BFKL and double-logarithmic resummations in $N = 4$ SUSY

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Content

1 Gluon reggeization

Mandelstam variables

$$s = (p_A + p_B)^2\,,\ t = (p_A - p_{A'})^2\,,\ u = (p_A - p_{B'})^2$$

Regge kinematics

$$s = 4E^2 \gg -t = |q|^2 \approx E^2\,\theta^2$$

QCD Born amplitude

$$M_{AB}^{A'B'}(s,t)|_{Born} = g\,T^c_{A'A}\,\delta_{\lambda_{A'}\lambda_A}\,\frac{2s}{t}\,g\,T^c_{B'B}\,\delta_{\lambda_{B'}\lambda_B}$$

Leading logarithmic approximation

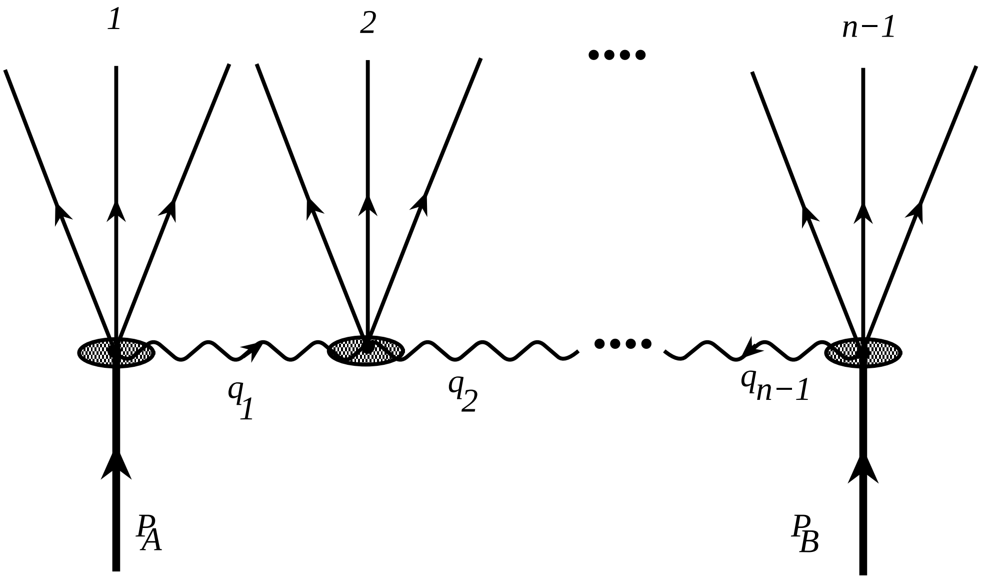
$$M_{AB}^{A'B'}(s,t) = M_{AB}^{A'B'}(s,t)|_{Born}\,s^{\omega(t)},$$

$$\alpha_s \ln s \sim 1\,,\ \alpha_s = \frac{g^2}{4\pi} \ll 1$$

Gluon Regge trajectory in LLA

$$\omega(-|q|^2) = -\frac{\alpha_s N_c}{4\pi^2}\int d^2k\,\frac{|q|^2}{|k|^2|q-k|^2} \approx -\frac{\alpha_s N_c}{2\pi}\,\ln\frac{|q^2|}{\lambda^2}$$

2 Inelastic amplitudes



Gluon production amplitude in LLA

$$M_{2\to 1+n} \sim \frac{s_1^{\omega_1}}{|q_1|^2}\, gT^{d_1}_{c_2c_1} C(q_2,q_1)\, \frac{s_2^{\omega_2}}{|q_2|^2} ... C(q_n,q_{n-1})\, \frac{s_n^{\omega_n}}{|q_n|^2}$$

Multi-Regge kinematics

$$s \gg s_r = (k_{r-1}+k_r)^2 \gg |q_r|^2\,,\ C(q_{r+1},q_r) = \frac{q_{r+1}q_r^*}{k_r^*}$$

3 BFKL equation (1975)

Impact parameter coordinates and momenta

$$\rho_k = x_k + iy_k\,,\ \rho_k^* = x_k - iy_k\,,\ p_k = i\frac{\partial}{\partial \rho_k}\,,\ p_k^* = i\frac{\partial}{\partial \rho_k^*}$$

Balitsky-Fadin-Kuraev-Lipatov equation

$$E\,\Psi(\vec{\rho}_1, \vec{\rho}_2) = H_{12}\,\Psi(\vec{\rho}_1, \vec{\rho}_2)\,,\ \Delta = -\frac{\alpha_s N_c}{2\pi}\,E$$

BFKL Hamiltonian

$$H_{12} = \ln\,|p_1 p_2|^2 + \frac{1}{p_1 p_2^*}(\ln|\rho_{12}|^2) p_1 p_2^*$$

$$+\frac{1}{p_1^* p_2}(\ln|\rho_{12}|^2) p_1^* p_2 - 4\psi(1)\,,\ \rho_{12} = \rho_1 - \rho_2$$

Möbius invariance and conformal weights (L. (1986))

$$\rho_k \to \frac{a\rho_k + b}{c\rho_k + d}\,,$$

$$m = \gamma + n/2\,,\ \widetilde{m} = \gamma - n/2\,,\ \gamma = 1/2 + i\nu$$

Bartels-Kwiecinski-Praszalowicz equation (1980)

$$E\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n) = H\,\Psi(\vec{\rho}_1, ..., \vec{\rho}_n)\,,\ H = \sum_{k<l}\frac{\vec{T}_k\vec{T}_l}{-N_c}\,H_{kl}$$

4 Integrability (1993)

Holomorphic factorization for $N_c \to \infty$ (L. (1988))

$$\Psi(\vec{\rho}_1, \vec{\rho}_2, ..., \vec{\rho}_n) = \sum_{r,s} a_{r,s}\, \Psi_r(\rho_1, ..., \rho_n)\, \Psi_s(\rho_1^*, ..., \rho_n^*)$$

Duality symmetry (L. (1998))

$$\rho_{r,r+1} \longrightarrow p_r \longrightarrow \rho_{r-1,r}$$

Simplest integral of motion

$$A = q_n = \rho_{12}\rho_{23}...\rho_{n1}\, p_1 p_2 ... p_n\,, \;\; [h, A] = 0$$

Transfer and monodromy matrices (L. (1993))

$$T(u) = tr\, t(u)\,,\;\; t(u) = L_1 L_2 ... L_n = \sum_{r=0}^{n} u^{n-r}\, q_r\,,$$

$$L_k = \begin{pmatrix} u + \rho_k\, p_k & p_k \\ -\rho_k^2\, p_k & u - \rho_k\, p_k \end{pmatrix}\,,\;\; \hat{l} = u\, \hat{1} + i\, \hat{P}$$

Yang-Baxter equation (L. (1993))

$$t_{r_1'}^{s_1}(u)\, t_{r_2'}^{s_2}(v)\, l_{r_1 r_2}^{r_1' r_2'}(v-u) = l_{s_1' s_2'}^{s_1 s_2}(v-u)\, t_{r_2}^{s_2'}(v)\, t_{r_1}^{s_1'}(u)$$

5 Pomeron in $N=4$ SUSY

BFKL kernel in two loops (F., L. (1998))

$$\omega = 4\,\hat{a}\;\chi(n,\gamma) + 4\;\hat{a}^2\,\Delta(n,\gamma)\,,\;\; \hat{a} = g^2 N_c/(16\pi^2)\,,$$

$$\chi(n,\gamma) = 2\Psi(1) - \Psi(\gamma + |n|/2) - \Psi(1-\gamma+|n|/2)$$

Non-analytic terms in QCD (K.,L. (2000))

$$\Delta_{QCD}(n,\gamma) = c_0\delta_{n,0} + c_2\delta_{n,2} + ...$$

Hermitian separability in $N=4$ SUSY (K.,L. (2000))

$$\Delta(n,\gamma) = \phi(M) + \phi(M^*) - \frac{\rho(M)+\rho(M^*)}{2\hat{a}/\omega}\,,\; M = \gamma + \frac{|n|}{2}\,,$$

$$\rho(M) = \beta'(M) + \frac{1}{2}\zeta(2)\,,\; \beta'(z) = \frac{1}{4}\left[\Psi'\left(\frac{z+1}{2}\right) - \Psi'\left(\frac{z}{2}\right)\right]$$

Maximal transcedentality (K.,L. (2002))

$$\phi(M) = 3\zeta(3) + \Psi''(M) - 2\Phi(M) + 2\beta'(M)\Big(\Psi(1) - \Psi(M)\Big)$$

$$\Phi(M) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+M}\left(\Psi'(k+1)\; - \;\frac{\Psi(k+1)-\Psi(1)}{k+M}\right)$$

6 Pomeron and graviton

BFKL Pomeron in a diffusion approximation

$$j = 2 - \Delta - D\,\nu^2$$

Anomalous dimension of twist-2 operators

$$\gamma = 1 + \frac{j-2}{2} + i\nu$$

Constraint from the conservation of $T_{\mu\nu}$

$$\gamma = (j-2)\left(\frac{1}{2} - \frac{1/\Delta}{1+\sqrt{1+(j-2)/\Delta}}\right)$$

AdS/CFT for the graviton Regge trajectry

$$j = 2 + \frac{\alpha'}{2}\,t\,,\ \ t = E^2/R^2\,,\ \ \alpha' = \frac{R^2}{2}\,\Delta$$

Gubser, Klebanov and Polyakov prediction

$$\gamma_{|z,j\to\infty} = -\sqrt{j-2}\,\Delta^{-1/2}_{|j\to\infty} = \sqrt{\pi j}\,z^{1/4}$$

Pomeron intercept at large α (K.,L.,O.,V.)

$$j = 2 - \Delta\,,\ \ \Delta = \frac{1}{\pi}\,z^{-1/2} \approx \frac{\sqrt{3}}{2\pi}z^{-1/2}\,,$$

See also: Polchinsky, Strassler, Brower, C-I Tan

7 One-loop anomalous dimension in $N=4$ SUSY

Anomalous dimension matrix (L. (1997))

$$\gamma_{gg} = -\frac{8}{j-1}+\frac{8}{j}-\frac{8}{j+1}+\frac{8}{j+2}+8S_1(j)\,,\ \gamma_{q\varphi} = -\frac{16}{j}\,,$$

$$\gamma_{gq} = -\frac{8}{j-1}+\frac{8}{j}-\frac{4}{j+1}\,,\ \gamma_{g\varphi} = -\frac{8}{j-1}+\frac{8}{j}\,,$$

$$\gamma_{qg} = -\frac{16}{j}+\frac{32}{j+1}-\frac{32}{j+2}\,,\ \gamma_{qq} = -\frac{16}{j}+\frac{16}{j+1}+8S_1(j),$$

$$\gamma_{\varphi g} = -\frac{24}{j+1}+\frac{24}{j+2}\,,\ \gamma_{\varphi q} = -\frac{12}{j+1}\,,\ \gamma_{\varphi\varphi} = 8S_1(j),$$

$$\widetilde{\gamma}_{gg} = -\frac{16}{j}+\frac{16}{j+1}+8S_1(j)\,,\ \widetilde{\gamma}_{gq} = -\frac{8}{j}+\frac{4}{j+1}\,,$$

$$\widetilde{\gamma}_{qg} = \frac{16}{j}-\frac{32}{j+1}\,,\ \widetilde{\gamma}_{qq} = \frac{8}{j}-\frac{8}{j+1}+8S_1(j)$$

Diagonalization in the Born Approximation

$$\left|\begin{array}{ccc} S_1(j-2) & 0 & 0 \\ 0 & S_1(j) & 0 \\ 0 & 0 & S_1(j+2) \end{array}\right|,\ \left|\begin{array}{cc} S_1(j-1) & 0 \\ 0 & S_1(j+1) \end{array}\right|$$

Integrable Heisenberg spin model (L. (1997))

8 Maximal transcedentality

Most transcendental functions (K.,L. (2002))

$$\gamma(j) = \hat{a}\gamma_1(j)+\hat{a}^2\gamma_2(j)+\hat{a}^3\gamma_3(j)+\ldots,\ \gamma_1(j+2) = -4S_1(j)$$

Two-loop dimension (K.,L.,V. (2003))

$$\frac{\gamma_2(j+2)}{8} = 2S_1\left(S_2+S_{-2}\right) - 2S_{-2,1} + S_3 + S_{-3}$$

Three-loop dimension (K.,L.,O.,V. (2004))

$$\gamma_3(j+2)/32 = -12\,\left(S_{-3,1,1}+S_{-2,1,2}+S_{-2,2,1}\right)$$

$$+6\,\left(S_{-4,1}+S_{-3,2}+S_{-2,3}\right) - 3\,S_{-5} - 2\,S_3\,S_{-2} - S_5$$

$$-2\,S_1^2\left(3\,S_{-3}+S_3-2\,S_{-2,1}\right) - S_2\left(S_{-3}+S_3-2\,S_{-2,1}\right)$$

$$+24\,S_{-2,1,1,1} - S_1\left(8S_{-4}+S_{-2}^2+4S_2S_{-2}+2S_2^2\right)$$

$$-S_1\left(3S_4-12S_{-3,1}-10S_{-2,2}+16S_{-2,1,1}\right)\,,$$

$$S_a(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,,\ S_{a,b,c,\cdots}(j) = \sum_{m=1}^{j}\frac{1}{m^a}\,S_{b,c,\cdots}(m)\,,$$

$$S_{-a}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a},\ S_{-a,b,\cdots}(j) = \sum_{m=1}^{j}\frac{(-1)^m}{m^a}\,S_{b,\cdots}(m),$$

$$\overline{S}_{-a,b,c\cdots}(j) = (-1)^j S_{-a,b,\ldots}(j)+S_{-a,b,\cdots}(\infty)\Big(1-(-1)^j\Big)$$

9 Meromorphic properties

BFKL prediction at $j = 1 + \omega \to 1$

$$\gamma(j) = 4\frac{\hat{a}}{\omega} + 0\frac{\hat{a}^2}{\omega} + 32\zeta_3\frac{\hat{a}^3}{\omega} - 16\left(32\zeta_3 + \frac{\pi^4}{9}\omega\right)\frac{\hat{a}^4}{\omega^4} + \ldots$$

Poles of $\gamma(j)$ at $j + r = \omega \to 0$

$$\frac{\gamma_1(j)}{4} = \frac{1}{\omega} - S_1(r+1) - \omega(S_2(r+1) + \zeta_2)) + \ldots,$$

$$\frac{\gamma_2(j)}{16} = \frac{1+(-1)^r}{2\omega^3} - S_1(r+1)\frac{1+(-1)^r}{\omega^2} - \frac{\zeta_2 + (-1)^r(\zeta_2 + 2S_2(r+1))}{2\omega} + \ldots,$$

$$\frac{\gamma_3(j)}{64} = \frac{1+(-1)^r}{\omega^5} - 2S_1(r+1)\,\frac{1+2(-1)^r}{\omega^4} - \frac{\frac{S_2}{2} - S_1^2 - S_{-2} + 2\zeta_2 + (-1)^r(\frac{5}{2}S_2 - 3S_1^2 + 2\zeta_2)}{\omega^3} + \ldots$$

Effective parameter of the expansion

$$x = \frac{\hat{a}}{\omega^2}$$

Different singularities for even and odd r

$$\gamma(j)_{j\to-(2k)} = \frac{\hat{a}}{\omega}\,a(x)\,,\ \gamma(j)_{j\to-(2k+1)} = \frac{4\hat{a}}{\omega} + O(\frac{\hat{a}^2}{\omega^2})$$

10 Double logarithms

Double-logarithmic approximation

$$A(s) = A_{Born}\, f(\rho)\,,\ \ f(\rho) = \sum_{k=0}^{\infty} c_k\, \rho^{2k}\,,\ \ \rho^2 = \frac{2\alpha}{\pi} \ln^2(s)$$

Example: $e\bar{e}$-backward scattering (GGLF, (1967))

$$f(\rho) = \int_{-i\infty}^{i\infty} \frac{dl}{2\pi i}\, e^{\rho\, l}\, \frac{d}{dl}\, \ln D_{-\frac{1}{4}}(l)\,,\ \ l = \frac{j}{\sqrt{2\alpha/\pi}}$$

Ansatz for ep structure functions

$$W(x, Q^2) = \int_{-i\infty}^{i\infty} \frac{d\omega}{2\pi i} x^{-\omega} \int_{-i\infty}^{i\infty} \left(\frac{Q^2}{\lambda^2}\right)^{\gamma} \phi_\omega(\gamma)$$

DL evolution equations in λ (KL (1986))

Positive signature in $N = 4$:

$$\gamma_+(\omega - \gamma_+) = 4\,\hat{a}$$

Negative signature in $N = 4$:

$$\gamma_-(\omega - \gamma_-) = 4\,\hat{a} - 8\frac{\hat{a}}{\omega}\gamma_8\,,$$

$$\gamma_8(\omega - \gamma_8) = 2\,\hat{a} + 2\hat{a}\frac{d}{d\omega}\gamma_8$$

11 Four-loop result (KLRSV)

$$\frac{\gamma_4}{256} = \mathbf{4S_{-7}} + \mathbf{6S_7} + 2(S_{-3,1,3} + S_{-3,2,2} + S_{-3,3,1} + S_{-2,4,1})$$

$$+3(-S_{-2,5} + S_{-2,3,-2}) + 4(S_{-2,1,4} - S_{-2,-2,-2,1} - S_{-2,1,2,-2}$$

$$-S_{-2,2,1,-2} - S_{1,-2,1,3} - S_{1,-2,2,2} - S_{1,-2,3,1}) + 5(-S_{-3,4} +$$

$$S_{-2,-2,-3}) + 6(-S_{5,-2} + S_{1,-2,4} - S_{-2,-2,1,-2} - S_{1,-2,-2,-2})$$

$$+7(-S_{-2,-5} + S_{-3,-2,-2} + S_{-2,-3,-2} + S_{-2,-2,3}) + 8(S_{-4,1,2}$$

$$+S_{-4,2,1} - S_{-5,-2} - S_{-4,3} - S_{-2,1,-2,-2} + S_{1,-2,1,1,-2}) +$$

$$9S_{3,-2,-2} - 10S_{1,-2,2,-2} + 11S_{-3,2,-2} + 12(S_{-2,2,-3} - S_{-6,1}$$

$$+S_{1,4,-2} + S_{4,-2,1} + S_{4,1,-2} - S_{-3,1,1,-2} - S_{-2,2,-2,1} -$$

$$S_{1,1,2,3} - S_{1,1,3,-2} - S_{1,1,3,2} - S_{1,2,1,3} - S_{1,2,2,-2} -$$

$$S_{1,2,2,2} - S_{1,2,3,1} - S_{1,3,1,-2} - S_{1,3,1,2} - S_{1,3,2,1} -$$

$$S_{2,-2,1,2} - S_{2,-2,2,1} - S_{2,1,1,3} - S_{2,1,2,-2} - S_{2,1,2,2} -$$

$$S_{2,1,3,1} - S_{2,2,1,-2} - S_{2,2,1,2} - S_{2,2,2,1} - S_{2,3,1,1} - S_{3,1,1,-2} -$$

$$S_{3,1,1,2} - S_{3,1,2,1} - S_{3,2,1,1}) + 13S_{2,-2,3} - 14S_{2,-2,1,-2}$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-72S_{1,1,1,-4} - 80S_{1,1,-4,1} - \zeta(\mathbf{3})\mathbf{S_1}(\mathbf{S_3} - \mathbf{S_{-3}} + \mathbf{2S_{-2,1}})$$

12 Singularities in 4-loops

BFKL prediction for γ in 4 loops

$$\lim_{j\to 1} \gamma_4(j) = -16 \left(32\zeta_3 + \frac{\pi^4}{9}\omega\right) \frac{\hat{a}^4}{\omega^4} + \ldots$$

Asymptotic Bethe anzatz result (KLRSV)

$$\lim_{j\to 1} \gamma_4(j) = -\frac{512}{\omega^7}$$

Dressing phase modification (Wrapping effect)

$$\zeta_3 \to \frac{47}{24}\,\zeta_3 - \frac{1}{4} S_{-3} + \frac{3}{4}\, S_{-2}\, S_1 + \frac{3}{8}\, S_1\, S_2$$

DL resummation at $j + 2r = \omega \to 0$

$$\gamma(j)(\omega - \gamma(j)) = 4\,\hat{a}\left(1 - \omega S_1 - \omega^2 (S_2 + \zeta_2)\right)$$

$$+16\hat{a}^2 (S_2 + \zeta_2 - S_1^2) + 4\,\hat{a}\,\gamma^2(j)(S_2 + S_{-2})$$

Agreement with the double-logarithmic prediction

$$\frac{1}{256}\,\gamma_4|_{j\to -2k+\omega} =$$

$$\frac{5}{\omega^7} - 20\frac{S_1}{\omega^6} + \frac{24\, S_1^2 - 14(S_2 + \zeta_2) + 4(S_2 + S_{-2})}{\omega^5} + \ldots$$

13 Discussion

1. Pomeron as a composite state of reggeized gluons.
2. Integrability of the BFKL dynamics.
3. Properties of the BFKL kernel in $N = 4$ SUSY.
4. Pomeron-graviton interplay
8. Maximal transcedentality.
9. Meromorphic properties of γ.
10.BFKL predictions .
11.Double-logarithmic asymptotics.
12.Next-to-leading DL predictions.
13.Inconsistency of the four-loop result.
14.Wrapping effects and BFKL and DL singularities .