Descending Plane Partitions and Permutations

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Abstract

The connections between alternating sign matrices and descending plane partitions had been pondered upon ever since it was discovered that they were counted by the same numbers. A more refined conjecture proposed by Mills, Robbins and Rumsey in 1983 states that the number of ASMs with k -1's is the same as that of descending plane partitions with k special parts. As a first step towards this understanding, we exhibit a natural bijection between descending plane partitions with no special part and permutations.

Definitions: ASMs

An *alternating sign matrix* (ASM) is a square matrix whose only nonzero entries are 1 and -1, in which all row and column sums are 1, and the nonzero entries alternate in sign in every row and column.

The number of alternating sign matrices of size n is given by

$$D(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}. \tag{1}$$

Proved by Zeilberger and then Kuperberg in 1996.

The number of alternating sign matrices counted where the 1 in the first row occurs at column k is given by

$$D(n,k) = \binom{n+k-2}{k-1} \frac{(2n-k-1)!}{(n-k)!} \prod_{j=0}^{n-2} \frac{(3j+1)!}{(n+j)!}. \tag{2}$$

Proved by Zeilberger in 1996. Recent proof by Ilse Fischer (2007) uses more general monotone triangles.

Descending Plane Partitions

A *descending plane partition* (DPP) is an array $a = (a_{ij})$ of positive integers defined for $j \geq i \geq 1$ that is written in the form

$$\begin{array}{ccccccc} a_{11} & a_{12} & \cdots & \cdots & \cdots & \cdots & a_{1,\mu_1} \\ & a_{22} & \cdots & \cdots & \cdots & a_{2,\mu_2} & \\ & & \cdots & \cdots & \cdots & & \\ & & a_{rr} & \cdots & a_{r,\mu_r} & & \end{array} \tag{3}$$

where,

1. $\mu_1 \geq \cdots \geq \mu_r$,

2. $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} > a_{i+1,j}$ whenever both sides are defined,

3. $a_{i,i} > \mu_i - i + 1$ for $i \leq i \leq r$,

4. $a_{i,i} \leq \mu_{i-1} - i + 2$ for $1 < i \leq r$.

A descending plane partition of *order n* is a descending plane partition all of whose entries are less than or equal to n.

Example

Here is a DPP of order 25.

$$\begin{array}{ccccccc} 12 & 12 & 11 & 9 & 8 & 5 & 1 \\ & 7 & 7 & 6 & 5 & 4 & \\ & & 5 & 5 & 3 & 3 & \\ & & & 3 & 2, & & \end{array} \tag{4}$$

and here is one more

$$. \tag{5}$$

Exercise: Check each axiom!

The number of DPPs is given by $D(n)$ in (1).

Understudied Objects!

Discrete Mathematics > Combinatorics > Partitions >

Descending Plane Partition

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$$\begin{array}{cccccc} 7 & 7 & 6 & 6 & 3 & 1 \\ & 6 & 5 & 4 & 2 & \\ & & 3 & 3 & & \\ & & & 2 & & \end{array}$$

A descending plane partition of order n is a two-dimensional array (possibly empty) of positive integers less than or equal to n such that the left-hand edges are successively indented, rows are nonincreasing across, columns are decreasing downwards, and the number of entries in each row is strictly less than the largest entry in that row. Implicit in this definition are the requirements that no "holes" are allowed in the array, all rows are flush against the top, and the diagonal element *must* be filled if *any* element of its row is filled. The above example shows a decreasing plane partition of order seven.

$$\begin{array}{cc} 3 & 3 \\ & 2 \end{array} \quad 3 \quad 3 \quad 3 \quad 2 \quad 3 \quad 1 \quad 3 \; 2 \; \emptyset$$

The sole descending plane partition of order one is the empty one ($\emptyset$), the two of order two are "2" and ($\emptyset$), and the seven of order three are illustrated above. In general, the number of descending plane partitions of order n is equal to the number of $+1$-bordered alternating sign matrices: 1, 2, 7, 42, 429, ... (Sloane's A005130).

SEE ALSO: Alternating Sign Matrix, Plane Partition

REFERENCES:

Andrews, G. E. "Plane Partitions (III): The Weak Macdonald Conjecture." *Invent. Math.* **53**, 193-225, 1979.

Bressoud, D. and Propp, J. "How the Alternating Sign Matrix Conjecture was Solved." *Not. Amer. Math. Soc.* **46**, 637-646.

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Timeline

- 1979, George Andrews invents DPPs to prove the so-called Weak ($q = 1$) Macdonald conjecture about the number of cyclically symmetric plane partitions in a cubic box $m \times m \times m$.

- 1982, Mills, Robbins and Rumsey prove the general Macdonald Conjecture.

- 1983, MRR make a number of far-reaching conjectures relating ASMs and DPPs.

- 1985, Gessel and Viennot publish their landmark paper (which implicitly contains the bijection in this paper).

- 1986, Robbins and Rumsey define the λ-determinant, which is one of the motivations for defining ASMs.

- 2004, P. Lalonde shows that the antiautomorphism of DPPs is Gessel-Viennot duality for a set of NILPs.

- 2006, C. Krattenthaler exhibits a bijection between DPPs and rhombus tilings of a hexagon with a triangle removed from the center.

Special Parts of a DPP

An entry $a_{i,j}$ of the descending plane partition a is called a *special part* if $a_{ij} \leq j - i$.

Back to the example:

$$\begin{array}{ccccccc} 12 & 12 & 11 & 9 & 8 & \underline{5} & \underline{1} \\ & 7 & 7 & 6 & 5 & \underline{4} & \\ & & 5 & 5 & 3 & \underline{3} & \\ & & & 3 & 2 & & \end{array} \tag{6}$$

Permutations: Ascents

A permutation π of the letters $\{1, \ldots, n\}$ has an *ascent* at position k with $1 \le k < n$, if $\pi_k < \pi_{k+1}$.

The number of permutations on n letters with k ascents is the Eulerian number $E(n,k)$ for $n \ge 0$, $0 \le k \le n-1$. The triangle begins

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 4 & & 1 & \\ 1 & & 11 & & 11 & & 1, \end{array} \tag{7}$$

and is given by

$$E(n,k) = (k+1)E(n-1,k) + (n-k)E(n-1,k-1), \tag{8}$$

with the initial condition $E(0,k) = 0$ if $k > 0$ and $E(0,0) = 1$.

Permutations: Inversions

The *inversion number* $I(\pi)$ of a permutation π on n letters is the number of pairs of elements i, j such that $i < j$ and $\pi_i < \pi_j$.

Another interpretation of $I(\pi)$ is the total number of elementary transpositions required to return π to the completely descending permutation $n(n-1)\dots21$. This is not the usual convention, but we will use this because we will count ascents.

Yet another interpretation is based on the corresponding matrix A_π of the permutation. Then $I(\pi)$ is the number of 0's which have a 1 to the right and above,

$$I(\pi) = \sum_{i<k, j<l} (A_\pi)_{ij}(A_\pi)_{kl}.$$

This formula can also be used for ASMs.

Conjecture 3 of MRR, 1983

Suppose that n, k, m, p are nonnegative integers, $1 < k < n$. Let $\mathcal{A}(n, k, m, p)$ be the set of alternating sign matrices such that
(i) the size of the matrix is $n \times n$,
(ii) the 1 in the top row occurs in position k,
(iii) the number of -1's in the matrix is m,
(iv) the number of inversions in the matrix is p.

On the other hand, let $\mathcal{D}(n, k, m, p)$ be the set of descending plane partitions such that
(I) no parts exceed n,
(II) there are exactly k - 1 parts equal to n,
(III) there are exactly m special parts,
(IV) there are a total of p parts.

Then $\mathcal{A}(n, k, m, p)$ and $\mathcal{D}(n, k, m, p)$ have the same cardinality.

The Main Result

There is a natural one-to-one correspondance between descending plane partitions of order n with k rows and no special part, and permutations of size n with k ascents.

In this result, k varies from 0 to $n-1$. The empty DPP, $a = \phi$ counts as a permutation with zero rows, and vacuously, with no special part. There is also exactly one permutation with zero ascents, namely $\pi = n(n-1)\cdots 21$.

The number of descending plane partitions of order n with k rows and no special parts is given by the Eulerian number $E(n, k)$.

Lemma: A single row

There is a natural one-to-one correspondance between descending plane partitions of order n with one row $a = (a_1, \ldots, a_m)$ and permutations $\beta\gamma$ of size n with a single ascent.

Essential idea:

Set $\gamma = (a_1, a_2 - 1, \ldots, a_m - (m-1))$ and β to be the remaining elements in decreasing order.

1. Part (2) of the Definition ensures that elements of γ are strictly decreasing,

2. Part (3) of the Definition ensures that $a_1 > m$, which implies $\gamma \neq (m, m-1, \ldots, 1)$ and thus $\beta \neq 0$ and moreover the last element of β is smaller than a_1.

What about inversions?

Ideally, we would like the number of inversion $I(\beta\gamma) = m$ in keeping with MRR's conjecture. However, this does not happen. We do however, obtain a concrete expression

$$I(\beta\gamma) = \sum_{i=1}^{m} a_i - m^2. \tag{9}$$

This is because γ_1 takes

$$\gamma_1 - m = a_1 - m$$

steps to reach its original position, γ_2 takes

$$\gamma_2 - (m-1) = a_2 - 1 - (m-1) = a_2 - m$$

steps and so on.

Good news: $I(\pi)$ depends on a, and not the order n.

Properties of the bijection

Assume a has length m. Then

1. γ has length m and β has length $n-m$,

2. $\beta_{n-m} = 1$ occurs if and only if either $m = 1$ or $a_m > m$. Assuming $1 < p < n$,

$$\beta_{n-m} = p \Leftrightarrow \forall i > m-p+1, a_i = m \text{ and } a_{m-p+1} > m.$$

3. $\beta_1 = n$ occurs if and only if $a_1 < n$. Assuming $0 < p < m$,

$$\beta_1 = n-p \Leftrightarrow \forall i \leq p, a_i = n \text{ and } a_{p+1} < n.$$

Lastly, $\beta_1 = n-m$ if and only if $a_1 = \cdots = a_m = p$.

Building a k-rowed DPP

1. Any row of a DPP is, by itself, also a valid DPP. Moreover, a row which is part of a DPP with no special part is also a DPP with no special part. The latter follows from the shifted position of successive rows.

2. Removing the last row from a DPP yields another valid DPP. Obviously, if the original DPP had no special part, neither will be new one.

Therefore, any k rowed DPP with no special parts can be built from a ($k-1$ rowed DPP <u>and</u> a single rowed DPP) <u>with</u> no special parts in the obvious way.

The Extension Lemma

Given a set S of positive integers of cardinality n, there exists a natural bijection between the DPP a with one row and no special part whose length m satisfies $m < n$ and $a_1 \leq n$, and a sequence of all the elements of S with one ascent.

Essential idea:

Construct the invertible map $\phi : S \to [n]$, which takes the smallest element of S to 1, the next to 2, and so on.

Now, the one rowed DPP gives rise to a sequence $\beta\gamma$ with one ascent. Use ϕ^{-1} to obtain the image of this sequence.

Illustration of the Essential Idea

Consider the DPP of order $n = 9$ with no special part

$$\begin{array}{ccccc} 7 & 7 & 6 & 5 & 5 \\ & 4 & 4 & 4 & \\ & & 3 & 2 & \end{array} \tag{10}$$

Then we start with the permutation 987654321.

$$\begin{array}{ccc} 77655 & \rightarrow & 98|53|76421 \\ 444 & \rightarrow & 9853|71|642 \\ 32 & \rightarrow & 985371|4|62 \end{array} \tag{11}$$

and we end up with the permutation 985371462, which has exactly three ascents.

Proof: DPP $\rightarrow$ Permutation

Start with a DPP with k rows $\alpha^{(1)}, \ldots, \alpha^{(k)}$ of length $m_1, \ldots, m_k$ respectively. We want to construct a permutation with k ascents.

Use induction on k. The case $k = 1$ is done. We assume that the first $k-1$ rows of a have been used to create a permutation with $k-1$ ascents, which we denote $\beta^{(1)} \ldots \beta^{(k)}$.

Let S be the set of integers in $\beta^{(k)}$ of cardinality m_{k-1}. $\alpha^{(k)}$ is a DPP with one row of length less that m_{k-1}, no special part whose first element is less than m_{k-1}.

Therefore, we use the extension lemma and create a sequence with one ascent from the elements of S, whose decreasing sequences we call $\gamma^{(k)}$ and $\gamma^{(k+1)}$.

Proof: DPP $\to$ Permutation (contd.)

In other words, we start with the permutation

$$\underbrace{\beta^{(1)}} \cdots \underbrace{\beta^{(k-1)}} \underbrace{\beta^{(k)}},$$

and end up with

$$\underbrace{\beta^{(1)}} \cdots \underbrace{\beta^{(k-1)}} \underbrace{\gamma^{(k)}} \underbrace{\gamma^{(k+1)}}.$$

Assume the extension lemmas for rows $\alpha^{(k-1)}$ and $\alpha^{(k)}$ utilized maps ϕ and ϕ' respectively.

Let p be the last entry in $\beta^{(k-1)}$. We want to show that the first entry of $\gamma^{(k)}$ is greater than p. Using the property Lemma on the row $\alpha^{(k-1)}$, we have that

$$a_{k-1,k+m_{k-1}-2} = \cdots = a_{k-1,k+m_{k-1}-\phi(p)} = m_{k-1}$$

and

$$a_{k-1,k-1} \geq \cdots \geq a_{k-1,k+m_{k-1}-\phi(p)-1} \geq m_{k-1} + 1.$$

Proof: DPP $\rightarrow$ Permutation (contd.)

In "$\phi(\beta^{(k)})$", the first $m_{k-1} - (\phi(p) - 1)$ letters of $\beta^{(k)}$ are greater than $\phi(p)$ and the last $p-1$ letters of $\beta^{(k)}$ are $p-1, \ldots, 1$.

For it to happen that $\gamma_1^{(k)} < p$, $\gamma_1^{(k)}$ must be one of the last $\phi(p)-1$ letters of $\beta^{(k)}$. This implies that the action of $\alpha^{(k)}$ forces at least all the first $m_{k-1} - (\phi(p)-1)$ letters into $\gamma^{(k+1)}$, which can only happen if

$$a_{k,k} = \cdots = a_{k,k+m_{k-1}-\phi(p)} = m_{k-1},$$

using the property lemma again. But this would imply

$$a_{k,k+m_{k-1}-\phi(p)} = a_{k-1,k+m_{k-1}-\phi(p)},$$

which violates the descent condition in the definition of the DPP.

Note that ϕ' did not enter the argument because the last $\phi(p)-1$ letters are mapped to $1, \ldots, \phi(p)$ by ϕ' also.

Proof: Permutation $\rightarrow$ DPP

We start with the permutation $\beta^{(1)} \ldots \beta^{(k+1)}$ with k ascents.

We read the permutation from the right, using the extension lemma on every successive pair of descending segments $\beta^{(j)}\beta^{(j+1)}$ to create the row $\alpha^{(j)}$ of the DPP.

Conditions (3) and (4) of the definition of the DPP as well as the "non-speciality" of every element are ensured by the construction. The only non-trivial item left to check is the strict columnwise descent.

Proof: Permutation $\to$ DPP (contd.)

With the previous notation for the permutation with k ascents, we denote the lengths of $\beta^{(k-1)}, \beta^{(k)}$ and $\beta^{(k+1)}$ being $m_{k-2} - m_{k-1}$, $m_{k-1} - m_k$ and m_k respectively so that the last three rows for the DPP, denoted $\alpha^{(k-2)}, \alpha^{(k-1)}$ and $\alpha^{(k)}$ have lengths m_{k-2}, m_{k-1} and m_k.

We will now show that the ascent condition on the permutation guarantees columnwise descent on the DPP. Suppose the last entry of $\beta^{(k)}$ is given by

$$\beta^{(k)}_{m_{k-1}-m_k} = p$$

and the first entry of $\beta^{(k+1)}$ by

$$\beta^{(k+1)}_1 = r, \qquad r > p.$$

Proof: Permutation → DPP (contd.)

Further, let the maps used in extension lemma for the rows $\alpha^{(k)}$ and $\alpha^{(k-1)}$ be denoted by ϕ and ϕ' respectively.

Here is a sketch.

$$\cdots\cdots[\qquad \beta^{(k-1)} \qquad]\quad \underbrace{[\qquad \beta^{(k)} \qquad p]\quad \underbrace{[r \qquad \beta^{(k+1)} \qquad]}_{\alpha^{(k)},\phi}}_{\alpha^{(k-1)},\phi'}$$

Proof: Permutation $\rightarrow$ DPP (contd.)

We now use the property lemma to conclude the following:

- From Part 2,

$$a_{k-1,k+m_{k-1}-2} = \cdots = a_{k-1,k+m_{k-1}-\phi'(p)} = m_{k-1},$$

  and $a_{k-1,k+m_{k-1}-\phi'(p)-1} \geq m_{k-1} + 1$.

- From Part 3,

$$a_{k,k} = \cdots = a_{k,k+m_{k-1}-\phi(r)-1} = m_{k-1},$$

  and either $m_{k-1} - m_k = r$ or $a_{k,k+m_{k-1}-\phi(r)} < m_{k-1} - 1$.

Proof: Permutation $\rightarrow$ DPP (contd.)

The ascent of the permutation implies $r > p$. This in turn implies $\phi(r) \geq \phi'(p)$ because it is possible that there are no elements between r and p.

A violation of the descent condition of the DPP would entail the overlapping of the parts of the $k-1$th and kth rows of a which equal m_{k-1}. This means

$$k + m_{k-1} - \phi'(p) \leq k + m_{k-1} - \phi(r) - 1,$$

which implies that $\phi(r) \leq \phi'(p) - 1$. But this is a contradiction. Therefore a permutation with k ascents gives rise to a DPP with k rows.

Inversions

The story is easily generalized.

If a permutation π has k ascents, then the inversion number is given by the corresponding descending plane partition a with k rows of sizes $m_1, \ldots, m_k$ as

$$I(\pi) = \sum_{i=1}^{k} \sum_{j=i}^{m_i+i-1} a_{i,j} - \sum_{i=1}^{k} m_i^2. \tag{12}$$

Essential idea:

Each row can only add to the inversion number.

The number of elementary transpositions required to reach the permutation given by the previous row is given by a formular parallel to that for the single row case.

Gessel-Viennot's bijection I: DPP

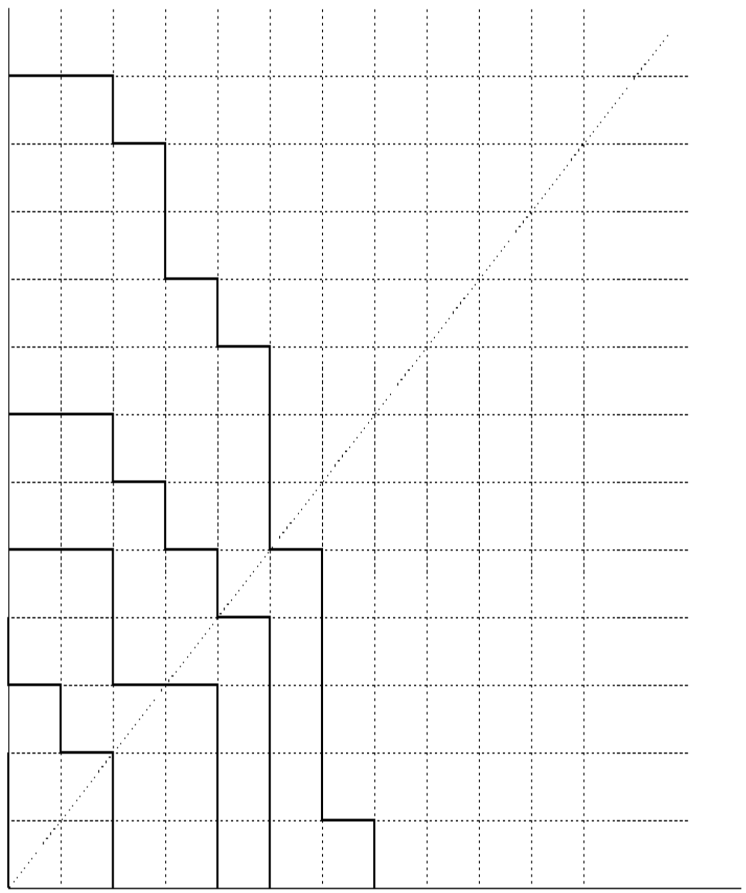
Every DPP of order n corresponds to set of nonintersecting lattice paths starting from some $\{(0, i)\}$ to $\{(j, 0)\}$ using horizontal and vertical steps where $0 \leq i, j \leq n$.

The first path starts from $(0, n)$. For every j, there is a horizontal step at height $a_{1,j}$ until one reaches $(k, 0)$ for some k.

The second path starts at $(0, k)$ and proceeds in a similar manner, and so on.

The last path is a downward path from some $(0, l)$ to the origin. Thus, we have $r + 1$ paths where r is the number of rows of a.

Example



Look at the horizontal steps to the right of the diagonal.

Gessel-Viennot's bijection II: Permutation

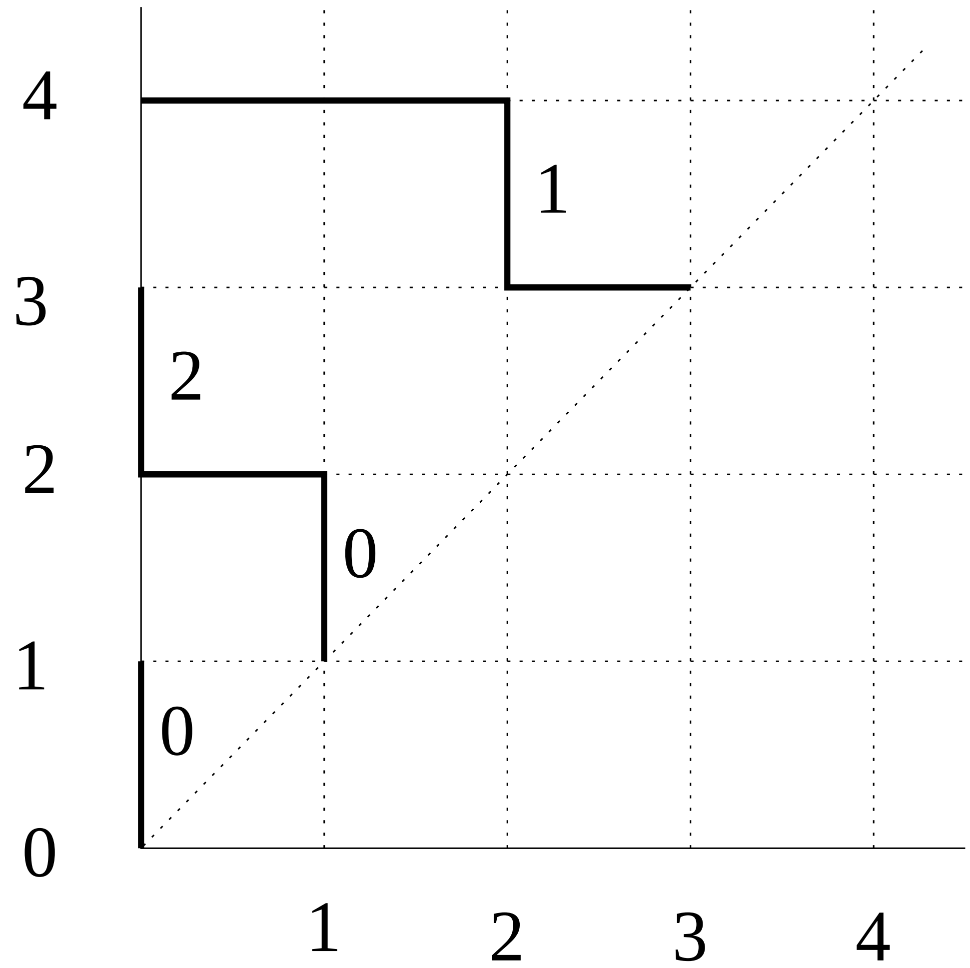
Every permutation π of size n corresponds to an inversion table $f : [n] \cup \{0\} \rightarrow \mathbb{N}$ such that $f(i) = \#\{j | 1 \leq j < i, \pi_j < \pi_i\}$.
Example: $\pi = 3142$, $f = 0021$.

For each permutation, lattice paths start at some $(0, i)$ and end at (j, j) using horizontal and vertical steps where $0 \leq i, j \leq n$.

For all i such that $f(i) \geq f(i+1)$, a path starts at $(0, i)$ and has a vertical step at $((k, j-1), (k, j))$ if $f(j) = j - k - 1$.

The Point: All special parts of a DPP correspond to horizontal steps to the right of the diagonal.

Correspondance Example



Paths for $\pi = 3142$.

The corresponding DPP is got by looking at the vertical positions of the horizontal steps and so we get

$$a = \begin{array}{ccc} 4 & 4 & 3 \\ & 2. & \end{array} \tag{13}$$

From our bijection we get that a corresponds to the permutation

$$2413,$$

which is exactly the reverse of the permutation we began with.

Open Questions

The bijection for arbitrary number of -1's. We can look at the set $M_k(n)$ of $n \times n$ ASMs with k -1's which corresponds to DPPs with k special parts.

It turns out that $\#M_k(n)$ has a natural representation as

$$\sum_{j=0}^{3k} c_{j,k} \, \frac{n!}{j!} \binom{n}{j},$$

which can be seen by considering ASMs as generalized permutations in the natural way (ongoing work with F. LeGac and R. Cori).

Possibly a kind of *generalized ascent* does the job.