Anomalous dimensions of high-spin operators beyond the leading order

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High-spin operators in gauge theory

Examples of Wilson operators with Lorentz spin N

- ✔ Twist-two operators in QCD

$$O_N^{(q)} = \bar{q}\gamma_+ D_+^{N-1} q(0)\,, \qquad O_N^{(g)} = \mathrm{tr}\left[F_{\mu+} D_+^{N-2} F_{\mu+}(0)\right]$$

- ✔ Twist-two operators in $\mathcal{N}=4$ SYM

$$O_N^{(L=2)}(0) = \mathrm{tr}\left[X(0) D_+^N X(0)\right]$$

$X(0) =$ holomorphic scalar field; $D_+ = \partial_+ + igA_+ -$light-cone component of the covariant derivative.

- ✔ Quasipartonic operators of higher twist L and Lorentz spin $N = n_1 + \ldots + n_{L-1}$

$$O_N^{(\{n\})}(0) = \mathrm{tr}\left[X(0) D_+^{n_1} X(0) \ldots D_+^{n_{L-1}} X(0)\right]$$

Satisfy the Callan-Symanzik equation

$$\left(\frac{\partial}{\partial \ln \mu} + \beta(\lambda)\frac{\partial}{\partial \ln \lambda}\right)\mathcal{O}_N^{(a)}(0) = -\gamma^{ab}(N)\mathcal{O}_N^{(b)}(0)\,,$$

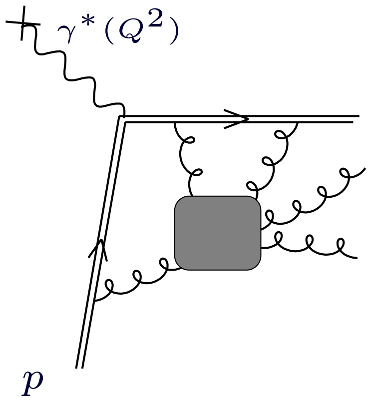
Anomalous dimensions are eigenvalues of the mixing matrix

$$\gamma(N) = \lambda\gamma_0(N) + \lambda^2\gamma_1(N) + \ldots\,,$$

Why the large N limit of the anomalous dimensions is interesting?

Classical example: deeply inelastic scattering of a fast parton (quark, gluon,...) off an external potential

$F(x, Q^2) \sim$



$$\Bigg\} \; M_X^2 = Q^2(1-x)/x\,, \qquad 0 \le x = \frac{Q^2}{s} < 1\,,$$

- ✔ OPE expansion: Operators of high spin $N \gg 1 \iff x \to 1$ asymptotics of the structure function

$$\int_0^1 dx\, x^N F(x, Q^2) = \sum_{L \ge 2} \frac{1}{Q^L} \langle p | O_N^{(L)}(0) | p \rangle_{\mu^2 = Q^2} \cdot c_N^{(L)}(\alpha_s(Q^2))$$

- ✔ For $x \to 1$ the final state has a small invariant mass $M_X^2 \to 0$ and is dominated by soft gluons $\implies$ Wilson loops $\implies$ cusp anomalous dimension
- ✔ The anomalous dimensions of Wilson operators with a large Lorentz spin $N \gg 1$ and twist $L = 2, 3, \ldots$ scale (at most) logarithmically with the spin

$$\gamma_N^{(L)} \sim \ln N \quad \implies \quad \langle p | O_N^{(L)}(0) | p \rangle_{\mu^2} \sim \exp\left(-\gamma_N^{(L)} \ln \mu^2\right) \qquad \textit{Sudakov asymptotics}$$

- ✔ The logarithmic (=Sudakov) scaling of anomalous dimensions is a universal feature of all gauge theories ranging from QCD to $\mathcal{N} = 4$ SYM theory. *What about subleading $1/N$ suppressed corrections?*

Anomalous dimensions at high spin: string theory

- Wilson operators in $\mathcal{N}=4$ SYM theory are dual to string excitations on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background

$$O_N^{(L)}(0) = \mathrm{tr}\left[X(0) D_+^{n_1} X(0) \dots D_+^{n_{L-1}} X(0)\right] \qquad \Longleftrightarrow$$

folded string spinning in $\mathrm{AdS}_3 \times S^1$ with large angular momenta N and L

- For operators $O_N^{(L)}(0)$ with large quantum numbers: $N =$ Lorentz spin, $L =$ isotopic $R-$charge, the anomalous dimension at strong coupling is given by energy of dual (semi)classical string configurations

$$E_{\mathrm{str}} - N - L = \gamma_N^{(L)}(\lambda)$$

- The string theory provides a definite prediction for $\gamma_N^{(L)}(\lambda)$ as a function of L and N in the $\mathcal{N}=4$ SYM theory in the strong coupling regime in the planar approximation

$$E_{\mathrm{str}} = \sqrt{\lambda} E_0\left(\frac{N}{\sqrt{\lambda}}, \frac{L}{\sqrt{\lambda}}\right) + E_1\left(\frac{N}{\sqrt{\lambda}}, \frac{L}{\sqrt{\lambda}}\right) + \dots$$

  with E_0 the classical string energy and E_1 one-loop quantum string correction.

- Large N asymptotics of the anomalous dimension (for $\sqrt{\lambda}\ln(N/L)/L \gg 1$) [Gubser, Klebanov, Polyakov'02] [Frolov, Tseytlin'02]

$$\gamma_N^{(L)}(\lambda) \sim \left(\frac{\sqrt{\lambda}}{\pi} - \frac{3\ln 2}{\pi} + \mathcal{O}(1/\sqrt{\lambda})\right) \ln N + \dots$$

Sudakov asymptotics

What about subleading $1/N$ corrections?

Warm up exercise: twist-two anomalous dimensions

Nonsinglet twist-two quark operator in QCD with the Lorentz spin N

$$\mathcal{O}_N^{(\mathrm{ns})}(0) = \bar{q}\,\gamma_+ D_+^{N-1} q(0)\,,$$

- ✔ One-loop anomalous dimension [Gross, Wilczek'73] [Georgi, Politzer'73]

$$\gamma_0(N) = C_F\left[4\psi(N+1) + 4\gamma - 3 - \frac{2}{N(N+1)}\right],$$

$\psi(x) = d\ln\Gamma(x)/dx$ Euler psi-function, $C_F = (N_c^2-1)/(2N_c)$ the $SU(N_c)$ Casimir

- ✔ Large N expansion

$$\gamma_0(N) = C_F\left[4\ln N + 4\gamma - 3 + 2N^{-1} - \tfrac{7}{3}N^{-2} + 2N^{-3} - \tfrac{59}{30}N^{-4} + \mathcal{O}(N^{-5})\right]$$

The anomalous dimension scales logarithmically $\sim \ln N$ and subleading corrections run in powers of N^{-1}

- ✔ Change the parameter of the expansion from N to $J^2 = N(N+1)$

$$\gamma_0(N) = C_F\left[2\,\ln J^2 + 4\gamma - 3 - \tfrac{4}{3}J^{-2} - \tfrac{2}{15}\,J^{-4} + \tfrac{16}{315}\,J^{-6} + \mathcal{O}(J^{-8})\right],$$

  - ✗ corrections run in *even* powers of J^{-1} only – the series is *parity-preserving !*
  - ✗ corrections to $\gamma_0(N)$ suppressed by odd powers of N^{-1} are not independent

Is this property accidental? Does it survive to higher loops? What about high-twist operators?

Twist-two anomalous dimensions to high loops

Large$-N$ expansion of the twist-two anomalous dimension in (super)YM

$$\gamma(N) = A(\lambda)\ln \bar{N} + B(\lambda) + N^{-1}\left[C(\lambda)\ln \bar{N} + D(\lambda) + \tfrac{1}{2}A(\lambda)\right] + \mathcal{O}(N^{-2}\ln^2 \bar{N}),$$

with $A, B, C, D, \ldots$ given by series in the coupling constant and $\ln \bar{N} = \ln N + \gamma$.

✔ The twist-two anomalous dimensions have *universal* logarithmic scaling $\gamma(N) \sim \ln \bar{N}$

$$A(\lambda) = 2\Gamma_{\rm cusp}(\lambda) = \textit{cusp anomalous dimension of Wilson loops}$$ [GK'88]

✔ Corrections to $\gamma(N)$ suppressed by powers of N^{-1} are more involved:

- ✗ They are not universal $\Longrightarrow$ the functions $B, D, \ldots$ depend on the quantum numbers of the operator
- ✗ They are enhanced logarithmically

$$\gamma(N) \sim \ldots + \sum_{p\geq 1} N^{-p}\left[a_{0p}(\ln\bar{N})^p + a_{1p}(\ln\bar{N})^{p-1} + \ldots + a_{pp}\right]$$

- ✗ Expansion of $\gamma(N)$ in inverse powers of $J^2 = N(N+1)$ involves all powers of J^{-1} starting from two loops $\Longrightarrow$ the series is not parity preserving and seem to have no internal structure ... but
- ✗ Three-loop calculation of $\gamma(N)$ in QCD revealed the existence of the following relations

$$C^{(3-\rm loop)} = \tfrac{1}{2}[A(\lambda)]^2$$ [Moch, Vogt, Vermaseren'04]

$$D^{(3-\rm loop)} = \tfrac{1}{2}A(\lambda)B(\lambda) - \tfrac{1}{2}A(\lambda)\beta(\lambda)$$ [Dokshitzer, Marchesini, Salam'06]

MVV: *"This calls for a further structural explanation of subleading N^{-1} corrections!"*

Conformal symmetry constraints

- ✔ The classical Yang-Mills Lagrangian is invariant under conformal transformations but this symmetry is broken on the quantum level unless the beta-function vanishes to all loops

- ✔ If the conformal symmetry were exact, the quasipartonic operators could be classified according to representations of the collinear $SL(2;\mathbb{R})$ subgroup of the full $SO(2,4)$ conformal group [Ohrndorf'82] [Makeenko'82]

  - ✗ Conformal invariance ensures that the operators belonging to different $SL(2;\mathbb{R})$ multiplets cannot mix under renormalization and their anomalous dimension depends on the conformal $SL(2;\mathbb{R})$ spin j

$$j = \tfrac{1}{2}(N + \Delta(\lambda))\,, \qquad N = \text{Lorentz spin}\,,\ \ \Delta = \text{scaling dimension}$$

  The scaling dimension receives anomalous contribution due to interaction $\Delta(\lambda) = N + L + \gamma(N) \Longrightarrow$ the conformal spin gets modified in higher loops as [Müller'94] [Belitsky,Müller'99]

$$j(\lambda = 0) = N + \tfrac{1}{2}L \quad \mapsto \quad j(\lambda) = N + \tfrac{1}{2}L + \tfrac{1}{2}\gamma(N)$$

  - ✗ The anomalous dimension of the quasipartonic operator is a function of $j(\lambda)$

$$\gamma(N) = f\left(N + \tfrac{1}{2}\gamma(N)\right)\,,$$

  twist-2: [Dokshitzer, Marchesini, Salam'05]
  twist-L: [Basso, GK'06]

  the scaling function $f(N)$ depends on twist L and other quantum numbers of the operator.

- ✔ In gauge theory with $\beta(\lambda) \neq 0$, the relation should be modified to incorporate the additional conformal symmetry breaking corrections. In the DREG/DRED scheme it reads [Basso, GK'06]

$$\gamma(N) = f\left(N + \tfrac{1}{2}\gamma(N) - \tfrac{1}{4}L\beta(\lambda)\right)$$

How to make use of this functional relation?

Parity preserving relation

- ✔ Perturbative solution to the functional relation

$$\gamma(N) = \lambda\gamma_0(N) + \lambda^2\gamma_1(N) + \dots \quad \Longrightarrow \quad f(N) = \lambda f_0(N) + \lambda^2 f_1(N) + \dots$$

with the coefficient functions related as

$$f_0(N) = \gamma_0(N)\,, \quad f_1(N) = \gamma_1(N) - \frac{1}{2}\gamma_0(N)\gamma_0'(N)\,, \quad \dots$$

The functions $\gamma(N)$ and $f(N)$ coincide to one loop but deviate starting from two loops

- ✔ All-loop solution (Lagrange-Bürmann formula)

$$f(N) = \sum_{k=1}^{\infty}\frac{1}{k!}\left(-\tfrac{1}{2}\partial_N\right)^{k-1}[\gamma(N)]^k = \gamma(N) - \frac{1}{4}\Big(\gamma^2(N)\Big)' + \frac{1}{24}\Big(\gamma^3(N)\Big)'' + \mathcal{O}(\gamma^4(N))$$

- ✔ Conformal symmetry allows one to relate the anomalous dimension $\gamma(N)$ to yet another function $f(N)$ but it does not tell us much about the properties of the latter. What is special about $f(N)$?

- ✔ In the $\mathcal{N}=4$ SYM, extract $f(N)$ from 'universal' three-loop anomalous dimension [Kotikov, Lipatov, Onischenko, Velizhanin'04] and expand it in powers of the conformal Casimir $J^2 = N(N+1)$ [with $\alpha = \frac{1}{2}\Gamma_{\rm cusp}(\lambda)$]

$$f(J) = 2\,\alpha\ln\bar{J}^2 - \Big[6\,\alpha^2\zeta_3 + \alpha^3\left(\tfrac{4}{3}\,\pi^2\zeta_3 - 20\,\zeta_5\right)\Big]J^0 + \Big[\tfrac{2}{3}\,\alpha + 2\,\alpha^2 + \left(\tfrac{2}{3}\,\pi^2 - \tfrac{2}{3}\,\pi^2\ln\bar{J}\right)\alpha^3\Big]J^{-2} + \dots$$

  - ✗ *In distinction with $\gamma(J)$, subleading corrections to $f(J)$ run in even powers of J^{-1} only!*
  - ✗ *Suppression of $\ln J$ terms inside subleading corrections: $f(J)\sim\alpha^3\ln J/J^2$ versus $\gamma(J)\sim\alpha\ln J/J$*

Parity preserving relations II

To verify the parity preserving relations:

(i) Take 2-, 3-, ... loop expression for the anomalous dimension (given by complicated expressions involving nested harmonic sums and various color factors)

(ii) Calculate the scaling function $f(N)$ using the Lagrange-Bürmann formula

(iii) Re-expand the scaling function in powers of $J^2 = N(N+1)$ and check the absence of odd powers of $1/J$

In our analysis, we used expressions for multi-loop anomalous dimensions of various twist-two operators in QCD and in SYM theories available in the literature. They include:

- ✔ Two-loop longitudinally polarized singlet distributions in QCD;
- ✔ Two-loop gluon linearly polarized distribution in QCD;
- ✔ Two-loop quark transversity distribution in QCD and its analogs in SYM theories with $\mathcal{N} = 0, 2$ supercharges;
- ✔ Three-loop nonsinglet unpolarized distributions in QCD;
- ✔ Three-loop singlet unpolarized distributions in QCD;
- ✔ Three-loop 'universal' distribution in $\mathcal{N} = 4$ SYM.
- ✔ All-loop anomalous dimensions of twist-two in QCD in the large β_0-limit

We found that in all cases the corresponding scaling functions $f(N)$ is given by the parity preserving series!

$$f(J) = \Gamma_{\text{cusp}}(\lambda) \ln \bar{J}^2 + f^{(0)} + f^{(1)}/J^2 + f^{(2)}/J^4 + \mathcal{O}(1/J^6)\,,$$

with universal leading $\ln J-$term and operator dependent coefficient functions $f^{(n)}(\ln J)$.

Magic relations

Let us make use of the parity preserving property of $f(J)$ and expand $\gamma(J)$ in powers of $1/J$.

- ✔ Perturbative solution

$$\gamma_0(J) = f_0(J)\,, \quad \gamma_1(J) = f_1(J) + \frac{1}{2} f_0(J) f_0'(J)\,, \quad \ldots$$

All-loop solution

$$\gamma(J) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\tfrac{1}{2}\partial_N\right)^{k-1} [f(J)]^k = f(J) + \frac{1}{4}\Big(f^2(J)\Big)' + \frac{1}{24}\Big(f^3(J)\Big)'' + \mathcal{O}(f^4(J))\,,$$

- ✔ Replace $f(J)$ by its leading term $f(J) = \Gamma_{\rm cusp}(\lambda) \ln J^2 + \ldots$

$$\gamma(J) = 2\Gamma_{\rm cusp}(\lambda) \ln J + 2[\Gamma_{\rm cusp}(\lambda)]^2 J^{-1} (\ln J + \ldots) - [\Gamma_{\rm cusp}(\lambda)]^3 J^{-2} \Big(\ln^2 J + \ldots\Big) + \mathcal{O}(\ln^3 J/J^3)$$

Leading $\sim (\ln J/J)^p$ power corrections can be resummed into

$$\gamma(J) = 2\,\Gamma_{\rm cusp}(\lambda) \ln \left(J + \Gamma_{\rm cusp}(\lambda) \ln J\right) \times [1 + \mathcal{O}(1/J)]$$

Main features [Basso, GK'06] [Beccaria, Dokshitzer, Marchesini'07]

- ✗ *The leading $\sim (\ln J/J)^p$ power corrections to the anomalous dimension are governed to all loops by the cusp anomalous dimension!*
- ✗ *The coefficient in front of $(\ln J/J)^p$ receives contribution starting from p loops only and it is determined by one-loop correction to the cusp anomaly:* $[\Gamma_{\rm cusp}(\lambda)]^p = \mathcal{O}(\lambda^p)$

Magic relations II

Large J expansion of the anomalous dimension (with $J^2 = N(N+1)$)

$$\gamma(J) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\tfrac{1}{2}\partial_N\right)^{k-1} [f\,(J)]^k = f(J) + \frac{1}{4}\Big(f^2(J)\Big)' + \frac{1}{24}\Big(f^3(J)\Big)'' + \mathcal{O}(f^4(J))$$

with $f(J)$ given by parity respecting series

$$f(J) = \Gamma_{\rm cusp}(\lambda)\ln \bar{J}^2 + f^{(0)} + f^{(1)}/J^2 + f^{(2)}/J^4 + \mathcal{O}(1/J^6)\,,$$

If $f(J)$ = even function of $1/J$, then its derivatives $\partial^p\left(f^{p+1}(J)\right)$ are even/odd function of $1/J$ for $p = \text{even/odd}$. Split $\gamma(J)$ into parity even/odd parts $\gamma_\pm(\ln\bar{J}, -J^{-1}) = \pm\gamma_\pm(\ln\bar{J}, J^{-1})$

$$\left.\begin{aligned} \gamma_+ &= f(J) + \mathcal{O}(\lambda^3) \\ \gamma_- &= \tfrac{1}{4}\partial_N [f(J)]^2 + \mathcal{O}(\lambda^4) \end{aligned}\right\} \quad\Longrightarrow\quad \gamma_- = \tfrac{1}{4}\Big(\gamma_+^2\Big)' + \tfrac{1}{48}\left(-\gamma_+\Big(\gamma_+^3\Big)'' + \tfrac{1}{4}\Big(\gamma_+^4\Big)''\right)' + \mathcal{O}(\lambda^4)$$

- ✔ *The coefficients in front of odd powers of $1/J$ in the large$-J$ expansion of $\gamma(J)$ can be expressed in terms of the coefficients accompanying smaller even powers of $1/J$ to less number of loops!*

$$\gamma(J) = \Gamma_{\rm cusp}(\lambda)\ln\bar{J}^2 + B(\lambda) + J^{-1}\Big[C(\lambda)\ln\bar{J}^2 + D(\lambda)\Big] + \mathcal{O}(J^{-2}\ln^2\bar{J}),$$

- ✔ *(Infinitely many) all-loop relations*

$$A = 2\Gamma_{\rm cusp}(\lambda)\,, \qquad C = [\Gamma_{\rm cusp}(\lambda)]^2\,, \qquad D = \Gamma_{\rm cusp}(\lambda)\,[B(\lambda) - \beta(\lambda)]\,, \quad \dots$$

Parity preserving relations for higher-twist anomalous dimensions

- ✔ High-twist operators in integrable $SL(2)$ sector of QCD [Braun, Derkachov, Manashov; Belitsky; GK'99]

$$\varepsilon_{jkl}\, q^j_\uparrow(z_1 n) q^k_\uparrow(z_2 n) q^l_\uparrow(z_3 n), \qquad \mathrm{tr}[G^\uparrow(z_1) G^\uparrow(z_2) \ldots G^\uparrow(z_L)]$$ *aligned helicity quark/gluon operators*

  One-loop dilatation operator = Hamiltonian of the $SL(2)$ spin chain of spin $s_{\rm q} = 1$ / $s_{\rm g} = \frac{3}{2}$

- ✔ One-loop anomalous dimension/scaling function for twist L and Lorentz spin N

$$f_0(N) = \gamma_0(N) = i(\ln Q(is))' - i(\ln Q(-is))'$$

  $Q(u) = \prod_{k=1}^N (u - u_k)$ satisfies the Baxter equation

$$(u + is)^L Q(u + i) + (u - is)^L Q(u - i) = t_L(u) Q(u)\,,$$

  Conserved charges $t_L(u) = 2u^L + q_2 u^{L-2} + \ldots + q_L$ with $q_2 \sim$ Casimir of the $SL(2)$

$$q_2 = -J^2 + Ls(s-1)\,, \qquad J^2 = (N + Ls)(N + Ls - 1) = \text{conformal spin}$$

- ✔ Large J expansion of $\gamma_0(N)$ $\quad\Longrightarrow\quad$ Asymptotic solution to the Baxter equation

  The anomalous dimensions of twist L occupy the band

$$2\Gamma_{\rm cusp}(\lambda) \ln J \le \gamma(N) \le L\Gamma_{\rm cusp}(\lambda) \ln J$$

  - ✗ To one-loop, only *minimal anomalous dimension* $\gamma_{\rm min}(N)$ has parity preserving property [GK'97]
  - ✗ Recent development: the same property holds true in $\mathcal{N} = 4$ SYM for twist-3 gaugino operators up to three loops and for twist-3 scalar operators up to four loops (at least) [Beccaria, Dokshitzer, Marchesini'07]

Parity preserving relations in the AdS/CFT

Quasipartonic operators of twist L and Lorentz spin $N = n_1 + \ldots + n_{L-1}$

$$\mathcal{O}_L^{\{n\}}(0) = \mathrm{tr}\left[X(0) D_+^{n_1} X(0) \ldots D_+^{n_{L-1}} X(0)\right]$$

Minimal anomalous dimension $\gamma_L(N)$ = energy of a (semi)classical folded string rotating on the AdS$_3 \times$S^1

$$\gamma_L(N) = E - N - L\,, \qquad E = \sqrt{\lambda} E_0\left(\frac{N}{\sqrt{\lambda}}, \frac{L}{\sqrt{\lambda}}\right) + E_1\left(\frac{N}{\sqrt{\lambda}}, \frac{L}{\sqrt{\lambda}}\right) + \ldots$$

For $\lambda \gg 1$ and $N/\sqrt{\lambda},\ L/\sqrt{\lambda} = \text{fixed}$, neglecting quantum correction E_1 (see [Frolov, Tirziu, Tseytlin'06])

$$\gamma_L(N) = L\left[\mathbb{K}(\tau)\, b - \mathbb{E}(\tau)\frac{\lambda'}{b} - 1\right]\,,$$

[Kazakov, Zarembo'04]

$$N/L = \frac{1}{2}\left[\mathbb{E}(\tau)\left(a + \frac{\lambda'}{b}\right) - \mathbb{K}(\tau)\left(b + \frac{\lambda'}{a}\right)\right]\,,$$

$\lambda' = g^2 N_c/(\pi L)^2$ the BMN coupling; the auxiliary variables τ, a and b parameterize classical rotating string solution

$$a = b/\sqrt{1-\tau}\,, \qquad b = \frac{1}{\mathbb{K}(\tau)}\left[\left(1 - \frac{\lambda'}{a^2}\right)\left(1 - \frac{\lambda'}{b^2}\right)\right]^{-1/2}.$$

The anomalous dimension $\gamma_L(N)/L$ is a function of N/L through parametric dependence of both functions on the auxiliary modular parameter τ.

Large spin limit

Asymptotics of the anomalous dimension for $N,\ L \gg 1$ is controlled by the parameter $\xi_{\rm str} = \frac{\lambda}{L^2}\ln^2\frac{N}{L}$

$$\gamma_N^{(L)}(\lambda) = \begin{cases} \lambda \dfrac{N}{2L^2} + \dots\,, & \text{for } 1 \ll N \ll L\,, \\ \dfrac{\lambda}{2\pi^2 L}\ln^2(N/L) + \dots\,, & \text{for } L \ll N \text{ and } \xi_{\rm str} < 1\,, \\ \dfrac{\sqrt{\lambda}}{\pi}\ln(N/\sqrt{\lambda}) + \dots\,, & \text{for } \xi_{\rm str} \gg 1\,, \end{cases}$$

[Frolov, Tseytlin'02]

- For $\xi_{\rm str} < 1$, the anomalous dimension has an expansion in powers of the BMN coupling

$$\gamma(\lambda) = L\left[\lambda'\gamma^{(0)}(N/L) + (\lambda')^2\gamma^{(1)}(N/L) + \dots\right]\,, \qquad \lambda' \equiv \lambda/(\pi L)^2$$

The first few coefficients are given by

$$\gamma^{(0)} = \frac{1}{2}\mathbb{K}(\tau)\left[(2-\tau)\mathbb{K}(\tau) - 2\mathbb{E}(\tau)\right]\,, \qquad \frac{N}{L} = \frac{1}{2}\frac{\mathbb{E}(\tau)}{\sqrt{1-\tau}\mathbb{K}(\tau)} - \frac{1}{2}$$

$$\gamma^{(1)} = \frac{1}{8}\mathbb{K}^3(\tau)\left[\left(4(2-\tau)\sqrt{1-\tau} - \tau^2\right)\mathbb{K}(\tau) - 8\sqrt{1-\tau}\,\mathbb{E}(\tau)\right]\,.$$

- For $\xi_{\rm str} \gg 1$, the leading asymptotics of $\gamma_N^{(L)}(\lambda)$ does not depend on the twist L and is the same as for $L = 2$ operators *(Sudakov universality)*

[Belitsky, Gorsky, GK'06]

$$\gamma(\lambda) = L\left[\sqrt{1 + \lambda' \ln^2(N/L)} - 1\right] \sim \frac{\sqrt{\lambda}}{\pi}\ln N$$

Scaling function in the AdS/CFT

The defining relation: $\gamma_L(N) = f_L\left(N + \frac{1}{2}\gamma_L(N)\right) \quad \Longrightarrow \quad f_L(N) = L\left[\lambda' f^{(0)} + (\lambda')^2 f^{(1)} + \dots\right]$

Observe a remarkable simplification of $f^{(1)}$ as compared to $\gamma^{(1)}$

$$f^{(0)} = \gamma^{(0)}\,, \qquad f^{(1)} = \gamma^{(1)} - \frac{1}{2}\gamma^{(0)}\partial_\alpha\gamma^{(0)} = -\frac{1}{8}\mathbb{K}^4(\tau)\tau^2\,.$$

Examine $\gamma_L(N)$ and $f_L(N)$ as functions of conformal spin $J^2 = (N + \frac{1}{2}L)(N + \frac{1}{2}L - 1) \sim \left(\alpha + \frac{1}{2}\right)L^2$

✔ $L \gg N \gg 1$:

$$\gamma^{(0)} = f^{(0)} = \frac{\pi^2}{2}\left[\alpha - \frac{1}{2}\alpha^2 + \dots\right],$$

Expansion of $f^{(1)}, f^{(2)}, \dots$ involves all powers of $\alpha = N/L$ and does not reveal any symmetry.

✔ $N \gg L \gg 1$:

$$\gamma^{(0)} = f^{(0)} = \ell^2\left[\frac{\ell-2}{2\ell} + 4\frac{\ell^2-\ell+1}{\ell^2}z^2 + \mathcal{O}\left(z^4\right)\right], \qquad \left(z = \frac{L}{J}\ln\frac{J}{L}, \quad \ell = \ln\frac{J}{L}\right)$$

$$\gamma^{(1)} = \ell^4\left[-\frac{1}{8} + \frac{2(\ell-2)}{\ell}z + \frac{2(\ell+1)}{\ell}z^2 + \mathcal{O}\left(z^3\right)\right]$$

$$f^{(1)} = \ell^4\left[-\frac{1}{8} + \frac{2(\ell+1)}{\ell}z^2 + \mathcal{O}\left(z^4\right)\right]$$

Expansion of $f^{(1)}, f^{(2)}, \dots$ only runs in even powers of $z \sim 1/J \quad \Longrightarrow \quad$ *The scaling function is given by the parity preserving series!*

Conclusion and outlook

- ✔ Recent multi-loop QCD calculations of twist-two anomalous dimensions revealed the existence of interesting structure of the subleading corrections suppressed by powers of the Lorentz spin.

- ✔ This structure is a manifestation of the ‘self-tuning’ property of the multi-loop anomalous dimensions – the anomalous dimension of Wilson operators is a function of their conformal spin which is modified in higher loops by an amount proportional to the anomalous dimension

$$\gamma(N) = f\left(N + \tfrac{1}{2}\gamma(N)\right)$$

- ✔ Parity property of the scaling function $f(N)$ leads to (infinite number of) relations between subleading corrections to multi-loop anomalous dimensions

- ✔ These relations hold true in QCD for twist-two anomalous dimensions to three loops and to all loops in the large β_0 limit.

- ✔ In the $\mathcal{N} = 4$ SYM, the same relations survive in the strong coupling regime for higher-twist scalar operators dual to a folded string rotating on the $\text{AdS}_3 \times \text{S}^1$.

- ✔ Going back from the anomalous dimensions $\gamma(N)$ to the dilatation operator $\mathbb{H}$ and conformal spin $\mathbb{J}$

$$\mathbb{H} = \mathbb{F}\big(\mathbb{J} + \tfrac{1}{2}\mathbb{H}\big)\,, \qquad [\mathbb{H},\mathbb{F}] = [\mathbb{J},\mathbb{F}] = 0$$

  The operator $\mathbb{F}$ is ‘more symmetric’ than $\mathbb{H}$ – its eigenvalues have the parity preserving property

  - ✗ What is a meaning of the operator $\mathbb{F}$ in spin chains and in gauge/string theory?

  - ✗ We made use of the parity preserving property of the function $f(N)$ but what is its origin?