Probing the higher loop dilatation operator of N=4 SYM

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Outline

- Anomalous dimensions, Integrability and such
- What type of calculations are easy
- Higher loop Hamiltonians
- The calculation of the 4-loop Hamiltonian in the $SU(2)$ sector
- Prospects at higher loops

Anomalous dimensions, Integrability and such

- anomalous dimensions=eigenvalues of dilatation operator

$$D(\lambda) = D_1 + \lambda D_2 + \lambda^2 D_3 + \dots$$

- (Long) String energies

$$E(\lambda) = \sqrt{\lambda} E_0 + E_1 + \frac{1}{\sqrt{\lambda}} E_2 + \dots$$

- AdS/CFT: $E(\lambda) = D(\lambda)$

▪ Comparison difficult because of strong / weak coupling

  ◇ successful if effective small couplings exist (large q-numbers)

Integrability may help with the extrapolation

- nontrivial qualitative relations between unrelated operators

- Known explicit higher loop anomalous dimensions in $\mathcal{N} = 4$ SYM:
  - BMN operators (2-loops) — Gross, Mikhailov, RR
  - single-magnon dispersion relation (all loops) — Gross, Mikhailov, RR; Santambrogio, Zanon
  - 2- and 3-loop cusp anomalous dimension — Kotikov, Lipatov, Onishchenko, Velizhanin; Bern, Dixon, Smirnov
  - 4-loop cusp anom. dim. tour de force — Bern, Czakon,Dixon, Kosower, Smirnov;
    - – further improvements — Cachazo, Spradlin, Volovich
- Using integrability in asymptotic regime – many more
  - ◇ Various long operators at
    - $1 - \text{loop}$ — Minahan, Zarembo;...
    - $2 - \text{loops}$ — Beisert, Kristjansen, Staudacher
    - higher — Beisert, Dippel, Staudacher
  - ◇ conjecturally all and to all orders — Beisert, Hernandez, Lopez; Beisert, Eden, Staudacher
  - ◇ remarkably: 4-loop cusp anom.dim.

    strong coupling NLO cusp — Benna, Benvenutti, Klebanov, Scardicchio

    analytic approaches at LO and NLO — Kostov, Serban, Volin; Beccaria, De Angelis, Forini; Casteill, Kristjansen

- Despite undisputed successes, questions remain:
  - sometimes used outside asymptotic regime (twist-2 $L \geq 4$)
    - ◇ when does wrapping become relevant?
      - recent attempts — Kotikov, Lipatov, Rej, Staudacher, Velizhanin
  - is the infinite spin anomalous dimension independent of twist and equal to the cusp anom dim?
  - integrability not proven at higher loops
    - ◇ though arguments from string side at quantum level — Berkovits; Mikhailov, Schafer-Nameki
  - is all information encoded in the 2-particle S-matrix?
    - ◇ if so, the 4-loop dressing phase contains $\zeta(3)$
- a curious consequence: beyond 4-loops all anomalous dimensions are transcendental even at finite length

Is this testable? Is the phase computable from first principles?

- Dilatation operator vs. scattering amplitudes

	amplitudes/AP kernel; $SL(2)$	dilatation operator
origin	IR	UV
operators	$L = 2$ $S \to \infty$/any spin	in principle all
	further applications	(mostly) dedicated

- Focus on the $SU(2)$ sector
  - "far" from $SL(2)$ sector; technically simplest; intuitive; extensively studied;

- Asymptotic Bethe ansatz in the $SU(2)$ sector:

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L = \prod_{k=1}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} e^{2i\theta(u_j, u_k)} \qquad u(p) = \frac{1}{2}\cot\frac{p}{2}\sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

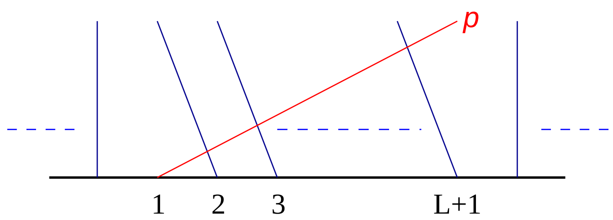
$$E - J = \sum_i \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p_i}{2}}$$

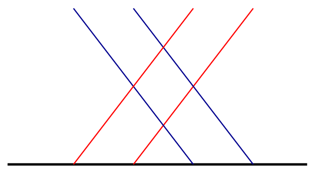
  - θ – bilinear in higher conserved charges

- Renormalization: use Lagrangian and off-shell Feynman rules
  - potential complications – the usual
  - main culprits – internal vector fields and fermions
    - 6 terms per gauge field vertex
    - many terms from fermion trace
- Can be avoided to some extent by an appropriate choice of states
  - symmetries and constraints go a long way
- Identify interactions that receive only scalar vertex contributions

Maximal shuffling:
- each vertex $\leftrightarrow$ permutation operator
- L-loops: interactions w/ L or $L+1$ sites
- no permutation of identical fields
- avoid states containing $\ldots(\phi Z)^n\ldots$

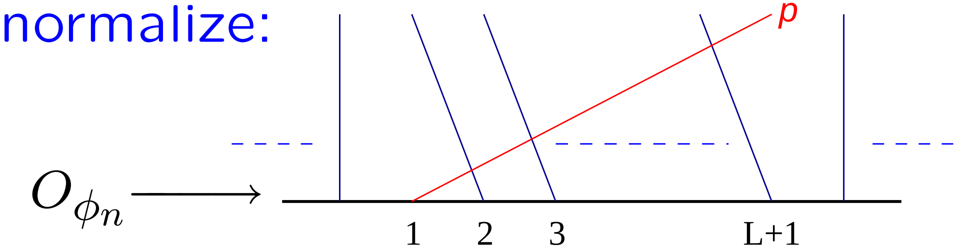
Examples:



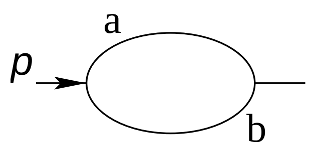


Single magnon operator in the $SU(2)$ sector: $O_{\phi_n} = \sum_l e^{il\phi_n} Z^l \Phi Z^{J-l}$

Renormalize:



$$\frac{L(a,b)}{(p^2)^{a+b-2+\epsilon}} =$$



$$O_{\phi_n} \mapsto I_L O_{\phi_n} \qquad \text{with} \qquad I_L = L(1,1)L(1,1+\epsilon)\ldots L(1,1+L\epsilon)\frac{1}{(p^2)^{L\epsilon}}$$

Assume smooth infinite length limit order by order and resum:

(Gross, Mikhailov, RR)

$$I(p) = \sum_{L=0}^{\infty} [\lambda(e^{i\phi_n} + e^{-i\phi_n} - 2)]^L I_L$$

- Saddle-point contribution: $e^{1-\sqrt{1-4\lambda(e^{i\phi_n/2}-e^{-i\phi_n/2})^2}}$
- Fluctuations: $\dfrac{1+\sqrt{1-4\lambda(e^{i\phi_n/2}-e^{-i\phi_n/2})^2}}{2}$

$$I^{\text{ren}} = e^{\frac{1}{\epsilon}\int_0^1 \frac{dt}{2t}\left[\sqrt{1-4\lambda(e^{i\phi_n/2}-e^{-i\phi_n/2})^2}-1\right]} I \quad \Rightarrow \quad \gamma_n = \sqrt{1+16\lambda\sin^2\frac{\phi_n}{2}}-1$$

The phase:

◇ Universal to all sectors; general structure: Arutyunov, Frolov, Staudacher; Beisert, Klose; Beisert, Tseytlin

$$\theta(u_1,u_2) = \sum_{r=2}^{\infty}\sum_{\nu=0}^{\infty} \beta_{r,r+1+2\nu}(g)\Big[q_r(u_1)q_{r+1+2\nu}(u_2) - q_r(u_2)q_{r+1+2\nu}(u_1)\Big]$$

$$q_{r+1}(p) = \frac{2}{r}\sin\frac{r\,p}{2}\left[\frac{\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}-1}{\frac{\lambda}{4\pi^2}\sin\frac{p}{2}}\right]^r \qquad g = \frac{\sqrt{\lambda}}{\pi}$$

- various suggestions for the origin of the phase:
  - relativistic 2d sigma model — Mann, Polchinski; Gromov, Kazakov, Viera, Sakai
  - fill to physical vacuum — Sakai, Satoh; Rej, Staudacher, Zieme

▪ constrained by crossing at strong coupling — Janik

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- strong coupling: $q \to \tilde{q}_r = g^{r-1}\, q_r \quad c_{r,s}(g) = g^{2-r-s}\beta_{r,s}(g) = \sum_n c_{r,s}^{(n)} g^{1-n}$

$$c_{r,s}^{(n)} = \frac{(1-(-)^{r+s})(r-1)(s-1)}{2(-2\pi)^n\Gamma(n-1)} \frac{\Gamma(\frac{1}{2}(s+r+n-3))\Gamma(\frac{1}{2}(s-r+n-1))}{\Gamma(\frac{1}{2}(s+r-n+1))\Gamma(\frac{1}{2}(s-r-n+3))}$$

Beisert, Hernandez, Lopez

- weak coupling: $\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$

• analytic continuation from strong coupling Beisert, Eden, Staudacher

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- weak coupling: $\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$

• analytic continuation from strong coupling – several possibilities: Beisert, Eden, Staudacher

L	no $\zeta(2n+1)$	with $\zeta(2n+1)$
4	$\beta_{2,3}^{(3)} = 2\zeta(3)$	$\beta_{2,3}^{(3)} = 4\zeta(3)$
5	$\beta_{2,3}^{(4)} = -20\zeta(5)$	$\beta_{2,3}^{(4)} = -40\zeta(5)$
6	$\beta_{2,3}^{(5)} = 210\zeta(7)$,	$\beta_{2,3}^{(5)} = 420\zeta(7)$,
	$\beta_{3,4}^{(5)} = 12\zeta(5)$, $\beta_{2,5}^{(5)} = -4\zeta(5)$	$\beta_{3,4}^{(5)} = 24\zeta(5)$, $\beta_{2,5}^{(5)} = -8\zeta(5)$

- Constraints on the L-loop Hamiltonian in the $SU(2)$ sector:
  - $SU(2)$ symmetry and structure of Feynman diagrams
  - vacuum energy and the energy of other BPS states
  - dispersion relation
  - $[L/2]$-particle S-matrix $\leftrightarrow$ $L=4$: need only 2-particle S-matrix
  - allow/introduce terms not affecting the eigenvalues
    - similarity transformations: $\mathcal{H}' = U^{-1}\mathcal{H}U$ & $U^\dagger \neq U^{-1}$

- past calculations:
  - spin chain Hamiltonian for AFS S-matrix — Beisert
    - imposes BMN scaling/existence of smooth continuum limit
  - 5-loop $SU(2)$ sector $\mathcal{H}$ assuming — Beisert, Dippel, Staudacher
    - integrability, smooth continuum limit, Feynman diagramatics
  - larger sectors

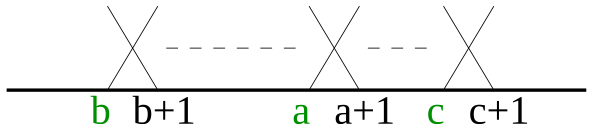
$$\mathcal{H}_0 = +\{\}$$

$$\mathcal{H}_1 = +2\{\} - 2\{1\}$$

$$\mathcal{H}_2 = -8\{\} + 12\{1\} - 2\big(\{1,2\} + \{2,1\}\big)$$

$$\begin{aligned}\mathcal{H}_3 = &+60\{\} - 104\{1\} + 4\{1,3\} + 24\big(\{1,2\} + \{2,1\}\big)\\ &- 4i\epsilon_2\{1,3,2\} + 4i\epsilon_2\{2,1,3\} - 4\big(\{1,2,3\} + \{3,2,1\}\big)\end{aligned}$$

$$\begin{aligned}\mathcal{H}_4 = &+\big(-560 - 4\beta_{2,3}\big)\{\}\\ &+\big(+1072 + 12\beta_{2,3} + 8\epsilon_{3a}\big)\{1\}\\ &+\big(-84 - 6\beta_{2,3} - 4\epsilon_{3a}\big)\{1,3\}\\ &- 4\{1,4\}\\ &+\big(-302 - 4\beta_{2,3} - 8\epsilon_{3a}\big)\big(\{1,2\} + \{2,1\}\big)\\ &+\big(+4\beta_{2,3} + 4\epsilon_{3a} + 2i\epsilon_{3c} - 4i\epsilon_{3d}\big)\{1,3,2\}\\ &+\big(+4\beta_{2,3} + 4\epsilon_{3a} - 2i\epsilon_{3c} + 4i\epsilon_{3d}\big)\{2,1,3\}\\ &+\big(4 - 2i\epsilon_{3c}\big)\big(\{1,2,4\} + \{1,4,3\}\big)\\ &+\big(4 + 2i\epsilon_{3c}\big)\big(\{1,3,4\} + \{2,1,4\}\big)\\ &+\big(+96 + 4\epsilon_{3a}\big)\big(\{1,2,3\} + \{3,2,1\}\big)\\ &+\big(-12 - 2\beta_{2,3} - 4\epsilon_{3a}\big)\{2,1,3,2\}\\ &+\big(+18 + 4\epsilon_{3a}\big)\big(\{1,3,2,4\} + \{2,1,4,3\}\big)\\ &+\big(-8 - 2\epsilon_{3a} - 2i\epsilon_{3b}\big)\big(\{1,2,4,3\} + \{1,4,3,2\}\big)\\ &+\big(-8 - 2\epsilon_{3a} + 2i\epsilon_{3b}\big)\big(\{2,1,3,4\} + \{3,2,1,4\}\big)\\ &- 10\big(\{1,2,3,4\} + \{4,3,2,1\}\big)\end{aligned}$$



- $\{...cba\} = ...P_{c,c+1}P_{b,b+1}P_{a,a+1}$
- β=phase parameters
- ϵ=similarity parameters

  $\mathcal{H} \to U(\epsilon)^{-1}\mathcal{H}U(\epsilon)$

• BMN scaling, continuum limits and such

LL action: Kruczenski; Kruczenski, Ryzhov, Tseytlin

– effective action of long wave length excitations above ferro. vac.

– action of excitations around fast string in $p_{\text{fast}} = \text{const}$ gauge

▪ General structure to 3-loops ($\tilde{\lambda} = \lambda/J^2$) Minahan, Tirziu, Tseytlin

$$\begin{aligned} \mathcal{L} &= \vec{C}(n)\cdot\partial_t\vec{n} - \frac{1}{4}\vec{n}(\sqrt{1-\tilde{\lambda}\partial_x^2}-1)\vec{n} + \frac{3\tilde{\lambda}^2}{128}(\partial_x\vec{n}\cdot\partial_x\vec{n})^2 \\ &+ \frac{\tilde{\lambda}^3}{64}\Big[\mathbf{a}(\lambda)(\partial_x\vec{n})^2(\partial_x^2\vec{n})^2 - \mathbf{b}(\lambda)(\partial_x\vec{n}\cdot\partial_x^2\vec{n})^2 - \mathbf{c}(\lambda)(\partial_x\vec{n}\cdot\partial_x\vec{n})^3\Big] \end{aligned}$$

gauge theory Minahan, Tirziu, Tseytlin	string theory Kruczenski, Ryzhov, Tseytlin
$\mathbf{a}(\lambda) = -\frac{7}{4}$	$\mathbf{a}(\lambda) = -\frac{7}{4}$
$\mathbf{b}(\lambda) = -\frac{23}{4} + b_1\lambda + \ldots$	$\mathbf{b}(\lambda) = -\frac{25}{4} + \frac{b_{-1}}{\sqrt{\lambda}} + \ldots$
$\mathbf{c}(\lambda) = \frac{12}{16} + c_1\lambda + \ldots$	$\mathbf{c}(\lambda) = \frac{13}{16} + \frac{c_{-1}}{\sqrt{\lambda}} + \ldots$

cf. Bethe ansatz: BDS vs. AFS

b_{-1}, c_{-1} – related to nonanalyticity at $\lambda \to \infty$ Minahan, Tirziu, Tseytlin

$$J\tilde{\lambda}^3\frac{1}{\sqrt{\lambda}} = \frac{\lambda^{5/2}}{J^5}$$

Beisert, Tseytlin; Schafer-Nameki, Zamaklar

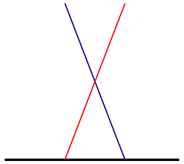
- Evidence for interpolating coefficients in spin chain $\mathcal{H}$
  - ◇ manifestation and resolution of "3-loop differences"

- b_1, c_1 related to $\beta_{2,3}$ and similarity trasformation parameters
  - ◇ changable but not removable by field redefinitions

- qualitative agreement with $\mathcal{H}$
  - ◇ similar structure with 3-loop LL terms

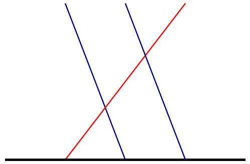
- Maximal shuffling interactions at 1-, 2-, 3- and 4-loops

1-loop :



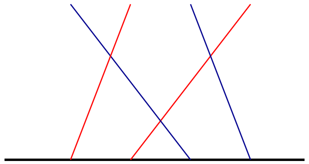
$\{1\}$

2-loops :

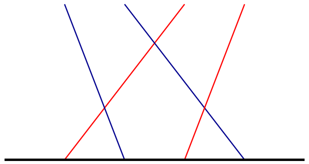


$\{2,1\}$

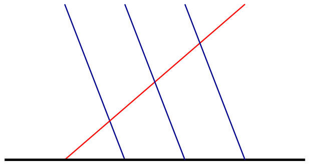
3-loops :



$\{1,3,2\}$

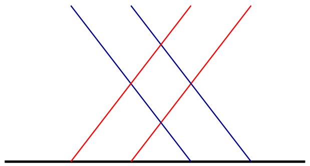


$\{2,1,3\}$

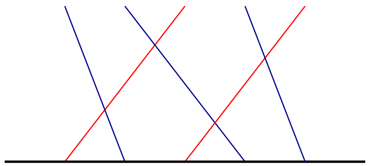


$\{3,2,1\}$

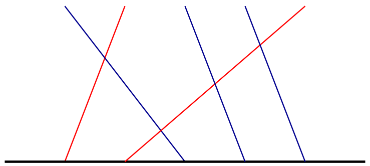
4-loops :



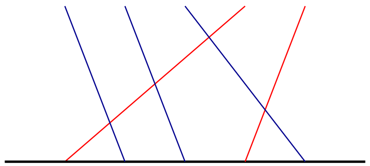
$\{2,1,3,2\}$



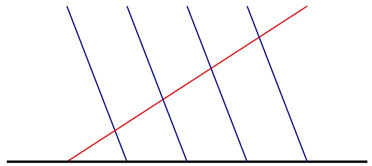
$\{2,1,4,3\}$



$\{1,4,3,2\}$

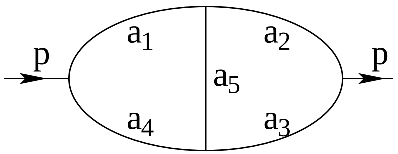


$\{3,2,1,4\}$



$\{4,3,2,1\}$

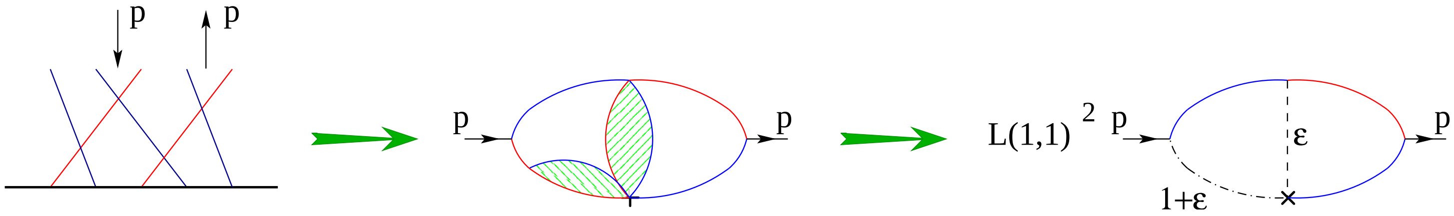
- The calculation:
  - regularization of IR divergences: keep some legs off-shell
  - reduce to "master integrals"
    - ◇ successive evaluation of 1-loop bubbles and use of partial integration reduces everything to 1- and 2-loop integrals



with $a_{1,2,3} = 1,\ a_4 = 1 + n\,\epsilon,\ a_5 = m\,\epsilon\ \ m, n \in \mathbb{Z}$

Example:

$\{2, 1, 4, 3\}$:



- The calculation:
  - regularization of IR divergences: keep some legs off-shell
  - reduce to "master integrals"

- Subtraction:

  counterterms/R-operation $\gamma = \lim_{\epsilon \to 0} \epsilon \mathbf{Z}^{-1} \dfrac{d\mathbf{Z}}{d \ln g_{YM}}$

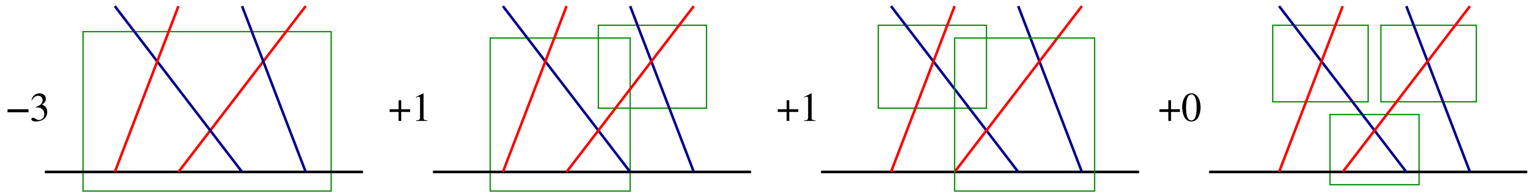
  ◇ effective/recursive subtraction rules Beisert, Kristjansen, Staudacher

- partition each diagram in non-overlapping connected sub-diagrams; interpret each member as renormalizing some operator; subtract each partition with an appropriate weight $L(-)^{\#\text{elem}}$

$\rightarrow$ a simple $1/\epsilon$ pole; Hamiltonian coefficient is $2(4\pi)^{2L}\text{Res}$

Example:

$\{1, 3, 2\}$:



- Coefficients of maximal shuffling terms: Beisert, McLoughlin, RR

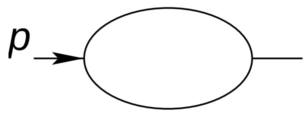
$$
\begin{array}{rr}
(-2)\,\{1\} & (-\ \ 4+\ \ 4\,\zeta(3))\{2,1,3,2\} \\
(-2)\,\{2,1\} & (+10-12\,\zeta(3))\{2,1,4,3\} \\
(+4)\,\{1,3,2\} & (+\ \ 2+\ \ 8\,\zeta(3))\{1,4,3,2\} \\
(-4)\,\{2,1,3\} & (-10+\ \ 4\,\zeta(3))\{3,2,1,4\} \\
(-4)\,\{3,2,1\} & (-10)\{4,3,2,1\}
\end{array}
$$

- phase and similarity trasformation coefficients

$$\beta_{2,3}^{(3)} = 4\zeta(3)$$

$$i\epsilon_2 = -1, \qquad \epsilon_{3a} = -2 - 3\zeta(3), \qquad i\epsilon_{3b} = -3 - \zeta(3)$$

- Could/Should this have been expected?

$L(1,1) \propto$ 

$\propto \frac{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{\Gamma(d-2)}$

$$= \frac{\left(4\pi e^{-\gamma}\right)^{\epsilon}}{16\pi^2\epsilon}\left(1+2\epsilon+\left(4-\frac{1}{12}\pi^2\right)\epsilon^2+\left(8-\frac{1}{6}\pi^2-\frac{7}{3}\zeta(3)\right)\epsilon^3+\ldots\right)$$

- $\zeta(3)$ can appear in an anom. dim. iff additional $1/\epsilon^3$ is present

$\longrightarrow$ 4-loops!

- What about twisted $\mathcal{N}=4$ SYM? phase vs. phases

β-deformation – similar to noncommutative deformation

$$\phi^i\phi^j \longrightarrow e^{i\alpha_{nm}h_n h_m}\phi^i\phi^j \qquad \alpha_{12}=\alpha_{23}=\alpha_{31}=\beta$$

Most general integrable noncommutative-type deformation Beisert, RR

- $$\phi^i\phi^j \longrightarrow e^{i\alpha_{AB}h_A h_B}\phi^i\phi^j$$

  h_A = Cartans of $GL(2,2|4)$
  α_{AB} antisymmetric $A=(\mu,m)$
  $\alpha_{\mu m}=0$

- 1-loop:
  - R-matrix twisted by $e^{i\alpha_{AB}h_A h_B}$

    $$\widetilde{R}(\boldsymbol{\alpha}) = e^{-i\alpha_{AB}h_A h_B}\, R(\text{undeformed})\, e^{-i\alpha_{AB}h_A h_B}$$

    - can show that it satisfies Yang-Baxter equation
    - applies to any theory with more than one global U(1) symmetry

- 1-loop H for sectors in fundamental representation – permutation operator is replaced by twisted permutation operator

- Higher loops? hints from Lagrangian realization as $\star$-product

$$\Phi_i \Phi_j \mapsto \Phi_i \star \Phi_j \equiv e^{i\alpha_{nm} h_n^{(i)} h_m^{(j)}} \Phi_i \Phi_j$$

e.g. Potential

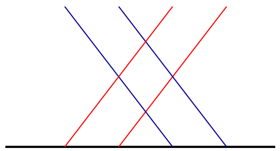
$$V = \left| e^{i\alpha_{12}}\phi^1\phi^2 - e^{-i\alpha_{12}}\phi^2\phi^1 \right|^2 + \left| e^{i\alpha_{23}}\phi^2\phi^3 - e^{-i\alpha_{23}}\phi^3\phi^2 \right|^2 + \left| e^{i\alpha_{31}}\phi^3\phi^1 - e^{-i\alpha_{31}}\phi^1\phi^3 \right|^2$$

- 2 loops – same as at 1-loop: replace $\mathcal{P}$ by $\mathcal{P}_{\boldsymbol{\alpha}}$

- higher loop – tedious, but same conclusion; planarity is crucial

  – clear for maximal shuffling interactions

  – also same momentum integrals



- Conclusion: dressing phase is not modified by the introduction of noncommutative-type deformations $\longrightarrow \beta_{2,3}^{(3)} = 4\zeta(3)$

Prospects at higher loops

- Higher-loop cusp extractable from 4-gluon scattering
  - 5-loop integrand known — Bern, Carrasco, Johansson, Kosower
  - extraction of the $1/\epsilon^2$ pole appears challenging

- Direct extraction of dressing phase from $SU(2)$-sector Hamiltonian
  - ◇ simpler integrals – lower-point functions
  - ◇ algebraic complications in relating Hamiltonian and phase
  - ◇ proof of integrability requires more than maximal shuffling

▪ Assuming integrability:

◇ only leading order term in the expansion of each phase coefficient is computable from maximal shuffling interactions

$\theta = \theta(\beta_{r,s}(\lambda)) \Rightarrow$ Hamiltonian depends on complete dressing phase coefficients $\beta_{r,s}(\lambda)$ rather than $\beta_{r,s}^{(l)}$

- subleading terms in expansion of β involve vectors and spinors

- test: all 8 5-loops maximal shuffling terms free of $\zeta(5)$

McLoughlin, RR

• How much to expect: look again at the expansion of $L(1,1)$:

$$L(1,1) \sim \frac{1}{\epsilon}\left(\ldots\zeta(2n+1)\left(a_{2n+1}\epsilon^{2n+1} + b_{2n+2}\epsilon^{2n+2}\right) + \ldots\right)$$

$\rightarrow$ expect $\zeta(2n+1)$ at $L = 2n+2$ in maximal shuffling interactions

- some tests:

$\mathcal{H}_6 = \cdots + (18 - 8\zeta(3) - 4\zeta(5))\{2,1,3,2,4,5\} + \ldots$

$\mathcal{H}_8 = \cdots - (19+10\zeta(3)-9\zeta(3)^2+24\zeta(5) - \zeta(7))\{2,1,3,2,4,5,6,7\} + \ldots$

Summary

- Proved integrability of the dilatation operator at four loops

- Four-loop anomalous dimensions in the $SU(2)$ sector consistent with conjectured BES/BHL dressing phase
  - Dilatation operator has transcentental coefficients consistent with the existence of interpolating functions
  - Insensitivity to wrapping

- Leading terms in the coefficients of the dressing phase are accessible to direct Feynman diagram calculations
  - related to maximal shuffling interactions
  - some contributions with obvious recursive structure