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The BES equation and its strong coupling limit

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References:

- ES ht/0603157, BES ht/0610251.
- BBKS ht/0611135, AABEK ht/0702028, work in progress.

- KMMZ ht/0402207, BDS ht/0405001, AFS ht/0406256, BK ht/0510124.
- B ht/0307015, S ht/0412188, BS ht/0504190.
- MVV hp/0403192, KLV ht/0301021, KLOV ht/0404092.
- BT ht/0509084, J ht/0603038, HL ht/0603204, BHL ht/0609044.
- BDS ht/0505205, BCDKS ht/0610248.
- CVS ht/0612309, BMcLR ht/07050321.
- GKP ht/0204051, FT ht/0204226.

0 Introduction

$\mathcal{N} = 4$ SYM

- The **AdS/CFT duality** relates $\mathcal{N} = 4$ SYM to IIB string theory on $AdS_5 \times S_5$. It is a weak/strong-coupling duality.
- The large N limit of the SYM theory can be described by **spin chains**.

Derivative operators

- Built from scalar fields X and **covariant derivatives**.
- The derivatives act as **magnons** moving on the chain of scalars.

Large spin all-loops anomalous dimension

- We start from an **all-loops conjecture** for the Bethe ansatz. A **large spin continuum limit** yields an **integral equation** for the density of Bethe roots.
- The energy grows logarithmically with the spin. It is given by sums of zeta values respecting a principle of **maximal transcendentality**.
- We discuss **dressing phases** (integrable modifications of the Bethe ansatz) that do not violate transcendentality.
- A **kernel from string theory** reverses the sign of certain contributions to the energy. At four loops, **agreement with field theory** is obtained.
- The integral equation is **split** into **two coupled equations** with simple kernels.
- The leading root distribution at **strong coupling** is obtained by analytic means. The corresponding energy matches the prediction by GKP.

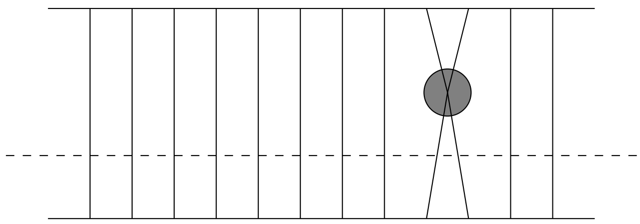
1 Spin Chain Picture in Gauge Theory

Derivative sector:

$$\{s_1, s_2, s_3, \ldots\} \; = \; Tr\Big((\mathcal{D}_z^{s_1} X)(\mathcal{D}_z^{s_2} X)(\mathcal{D}_z^{s_3} X)\ldots\Big)$$

- X is a complex scalar field of the $\mathcal{N} = 4$ SYM theory with $SU(N)$ gauge group. $\mathcal{D}_\mu \; = \; \partial_\mu + \, i\, g_{YM}\, A_\mu$.
- The operators carry traceless symmetric Lorentz representation of spin $s = s_1 + s_2 + s_3 + \ldots$; project $z = x_1 + ix_2$.

- Loop diagrams define a Hamiltonian that can transfer derivatives from one site to another. Free lines do not (as long as we look at a certain tensor component).
- In the large N limit this defines a nearest neighbour interaction.



Two-site Hamiltonian.

We may view the derivatives as "magnons" moving on the sites of a spin chain.

At one loop (B):

$$\mathcal{H}^{(0)} \; = \; \sum_{i=1}^{L} \mathcal{H}_i^{(0)}$$

$$\mathcal{H}_i^{(0)}(\{s_1,\, s_2\} \;\rightarrow\; \{s_1,\, s_2\}) \;\; = \;\; h(s_1) \, + \, h(s_2)\,,$$

$$\mathcal{H}_i^{(0)}(\{s_1,\, s_2\} \;\rightarrow\; \{s_1 - d,\, s_2 + d\}) \;\; = \;\; -\frac{1}{|d|}$$

2 Bethe Equations

- The one-loop Hamiltonian above defines the Heisenberg XXX chain with spin $-\frac{1}{2}$.

The dynamics of the system is captured by the **Bethe ansatz**

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L = \prod_{j\neq k}\left(\frac{u_k-u_j-i}{u_k-u_j+i}\right), \quad j,k\in\{1,\ldots,s\},$$

$$\prod_{k=1}^{s}\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right) = 1, \qquad E = \sum_{k=1}^{s}\left(\frac{i}{u_k+\frac{i}{2}}-\frac{i}{u_k-\frac{i}{2}}\right).$$

All-loops conjecture (S,BS):

$$u\pm\frac{i}{2} = x^{\pm}+\frac{g^2}{2x^{\pm}}, \qquad g = \frac{\sqrt{\lambda}}{4\pi}$$

The deformed system is

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{j\neq k}\frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+},$$

$$\prod_{k=1}^{s}\left(\frac{x_k^+}{x_k^-}\right) = 1, \qquad E(g) = \sum_{k=1}^{s}\left(\frac{i}{x_k^+}-\frac{i}{x_k^-}\right).$$

3 One-Loop Large Spin Limit

- The $L = 2$ case is exactly solvable for any (even) spin; the u_k are the zeroes of certain Hahn polynomials.
- The roots are real and symmetrically distributed around zero. The density peaks at the origin, there is no gap.
- The outermost roots grow as $\max\{|u_k|\} \rightarrow s/2$.
- The mode numbers are ∓ 1 for negative/positive roots.
- For $L > 2$ there is more than one state. However, for the lowest state the root distribution is again real and symmetric with $n = \text{sign}(u)$.

We take the logarithm of the Bethe equations

$$-i\,L\,\log\left(\frac{u_k + \frac{i}{2}}{u_k - \frac{i}{2}}\right) = 2\,\pi\,n_k - i\sum_{j\neq k}\log\frac{u_k - u_j - i}{u_k - u_j + i}\,,$$

rescale $u \rightarrow s\,\bar{u}$, expand in 1/s, and take a continuum limit:

$$0 = 2\,\pi\,\epsilon(\bar{u}) - 2\,⨍_{-1/2}^{1/2} d\bar{u}'\,\frac{\bar{\rho}_0(\bar{u}')}{\bar{u} - \bar{u}'}$$

One may solve by an inverse Hilbert transform:

$$\bar{\rho}_0(\bar{u}) = \frac{1}{\pi}\log\frac{1 + \sqrt{1 - 4\,\bar{u}^2}}{1 - \sqrt{1 - 4\,\bar{u}^2}} = \frac{2}{\pi}\,\text{arctanh}\left(\sqrt{1 - 4\,\bar{u}^2}\right)$$

The one-loop energy is:

$$E_0 = \frac{1}{s}\int_{-\frac{1}{2}}^{\frac{1}{2}} d\bar{u}\,\frac{\bar{\rho}_0(\bar{u})}{\bar{u}^2 + \frac{1}{4\,s^2}} = 4\,\log(s) + \mathcal{O}(s^0)$$

4 Asymptotic All-Loops Large Spin Limit

In the following we assume even spin s and label the roots by

$$j,k \in \{\pm\frac{1}{2},\pm\frac{3}{2},\ldots,\pm\frac{s-1}{2}\}\,.$$

We begin by rewriting the asymptotic all-loop Bethe equations:

$$\left(\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\right)^L \left(\frac{1+g^2/2(x_k^-)^2}{1+g^2/2(x_k^+)^2}\right)^L =$$

$$= \prod_{j\neq k}\frac{u_k-u_j-i}{u_k-u_j+i}\left(\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\right)^2$$

Alternatively:

$$2L\arctan(2u_k)+iL\log\left(\frac{1+g^2/2(x_k^-)^2}{1+g^2/2(x_k^+)^2}\right)=2\pi\tilde{n}_k-$$

$$-2\sum_{j\neq k}\arctan\left(u_k-u_j\right)+2i\sum_{j\neq k}\log\left(\frac{1-g^2/2x_k^+x_j^-}{1-g^2/2x_k^-x_j^+}\right)$$

For each L the lowest state has mode numbers

$$\tilde{n}_k = k+\frac{L-2}{2}\,\epsilon(k)\,.$$

As $s\to\infty$ we introduce a smooth continuum variable $x=\frac{k}{s}$. The excitation density is $\rho(u)=\frac{dx}{du}$.

We divide the logarithmic Bethe equation by s, replace the sums by integrals, and differentiate w.r.t. u. We do **not** rescale u by $1/s$.

$$\frac{L}{s}\frac{1}{u^2+\frac{1}{4}} + \frac{iL}{s}\frac{d}{du}\log\left(\frac{1+g^2/2(x^-(u))^2}{1+g^2/2(x^+(u))^2}\right) =$$
$$= 2\pi\rho(u) + \frac{2\pi}{s}(L-2)\delta(u) - 2\int_{-b}^{b} du' \frac{\rho(u')}{(u-u')^2+1}$$
$$+ 2i\int_{-b}^{b} du' \rho(u') \frac{d}{du}\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right)$$

- The L dependent terms drop in the large spin limit.

Split

$$\rho(u) = \rho_0(u) - g^2\,\frac{E_0}{s}\,\sigma(u)\,.$$

Final integral equation:

$$\begin{aligned} 0 \;=\; & 2\,\pi\,\sigma(u) \\ & -2\int_{-\infty}^{\infty} du' \frac{\sigma(u')}{(u-u')^2+1} \\ & -\left(\frac{1}{2}\frac{d}{du}\right)\left[\frac{1}{x^+(u)}+\frac{1}{x^-(u)}\right] \\ & +2i\int_{-\infty}^{\infty} du' \sigma(u') \frac{d}{du}\log\left(\frac{1-g^2/2x^+(u)x^-(u')}{1-g^2/2x^-(u)x^+(u')}\right) \end{aligned}$$

- This is an asymptotic result, because L needs to grow with the order in g^2 to avoid “wrapping”.
- The final formula is L independent. ”Wrapping” is thus absent.

5 Weak Coupling and Transcendentality

We introduce the Fourier transform $\hat{\sigma}(t)$ of the fluctuation density $\sigma(u)$

$$\hat{\sigma}(t) = e^{-\frac{t}{2}} \int_{-\infty}^{\infty} du\, e^{-itu}\, \sigma(u)\,.$$

The integral equation becomes

$$\hat{\sigma}(t) \;=\; \frac{t}{e^{\,t}-1} \Bigg[\frac{J_1(2\,g\,t)}{2\,g\,t} - - 4\,g^2 \int_0^{\infty} dt'\; \hat{K}(2\,g\,t, 2\,g\,t')\; \hat{\sigma}(t') \Bigg]\,,$$

with the non-singular kernel

$$\hat{K}(t,\, t') \;=\; \frac{J_1(t)\, J_0(t') \;-\; J_0(t)\, J_1(t')}{t \;-\; t'}\,.$$

The energy is

$$f(g) \;=\; \frac{E(g)}{\log(s)} \;=\; 8\,g^2 \;-\; 64\,g^4 \int_0^{\infty} dt\, \hat{\sigma}(t)\, \frac{J_1(2\,g\,t)}{2\,g\,t}\,.$$

The integral equation is of Fredholm II type. One may solve by iteration:

$$\hat{\sigma}(t) = \frac{1}{2}\,\frac{t}{e^t\,-\,1} - g^2\left(\frac{1}{4}\,\frac{t^3}{e^t\,-\,1} + \zeta(2)\,\frac{t}{e^t\,-\,1}\right) + \ldots\,,$$

where we have used

$$\zeta(n+1) = \frac{1}{n!} \int_0^{\infty} \frac{dt\; t^n}{e^t\,-\,1}\,.$$

We find

$$f(g) \;=\; 8\,g^2 \;-16\,\zeta(2)\,g^4 \;+\; \Big(4\,\zeta(2)^2 \;+\; 12\,\zeta(4)\Big)\,8\,g^6$$
$$-\Big(4\,\zeta(2)^3 + 24\,\zeta(2)\zeta(4) - 4\,\zeta(3)^2 + 50\,\zeta(6)\Big)\,16\,g^8 + \ldots$$

or, alternatively:

$$f(g) = 8g^2 - \frac{8}{3}\pi^2 g^4 + \frac{88}{45}\pi^4 g^6 - \left(\frac{73}{630}\pi^6 - 4\zeta(3)^2\right)16g^8 + \ldots$$

- Agrees with KLOV up to three loops (in the large spin limit their harmonic sums become zeta functions).

The result obeys a principle of **uniform transcendentality**:

The l-loop contributions have degree of transcendentality 2 l - 2.

6 Dressing Kernels

The higher-loop Bethe equations may receive corrections:

$$\left(\frac{x_k^+}{x_k^-}\right)^L = \prod_{\substack{j=1\\ j\neq k}}^{S} \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/x_k^+x_j^-}{1-g^2/x_k^-x_j^+}\,\exp\big(2i\theta(u_k,u_j)\big),$$

Dressing phase (AFS,BK):

$$\theta(u_k,u_j) = \sum_{r\geq 2,\nu\geq 0} \beta_{r,r+1+2\nu}(g)\big(q_r(u_k)q_{r+1+2\nu}(u_j) - q_r(u_j)q_{r+1+2\nu}(u_k)\big)$$

At weak coupling:

$$\beta_{r,r+1+2\nu}(g) = \sum_{\mu=\nu}^{\infty} g^{2r+2\nu+2\mu}\,\beta_{r,r+1+2\nu}^{(r+\nu+\mu)}$$

In the integral equation we have to use

$$\hat{K}(t,t') = \hat{K}_{\mathrm{m}}(t,t') + \hat{K}_{\mathrm{d}}(t,t')$$

with the dressing kernel

$$\hat{K}_{\mathrm{d}}(t,t') = \frac{4}{t\,t'} \sum_{\rho\geq 1,\nu\geq 0,\mu\geq\nu} g^{2\mu+1}(-1)^{\nu}\Big(\beta_{2\rho,2\rho+1+2\nu}^{(2\rho+\nu+\mu)}\,J_{2\rho+2\nu}(t)J_{2\rho-1}(t') + \beta_{2\rho+1,2\rho+2\nu+2}^{(2\rho+1+\nu+\mu)}\,J_{2\rho}(t)J_{2\rho+1+2\nu}(t')\Big)\,.$$

- KLOV: There is no three-loop correction in weakly coupled field theory. We must choose

$$\beta_{2,3}^{(2)} = 0\,.$$

7 Dressing Respecting Transcendentality

Four-loop term of the scaling function when dressing is included:

$$f(g) = \ldots - 16\left(\frac{73}{630}\,\pi^6 - 4\,\zeta(3)^2 + 2\,\beta_{2,3}^{(3)}\,\zeta(3)\right) g^8 + \ldots .$$

- Transcendentality is preserved, if $\beta_{2,3}^{(3)}$ is a rational number times $\zeta(3)$ (or π^3).

Generally:

$\beta_{r,s}^{(\ell)}$ should have degree of transcendentaliy $2\ell + 2 - r - s$.

Can we eliminate all odd zetas from the scaling function?

Impose:

- Each coefficient $\beta_{r,s}^{(\ell)}$ contains exactly one zeta function.
- A constraint from Feynman graphs:

$$\beta_{r,s}^{(\ell)} = 0\,, \qquad \ell < r + s - 2.$$

The coefficients are uniquely determined:

$$\begin{aligned}
&\beta_{2,3}^{(3)} \to +2\,\zeta(3),\\
&\beta_{2,3}^{(4)} \to -20\,\zeta(5),\\
&\beta_{2,3}^{(5)} \to +210\,\zeta(7), \quad \beta_{3,4}^{(5)} \to +12\,\zeta(5), \quad \beta_{2,5}^{(5)} \to -4\,\zeta(5),\\
&\beta_{2,3}^{(6)} \to -2352\,\zeta(9), \quad \beta_{3,4}^{(6)} \to -210\,\zeta(7), \quad \beta_{2,5}^{(6)} \to +84\,\zeta(7).
\end{aligned}$$

The scaling function simplifies to

$$\begin{aligned}
f_0(g) = {} & 8\,g^2 - \frac{8}{3}\,\pi^2\,g^4 + \frac{88}{45}\,\pi^4\,g^6 - 16\,\frac{73}{630}\pi^6\,g^8 + 32\,\frac{887}{14175}\,\pi^8 g^{10}\\
& - 64\,\frac{136883}{3742200}\,\pi^{10} g^{12} + 128\,\frac{7680089}{340540200}\,\pi^{12} g^{14} \mp \ldots .
\end{aligned}$$

8 An "Analytic Continuation"

For perturbative string theory write the dressing phase as

$$\theta(u_k, u_j) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s}(g) \big(\tilde{q}_r(u_k)\, \tilde{q}_s(u_j) - \tilde{q}_s(u_k)\, \tilde{q}_r(u_j)\big) \,.$$

The charges $\tilde{q}_r(u)$ are normalized as $\tilde{q}_r(u) = g^{r-1} q_r(u)$ so that

$$c_{r,s}(g) = g^{2-r-s} \beta_{r,s}(g).$$

The strong-coupling expansion of $c_{r,s}$ within string theory is

$$c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}.$$

Proposal for the all-order strong-coupling expansion:

$$c_{r,s}^{(n)} = \frac{\big(1-(-1)^{r+s}\big)\zeta(n)}{2(-2\pi)^n\,\Gamma(n-1)}\,(r-1)(s-1) \;* \\ * \;\frac{\Gamma[\frac{1}{2}(s+r+n-3)]\,\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s+r-n+1)]\,\Gamma[\frac{1}{2}(s-r-n+3)]}\,,$$

Singular for $n = 0, 1$, when

$$c_{r,s}^{(0)} = \delta_{r+1,s}\,, \qquad c_{r,s}^{(1)} = -\frac{\big(1-(-1)^{r+s}\big)}{\pi}\,\frac{(r-1)(s-1)}{(s+r-2)(s-r)}\,.$$

(The latter are the AFS and BT,HL terms, respectively.)

Based on:

- $n = 0, 1$: available data
- for even n: crossing symmetry (J,BHL)
- for odd n: natural choice

> Can we **interpolate to weak coupling** in order to recompute $f(g)$ with this dressing kernel?

Analogy to the digamma function:

$\Psi(z) = \partial_z \log \Gamma(z)$ has the asymptotic expansion $(z >> 0)$

$$\Psi(1+z) = \log z + \sum_{n=1}^{\infty} \frac{c_n}{z^n}\,, \qquad c_n = -\frac{\mathrm{B}_n}{n} = (-1)^n \zeta(1-n)\,,$$

while the expansion around $z = 0$ reads

$$\Psi(1+z) = -\gamma_{\mathrm{E}} + \sum_{k=1}^{\infty} \tilde{c}_k\, z^k, \qquad \tilde{c}_k = -(-1)^k \zeta(1+k)\,.$$

The expansion coefficients for large and small z are almost the same!

$$c_n = -\tilde{c}_{-n}$$

$c_{r,s}(g)$ has the weak coupling expansion

$$c_{r,s}(g) = -\sum_{n=1}^{\infty} c_{r,s}^{(-n)} g^{1+n}.$$

We use the identities

$$\zeta(1-z) = 2(2\pi)^{-z} \cos(\tfrac{1}{2}\pi z)\, \Gamma(z)\, \zeta(z) \quad \text{and} \quad \Gamma(1-z) = \frac{\pi}{\sin(\pi z)\, \Gamma(z)}$$

to obtain

$$c_{r,s}^{(n)} = \frac{\left(1-(-1)^{r+s}\right) \cos(\frac{1}{2}\pi n)\, (-1)^{s-1-n}\, \zeta(1-n)}{\Gamma[\frac{1}{2}(5-n-r-s)]\, \Gamma[\frac{1}{2}(3-n+r-s)]} * \\ * \frac{\Gamma(2-n)\, \Gamma(1-n)\, (r-1)(s-1)}{\Gamma[\frac{1}{2}(3-n-r+s)]\, \Gamma[\frac{1}{2}(1-n+r+s)]}\,.$$

- Only even n contribute.
- The constraint from Feynman graphs is satisfied.
- The degree of transcendentality is correct.
- Strong argument in BES, v2 contains a proof for $c_{2,3}$.
  General proof in KL `ht/0611204`.

8 String Phase and Scaling Function

The weak coupling expansion of the string theory dressing phase yields the kernel

$$
\begin{aligned}
&\beta_{2,3}^{(3)} = +4\,\zeta(3), \\
&\beta_{2,3}^{(4)} = -40\,\zeta(5), \\
&\beta_{2,3}^{(5)} = +240\,\zeta(7), \quad \beta_{3,4}^{(5)} = +24\,\zeta(5), \quad \beta_{2,5}^{(5)} = -8\,\zeta(5), \\
&\beta_{2,3}^{(6)} = -4704\,\zeta(9), \quad \beta_{3,4}^{(6)} = -420\,\zeta(7), \quad \beta_{2,5}^{(6)} = +168\,\zeta(7).
\end{aligned}
$$

- These are exactly twice the values we obtained above by requiring the absence of odd zeta functions!

The scaling function becomes

$$
\begin{aligned}
f_+(g) = 8g^2 - \frac{8}{3}\,\pi^2 g^4 + \frac{88}{45}\,\pi^4 g^6 - 16\left(\frac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \\
+ 32\left(\frac{887}{14175}\,\pi^8 + \frac{4}{3}\,\pi^2\zeta(3)^2 + 40\,\zeta(3)\,\zeta(5)\right) g^{10} \\
- 64\left(\frac{136883}{3742200}\,\pi^{10} + \frac{8}{15}\,\pi^4\zeta(3)^2 + \frac{40}{3}\,\pi^2\zeta(3)\,\zeta(5)\right. \\
\left. + 210\,\zeta(3)\,\zeta(7) + 102\,\zeta(5)^2\right) g^{12} + \ldots\,.
\end{aligned}
$$

$f_+(g)$ is obtained from $f(g)$ (trivial dressing phase) by multiplying all odd zeta functions by the imaginary unit i.

9 Agreement with Field Theory

In parallel to our effort, BCDKS have completed a direct computation of the scaling function $f(g)$ at four loops. Their calculation uses unitarity methods and conformal invariance to predict a set of integrals which are evaluated with the help of the MB representation. The exponentiation of infrared singularities is a stringent check.

BCDKS find

$$\begin{aligned} f(g) &= \ldots - 64 \times (29.335 \pm 0.052)\, g^8 + \ldots \\ &= \ldots - \big(3.0192 \pm 0.0054\big) \times 10^{-6} \lambda^4 + \ldots . \end{aligned}$$

Recall our value:

$$\begin{aligned} f_+(g) &= \ldots - 16 \left(\frac{73}{630} \pi^6 + 4\, \zeta(3)^2 \right) g^8 + \ldots \\ &\approx \ldots - 3.01502 \times 10^{-6} \lambda^4 + \ldots . \end{aligned}$$

> The four-loop value calculated by Bern, Czakon, Dixon, Kosower and Smirnov matches the fourth term in $f_+(g)$.

- BCDKS independently guessed the sign-flipped scaling function $f_+(g)$. They checked compatibility with the KLV approximation to rather high order.
- CSV `ht/0612309` have improved the error bar of the BCDKS result by three orders of magnitude.
- BMcLR constructed the four-loop dilatation operator of the $su(2)$ sector from Feynman graphs. They confirm

$$\beta_{2,3}^{(3)} = 4\, \zeta(3) .$$

10 Magic Kernels

We decompose the "main scattering" kernel into two parts

$$\hat{K}_{\rm m}(t,t') = \hat{K}_0(t,t') + \hat{K}_1(t,t')\,,$$

even and odd, respectively, under $t \to -t$, $t' \to -t'$:

$$\hat{K}_0(t,t') = \frac{t\,J_1(t)J_0(t') - t'\,J_0(t)J_1(t')}{t^2 - t'^2}\,,$$

$$\hat{K}_1(t,t') = \frac{t'\,J_1(t)J_0(t') - t\,J_0(t)J_1(t')}{t^2 - t'^2}\,.$$

The component $\hat{K}_1$ causes the odd zeta contributions to $f(g)$:

$$\hat{\sigma}_0(t) = \frac{t}{e^t - 1}\Big[\hat{K}_0(2\,g\,t,0) - 4\,g^2\int_0^\infty dt'\;\hat{K}_0(2\,g\,t,2\,g\,t')\,\hat{\sigma}_0(t')\Big]$$

leads to $f_0(g)$.

- Replace $\hat{K}_0$ by $\hat{K}_{\rm m} - \hat{K}_1$. Note $\hat{K}_1(t,0) = 0$.
- Substitute $\hat{\sigma}_0$ into the convolution on $\hat{K}_1$.

$$\hat{\sigma}_0(t) = \frac{t}{e^t - 1}\Big[\hat{K}_{\rm m}(2\,g\,t,0) + \hat{K}_{\rm c}(2\,g\,t,0) - 4g^2\int_0^\infty dt'\,\Big(\hat{K}_{\rm m}(2\,g\,t,2\,g\,t') + \hat{K}_{\rm c}(2\,g\,t,2\,g\,t')\Big)\,\hat{\sigma}_0(t')\Big]$$

with the function $\hat{K}_{\rm c}$

$$\hat{K}_{\rm c}(t,t') = 4\,g^2\int_0^\infty dt''\,\hat{K}_1(t,2\,g\,t'')\,\frac{t''}{e^{t''} - 1}\,\hat{K}_0(2\,g\,t'',t').$$

- $\hat{K}_{\rm c}$ has the form of a dressing kernel.
- At weak coupling the coefficients $\beta_{r,s}^{(l)}$ can easily be derived in closed form. The kernel satisfies the constraint from Feynman graphs.

Flipping odd zeta contributions:

Any Fredholm II equation

$$\sigma(t) \;=\; P(t) \;-\; 4g^2 \int dt'\, K(t,t')\, \sigma(t')$$

may be solved by iteration:

$$\sigma \;=\; \sum_{n=0}^{\infty} (-4g^2)^n (K\,*)^n\, P$$

(The star denotes a convolution.)

In our case:

$$K \;=\; K_0 \;+\; K_1 \;+\; 8g^2\, K_1 * K_0$$

- The resolvent is a sum of words consisting of K_0, K_1.
- The dressing kernel changes the sign of appending K_1 after K_0.
- In the perturbative expansion, each beginning and each end of a K_1 string in a word produces an odd zeta function, because K_1 is odd in both arguments, while K_0 is even.

With or without dressing kernel, the scaling function contains terms with $2\,m$ odd zeta functions. The relative sign of such terms is $(-1)^m$.

11 Numerics by BBKS

The fact that

$$\hat{K}_0(t,t') = \sum_{n=1}^{\infty} 2\,(2n-1)\,\frac{J_{2n-1}(t)}{t}\,\frac{J_{2n-1}(t')}{t'}$$

$$\hat{K}_1(t,t') = \sum_{n=1}^{\infty} 2\,(2n)\,\frac{J_{2n}(t)}{t}\,\frac{J_{2n}(t')}{t'}$$

led BBKS to expand in terms of the eigenfunctions

$$F_n = \frac{J_n(2\,g\,t)}{2\,g\,t}\,.$$

Let

$$s(t) = \frac{e^t-1}{t}\,\hat{\sigma}(t) = \sum_{n=1}^{\infty} s_n(g)\,F_n\,,$$

$$Z_{mn}(g) = \int_0^{\infty} dt\,\frac{J_m(2\,g\,t)\,J_n(2\,g\,t)}{t\,(e^t-1)}\,.$$

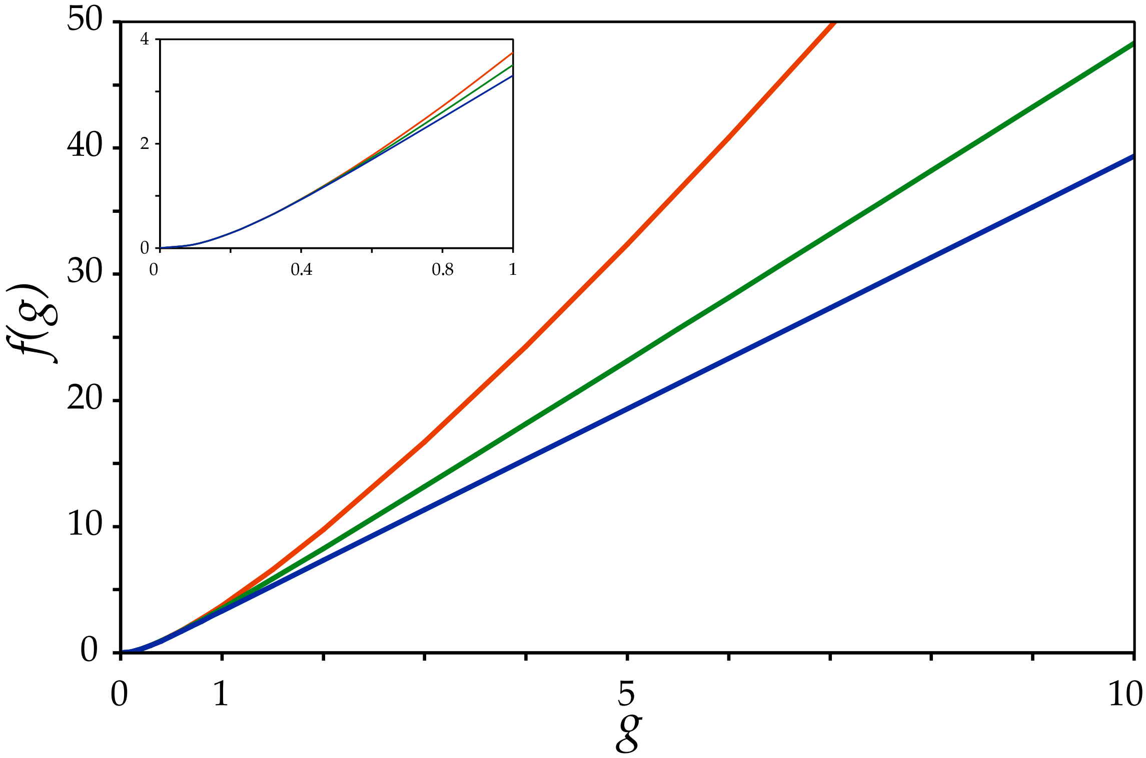
The integral equation becomes (dressing included)

$$\sum_{n=1}^{\infty} s_n\,F_n = F_1 + 8\sum_{n=1}^{\infty} n\,Z_{2n,1}\,F_{2n} - 2\sum_{m,n=1}^{\infty} n\,Z_{nm}\,s_m\,F_n - 16\sum_{l,m,n=1}^{\infty} n\,(2m-1)\,Z_{2n,2m-1}\,Z_{2m-1,l}\,s_l\,F_{2n}$$

which BBKS make into a matrix problem by taking the F_n out and truncating the sums. The energy is

$$f_+(g) = 8\,g^2\,s_1\,.$$

Plot and extrapolation taken from BBKS.



$f(g)$, $f_0(g)$, $f_+(g)$.

- The matrix rank should not go much beyond the value for g. The numerics is reliable up to $g \approx 20$.
- The transition to the linear regime happens around $g \approx 1$. Extrapolation is well behaved.

Strong coupling behaviour of $f_+(g)$:

$$f_+(g) = 4.000000\, g - 0.661907 - 0.0232\, g^{-1} + \ldots$$

Error: $\pm\{1, 2, 1\}$ in the last digit displayed.

Exact result: GKP, FT

$$f_+(g) = 4\, g - \frac{3\log(2)}{\pi} - ?$$

12 BES: Dressing is Nesting

Odd rows:

$$0 \;=\; s_{2m-1} \;-\; \delta_{m,1} \;+\; 2\,(2m-1)\,Z_{2m-1,r}\,s_r$$

Even rows:

$$0 \;=\; s_{2n} - 8\,n\,Z_{2n,1} + 2\,(2n)\,Z_{2n,r}\,s_r + 16\,n\,(2m-1)\,Z_{2n,2m-1}\,Z_{2m-1,r}\,s_r$$

or

$$0 \;=\; s_{2n} \;+\; 2\,(2n)\,Z_{2n-1,2m-1}\,s_{2m-1} \;-\; 2\,(2n)\,Z_{2n-1,2m}\,s_{2m}$$

Multiply by F_n and sum over the free index:

$$\hat{\sigma}_o(t) \;=\; \frac{t}{e^t-1}\left[\frac{J_1(2\,g\,t)}{2\,g\,t} - 4g^2\int_0^\infty dt'\,\hat{K}_0(2\,g\,t,2\,g\,t')\,(\hat{\sigma}_e+\hat{\sigma}_o)(t')\right]$$

$$\hat{\sigma}_e(t) \;=\; \frac{t}{e^t-1}\left[-4g^2\int_0^\infty dt'\,\hat{K}_1(2\,g\,t,2\,g\,t')\,(\hat{\sigma}_e-\hat{\sigma}_o)(t')\right]$$

where

$$\hat{\sigma}_o(t) \;=\; \frac{t}{e^t-1}\sum_{n=1}^{\infty} s_{2n-1}\,F_{2n-1}\,, \qquad \hat{\sigma}_e(t) \;=\; \frac{t}{e^t-1}\sum_{n=1}^{\infty} s_{2n}\,F_{2n}\,.$$

Rescale

$u \to u/\epsilon\,,\;\; \epsilon = 1/(2g)\,,\;\; \sigma_{o,e} \to \epsilon^2\,\sigma_{o,e}\,. \qquad \sigma^{\pm} = \sigma_e \pm \sigma_o\,.$

Odd:

$$2\pi\,\sigma_o(u) - 2\int_{-\infty}^{\infty}\frac{du'\,\sigma_o(u')\,\epsilon}{(u-u')^2+\epsilon^2} + \int_{-\infty}^{\infty} du'\,\tilde{K}_0(u,u')\left(\sigma^+(u') - \frac{1}{\pi}\right) = 0$$

Even:

$$2\pi\,\sigma_e(u) - 2\int_{-\infty}^{\infty}\frac{du'\,\sigma_e(u')\,\epsilon}{(u-u')^2+\epsilon^2} + \int_{-\infty}^{\infty} du'\,\tilde{K}_1(u,u')\,\sigma^-(u') = 0$$

13 Strong Coupling Limit in the u-Picture

Backward Fourier transform of the BBKS numerical analysis:

- For $g \to \infty$, u scales with $2g$. Rescale and expand in $\epsilon = 1/(2g)$.

Further:

- The kernels $\tilde{K}_{0,1}$ are given in terms of the $x^{\pm}(u)$ functions. Their square root branch cut forces to distinguish the regimes $|u| \lessgtr 1$.
- The FT of F_{2n-1} tends to zero outside the interval $|u| < 1$, so that

$$\sigma_o^{\ell}(u) \;=\; 0 \quad : \quad |u| \;>\; 1\,.$$

Odd$^{\ell}$, independent variable $|u| < 1$:

$$\sigma^{+\,\ell}(u') \;=\; \frac{1}{\pi} \quad : \quad |u'| \;<\; 1$$

Odd$^{\ell}$, $|u| > 1$: Not new

Even$^{\ell}$, $|u| < 1$:

$$\left(\int_{-\infty}^{-1} + \int_{1}^{\infty}\right) du' \sqrt{1 - \frac{1}{(u')^2}}\, \frac{u'}{u - u'}\, \sigma^{-\,\ell}(u')$$
$$=\; -\alpha \;+\; \pi\sqrt{1-u^2}\,\sigma^{-\,\ell}(u)$$

Here

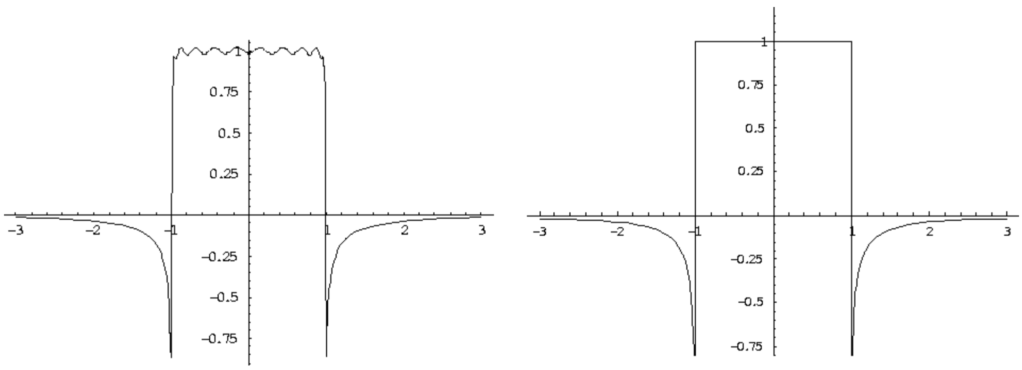
$$\alpha \;=\; \int_{-\infty}^{\infty} du\, \sigma^{-\,\ell}(u)\,.$$

Even$^{\ell}$, $|u| > 1$: Identically zero

Even$^{s\ell}$, $|u| > 1$:

$$-\alpha \;=\; u\sqrt{1-\frac{1}{u^2}}\int_{-\infty}^{\infty} du'\,\frac{1}{u-u'}\,\sigma^{+\,\ell}(u') \;+\; \left(\int_{-\infty}^{-1}+\int_{1}^{\infty}\right) du'\,\sqrt{1-\frac{1}{(u')^2}}\,\frac{u'}{u-u'}\,\sigma^{-\,\ell}(u')$$

- Equation on $\sigma_e(u)\ :\ |u| > 1$
- Put $u = \coth(x)$ and solve by Fourier transform (like KSV).
- **Even**$^{\ell}$ fixes $\sigma_e(u)\ :\ |u| < 1$.
- **Odd**$^{s\ell}$ yields $\sigma_o(u) = 0\ :\ |u| > 1$ and $\sigma^{+\,s\ell}(u) = 0\ :\ |u| < 1$.
- Solution also in AABEK, KSV `ht/0703031`, BdAF `ht/0703131`



$$\pi\sigma(u) = \left(1-\frac{1}{2}\theta(|u|-1)\left[\left(\frac{u+1}{u-1}\right)^{\frac{1}{4}}+\left(\frac{u-1}{u+1}\right)^{\frac{1}{4}}\right]\right)+\dots$$

- The solution reproduces the GKP value for the leading energy.
- We found an **algebraic function** carrying $\log(s)$ as a coefficient.
- A **gap opens**:

$$\sigma(u) \;=\; 1/\pi \;+\; 0\,\epsilon \;+\dots \quad : \quad |u| < 1\,.$$

14 NLO - Work in Progress (AABEK)

In the BBKS t-Picture the leading solution corresponds to

$$s^{\ell}_{2n-1} = \frac{(-1)^{n-1}(2n-1)!!}{2^n(n-1)!}, \qquad s^{\ell}_{2n} = s^{\ell}_{2n-1}.$$

Note: These are the Taylor coefficients of $(1+x)^{-3/2}$.

- In the u-picture, the higher orders contain non-integrable singularities. An order-by-order treatment is perhaps impossible.
- The branch points $u = \pm 1$ become relevant, c.f. KSV.
- In the t-picture, expanded in powers of g^{-1}, we meet divergent sums as the matrix rank is taken to infinity.

Nonetheless, the t-picture NLO matrix equations are solved by

$$\begin{aligned} s^{s\ell}_{2m-1} &= (-1)^m \frac{2m-1}{2}\left[\frac{b}{6}\frac{(2m-1)!!}{(2m-4)!!} - 2c\frac{(2m-1)!!}{(2m-2)!!}\right], \\ s^{s\ell}_{2m} &= (-1)^m\, m\left[\frac{b}{6}\frac{(2m+1)!!}{(2m-2)!!} + 2c\frac{(2m-1)!!}{(2m-2)!!}\right] \end{aligned}$$

with b and c left undetermined. The numerical best-fit is

$$b = \frac{1}{2} + \frac{3\log(2)}{\pi}, \qquad c = -\frac{3\log(2)}{8\pi}.$$

- The logarithm may be an accumulative effect of all higher orders, c.f. KSV.
- CK `ht/07050890` derived the NLO energy directly from the dressed Bethe ansatz.

15 Conclusions

- We have discussed the all-loops Bethe ansatz for the derivative operator sector. The energy of the lowest lying state scales logarithmically with the total spin s as the number of derivatives becomes large.
- We have shown how the one-loop logarithm carries over to the higher order contributions. The coefficient of $\log(s)$ is the "scaling function" $f(g)$.
- The weak coupling (gauge theory) Bethe ansatz is fixed up to four loops by current data. It contains a "dressing factor" which becomes relevant at four loops and beyond.
- At strong-coupling (string theory) the dressing phase had been conjectured on grounds of calculational data paired with crossing symmetry constraints. We have presented the weak coupling expansion of this string theory dressing phase and discussed its effect on the scaling function.
- The four-loop term of the result $f_+(g)$ agrees with field theory calculations!
- Our result explains the string theory/field theory "discrepancies" within the AdS_5/CFT_4 duality. It supports the original form of the AdS/CFT conjecture whereas the weak coupling dressing phase breaks perturbative BMN scaling at four loops and beyond.
- We have "unnested" the dressing phase on the expense of introducing an auxiliary density, in the spirit of RSZ `ht/0702151` and SS `ht/0703177`.
- We have analytically derived the strong coupling limit of the root density of the BES equation, and presented some preliminary results on the NLO correction. A systematic understanding of the higher orders is still lacking.

16 Outlook

- The "unnesting" of the dressing phase might signal the possibility of introducing another level into the nested Bethe ansatz (c.f. BS) describing the spectrum of operators.
- We need to understand the strong-coupling behaviour of the scaling function by analytic means. The two-loop energy can hopefully be calculated from the AdS sigma model, c.f. RTT `ht/07043638`.
- The scaling function $f(g)$ escapes the problem of "wrapping" because of the L independence of the underlying integral equation. This is not so for "short" operators. The calculation of the four-loop anomalous dimension of the Konishi field would help to understand wrapping effects. We plan to draw upon a method developed for the calculation of a class of three-loop anomalous dimensions.
- The scaling function, which we obtained from the Bethe ansatz, also occurs as a coefficient in the iteration relation for MHV amplitudes proposed by BDS. We should try to understand how the recursive structure of these amplitudes is related to integrability.
- Four-point functions of BPS operators seem to show iterative patterns, too. We will attempt a calculation of the three-loop four-point function of the stress tensor multiplet in $\mathcal{N} = 4$ SYM. Hopefully, we will discover a guiding principle like the "rung rule" of BCDKS.