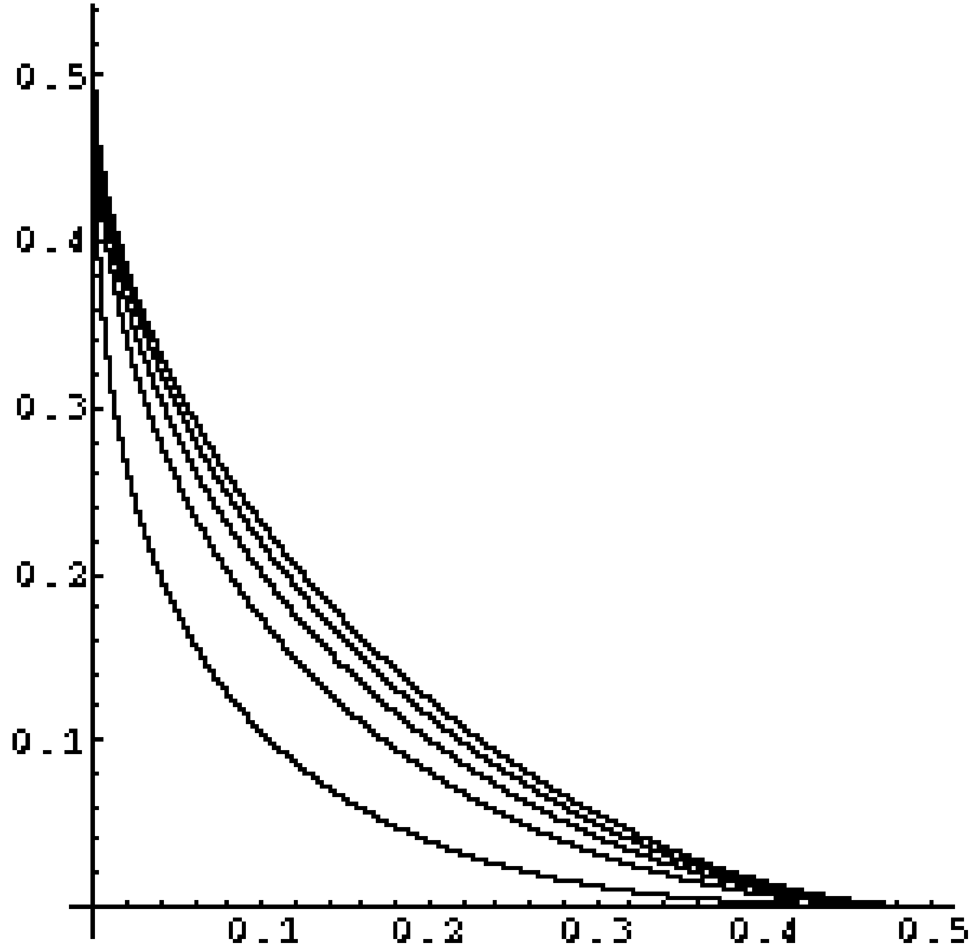
$$\begin{aligned} x &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\ &\qquad \times\left\{\left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta,2\eta)\right. \\ &\qquad\qquad \left.-\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta,2\eta)\right\}, \\ y &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\ &\qquad \times\left\{\left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta+\lambda-\eta,2\eta)\right. \\ &\qquad\qquad \left.-\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta+\lambda-\eta,2\eta)\right\}. \end{aligned}$$

Parametric form of limit shape for generic Δ, with parameter $\zeta\in[0.\pi-\lambda-\eta]$, and

$$\gamma:=\frac{\pi}{\pi-2\eta}, \qquad \Phi(\mu,\nu)=\frac{\sin(\nu)}{\sin(\mu)\sin(\mu+\nu)}, \quad \Psi(\mu,\nu)=\frac{\sin\nu\sin(2\mu+\nu)}{\sin^2\mu\sin^2(\mu+\nu)}.$$

NB: γ rational $\Longrightarrow$ algebraic curve

γ irrational $\Longrightarrow$ non-algebraic curve



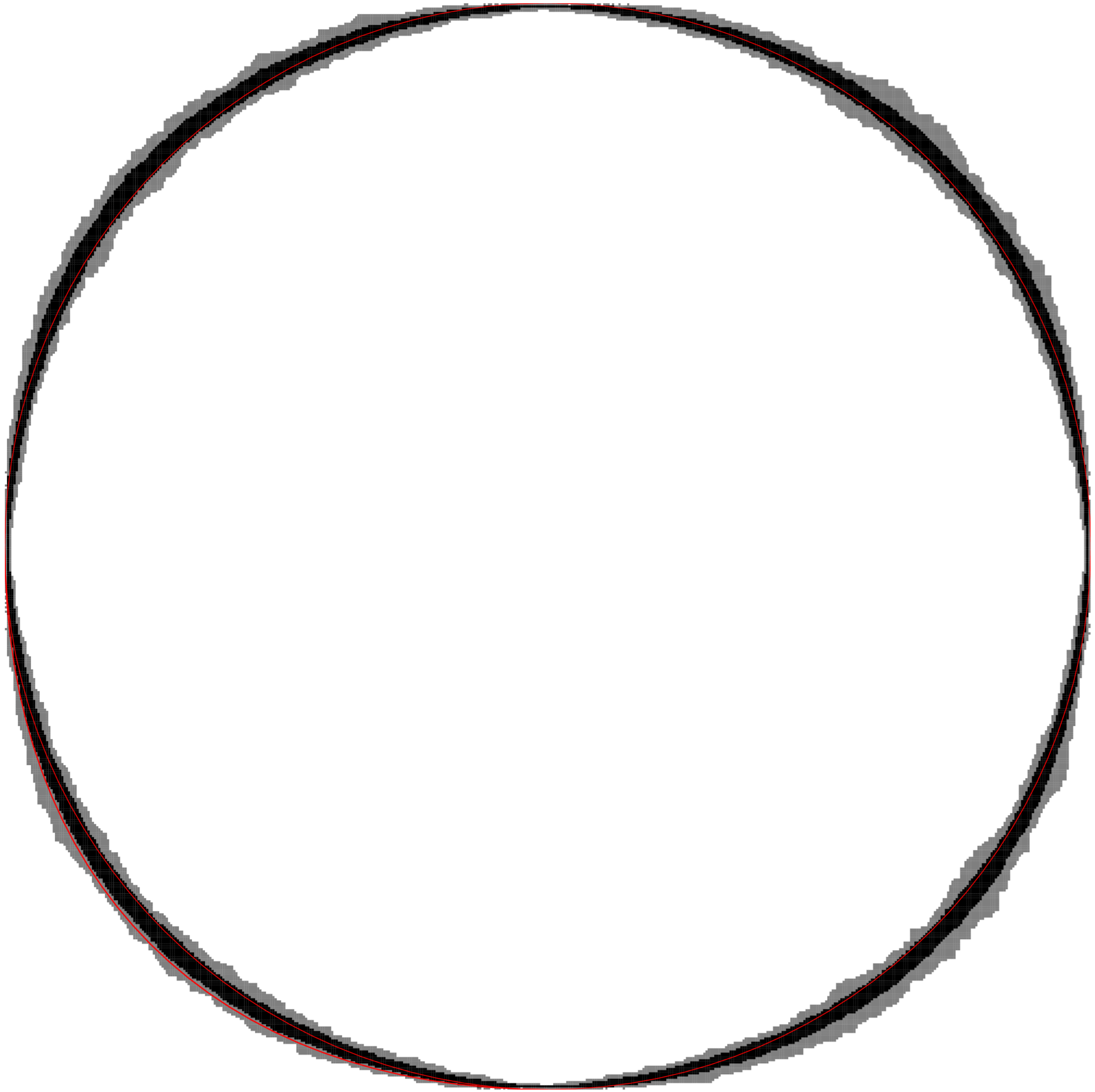
Limit shapes for $\Delta = 0.9,\ 0.5,\ 0,\ -0.5,\ -0.9.$

$$(2x-1)^2 + (2y-1)^2 - 4xy = 1, \qquad x, y \in [0, \tfrac{1}{2}].$$

ASMs: N=500

199 samples

$\Delta = 1/2$



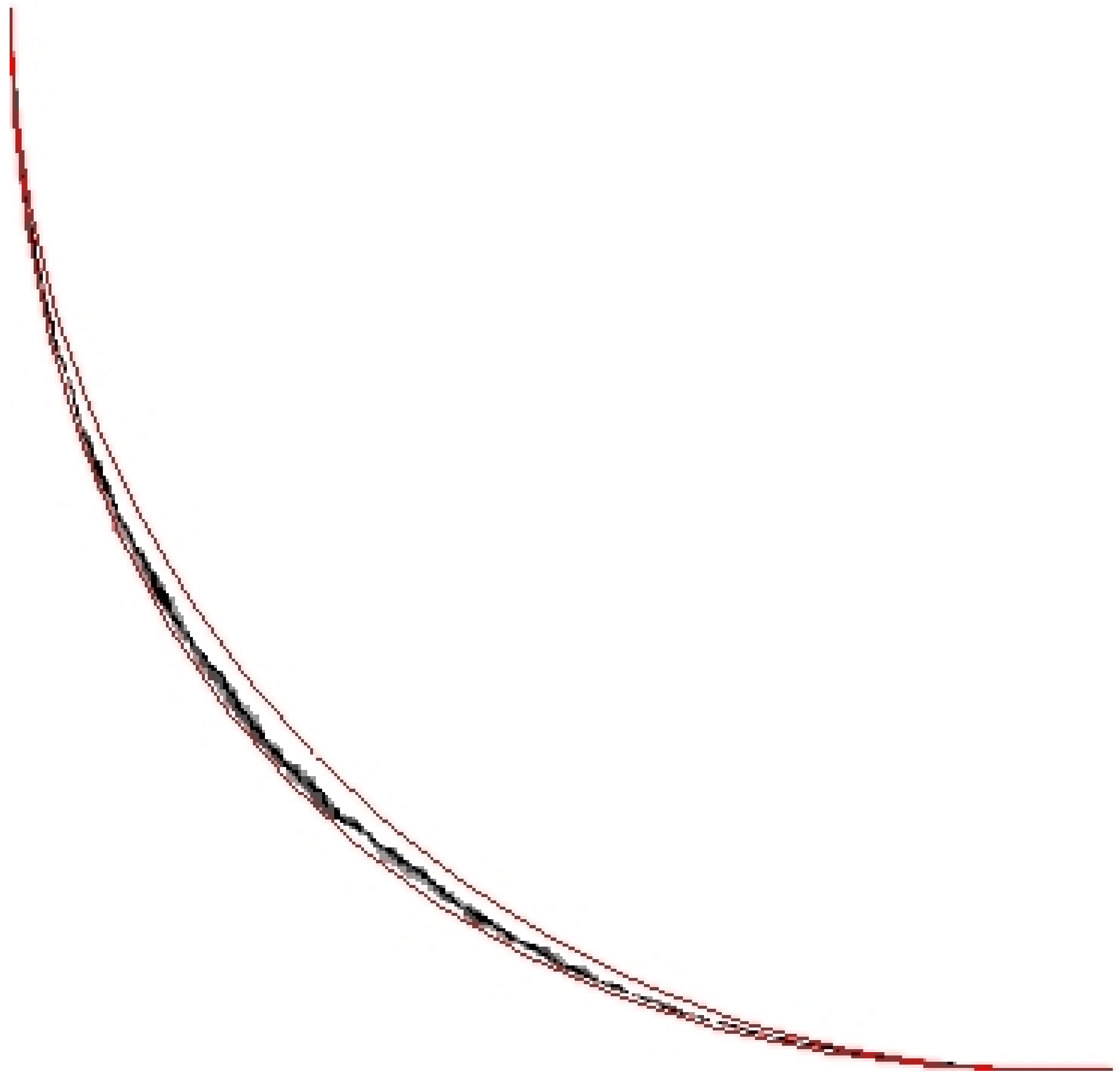
Ben Wieland (January 2008)

http://www.math.brown.edu/~wieland

ASMs: N=1500

10 samples

$\Delta = 1/2$



Ben Wieland (April 2008)

http://www.math.brown.edu/~wieland

What about fluctuations?

Fluctuations of the limit shape are driven by the evaporation of SPE solutions from the logarithmic well (Penner potential of Random Matrices), just like in the $\Delta = 0$ case. From universality considerations, the Airy process of Arctic Circle [Johansson'05] is again expected.

Alternating Sign Matrices

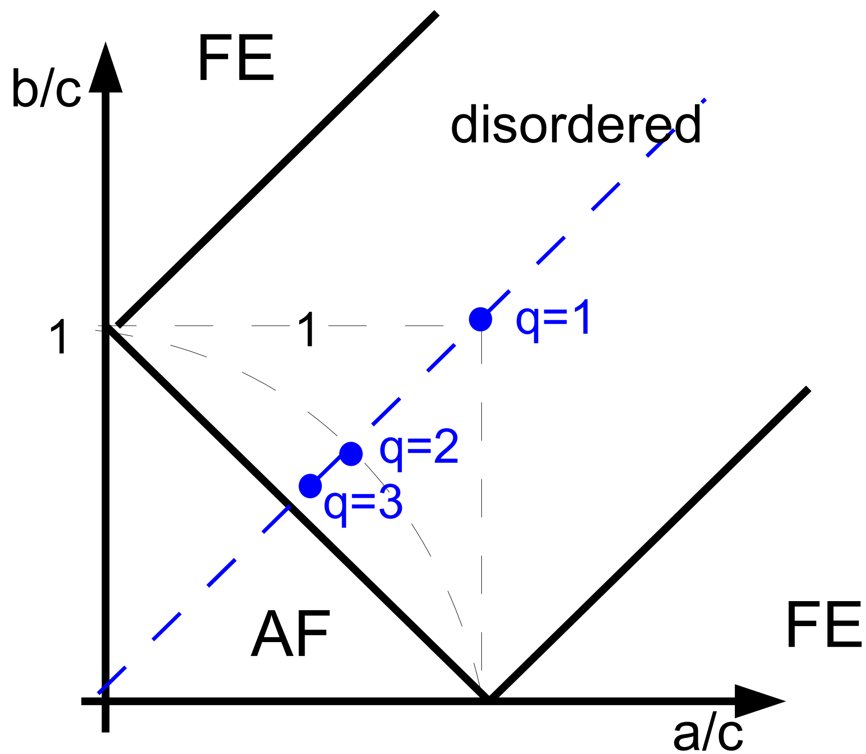
$N \times N$ matrix with entries $\in \{0, 1, -1\}$ and such that:

- non-zero entries alternate in sign;
- for each line or column, sum of entries equals 1.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

ASMs $q-$enumeration: assign weight q^k to an ASM with k "-1" entries.

It corresponds to the domain-wall six-vertex model when $t = 1$ and $\Delta = 1 - \frac{q}{2}$



What about the $q \to 0$ limit?

For finite N, ASMs $\overrightarrow{q \to 0}$ `permutation matrices'.

We expect a trivial arctic curve, degenerating to the boundary of the unit square

What about the $q \to 0$ limit?

For finite N, ASMs $\underset{q \to 0}{\longrightarrow}$ `permutation matrices'.

We expect a trivial arctic curve, degenerating to the boundary of the unit square

From general formula we get instead:

$$x + y = \frac{1}{2} - \frac{1}{\pi} \cos \pi (x - y) \,, \qquad\qquad x, y \in [0, \frac{1}{2}]$$

NB: $N \to \infty$ and $q \to 0$ do not commute.

More generally, for the DW 6VM, as $b - a$ varies over the interval $(-1, 1)$, the arctic curve is deformed continuously from one diagonal of the unit square to the other.

What about the $q \to 0$ limit?

For finite N, ASMs $\overrightarrow{q\to 0}$ `permutation matrices'.

We expect a trivial arctic curve, degenerating to the boundary of the unit square

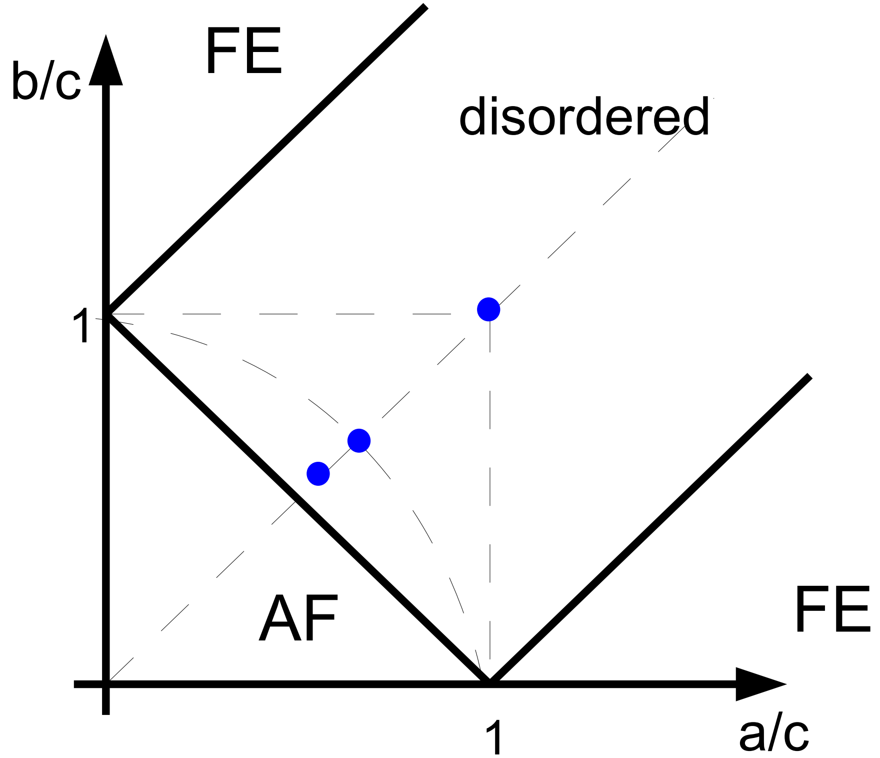
From general formula we get instead:

$$x + y = \frac{1}{2} - \frac{1}{\pi} \cos \pi(x - y)\,, \qquad\qquad x, y \in [0, \frac{1}{2}]$$

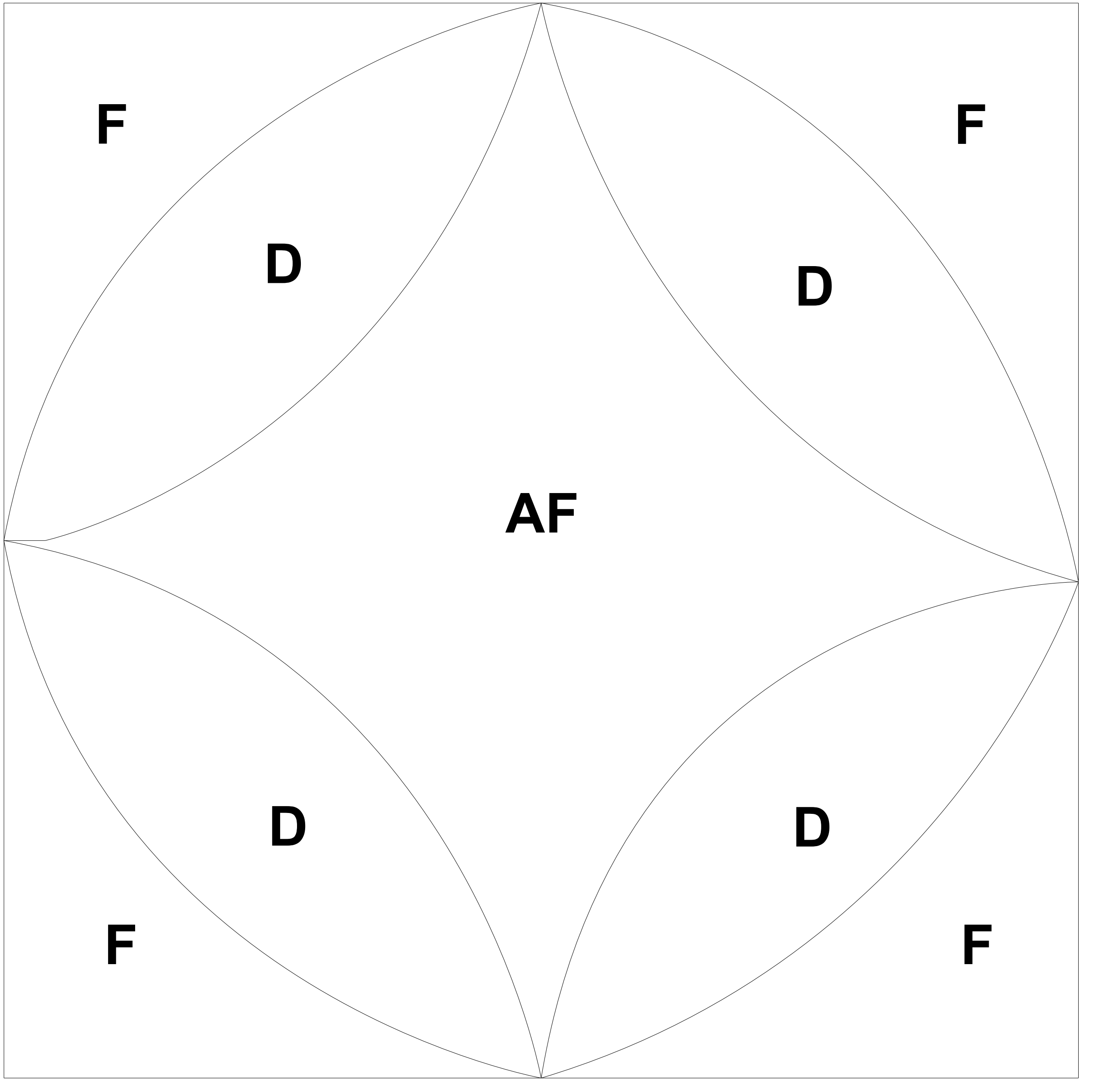
NB: $N \to \infty$ and $q \to 0$ do not commute.

Question: What does this curve describe?
Of which model is it the Arctic curve?

Antiferromagnetic regime ($\Delta < -1$)



Antiferromagnetic regime ($\Delta < -1$)



Antiferromagnetic regime ($\Delta < -1$)

The whole construction given above is still valid.

We only need to evaluate $\lim_{N\to\infty} \partial_z \ln h_N(z)$

For $\Delta \to -\infty$ this is possible. In fact this is another particular case (together with the $\Delta = 0$ case) were the condensation hypothesis has been proved.

Antiferromagnetic regime ($\Delta < -1$)

The whole construction given above is still valid.

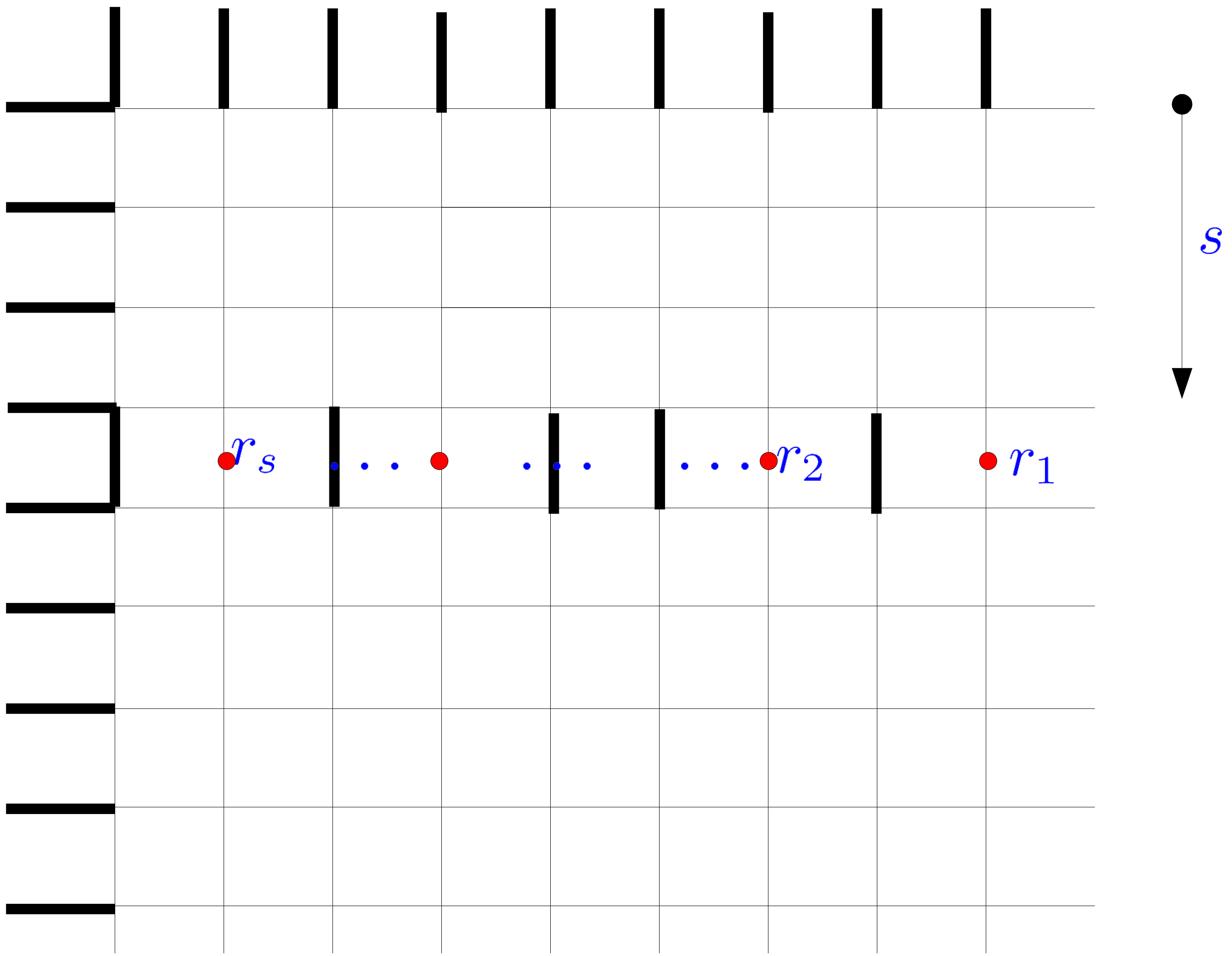
We only need to evaluate $\lim_{N\to\infty} \partial_z \ln h_N(z)$

For $\Delta \to -\infty$ this is possible. In fact this is another particular case (together with the $\Delta = 0$ case) were the condensation hypothesis has been proved.

Question: How to evaluate

$\lim_{N\to\infty} \partial_z \ln h_N(z)$ for $\Delta \in (-\infty, -1)$?

Free Boundaries



$Z_{N,s}(\mathbf{r})$

Free Boundaries

Such quantity is suitable for going beyond the Arctic curve and investigate the limit shape of the heigth function of the DW 6VM.

Natural question: what is the asymptotic distribution of these s full lines on the `free' boundary, as

$$N, s \to \infty \qquad \frac{s}{N} = y \quad ?$$

NB: for a complete treatment of the analogous problem in the case of dimers on the hexagonal lattice see [Di Francesco-Reshetikhin'09]

Free Boundaries

For $Z_{N,s}(\mathbf{r})$ the following Multiple Integral Representation holds:

$$Z_{N,s}(\mathbf{r}) = cost. \prod_{j=1}^{s} t^{r_j} \left(\frac{1}{2\pi i}\right)^s \oint_{C_0} \cdots \oint_{C_0} d^s z \, h_N^{(s)}(z_1, \dots, z_s) \prod_{j=1}^{s} \frac{1}{z_j^{r_j}} \times \prod_{1 \le j < k \le s} \frac{(z_j - z_k)}{z_j z_k - 2\Delta t z_j + t^2} \, .$$

NB: specializing the parameter to the ice point (ASMs, $\Delta = \frac{1}{2}\,,\ t = 1$) this formula generalize to arbitrary s recent results for doubly refined enumeration of monotonous triangle and trapezoids [Fischer-Romik'09]

Free Boundaries

For $Z_{N,s}(\mathbf{r})$ the following Multiple Integral Representation holds:

$$Z_{N,s}(\mathbf{r}) = cost. \prod_{j=1}^{s} t^{r_j} \left(\frac{1}{2\pi i}\right)^s \oint_{C_0} \cdots \oint_{C_0} d^s z \, h_N^{(s)}(z_1, \ldots, z_s) \prod_{j=1}^{s} \frac{1}{z_j^{r_j}}$$
$$\times \prod_{1 \leq j < k \leq s} \frac{(z_j - z_k)}{z_j z_k - 2\Delta t z_j + t^2} .$$

NB: specializing the parameter to the ice point (ASMs, $\Delta = \frac{1}{2}$, $t = 1$) this formula generalize to arbitrary s recent results for doubly refined enumeration of monotonous triangle and trapezoids [Fischer-Romik'09]

Question: How to investigate the scaling limit of this representation?