Mass Spectrum in large N QCD: 30 years later

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1 Introduction

In this talk I am going to reflect on relation between CFT and Regge trajectories, speculating how to get from perturbative QCD to Regge trajectories using analytic methods buried in my old paper. I also will discuss the meaning of the magic coincidence between the regularized $AdS \times S^5$string theory and Padé regularized CFT.

2 Padé regularization

By starting with Padé approximation for 2-point propagators of gauge invariant operators at large Euclidean momenta and taking the limit when the order of approximation goes infinity together with position in Euclidean space where we approximate we arrive at approximation by infinite number of resonances. Unitarity is preserved by this approximation.

3 Conformal limit

In case of power behavior of perturbative propagators we find exact formulas for Padé approximants. In the limit of infinite number of resonances they reduce to ratio of Bessel functions with index related to anomalous dimension of the operator. The mass spectrum is determined by roots of Bessel functions.

4 Comparison with String Model

Using conjectured analogy between $N = 4$ SYM theory and certain string theory several authors recently reproduced my formulas for the mass spectrum. I am trying to explain why this miracle happens and what are implications for QCD. The short answer is that mass spectrum follows from general analytic properties of CFT as UV limit of the theory of free particles. Specific form of string theory in the AdS/CFT analogy is irrelevant.

5 Large N QCD and extrapolation of the mass spectrum

Masses in Padé regularized theory depend on the infrared cutoff R having the meaning of spacial box (though none of Lorentz symmetries including translation invariance is broken). We discuss how to extrapolate to $R = \infty$ using perturbation theory and nonperturbative corrections from SVZ vacuum condensates.

6 Padé perturbation theory

Explicit formulas for transforming ordinary perturbation expansion into the set of linear equations for coefficients of the mass spectrum equation are derived. This equation starts with the Bessel function equal zero and gets corrections to the expansion coefficients in terms of mass.

7 Introduction

Large N limit of QCD is a remarkable approximation, with many features we observe in hadron world. In this limit mesons must lie on infinitely rising Regge trajectories, corresponding to families of free particles with arbitrary spins. Decays of these particles show up in higher orders of $1/N$ expansion. These particles are composed of quark-antiquark pair plus some glue, permanently confining these quarks. In fact, quark confinement makes clear mathematical sense only in that limit, otherwise quark and antiquark being separated far enough will create extra quark-antiquark pair transforming them into pair of mesons. In the $N = \infty$ limit this does not happen, so that quark and antiquark can be separated arbitrarily far without creating extra pair from vacuum. In fact, the higher spin mesons correspond exactly to this picture of large separation of quark and antiquark. Apparently, this expansion contains some small factor in front of $1/N$ so that it works well numerically.

Another mathematical toy, also close to reality is a Conformal Field Theory (CFT). The simplest version of CFT is a free quark-gluon theory, in the limit of massless quarks. In reality, quarks are only asymptotically free, at distances less than $10^{-13}cm$, where quark-gluon interaction is small. To the first order in this interaction, QCD is a CFT with nontrivial anomalous dimensions of gauge invariant composite fields. The β function, violating conformal symmetry, shows up in the next order, when effective coupling starts running with logarithm of spatial or momentum scale.

So, we have two complementary pictures, describing QCD in small distances (CFT) and large distances (Regge trajectories). Ultimately, one would like to prove the Regge trajectories starting from asymptotic freedom. This task, even at $N = \infty$ proved to be beyond our present mathematical capabilities. Maybe this hard problem will stay for another century, like Fermat Theorem did. More modest approach would be to take this picture for granted, and check its consistency with asymptotic freedom. In practice, one would like to have an algorithm for computation of meson spectrum based on these two properties.

I made some advances in that direction 30 years ago, by noticing that the conformal limit of QCD is "dual" to the Regge trajectories in a following sense. If we assume existence of such trajectories and try to compute masses by requesting that sum of the pole terms for the 2-point function of conformal fields asymptotically reproduces CFT at large momenta – we, in fact, get quite sensible mass spectrum, in qualitative and even quantitative agreement with experiment. It turned out, that sum of positive pole terms with positive masses and positive residues represents a very special case of meromorphic function. Requesting that it approaches the power or logarithmic asymptotics with maximal speed (faster then any negative power of momenta) - fixes the spectrum almost uniquely.

This observation was then elaborated in famous QCD sum rules by Shifman, Vainstein, Zakharov and Voloshin. They introduced CFT breaking by powerlike terms due to VEV of various composite field such as quark condensate or vacuum density of YM Lagrangean. They did not use my analytic methods for computation of mass spectrum from maximal convergence principle, but rather compared sums of empirical pole terms with free quark loop and, indeed, observed remarkable agreement at the momentum scale, where the powelike terms from vacuum condensates started to kick in.

For many years, this served as a happy end of the story. The initial motivation based on maximal convergence as well as exact analytical solution for mass spectrum I found (roots of Bessel functions with indexes equal to dimension of conformal operator) was totally forgotten by everybody including myself.

Recently, another progress was made in this field. This progress is mostly a mathematical in its nature (like almost all which is happening in Modern Theoretical Physics), but it may be a beginning of the solution of our Fermat Theorem. At least, some beautiful unexpected analogies were found, and some hopes of exact solution of Large N QCD were revived.

What is especially exciting for me personally, is that for the first time, there is a CFT in 4 dimension with calculable anomalous dimensions. I dreamed about such theory in late sixties, even before discovery of asymptotically free QCD. This theory: $\mathcal{N} = 4$ SYM theory is degenerate in the sense that β function vanishes identically in virtue of SUSY. However, this theory is far from trivial. In the Large N limit it is dual (or at least is believed to be dual) to certain "string theory", namely $AdS \times S^5$string theory,created for that purpose few years ago by Maldacena, Polyakov and Klebanov. Bad news about this theory is that it does not have any string tension, so that there is no mass spectrum. This is, of course, to be expected, as CFT does not have a mass spectrum either.

Still, the $AdS \times S^5/CFT$ duality gets very close to desired string picture. Just perturb it in some subtle way, and the tension will arise. Several people observed that regularizing this string theory along the 5th (scaling) coordinate will produce mass spectrum. Oh-miracle- it exactly reproduced my Bessel roots. Here is where it hits a brick wall, though. Any known perturbations of this dual theory, including above mentioned 5th coordinate regularization, break one of its symmetries, destroying analogy and

thus killing the purpose.
In this talk I am going to reflect on relation between CFT and Regge trajectories, speculating how to get from perturbative QCD to Regge trajectories using analytic methods buried in my old paper. I also will discuss the meaning of the magic coincidence between the regularized $AdS \times S^5$string theory and Padé regularized CFT.

8 Padé regularization

Padé approximation was originally invented centuries ago to improve Taylor expansion of the function by replacing it by a rational function with the same first terms of taylor expansion. Here, we are going to use the methods of Padé approximation to achieve a different goal of restoring correct analytical properties of asymptotic expansion at large variable. We are going to take the limit of large order N of rational function when this function becomes meromorphic, because this is exactly the correct analytic property of large N QCD.
There is more than that in Padé approximation in this particular case. The functions $G(t)$ under considerations (2-point function of gauge invariant operators made of quark, antiquark and gluon field) are so called Stiltjes functions of momentum square $t = p^2$. They are analytic in the plane, cut along positive axis, with positive discontinuity along this axis $\rho(t) = \Im G(t + \imath 0) > 0$. In perturbation theory, this discontinuity has the meaning of cross section of creation of quark-antiquark pair plus gluons. In the world of confined quarks in the large N limit this discontinuity must be equal to the sum of infinite number of infinitely narrow resonances coming from transitions into mesons made of quarks:

$$\rho(t) = \sum Z_k \pi \delta(t - m_k^2).$$

In perturbative QCD the spectral function $\rho(t)$ is proportional to some power of t times some function of effective coupling constant $\lambda(\log(t))$, satisfying the RG equation $\lambda^{'} = \beta(\lambda) = -a\lambda^2 - b\lambda^3$. As it turns out, Padé approximation in a certain limit of large order N, provides an ideal tool to match these contradicting asymptotic formulas.
There are theorems which guarantee that Padé approximation of Stiltjes function preserves its analytic properties. In our case this means that the discontinuity $\rho(t)$ will be approximated by sum of positive delta terms with real masses m_k as it should, for any finite rank of approximation.
From the physical point of view this means that it provides us with the theory of free mesons approximating perturbative QCD. We are going to use this not as an approximation, but rather as a **definition** of large N QCD. Then we will transform perturbation theory into a systematic method of computation of meson masses.
We start from some large Euclidean momentum $p^2 \to -\Lambda$ where perturbation theory holds, and expand in Taylor series of $x = p^2 + \Lambda$. This expansion can be replaced by an ordinary Padé approximant:

$$G(-\Lambda + x) = \frac{P(-\Lambda + x)}{Q(-\Lambda + x)} + O\left(x^{(2N+1)}\right);$$

The standard theory of Padé approximants provides us with linear set of equations for $Q(t)$

$$\int_0^\infty dt Q(t)\rho(t)\left(1+t/\Lambda\right)^{(r-2N-1)} = 0; r = 0, \dots N-1,$$

as well as explicit expression for $P(t)$:

$$P(t) = Q(t)G(t) - \int_0^\infty ds Q(s)\rho(s)\left(1+s/\Lambda\right)^{-(2N+1)} \frac{(1+t/\Lambda)^{(2N+1)}}{\pi(s-t)}$$

9 Conformal limit

In case we simply replace $\rho(t) \rightarrow \sigma t^\nu$ corresponding to its asymptotics, the solution of Padé equations is given by so called Jacobi polynomials.
In the limit of infinite order, with fixed ratio $R = \frac{N}{\sqrt{\Lambda}}$ it reduces to the Bessel function

$$Q(t) \to u^{-\nu/2} J_\nu \left(2\sqrt{u}\right) = (-u)^{-\nu/2} I_\nu \left(2\sqrt{-u}\right), u = tR^2.$$

The approximant reduces to the ratio of two Bessel functions:

$$G(t) = \frac{\sigma}{\sin(\pi\nu)} \left[\left((-t)^\nu \frac{I_{-\nu}(2R\sqrt{-t})}{I_\nu(2R\sqrt{-t})} \right) - (-t)^{\nu_0} \right]$$

Here $\nu_0 + 2 = \Delta_0$ is the normal dimension for the composite operator which 2-point function we are considering, and $\nu+2 = \Delta$ is its anomalous dimension. In the free QCD limit they coincide and become integer, in which case the logarithms appear after removing $0/0$ uncertainty. For example at $\nu \to 1$, corresponding to conserved vector current $\bar{q}\gamma_\mu q$ with dimension $\Delta = \Delta_0 = 3$:

$$G(q^2) = \sigma q^2 \frac{J_0(2Rq)\log(q^2) - \pi Y_0(2Rq)}{J_0(2Rq)}$$

In the deep Euclidean asymptotics $t \to -\infty$, the Bessel functions exponentially approach power limits so that

$$\frac{I_{-\nu}(2R\sqrt{-t})}{I_{\nu}(2R\sqrt{-t})} \rightarrow 1 + O\left(\exp\left(-4R\sqrt{-t}\right)\right)$$

and we recover the perturbation theory. Thus, our limit of Padé approximant provides meromorphic function, approaching given asymptotic behavior with exponential accuracy at large argument. In real world we must tend $R \rightarrow \infty$ to recover original theory. This limit can only be taken after some rearrangement of perturbation theory, which we consider below. In the zeroth order the mass spectrum corresponds to roots of Bessel functions with integer index, linearly growing with angular momentum. It looks as shown at Figure 5.

The one-loop computation introduces anomalous dimensions, depending on spin and other quantum numbers. The resulting Regge trajectories look as shown at Figure 10, for two values of effective coupling constant λ at confinement scale.

We observe good fit to experimental masses for $\lambda = 0.83$.

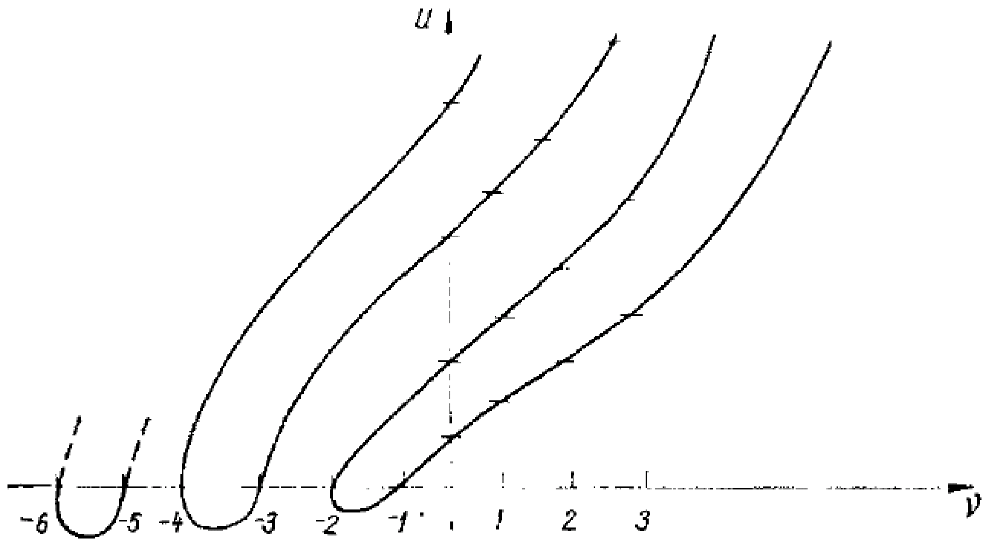

FIGURE 5

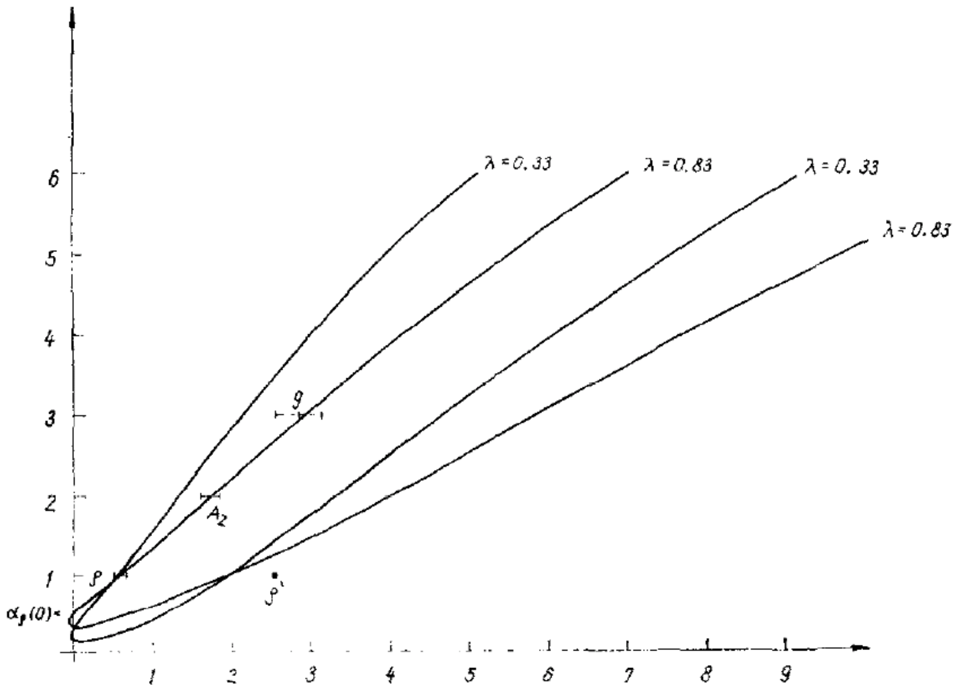

FIGURE 10

10 Comparison with String Model

Conjectured correspondence (Maldacena,Polyakov,Klebanov) between $AdS \times S^5$string theory and $\mathcal{N} = 4$ SYM theory in 4 dimensions goes as follows. Every gauge invariant operator $\mathcal{O}(x)$ in SYM theory corresponds to a field in anti deSitter space of $4 + 1 = 5$ dimensions, with the metric

$$ds^2 = \frac{\eta_{\mu\nu} dx^\mu dx^\nu - dz^2}{z^2}$$

where $\eta_{\mu\nu}$ =Diag$(1, -1, -1, -1)$. In SYM theory, this is a quantum field with the source $\phi_0(x)$ and corresponding VEV

$$\left\langle e^{\imath \int d^4x \phi_0(x) \mathcal{O}(x)} \right\rangle_{SYM} = \left\langle e^{\imath S_{AdS}[\phi]} \right\rangle_{\phi(x,z=0)=\phi_0(x)}$$

where the source $\phi_0(x)$ serves as a boundary value for the field $\phi(x, z)$ in deSitter space at its singular point $z = 0$.

In general, computation of the RHS of this equation involves quantum string effects which are as formidable as those of 4D field theory, but correspondence is fascinating on a purely theoretical level. The weak coupling of string theory (i.e. classical gravity) corresponds to unphysical large coupling limit of SYM theory. The spectrum of anomalous dimensions is known for arbitrary coupling constants in SYM theory, thanks to recent breakthrough, by Beisert,Staudacher and others. In case of conserved vector current the answer is especially simple, as the anomalous dimension is not renormalized.

In this case (see recent work of Erlich,Kribs and Low) one have to minimize the classical action

$$S_{AdS} = -\int d^4x dz \sqrt{-g} \frac{1}{4g_5^2} F_{MN} F^{MN},$$

where the capital roman letters $M, N = 0, 1, 2, 3, z$.The boundary value for the abelian gauge field A_M at $z = \varepsilon \to 0$ should coincide with the current $J_\mu = \bar{q}\gamma_\mu q$ in our four dimensions. In coordinate space the problem is to find the propagator of this abelian gauge field $A_M(x, z)$ and set one end to the boundary $z = \varepsilon$ of AdS space. In 4-momentum space q we have to solve the classical YM equation, corresponding to AdS metric:

$$z\partial_z \left(\frac{1}{z} \partial_z V(q, z) \right) + q^2 V(q, z) = 0,$$

This produces the Bessel functions

$$V(q, z) = qz \left(Y_0(q\varepsilon) J_1(qz) - J_0(q\varepsilon) Y_1(qz) \right)$$

After straightforward computation of the classical action S_{AdS} using the AdS/CFT correspondence we find:

$$\int d^4 x e^{\imath q x} \langle J_\mu(x) J_\nu(0) \rangle = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \Sigma(q^2)$$

with

$$\Sigma(q^2) = -\frac{1}{g_5^2} \left[\frac{1}{z} \frac{\partial_z V(q,z)}{V(q,z)} \right]_{z=\varepsilon} \to \frac{q^2}{2 g_5^2} \frac{\log(q) J_0(q\varepsilon) - \pi Y_0(q\varepsilon)}{J_0(q\varepsilon)}$$

This is exactly the same result as we obtained from Padé regularization. Comparing the normalization constant we find that

$$g_5^2 = \frac{12\pi^2}{N_c}$$

in the units of AdS curvature.
In the string model, the propagator comes about as a result of solution of classical equations of motion in five dimensional curved space. In the Padé theory, it follows from much more general principles. In particular, for large string coupling when explicit solution is not known, we know from the work of Beisert,Eden and Staudacher explicit spectrum of anomalous dimensions (to be more precise, we know some one-dimensional integral equation, solution of which gives this spectrum, which is as good

as exact formula for the computational purposes). The same work gives further arguments in favor of the AdS/CFT correspondence.
We therefore, can rely on Padé theory to predict the following spectrum of masses based on regularization of deSitter space at $z = \varepsilon$.

$$J_{\Delta(g)-2}(M\varepsilon) = 0$$

The dilatation operator spectrum and the mass spectrum are related as index and argument of the root of Bessel function. The only assumption made in this prediction is the fastest convergence of propagator to its conformal asymptotics.
Note that in general, there are many conformal operators with the same normal dimension, all mixed together by renormalization, so that finding the spectrum involves diagonalization of the Dilatation operator. This was done by means of Bethe Anzatz, using remarkable analogy with Heisenberg spin chain. The basic operators are traces of products of scalar fields $\Phi_i(x)$ where $i = 1, ...6$ which fields transform as color tensors (like the YM field strength $F_{\mu\nu}(x)$). These indices $i = 1, ...6$ correspond to various "spin" values of this quantum Heisenberg chain, and dilatation operator, whose eigenvalues are anomalous dimensions, reduces to certain (long range) quantum Hamiltonian of one-dimensional chain with periodic boundary conditions. So, this is an internal symmetry space, nothing to do to the color space not to mention physical space.

For example, for the lowest dimension operator, made of five scalars has dimension

$$\Delta = 5 + 8g^2 - 24g^4 + 136g^6 - 16\left(\frac{115}{2} + 8\varsigma(3)\right)g^8 + O(g^{10});$$

where

$$g = \frac{\sqrt{\lambda}}{4\pi}$$

and $\lambda = N_c g_{gauge}^2$ is the 't Hoofts coupling constant in the limit of infinite number of colors N_c. Higher terms in this expansion are all available, as well as exact set of integral equations coming from Bethe Anzatz solution of the 1D quantum Heisenberg chain. We have to note, however, that there is no physical mechanism in this $\mathcal{N} = 4$ SYM theory which would in fact produce a mass spectrum. The regularization – Padé or deSitter remains a purely phenomenological device, while we are waiting for the real theory.

11 Large N QCD and extrapolation of the mass spectrum

In the large N QCD we all believe that there is a physical mechanism leading to quark confinement in the large N limit and beyond. This mechanism has to do with properties of QCD vacuum – it repels the

color flux, squeezing it into tubes connecting quarks. In the large N limit these tubes are supposedly infinitely thin, and act as some kind of strings connecting quarks and confining them into mesons and baryons.
Leaving aside the hard task of rigorous proof of quark confinement, let us take it for granted and try to see how we can use it with our Padé regularization to compute the hadron spectrum.
Mathematically, we have to do the following. We have to add the terms, breaking conformal symmetry in QCD (β function plus vacuum condensates) to the spectral density $\rho(t)$ and perturb above Bessel solution by these terms. Then, we will have to extrapolate resulting masses to $R \to \infty$. From dimensional arguments, keeping in mind that R has dimension of length, we will have the following spectrum

$$m_k = \frac{F_k(\mu R)}{R}$$

where μ is the quark confinement scale. The function $F_k(\mu R)$ is known at small arguments from perturbative QCD. The running coupling constant $\lambda(\log(\mu R))$ will produce corrections in inverse powers of $\log(\mu R)$ while the SVZ vacuum condensates will produce powerlike corrections $(\mu R)^a$. The quark confinement then simply means that $F_k(\mu R)$ grow linearly at large arguments to cancel the R in denominator of the mass spectrum formula.
Convenient method of such extrapolation is given by the following Legendre transform:

$$m_k = \lim_{\gamma \nearrow 1} \min_R \left(\frac{F_k(\mu R)}{R^\gamma} \right)$$

Minimization with respect to R at $0 < \gamma < 1$ will find a minimum at finite R because $F_k(\mu R)$ starts with constant at $R = 0$ and asymptotically grows linearly, which is faster than denominator. So, it goes to $+\infty$ both at $R \to 0$ and $R \to +\infty,$ having a minimum somewhere in between. The position $R_*(\gamma) = X(\gamma)/\mu$ of this minimum can be computed as a function of γ using QCD perturbation expansion as well as powerlike corrections coming from vacuum condensates. From dimensional arguments we will have:

$$m_k = \lim_{\gamma \nearrow 1} \mu^\gamma \Phi_k(\gamma) = \mu \Phi_k(1)$$

where

$$\Phi_k(\gamma) = \min_R \left(\frac{F_k(\mu R)}{(\mu R)^\gamma} \right) = \frac{F_k(X(\gamma))}{(X(\gamma))^\gamma}$$

is a universal dimensionless function. This phenomenon was called dimensional transmutation in the seventies.

The QCD perturbation expansion was considered in my old paper, - it goes in powers of $\alpha = \sqrt{\gamma}$with calculable coefficients. The first two terms I computed did not show any convergence. With modern knowledge of 3-and 4-loops in QCD one can get two more terms of this expansion, which seems like a straightforward task for a hard-working Ph.D. student.
The vacuum condensate corrections were considered only at a phenomenological level by SVZ, without using Padé equations. They observed, that matching the sum of pole terms with the first quark loop plus the power terms from vacuum condensate does provide correct mass scale as well as their relative ratios. From my point of view this is a particular case of minimization procedure: they balanced derivatives of the quark loop and the power term and used resulting scale R as a mass scale in the sum rules, similar to the Padé equations.
So, it seems that the minimization procedure is sensible at phenomenological level, but the perturbative corrections are small compared to the powerlike vacuum condensate corrections. The systematic procedure for this Padé perturbation theory will be described below.

12 Padé perturbation theory

In order to perturb Padé equations around the powerlike spectral function $\rho(t) = t^{\nu}$ we have to find the Greens function from the following set of equations

$$\int_0^\infty dt t^\nu \left(1 + t/\Lambda\right)^{(r-2N-1)} G(t,s) = s^\nu \left(1 + s/\Lambda\right)^{(r-2N-1)}; r = 0, \ldots N-1.$$

The solution is known for the last 30 years

$$\begin{aligned} G(t,s) &= \oint_C \frac{d\omega}{2\pi\imath} \oint_{C'} \frac{d\omega'}{2\pi\imath} \frac{1}{\omega' - \omega} \frac{f(\omega)}{f(\omega')} \frac{(1 + t/\Lambda)^\omega}{(1 + s/\Lambda)^{\omega'}}; \\ f(\omega) &= \frac{\Gamma(2N + 1 - \omega)\Gamma(-\omega)}{\Gamma(N + 1 - \nu - \omega)\Gamma(N + 1 - \omega)} N^{2(1-\nu)}. \end{aligned}$$

Here contour C encloses the poles of $f(\omega)$ which are located at $\omega = 0, \ldots N$ and C' encloses its zeroes, which are located at $\omega = N + k - \nu; k = 1, \ldots\infty$. By taking residues at poles of $f(\omega)$ we observe that $G(t,s)$ is an N-degree polynomial with t-dependent coefficients.
Now we split the spectral density into leading conformal part plus sum of higher negative power terms

$$\rho(t) = \sigma t^\nu \left(1 + \sum_k \epsilon_k t^{-a_k}\right)$$

and replace the factor $s^{\nu}\,(1+s/\Lambda)^{(r-2N-1)}$ in Padé equation by the LHS of the above equation for $G(t,s)$. We find

$$0 = \int\limits_0^\infty dt t^{\nu}\,(1+t/\Lambda)^{(r-2N-1)}\,S(t);$$
$$S(t) = \int\limits_0^\infty ds Q(s)(1+\sum_k \epsilon_k s^{-a_k}))G(t,s).$$

In other words, this $S(t)$ by construction satisfies Padé equations for conformal theory, solution to which we know. This is the Jacobi polynomial, which in our notations read:

$$S(t) = \oint\limits_C \frac{d\omega}{2\pi\imath} f(\omega)\,(1+t/\Lambda)^{\omega}$$

Now we can go to the limit $N \to \infty$ when Jacobi polynomial reduces to Bessel functions. This can easier be done directly for expansion coefficients:

$$Q(s) = \sum_{n=0}^{\infty} q_n \left(sR^2\right)^n$$

The resulting linear equations for these coefficients read

$$q_n^{(0)} = q_n + \sum_{m=0}^{\infty} F_{nm} q_m$$

where

$$q_n^{(0)} = \frac{(-1)^n}{n!\Gamma(\nu + n + 1)}$$

are the Bessel expansion coefficients and

$$F_{nm} = \frac{(-1)^n}{n!\Gamma(\nu + n)} \sum_k \epsilon_k R^{2a_k} \frac{\Gamma(\nu - a_k + m)}{(a_k + n - m)\Gamma(a_k - m)}$$

As this equation is linear one can write a formal solution

$$\overrightarrow{q} = \left(1 + \widehat{F}\right)^{-1} \overrightarrow{q_0}$$

Higher powers of $\widehat{F}$ matrix involve calculable (hypergeometric) sums

$$S(n, l|a, b) = \sum_{m=0}^{\infty} \frac{\Gamma(\nu - a + m)}{\Gamma(a - m)} \frac{(-1)^m}{m!\Gamma(\nu + m)} \frac{1}{(a + n - m)(b + m - l)}$$

which converge as

$$\frac{\sin(\pi a)}{\pi} \sum_m m^{-2a} \frac{1}{(a + n - m)(b + m - l)}$$

In particular, the second power

$$\left(\widehat{F}^2\right)_{nl} = \frac{(-1)^n}{n!\Gamma(\nu + n)} \sum_{k,q} \epsilon_k R^{2a_k} \epsilon_q R^{2a_q} S\left(n, l|a_k, a_q\right) \frac{\Gamma(\nu - a_q + l)}{\Gamma(a_q - l)}$$

In real life there is one more complication, namely the logarithmic corrections to all power terms. This can be accounted by replacing constants ϵ_k by linear operators, depending on $\hat{d} = \frac{d}{da_k}$ (which produces $-\log(t)$)

$$\epsilon_k \implies \epsilon_k(\hat{d})$$

Using Callan-Symanzik equations like I did in the old paper, one can reduce these logarithmic corrections to the same matrix F_{nm} differentiated by a_k.
Final complication is that the terms violating conformal invariance, are no longer diagonal in terms of dimension of operators, so that there will be off-diagonal terms. The above equations will become matrix equations in space of conformal operators, with ϵ_k having off-diagonal terms, and σ being block-diagonal in terms of operator dimension. Note that the one-loop corrections exponentiate, as in the old paper, so that both ν and a_k have non-integer parts proportional to effective coupling at scale R .
My main message is that relation between ordinary high-energy expansion for 2-point function and equation for the mass spectrum is bilinear. All one has to do is to compute some number of terms in $\widehat{F}$ matrix. The expansion for Bessel functions converge, so that vectors $\overrightarrow{q}, \overrightarrow{q}_0$ are normalizable. Their scalar products at momentum vector $\overrightarrow{T}(x) = \{1, x, x^2, ...\}$ must also converge

$$\overrightarrow{T}(tR^2)\overrightarrow{q}_0 = \left(R\sqrt{t}\right)^{-\nu} J_\nu\left(2R\sqrt{t}\right), \overrightarrow{T}(tR^2)\overrightarrow{q} = Q(t)$$

The mass spectrum m is to be found from

$$\overrightarrow{T}(m^2R^2)\left(1+\widehat{F}\right)^{-1}\overrightarrow{q}_0 = 0$$

or, in the form suitable for iterations

$$(mR)^{-\nu}J_\nu\left(2mR\right) = \overrightarrow{T}\left(m^2R^2\right)\frac{\widehat{F}}{1+\widehat{F}}\overrightarrow{q}_0$$

If anybody still cares about computation of mass spectrum in large N QCD, the above equations seems to provide the proper toolset. We know from old work that zeroth approximation is close to reality, and we know enough about QCD perturbation theory by now to be able to find several terms – if not analytically, then by computer.