# The strong coupling limit of the scaling function from the quantum string Bethe ansatz

Charlotte Kristjansen
NBI & NORDITA

Based on: P.-Y. Casteill & C.K., ArXiv: 0705.8090 [HEP-TH]

- The scaling function and its weak/strong coupling limits
- A string computation by Frolov, Tirziu & Tseytlin
- Extracting the strong coupling behaviour from the Bethe equations

# AdS/CFT

$\mathcal{N}=4$ SYM $\longleftrightarrow$ IIB strings on $AdS_5 \times S^5$

Operator $\mathcal{O}$ $\longleftrightarrow$ string state $|\mathcal{O}\rangle$

$$\Delta_{\mathcal{O}} = E_{|\mathcal{O}\rangle}$$

Parameters : $\lambda, N$

$N \to \infty$

Planar gauge theory $\longleftrightarrow$ free strings

Perturbation theory

$\lambda$ small $\qquad\qquad$ $\lambda$ large

# The scaling function

$\mathcal{O} = \mathrm{Tr}(D^S Z^J) + \ldots$

$J = 2, \quad S \to \infty$

$$\Delta_{\mathcal{O}} = S + \underbrace{f(\lambda)}_{\text{scaling function}} \log S + O(S^0) \qquad (*)$$

AdS/CFT : $\Delta_{\mathcal{O}} = E_{|\mathcal{O}\rangle}$

## Universality

gauge

- (*) holds to any order in perturbation theory [Korchemsky '89]
- (*) holds for J finite, $S \to \infty$, perturbatively [Belitsky, Gorsky Korchemsky '06]

string

- (*) holds for J finite, $S \to \infty$, $\lambda \to \infty$ [Gubser, Polyakov, Klebanov '02]
- (*) holds for $1 \ll J \ll S$, $\lambda \to \infty$ [Frolov, Tseytlin '02]

$$\Delta - S - J = f(\lambda) \log\left(\frac{S}{J}\right)$$ [Frolov, Tirziu Tseytlin '06]

# The scaling function

## Gauge theory :

$$f(g) = 4g^2 - \frac{2}{3}\pi^2 g^2 + \frac{11}{45}\pi^4 g^6 - \left(\frac{73}{630}\pi^6 + 4\zeta_3^2\right)g^8 + O(g^{10}) \; ; \quad g^2 = \frac{\lambda}{8\pi^2}$$

1-loop : Gross & Wilczek '73, Georgi & Politzer '74

2-loop : Kotikov & Lipatov '04 ; Kotikov, Lipatov & Velizhanin '03

3-loop : Kotikov, Lipatov, Onishchenko, Velizhanin '04

4-loop : Bern, Czakon, Dixon, Kosower, Smirnov '06

## String Theory

$$f(\lambda) = \frac{\sqrt{\lambda}}{\pi} - \frac{3\log 2}{\pi} + O\left(\frac{1}{\sqrt{\lambda}}\right)$$

classical — [Gubser, Polyakov, Klebanov '02]

one-loop — [Frolov, Tseytlin '02]

Can we find an interpolating function as predicted by AdS/CFT

Natural framework for investigating this:

Bethe equations

# The scaling function

## An integral equation for the scaling function (BES)

[Eden & Staudacher '06] [Beisert, Eden, Staudacher '06]

Start from the gauge theory asymptotic Bethe ansatz

- Assume $J$ large enough for asymptoticity to hold (but $J \ll S$)
- Make use of gauge theory phase factor

Produce (in principle) all loop perturbative prediction for $f(g)$

## Strong coupling analysis of the BES equation

- Leading order strong coupling result $\frac{\sqrt{\lambda}}{\pi}$ reproduced

  Numerically:
  [Benna, Benvenuti, Klebanov & Scardicchio '06]
  [Beccaria, De Angelis, Forini '07]

  Analytically:
  [Kotikov, Lipatov '06], [Alday, Arutyunov, Benna, Eden & Klebanov '07]
  [Kostov, Serban, Volin '07]

- Next to leading order strong coupling result
  $-\frac{3\log(2)}{\pi}$ reproduced

  Numerically only: [Benna, Benvenuti, Klebanov, Scardicchio]

- Numerical prediction for $\frac{1}{\sqrt{\lambda}}$ term
  [Benna, Benvenuti, Klebanov, Scardicchio '06]

# Our approach

$$
\begin{tikzcd}
\text{Bethe ansatz} \quad \sigma = \sigma_{gauge} \arrow[r] \arrow[d, Leftrightarrow] & \text{BES equation} \arrow[d, dashed, "\text{Difficult}"] \\
\text{Bethe ansatz} \quad \sigma = \sigma_{string} \arrow[r, "\text{easy}"'] & \text{Strong coupling limit}
\end{tikzcd}
$$

Strategy :

- consider string with angular momenta $(S, J)$ in $AdS_3 \times S^1$

$$1 \ll J \ll S, \quad z = \frac{\sqrt{\lambda}}{\pi J} \log\left(\frac{S}{J}\right) \text{ fixed} \quad (*)$$

- Solve Bethe equations to all orders in $z$

- Expand the resulting scaling function for $z \to \infty$

  => obtain analytically the two leading orders in the $\frac{1}{\sqrt{\lambda}}$

Bethe equation equivalent of a string theory computation by Frolov, Tirziu & Tseytlin in the limit $(*)$ suggested by Belitsky, Gorsky, Korchemsky.

# The string computation

[Frolov, Tirziu, Tseytlin '06]

Consider string with angular momenta $(S, J)$ in $AdS_3 \times S^1$

$$1 << J << S \quad , \quad z = \frac{\sqrt{\lambda}}{\pi J} \log\left(\frac{S}{J}\right) \quad \text{fixed}$$

Calculate classical energy solving classical e.o.m.

$$E_0 = S + J\sqrt{1+z^2} \qquad (1)$$

strong coupling expansion $z \to \infty$

$$E_0 = S + \frac{\sqrt{\lambda}}{\pi} \log\left(\frac{S}{J}\right)$$

Calculate 1-loop string energy summing over fluctuations

$$E_1 = \frac{J}{\sqrt{\lambda}} \frac{1}{\sqrt{1+z^2}} \Big\{ z\sqrt{1+z^2} - (1+2z^2)\log\left[z+\sqrt{1+z^2}\right]$$

$$- z^2 + 2(1+z^2)\log\left[1+z^2\right] - (1+2z^2)\log\left[\sqrt{1+z^2}\right]\Big\} \qquad (2)$$

Strong coupling expansion, $z >> 1$

$$E_1(z >> 1) = -\frac{3\log(2)}{\pi} \log\left(\frac{S}{J}\right)$$

Aim: reproduce (1) and (2) using the quantum string Bethe ansatz

# Analytic vs. non-analytic in $\lambda$

$$\mathcal{Z} = \frac{\sqrt{\lambda}}{\pi J} \log\left(\frac{S}{J}\right)$$

$$(E_1)^{\text{analytic}}_{\text{string}} = \frac{J}{\sqrt{\lambda}}\left(\mathcal{Z} - \frac{1+2\mathcal{Z}^2}{\sqrt{1+\mathcal{Z}^2}}\log\left[\mathcal{Z}+\sqrt{1+\mathcal{Z}^2}\right]\right)$$

$$(E_1)^{\text{non-analytic}}_{\text{string}} = \frac{J}{\sqrt{\lambda}}\frac{1}{\sqrt{1+\mathcal{Z}^2}}\Big(-\mathcal{Z}^2 + 2(1+\mathcal{Z}^2)\log\left[1+\mathcal{Z}^2\right]$$
$$-(1+2\mathcal{Z}^2)\,\mathrm{Log}\left[\sqrt{1+2\mathcal{Z}^2}\right]\Big)$$

Strong coupling behaviour

$$(E_1)^{\text{analytic}}_{\text{string}} \sim \left(\frac{-2\log\mathcal{Z} + 1 - 2\log 2}{\pi}\right)\log\left(\frac{S}{J}\right) \quad \text{as } \mathcal{Z}\to\infty .$$

$$(E_1)^{\text{non-analytic}}_{\text{string}} \sim \left(\frac{2\log\mathcal{Z} - 1 - \log(2)}{\pi}\right)\log\left(\frac{S}{J}\right) \quad \text{as } \mathcal{Z}\to\infty$$

The splitting into analytic and non-analytic part is natural from the Bethe ansatz point of view.

OBS: The $\dfrac{-3\log(2)}{\pi}$ is due to a non-trivial cancellation between the analytic and the non-analytic part

# The string Bethe equations in SL(2) sector

$$\left(\frac{x_k^+}{x_k^-}\right)^J = \prod_{j\neq k}^{S} \left(\frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\right) \frac{1 - g^2/2x_k^+ x_k^-}{1 - g^2/2x_k^+ x_k^-} \, \sigma^2(x_k, x_j) \quad , \quad g^2 = \frac{\lambda}{8\pi^2}$$

$j, k$ label excitations

$$\exp(ip) = \frac{x^+}{x^-} \; ; \quad \prod_{k=1}^{S} \left(\frac{x_k^+}{x_k^-}\right) = 1 \; , \quad E = \frac{\lambda}{8\pi^2}\, i \sum_{k=1}^{S} \left(\frac{1}{x_k^+} - \frac{1}{x_k^-}\right)$$

To string one-loop order

$$\sigma = \underbrace{\sigma_{AFS}}_{\text{classical}} + \underbrace{\frac{1}{\sqrt{\lambda}}\,\sigma_{HL}}_{\text{one-loop}}$$

[Arutynov, Frolov, Staudacher '04] [Hernandez + Lopez '06]

Classical energy $E_0$ :

- $\sigma = \sigma_{AFS}$
- consider thermodynamical limit $J \to \infty$

One-loop energy $E_1$ :

- add $\frac{1}{\sqrt{\lambda}}\,\sigma_{HL}$ : the source of non-analytic corrections
- include $\frac{1}{J}$ corrections to the thermodynamical limit : the source of analytic corrections

OBS : $1 \ll J \ll S \; , \quad z = \frac{\sqrt{\lambda}}{\pi J} \log\left(\frac{S}{J}\right)$

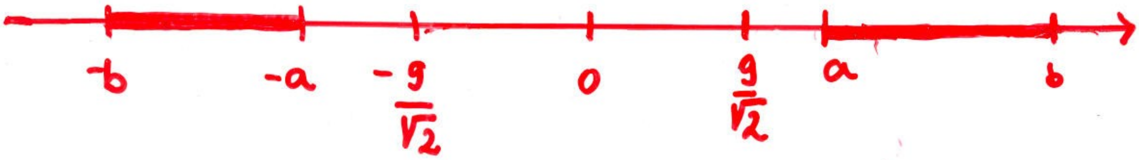
# Bethe equations at leading order

At the classical level

$$\frac{2}{J}\sum_{j\neq k}^{S}\frac{1}{(x_k-x_j)\left(1-g^2/2x_j^2\right)} = -2\pi m_k + \frac{1}{x_k\left(1-g^2/2x_k^2\right)}$$

$$-\frac{2}{J}\,\frac{g^2}{2x_k^2}\,\frac{1}{1-g^2/2x_k^2}\sum_{j\neq k}\frac{1}{1-g^2/2x_j^2}\,\frac{1}{x_j}$$

Assume symmetric distribution of roots:
( => recover directly $g=0$ result of Beisert, Frolov, Staudacher, Tseytlin '03] )
( => last term above vanishes)




Introduce resolvant :

$$G(x) = \frac{1}{J}\sum_{j=1}^{S/2}\frac{1}{x-x_j}\,\frac{1}{1-g^2/2x_j^2} \equiv \int_a^b dy\,\frac{\rho(y)}{x-y}$$

Assume that $G(x)$ and $\rho(x)$ have well-defined $\frac{1}{J}$ expansions

$$G(x) = G_0(x) + \frac{1}{J}G_1(x) + \dots$$

$$\rho(x) = \rho_0(x) + \frac{1}{J}\rho_1(x)$$

Normalization conditions:

$$\int_a^b dy\,\rho(y)\left\{1-\frac{g^2}{2y^2}\right\} = \frac{S}{2J}$$

$$\int_a^b dy\,\rho(y) = \frac{S}{2J} + \frac{E-S-J}{4J}$$

# Equation for the resolvant



Leading order equation:

$$G_0(x+i0) + G_0(x-i0) - 2\,G_0(-x) = V_0(x) \,, \qquad x \in [a,b]$$

$$V_0(x) = 2\pi n - \frac{1/x}{1 - g^2/2x^2}$$

Saddle point equation of the $O(n)$ model on a random lattice for $n=-2$ [Kostov '89]

Same structure at higher orders in $\frac{1}{J}$ (if we solve the model perturbatively

$$G(x+i0) + G(x-i0) - 2\,G(-x) = V(x)$$

Solution well-known

[Kostov, Staudacher '92] [Eynard, Kristjansen '95]

# Correction terms I

The HL phase:

$$V_1(x) = \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{S} \sum_{r=2}^{\infty} \sum_{m=0}^{\infty} \left(\frac{\lambda}{16\pi^2}\right)^{m+r} \cdot C_{r,2m+r+1} \left[ q_r(x)\, q_{2m+r+1}(x_j) - q_r(x_j)\, q_{2m+r+1}(x) \right]$$

$$C_{r,2m+r+1} = -8 \frac{(r-1)(2m+r)}{(2m+2r-1)(2m+1)} \quad , \quad q_r(x) = \frac{-i}{J} \frac{1}{1 - g^2/2x^2} \frac{1}{x^r}$$

$\sum_j$ can be replaced by $\int dy\, \rho_0(y)$ at this order

Odd moments $\int dy\, \rho_0(y) \frac{1}{y^{2n+1}}$ vanish

$\sum_{r,s}$ can be carried out

$$V_1^{\text{non-analytic}}(x) =$$

$$-\frac{2}{J} \frac{1}{\pi} \frac{1}{1 - g^2/2x^2} \cdot \int_a^b dy\, \rho_0(y) \left( - \frac{\sqrt{2} g x \left(y^2 + \frac{g^2}{2}\right)}{(x^2 - y^2)\left(x^2 y^2 - \frac{g^4}{4}\right)} \right.$$

$$+ \left( \frac{x^2 + y^2}{(x^2 - y^2)^2} + \frac{g^2}{2} \frac{x^2 y^2 + \frac{g^4}{4}}{\left(x^2 y^2 - \frac{g^4}{4}\right)^2} \right) \mathrm{Log}\left[ \frac{x + \frac{\sqrt{g}}{2}}{x - \frac{\sqrt{g}}{2}} \right]$$

$$\left. + 2xy \left( \frac{1}{(x^2 - y^2)^2} + \frac{\frac{g^4}{4}}{x^2 y^2 - \frac{g^4}{4}} \right) \mathrm{Log}\left[ \frac{y - \frac{\sqrt{g}}{2}}{y + \frac{\sqrt{g}}{2}} \right] \right)$$

# Correction terms II

The anomaly term, first correction to the thermodynamical limit

Analytic in $\lambda$

Originates from the repulsion term in the string Bethe equations

$$\sum_{j\neq k}^{S} -i\log \frac{x_k^- - x_j^+}{x_k^+ - x_j^-}\,\frac{1-g^2/2x_k^- x_j^+}{1-g^2/2x_k^+ x_j^-} = -i\sum_{j=1}^{S}\log\frac{u_k-u_j+i}{u_k-u_j-i}$$

$$u(x) = x + \frac{g^2}{2x}\,, \qquad \rho(x) = \frac{dn}{du(x)}\,,$$

Naive expansion of the logarithm breaks down if $u_k - u_j \sim O(1)$

Use instead

$$u_j - u_k = \frac{j-k}{\rho(x_k)} - \frac{(j-k)^2}{J}\,\frac{\rho_0'(x_k)}{(\rho_0(x_k))^3}\,\frac{1}{1-g^2/2x_k^2} + O\left(\frac{1}{J^2}\right)$$

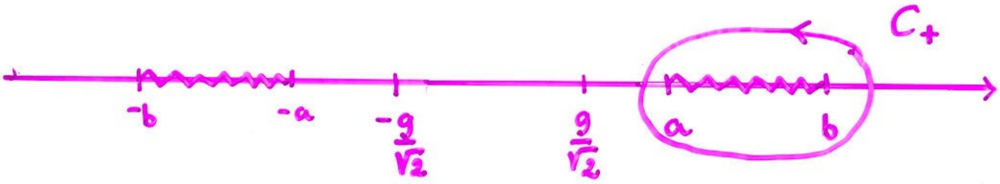
$$V_1^{\text{analytic}}(x) = \frac{1}{J}\,\frac{1}{1-g^2/2x^2}\left(\pi\rho_0'(x)\right)\left(\coth(\pi\rho_0(x)) - \frac{1}{\pi\rho_0(x)}\right)$$

# General solution

$$G(x+i0) + G(x-i0) - 2G(-x) = V'(x) \qquad ; \; x \in [0,b]$$

[Eynard, Kristjansen '95]

$$G_-(x) = \frac{1}{2}\oint_{C_+} \frac{dy}{2\pi i} \frac{V'(y)}{x^2-y^2} \left\{ \frac{(x^2-a^2)^{1/2}\,(x^2-b^2)^{1/2}}{(y^2-a^2)^{1/2}\,(y^2-b^2)^{1/2}} \right\}$$




Boundary conditions (follows from the normalization cond)

$$\oint_{C_+} \frac{dy}{2\pi i} \frac{V'(y)}{(y^2-a^2)^{1/2}(y^2-b^2)^{1/2}} = 0$$

$$\oint_{C_+} \frac{dy}{2\pi i} \frac{V'(y)\cdot y^2}{(y^2-a^2)^{1/2}(y^2-b^2)^{1/2}} + \frac{g^2}{2}\oint \frac{dy}{2\pi i} \frac{V'(y)\,ab}{(y^2-a^2)^{1/2}(y^2-b^2)^{1/2}} = \frac{S}{J}$$

$$E = J + S - 2J g^2 G_-(0)$$

$$G(x) = G_+(x) + x\,G_-(x) \qquad , \quad G_\pm(-x) = G_\pm(x)$$

# Leading order solution

$$V(x) = V_0(x) = 2\pi n - \frac{1/x}{1 - g^2/2x^2}$$

$$E_0 = S + J \frac{1}{\sqrt{(1-\hat{g}^2)(1-k^2\hat{g}^2)}} \left[ 1 - \hat{g}^2 \frac{E(k')}{K(k')} \right] ,$$

$$\hat{g} = \frac{g}{\sqrt{2}a} , \quad k = \frac{a}{b} , \quad k' = (1-k^2)^{1/2}$$

Limit

$$1 << J << S \qquad z = \frac{\sqrt{\lambda}}{\pi J} \log\left(\frac{S}{J}\right) \text{ fixed}$$

$$\Downarrow$$

$$k \to 0, \quad a \to 0, \quad b \to \infty$$

Boundary conditions

(1) $\log\left(\frac{S}{J}\right) \sim \log\left(\frac{1}{k}\right)$

(2) $\hat{g}^2 = \frac{z^2}{z^2+1} < 1 \qquad \left( \Rightarrow \frac{g^2}{2} < a^2 \right)$

$$E_0 = S + J \frac{1}{\sqrt{1 - \frac{z^2}{1+z^2}}} = S + J\sqrt{1+z^2}$$

$$= S + \frac{\sqrt{\lambda}}{\pi} \log\left(\frac{S}{J}\right) + \dots$$

# Next to leading order

Need to know $\rho_0(x)$ and $\rho_0'(x)$

$$\rho_0(x) = \frac{x}{i\pi}\left\{ G_-(x-i0) - G_-(x+i0) \right\}$$

$$= \frac{x\sqrt{x^2-a^2}}{2\pi b\sqrt{b^2-x^2}}\left( \frac{b}{x^2-\frac{g^2}{2}} \frac{\sqrt{b^2-\frac{g^2}{2}}}{\sqrt{a^2-\frac{g^2}{2}}} - 4\Pi\left(1-\frac{x^2}{b^2}, k'\right)\right)$$

OBS: Exact formula

In the limit

$k\to 0$, $a\to 0$, $b\to\infty$ ; $z = \frac{\sqrt{\lambda}}{\pi J}\log\left(\frac{s}{J}\right)$ fixed,

$\omega = \frac{x}{a}$

$$\rho_0(\omega) \approx \frac{2}{\pi}\frac{\omega\sqrt{\omega^2-1}}{\omega^2-\hat{g}^2}\log\left(\frac{s}{J}\right)$$

$$\rho_0'(\omega) \approx \frac{2}{\pi}\frac{(1-2\hat{g}^2)\omega^2+\hat{g}^2}{\sqrt{\omega^2-1}\,(\omega^2-\hat{g}^2)^2}\log\left(\frac{s}{J}\right)$$

can be used in integrals where the remaining part of the integrand vanishes sufficiently fast at $\infty$

# Next to leading order

Exact solution formula

$$G_{1-}(x) = \frac{1}{2}\oint_{C_+} \frac{dy}{2\pi i} V_1(y)\left[\frac{1}{x^2-y^2} - \frac{\frac{g^2}{2ab}}{1-\frac{g^2}{2ab}}\,\frac{1}{y^2}\right]\left\{\frac{(y^2-a^2)^{1/2}\,(y^2-b^2)^{1/2}}{(x^2-a^2)^{1/2}\,(x^2-b^2)^{1/2}}\right\}$$

$$E_1 = -2g^2 G_{1-}(0)$$

Taking the limit

$$1 \ll J \ll S \quad , \quad z = \frac{\sqrt{\lambda}}{\pi J}\log\left(\frac{S}{J}\right) \text{ fixed}$$

$$y = \frac{\omega}{a} \quad , \quad V_1(x) = \frac{1}{a} V_1(\omega)$$

$$E_1 = -2\hat{g}^2 \int_1^\infty \frac{d\omega}{\pi} V_1(\omega) \frac{(\omega^2-1)^{1/2}}{\omega^2} \quad , \quad \hat{g}^2 = \frac{z^2}{z^2+1}$$

provided $V_1(\omega)$ behaves well as $\omega \to \infty$

# The analytic term

$$V_1^{\text{analytic}}(x) = \frac{1}{1 - g^2/2x^2}\left(\pi \rho_0'(x)\right)\left(\coth\left(\pi\rho_0(x)\right) - \frac{1}{\pi\rho_0(x)}\right)$$

Taking the limit

$$1 \ll J \ll S \quad , \quad Z = \frac{\sqrt{\lambda}}{\pi J}\log\left(\frac{S}{J}\right) \text{ fixed}$$

$$\rho_0(x) \longrightarrow \rho_0(\omega) = \frac{2}{\pi}\,\frac{\omega\sqrt{\omega^2-1}}{\omega^2-\hat{g}^2}\log\left(\frac{S}{J}\right) \quad ; \quad \omega = \frac{x}{a}$$

$$\coth\left(\pi\rho_0(x)\right) \rightarrow 1 \ , \quad \frac{1}{\pi\rho_0(x)} \rightarrow 0$$

$$(E_1)^{\text{analytic}}_{\text{Bethe}} = -2\hat{g}^2 \oint_{C_+} \frac{d\omega}{2\pi i}\,\frac{(\omega^2-1)^{1/2}}{\omega^2}\, V_1^{\text{analytic}}(\omega)$$

$$= -4\frac{J}{\sqrt{\lambda}}\,\hat{g}^2 \int_1^\infty d\omega\, \frac{(1-2\hat{g}^2)\omega^2 + \hat{g}^2}{(\omega^2-\hat{g}^2)^3} \quad , \quad \hat{g} = \frac{z^2}{z^2+1} < 1$$

$$\approx \frac{J}{\sqrt{\lambda}}\left(Z - \frac{1+2z^2}{\sqrt{1+z^2}}\,\mathrm{Log}\left[z + \sqrt{1+z^2}\right]\right)$$

$$= (E_1)^{\text{analytic}}_{\text{string}}$$

$$V_1^{\text{non-analytic}}(\omega)$$

$$= \frac{1}{2\pi}\,\frac{\omega^2}{\omega^2-\hat{g}^2}\int_1^\infty d\nu\, g_0(\nu)\Big\{\Delta\phi(\omega,\nu)+\Delta\phi(\omega,-\nu)$$

$$-\Delta\phi(-\omega,\nu)-\Delta\phi(-\omega,\nu)\Big\}$$

$$(E_1)^{\text{non-analytic}}_{\text{Bethe}} = -2\hat{g}^2\oint_{C_+}\frac{d\omega}{2\pi i}\,\frac{(1-\omega^2)^{1/2}}{\omega^2}\,V_1^{\text{non-analytic}}(\omega)$$

Exchange order of integration

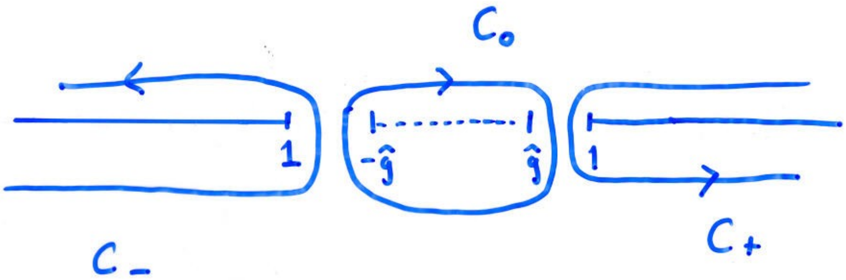
$$= -\frac{2\hat{g}^2}{\pi}\int_1^\infty d\nu\, g_0(\nu)\oint_{C_+}\frac{d\omega}{2\pi i}\,\frac{(1-\omega^2)^{1/2}}{(\omega^2-\hat{g}^2)}\Big\{A(\omega,\nu)+B(\omega,\nu)\Big\}$$

$A(\omega,\nu)$ : Only poles in $\omega$, at $\omega=\pm\nu$, $\omega=\pm\frac{\hat{g}^2}{\nu}$

$B(\omega,\nu)$ : Poles at $\omega=\pm\nu$, $\omega=\pm\frac{\hat{g}^2}{\nu}$ and a log cut $[-\hat{g},\hat{g}]$

Integrand odd in $\omega$ $\Rightarrow$

$$\oint_{C_+} = \frac{1}{2}\oint_{C_+} + \frac{1}{2}\oint_{C_-} = +\frac{1}{2}\int_{C_0}$$



# The non-analytic term

The A-term : trivial – only poles

The B-term : almost trivial – poles and a logarithm

$$B(\omega,\nu) = \frac{1}{2}\left(\frac{\omega^2+\nu^2}{(\omega^2-\nu^2)^2} + \frac{\hat{g}^2(\nu^2\omega^2+\hat{g}^4)}{(\nu^2\omega^2-\hat{g}^4)^2}\right)\operatorname{Log}\left[\frac{\omega-\hat{g}}{\omega+\hat{g}}\right]$$

$$= \hat{B}(\omega,\nu)\operatorname{Log}\left[\frac{\omega-\hat{g}}{\omega+\hat{g}}\right]$$

$$I_B = +\frac{1}{2}\oint_{C_0}\frac{d\omega}{2\pi i}\,\frac{\sqrt{1-\omega^2}}{\omega^2-\hat{g}^2}\,\hat{B}(\omega,\nu)\operatorname{Log}\left[\frac{\omega-\hat{g}}{\omega+\hat{g}}\right]$$

$$= \frac{1}{2}\int_{-\hat{g}}^{\hat{g}} d\omega\,\frac{\sqrt{1-\omega^2}}{\omega^2-\hat{g}^2}\,\hat{B}(\omega,\nu)$$ elementary integrals

Putting everything together

$$(E_1)^{\text{non-analytic}}_{\text{Bethe}} = \int_1^\infty d\nu \oint_{C_0} d\omega \ldots$$

$$= \frac{J}{\sqrt{\lambda}}\,\frac{1}{\sqrt{1+z^2}}\left\{-z^2 + 2(1+z^2)\operatorname{Log}(1+z^2) - (1+2z^2)\operatorname{Log}\left(\sqrt{1+2z^2}\right)\right\}$$

$$= (E_1)^{\text{non-analytic}}_{\text{string}}$$

# Summary

$(E_1)_{Bethe} = (E_1)_{string}$ at the functional level

in the limit

$1 \ll J \ll S$ , $Z = \frac{\sqrt{\lambda}}{\pi J} \log\left(\frac{S}{J}\right)$

$$E_1 = \frac{-3\log 2}{\pi} \log\left(\frac{S}{J}\right) \quad \text{as} \quad Z \to \infty$$

Subtle cancellations between the analytic term (anomaly) and non-analytic term (HL-phase) as $Z \to \infty$

$$(E_1)^{\text{analytic}} = \left(\frac{-2\log Z + 1 - 2\log 2}{\pi}\right) \log\left(\frac{S}{J}\right) \quad \text{as } Z \to \infty$$

$$(E_1)^{\text{non-analytic}} = \left(\frac{2\log Z - 1 - \log 2}{\pi}\right) \log\left(\frac{S}{J}\right) \quad \text{as } Z \to \infty$$

# Conclusion & Outlook

- The expected strong coupling behaviour of the scaling function follows from the Bethe eqns.

- An additional consistency check of the phase factor (at next to leading order)

- A method which allows us (in principle) to go to any order in the strong coupling expansion

  Numerical prediction exists for the $\frac{1}{\sqrt{\lambda}}$ term
  [Benna, Benvenuti, Klebanov & Scardicchio '06]

  Analytical string computation on the way? Hard
  [Roiban, Tirziu, Tseytlin '07]

? Does this explain why it is so hard to get the $\frac{-3\log 2}{\pi}$ from the BES equation