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Zamolodchikov-Faddeev Algebra for $\mathrm{AdS}_5 \times \mathrm{S}^5$ Superstring

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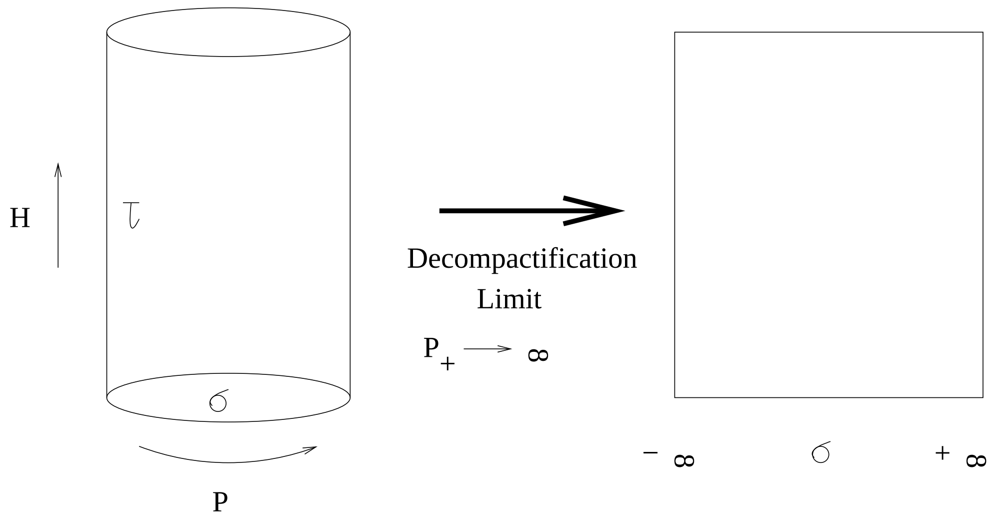
Plan

- Gauge-fixed Sigma-model and Decompactification Limit
- S-matrix and its Symmetries
- Zamolodchikov-Faddeev Algebra
- Crossing Symmetry
- The S-matrix and its Properties

Decompactification limit



Light-cone gauge-fixed string sigma-model in the limit $P_+ \to \infty$

The Hamiltonian $\mathrm{H} \sim P_-$ expands in powers of fields

$$\mathrm{H} = \int_{-P_+}^{P_+} \mathrm{d}\sigma \Big(\mathcal{H}_2 + \frac{1}{\sqrt{\lambda}} \mathcal{H}_4 + \frac{1}{\lambda} \mathcal{H}_6 + \cdots \Big) \quad \overset{P_+ \to \infty}{\Longrightarrow} \quad \text{massive theory on 2dim plane}$$

$$\mathrm{P} \ \sim \text{the generator of rigid } \sigma\text{−rotations is non − vanishing (theory is off − shell)}$$

Off-shell Symmetry Algebra

Symmetry algebra of H in the infinite-volume limit:

$$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2) \oplus \mathrm{H} \oplus \mathrm{P}$$

[Frolov, Plefka, Zamaklar and G.A. hep-th/0609157]

[In Gauge Theory: Beisert hep-th/0511082]

One copy of the centrally extended $\mathfrak{psu}(2|2)$ algebra contains

$\mathbf{R}_{\alpha}{}^{\beta}, \mathbf{L}_{b}{}^{a}$ generate two $\mathfrak{su}(2)$ subalgebras

$\mathbf{Q}_{\alpha}{}^{a}, \ \ \mathbf{Q}^{\dagger\,\alpha}_{a}$ are supersymmetry generators

$\mathbf{H}, \mathbf{C}, \mathbf{C}^{\dagger}$ are three central elements

Algebra relations

$$
\begin{aligned}
\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}^{\dagger\beta}_b\} &= \delta^a_b \mathbf{R}_\alpha{}^\beta + \delta^\beta_\alpha \mathbf{L}_b{}^a + \frac{1}{2}\delta^a_b\delta^\beta_\alpha \mathbf{H}\,, \\
\{\mathbf{Q}_\alpha{}^a, \mathbf{Q}_\beta{}^b\} &= \epsilon_{\alpha\beta}\epsilon^{ab}\,\mathbf{C}\,, \qquad \{\mathbf{Q}^{\dagger\,\alpha}_a, \mathbf{Q}^{\dagger\beta}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta}\,\mathbf{C}^\dagger
\end{aligned}
$$

Central charge

$$\mathbf{C} = \tfrac{1}{2} i g (e^{i\mathbf{P}} - 1) e^{2i\xi}$$

Here $\mathbf{P}$ is the operator of total momentum

The phase ξ is related to the value of $x_-(-\infty)$

The algebra admits a $\mathrm{U}(1)$-automorphism

$$\mathbf{Q} \to e^{i\xi}\mathbf{Q}\,, \quad \mathbf{C} \to e^{2i\xi}\mathbf{C}$$

Degrees of Freedom and Scattering

The symmetry algebra of H in the infinite-volume limit contains

$$\mathfrak{psu}(2|2) \oplus \mathfrak{psu}(2|2)$$

In the light-cone gauge there are 16 physical degrees of freedom

$$16 = 8 \text{ bosons} + 8 \text{ fermions} \sim \underbrace{X_{i\dot{i}}}, \qquad i, \dot{i} = 1, \ldots, 4$$

$$16 \times 16 \xrightarrow{S} 16 \times 16$$

Size of the full S-matrix

$$S_{256\times256} \sim \underbrace{S_{ij}^{kl}}_{16\times16} \otimes \underbrace{S_{\dot{i}\dot{j}}^{\dot{k}\dot{l}}}_{16\times16}$$

One copy, S_{ij}^{kl}, scatters fundamental irreps of $\mathfrak{psu}(2|2)$

Assume that quantum string is integrable and world-sheet scattering is factorized

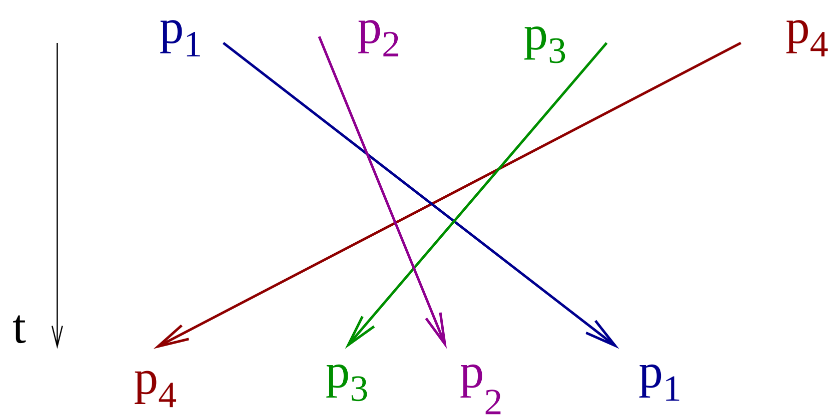
Use symmetries to constrain the S-matrix

The S-matrix

Introduce the *in*-basis and the *out*-basis as

$$|p_1, p_2, \cdots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} = A^\dagger_{i_1}(p_1)\cdots A^\dagger_{i_n}(p_n)|0\rangle\,, \quad p_1 > p_2 > \cdots > p_n$$

$$|p_1, p_2, \cdots, p_n\rangle^{(out)}_{i_1,\ldots,i_n} = A^\dagger_{i_n}(p_n)\cdots A^\dagger_{i_1}(p_1)|0\rangle\,, \quad p_1 > p_2 > \cdots > p_n$$



In the scattering process the in-state goes to the out-state

$$|p_1, \ldots, p_n\rangle^{(in)}_{i_1,\ldots,i_n} \to |p_1, \ldots, p_n\rangle^{(out)}_{i_1,\ldots,i_n}$$

Scattering

$$|p_1,...,p_n\rangle^{(in)}_{i_1,...,i_n} \rightarrow |p_1,...,p_n\rangle^{(out)}_{i_1,...,i_n}$$

We expand initial states on a basis of final states

In particular, the two-particle in- and out-states are related by

$$|p_1,p_2\rangle^{(in)}_{i,j} = \mathbf{S}\cdot|p_1,p_2\rangle^{(out)}_{i,j} = \underbrace{S^{kl}_{ij}(p_1,p_2)}_{\text{two-body S-matrix}} \quad |p_1,p_2\rangle^{(out)}_{k,l}$$

or by using the explicit basis

$$A^\dagger_i(p_1)A^\dagger_j(p_2)|0\rangle = \mathbf{S}\cdot A^\dagger_j(p_2)A^\dagger_i(p_1)|0\rangle = S^{kl}_{ij}(p_1,p_2)A^\dagger_l(p_2)A^\dagger_k(p_1)|0\rangle$$

The conventional Zamolodchikov algebra

$$A_i^\dagger(p_1)A_j^\dagger(p_2) = S_{ij}^{kl}(p_1,p_2)A_l^\dagger(p_2)A_k^\dagger(p_1)$$

- In absence of interactions $S_{ij}^{kl} = \pm\delta_i^k\delta_j^l \Leftarrow$ graded unity
- In many known cases, for $p_1 = p_2$ the S-matrix turns into to the "minus permutation". This reflects the absence of two-soliton state with equal momenta.

Generally, one could define a "twisted" Zamolodchikov algebra

$$A_i^\dagger(p_1)A_j^\dagger(p_2) = S_{ab}^{kl}(p_1,p_2)A_n^\dagger(p_2)A_m^\dagger(p_1)U_{ij,kl}^{ab,mn}$$

where U is a tensor operator which leaves the vacuum invariant

$$U_{ij,kl}^{ab,mn}|0\rangle = \delta_i^a\delta_j^b\delta_k^m\delta_l^n|0\rangle$$

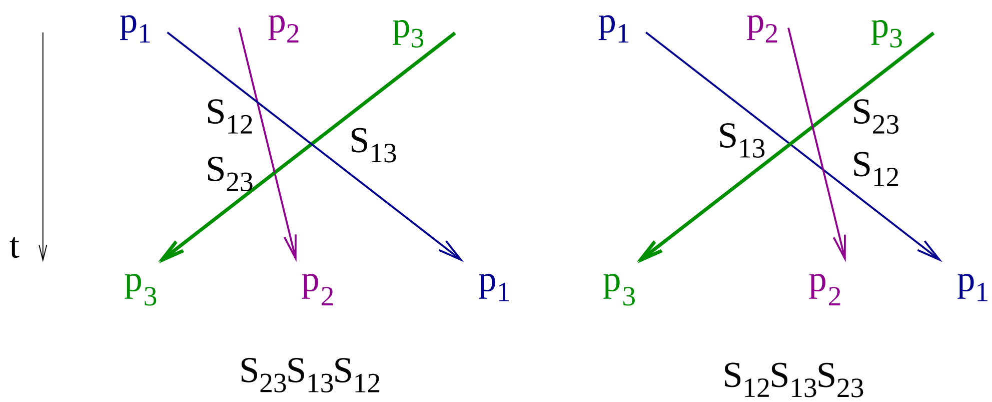
Introduce $\underbrace{A^\dagger}_{\text{row}} = A_i^\dagger(p)E^i\,, \qquad \underbrace{A}_{\text{column}} = A^i(p)E_i$

Yang-Baxter Equation

$$A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger S_{12}$$

Two different ways of reordering $A_1^\dagger A_2^\dagger A_3^\dagger$ to $A_3^\dagger A_2^\dagger A_1^\dagger$ give

$$A_1^\dagger A_2^\dagger A_3^\dagger = A_3^\dagger A_2^\dagger A_1^\dagger S_{12} S_{13} S_{23} \qquad A_1^\dagger A_2^\dagger A_3^\dagger = A_3^\dagger A_2^\dagger A_1^\dagger S_{23} S_{13} S_{12}$$



Absence of new cubic relations implies the Yang-Baxter equation

$$S_{23}(p_2, p_3) S_{13}(p_1, p_3) S_{12}(p_1, p_2) = S_{12}(p_1, p_2) S_{13}(p_1, p_3) S_{23}(p_2, p_3)$$

Symmetries of the S-matrix

The Hamiltonian H commutes with generators $\mathbf{J^a}$ of c.e. $\mathfrak{psu}(2|2)$

$$\mathbf{J^a} \cdot |0\rangle = 0$$

$$\mathbf{J^a} \cdot A_i^\dagger(p)|0\rangle = J^{\mathbf{a}}{}_i^j(p) A_j^\dagger(p)|0\rangle$$

$$\mathbf{J^a} \cdot A_i^\dagger(p_1) A_j^\dagger(p_2)|0\rangle = J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2) A_k^\dagger(p_1) A_l^\dagger(p_2)|0\rangle$$

The invariance condition for the S-matrix is derived from

$$\mathbf{J^a} \cdot A_i^\dagger(p_1) A_j^\dagger(p_2)|0\rangle = S_{ij}^{kl}(p_1,p_2)\, \mathbf{J^a} \cdot A_l^\dagger(p_2) A_k^\dagger(p_1)|0\rangle$$

This is the following condition

$$S_{12}(p_1,p_2) J_{12}^{\mathbf{a}}(p_1,p_2) = J_{21}^{\mathbf{a}}(p_2,p_1) S_{12}(p_1,p_2)$$

Fundamental Representation of c.e. $\mathfrak{psu}(2|2)$

[Beisert hep-th/0511082]

Introduce a basis of the 4dim fundamental representation

$$|e_i\rangle = \begin{cases} |e_a\rangle, & a = 1, 2 \\ |e_\alpha\rangle, & \alpha = 3, 4 \end{cases}$$

Realization of the supersymmetry generators by 4×4 matrices

$$\begin{aligned}
Q_3{}^1 &= aE_3{}^1 + bE_2{}^4 \\
Q_4{}^1 &= aE_4{}^1 - bE_2{}^3 \\
Q_3{}^2 &= aE_3{}^2 - bE_1{}^4 \\
Q_4{}^2 &= aE_4{}^2 + bE_1{}^3
\end{aligned}
\qquad
\begin{aligned}
\bar{Q}_1{}^3 &= cE_4{}^2 + dE_1{}^3 \\
\bar{Q}_1{}^4 &= -cE_3{}^2 + dE_1{}^4 \\
\bar{Q}_2{}^3 &= -cE_4{}^1 + dE_2{}^3 \\
\bar{Q}_2{}^4 &= cE_3{}^1 + dE_2{}^4
\end{aligned}
\qquad ad - bc = 1$$

Central charges: $\quad \mathrm{H} = ad + bc\,, \qquad \mathrm{C} = ab\,, \qquad \mathrm{C}^\dagger = cd$

Rep. is unitary if

$$d^* = a\,, \qquad c^* = b$$

Parameters of the irrep are combined into a matrix

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SU}(1,1)$$

Not all values of the central charges are allowed since

$$\mathrm{H}^2 - 4\mathrm{C}\mathrm{C}^* = 1$$

Central charges are parametrized by a real H and by a phase of C

$$\text{An automorphism} \quad h \to \begin{pmatrix} e^{i\varphi}a & e^{-i\varphi}b \\ e^{i\varphi}c & e^{-i\varphi}d \end{pmatrix}$$

does not affect the charges and reflects a choice of the basis

The space of central charges is the two-sheeted hyperboloid

$$\mathrm{SU}(1,1)/\mathrm{U}(1)$$

Fundamental Unitary Irrep $V(p,\zeta)$

$$a = \sqrt{\frac{g}{2}}\eta\,, \quad b = \sqrt{\frac{g}{2}}\frac{i\zeta}{\eta}\Big(\frac{x^+}{x^-} - 1\Big)\,, \quad c = -\sqrt{\frac{g}{2}}\frac{\eta}{\zeta x^+}\,, \quad d = \sqrt{\frac{g}{2}}\frac{x^+}{i\eta}\Big(1 - \frac{x^-}{x^+}\Big)$$

The constraint $ad - bc = 1$ implies that $x^\pm$ satisfy

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}$$

Comparing

$$C = ab = \tfrac{1}{2} ig\zeta\Big(\frac{x^+}{x^-} - 1\Big) \quad \Longleftrightarrow \quad \underbrace{C = \tfrac{1}{2} ige^{2i\xi}(e^{ip} - 1)}_{\text{string sigma-model}}\,, \qquad \xi = x_-(-\infty)$$

$$\frac{x^+}{x^-} = e^{ip}\,, \qquad \zeta = e^{2i\xi}$$

Central charge H gives the (BDS) dispersion law

$$\mathrm{H}^2 = 1 + 4g^2\sin^2(\frac{p}{2}) \equiv \omega(p)^2$$

The parameter η reflects a freedom in the choice of the basis

Unitarity requires that p is real and

$$\zeta = e^{2i\xi}\,, \quad \eta = \sqrt{ix^- - ix^+}e^{i(\xi+\varphi)}$$

where φ and ξ are real parameters.

To summarize:

- Central charges depend on p and ξ only
- The phase φ correspond to a choice of the basis

The most symmetric choice correspond to φ*=0* since $Q^a_\alpha \sim e^{i\xi}$
We call it the string choice . It leads to the standard YB and ZF

Other choices are also possible. They lead to twisted YB and ZF

Representation of c.e. $\mathfrak{psu}(2|2)$ in the Fock Space

The central charges P and H are additive

$$\mathrm{P}|A_{i_1}(p_1)\dots A_{i_n}(p_n)\rangle = \sum_{k=1}^{n} p_k|A_{i_1}(p_1)\dots A_{i_k}(p_n)\rangle$$

$$\mathrm{H}|A_{i_1}(p_1)\dots A_{i_n}(p_n)\rangle = \sum_{k=1}^{n} \underbrace{\omega(p_k)}_{\text{dispersion}} |A_{i_1}(p_1)\dots A_{i_k}(p_n)\rangle$$

P and H belong to the commutative subalgebra

$$\mathrm{I}_q = \int q(p) A_i^\dagger(p) A^i(p)$$

Commutation relations

$$\mathrm{P}A^\dagger = pA^\dagger + A^\dagger\mathrm{P} \qquad \mathrm{P}A = -pA + A\mathrm{P}$$

Representation of c.e. $\mathfrak{psu}(2|2)$ in the Fock Space

What about additivity of $\mathbf{C}$?
States $|A_i^\dagger(p)\rangle$ depend on momentum p only. Identify

$$|A_i^\dagger(p)\rangle \equiv |e_i\rangle, \quad \text{basis of } V(p,1)$$

$$\mathbf{C}|A_i^\dagger(p)\rangle = \tfrac{1}{2}ig\,(e^{ip}-1)|A_M^\dagger(p)\rangle \quad \leftarrow \quad \zeta = e^{2i\xi} = 1$$

Further, we would like to identify

$$|A_{i_1}^\dagger(p_1)A_{i_2}^\dagger(p_2)\rangle \sim V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)$$

which leads to

$$\mathbf{C}|A_{i_1}^\dagger(p_1)A_{i_2}^\dagger(p_2)\rangle = \tfrac{1}{2}ig(e^{i\mathbf{P}}-1)|A_{i_1}^\dagger(p_1)A_{i_2}^\dagger(p_2)\rangle$$

On the other hand, the additivity of $\mathbf{C}$ implies

$$\mathbf{C}\, V(p_1,\zeta_1)\otimes V(p_2,\zeta_2) = \frac{1}{2} ig \left(\zeta_1(e^{ip_1}-1)+\zeta_2(e^{ip_2}-1)\right) V(p_1,\zeta_1)\otimes V(p_2,\zeta_2)$$

which result in

$$e^{i(p_1+p_2)}-1 = \zeta_1(e^{ip_1}-1)+\zeta_2(e^{ip_2}-1)$$

Two solutions for ζ_k lying on the unit circle

$$\{\zeta_1 = e^{ip_2}\,,\ \zeta_2 = 1\}\,, \quad \text{or} \quad \{\zeta_1 = 1\,,\ \zeta_2 = e^{ip_1}\}$$

The first solution can be interpreted as the *braiding relation*

$$\mathbf{C}\, A_i^\dagger(p) = C(p)A_i^\dagger(p)\, e^{i\mathbf{P}} + A_i^\dagger(p)\,\mathbf{C}$$

The second solution corresponds to

$$\mathbf{C}\, A_i^\dagger(p) = C(p)A_i^\dagger(p) + e^{ip}A_i^\dagger(p)\,\mathbf{C}$$

Braiding

Two-particle representation given by the standard coproduct

$$J^{\mathbf{a}}{}_{ij}^{kl}(p_1,p_2) = J^{\mathbf{a}}{}_{i}^{k}(p_1;c_1)\delta_j^l + \underbrace{(-1)^{\epsilon(i)\epsilon(\mathbf{a})}}_{\text{statistics}}\delta_i^k J^{\mathbf{a}}{}_{j}^{l}(p_2;c_2)\,,$$

where c_i denote central charges on “one-particle” representations

The coproduct can be reinterpreted as a non-trivial braiding between symmetry generators and ZF oscillators

$$\mathbf{J}^{\mathbf{a}} A_i^\dagger(p) = J^{\mathbf{b}}{}_{m}^{k}(p) A_k^\dagger(p) \Theta_{\mathbf{b}i}^{\mathbf{a}m}(p;\mathbf{P}) + (-1)^{\epsilon(i)\epsilon(\mathbf{a})} A_m^\dagger(p) \widetilde{\Theta}_{\mathbf{b}i}^{\mathbf{a}m}(p;\mathbf{P}) \mathbf{J}^{\mathbf{b}}$$

Conditions on the braiding factors Θ are

$$J^{\mathbf{b}}{}_{m}^{k}(p_1)\Theta_{\mathbf{b}i}^{\mathbf{a}m}(p_1;p_2) = J^{\mathbf{a}}{}_{i}^{k}(p_1;c_1)\,, \quad \widetilde{\Theta}_{\mathbf{b}i}^{\mathbf{a}m}(p_1;p_2) J^{\mathbf{b}}{}_{j}^{k}(p_2) = \delta_i^m J^{\mathbf{a}}{}_{j}^{k}(p_2;c_2)$$

Invariant S-matrix

$$S_{12}(p_1,p_2)J^{\mathbf{a}}_{12}(p_1,p_2) = J^{\mathbf{a}}_{21}(p_2,p_1)S_{12}(p_1,p_2)$$

Let $J(p;\zeta,\varphi)$ be a generator of the fund. unitary irrep of c.e. $\mathfrak{psu}(2|2)$

$$S_{12}(p_1,p_2)\big(J(p_1;e^{ip_2},\varphi_1)\otimes\mathbb{I}+\Sigma\otimes J(p_2;1,\varphi_2)\big) = \big(J(p_1;1,\tilde{\varphi}_1)\otimes\Sigma+\mathbb{I}\otimes J(p_2;e^{ip_1},\tilde{\varphi}_2)\big)S_{12}(p_1,p_2)$$

Here $\Sigma = \mathrm{diag}(1,1,-1,-1)$ takes care of statistics

For the string symmetric choice leading to the canonical S-matrix one takes

STRING THEORY BASIS : $\varphi_1 = \varphi_2 = \tilde{\varphi}_1 = \tilde{\varphi}_2 = 0$

For the “spin chain” choice *(the Beisert S-matrix, hep-th/0511082)* one takes

SPIN CHAIN BASIS: $\varphi_2 = \tilde{\varphi}_1 = 0\,, \quad \varphi_1 = -\frac{p_2}{2}\,, \quad \tilde{\varphi}_2 = -\frac{p_1}{2}$

Introduce $\eta = \sqrt{ix^- - ix^+}$ and

$$\eta_1 = \eta(p_1)e^{\frac{i}{2}p_2}\,, \quad \eta_2 = \eta(p_2)\,, \quad \tilde{\eta}_1 = \eta(p_1)\,, \quad \tilde{\eta}_2 = \eta(p_2)e^{\frac{i}{2}p_1}$$

$$
\begin{aligned}
S(p_1,p_2) = {} & \frac{x_2^- - x_1^+}{x_2^+ - x_1^-}\,\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\Big(E_1^1\otimes E_1^1 + E_2^2\otimes E_2^2 + E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1\Big)\\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_2^- + x_1^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\,\frac{\eta_1\eta_2}{\tilde\eta_1\tilde\eta_2}\Big(E_1^1\otimes E_2^2 + E_2^2\otimes E_1^1 - E_1^2\otimes E_2^1 - E_2^1\otimes E_1^2\Big)\\
& - \Big(E_3^3\otimes E_3^3 + E_4^4\otimes E_4^4 + E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3\Big)\\
& + \frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^- + x_2^+)}{(x_1^- - x_2^+)(x_1^- x_2^- - x_1^+ x_2^+)}\Big(E_3^3\otimes E_4^4 + E_4^4\otimes E_3^3 - E_3^4\otimes E_4^3 - E_4^3\otimes E_3^4\Big)\\
& + \frac{x_2^- - x_1^-}{x_2^+ - x_1^-}\,\frac{\eta_1}{\tilde\eta_1}\Big(E_1^1\otimes E_3^3 + E_1^1\otimes E_4^4 + E_2^2\otimes E_3^3 + E_2^2\otimes E_4^4\Big)\\
& + \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{\eta_2}{\tilde\eta_2}\Big(E_3^3\otimes E_1^1 + E_4^4\otimes E_1^1 + E_3^3\otimes E_2^2 + E_4^4\otimes E_2^2\Big)\\
& + i\,\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)\tilde\eta_1\tilde\eta_2}\Big(E_1^4\otimes E_2^3 + E_2^3\otimes E_1^4 - E_2^4\otimes E_1^3 - E_1^3\otimes E_2^4\Big)\\
& + i\,\frac{x_1^- x_2^- (x_1^+ - x_2^+)\eta_1\eta_2}{x_1^+ x_2^+ (x_1^- - x_2^+)(1 - x_1^- x_2^-)}\Big(E_3^2\otimes E_4^1 + E_4^1\otimes E_3^2 - E_4^2\otimes E_3^1 - E_3^1\otimes E_4^2\Big)\\
& + \frac{x_1^+ - x_1^-}{x_1^- - x_2^+}\,\frac{\eta_2}{\tilde\eta_1}\Big(E_1^3\otimes E_3^1 + E_1^4\otimes E_4^1 + E_2^3\otimes E_3^2 + E_2^4\otimes E_4^2\Big)\\
& + \frac{x_2^+ - x_2^-}{x_1^- - x_2^+}\,\frac{\eta_1}{\tilde\eta_2}\Big(E_3^1\otimes E_1^3 + E_4^1\otimes E_1^4 + E_3^2\otimes E_2^3 + E_4^2\otimes E_2^4\Big)
\end{aligned}
$$

Symmetry Generators and ZF Operators

STRING THEORY BASIS:

$$\mathbf{L}_a{}^b A^\dagger(p) = A^\dagger(p)\, L_a^b + A^\dagger(p)\, \mathbf{L}_a{}^b\,,$$
$$\mathbf{R}_\alpha{}^\beta A^\dagger(p) = A^\dagger(p)\, R_\alpha^\beta + A^\dagger(p)\, \mathbf{R}_\alpha{}^\beta\,,$$
$$\mathbf{Q}_\alpha{}^a A^\dagger(p) = A^\dagger(p)\, Q_\alpha^a(p)\, e^{i\mathbf{P}/2} + A^\dagger(p)\, \Sigma\, \mathbf{Q}_\alpha{}^a\,,$$
$$\mathbf{Q}^\dagger_a{}^\alpha A^\dagger(p) = A^\dagger(p)\, \overline{Q}_a^\alpha(p)\, e^{-i\mathbf{P}/2} + A^\dagger(p)\, \Sigma\, \mathbf{Q}^\dagger_a{}^\alpha\,,$$

SPIN CHAIN BASIS:

$$\mathbf{L}_a{}^b A^\dagger(p) = A^\dagger(p)\, L_a^b + A^\dagger(p)\, \mathbf{L}_a{}^b\,,$$
$$\mathbf{R}_\alpha{}^\beta A^\dagger(p) = A^\dagger(p)\, R_\alpha^\beta + A^\dagger(p)\, \mathbf{R}_\alpha{}^\beta\,,$$
$$\mathbf{Q}_\alpha{}^a A^\dagger(p) = A^\dagger(p)\, Q_\alpha^a(p)\, \Theta(\mathbf{P}) + A^\dagger(p)\, \Sigma\, \mathbf{Q}_\alpha{}^a\,,$$
$$\mathbf{Q}^\dagger_a{}^\alpha A^\dagger(p) = A^\dagger(p)\, \overline{Q}_a^\alpha(p)\, \overline{\Theta}(\mathbf{P}) + A^\dagger(p)\, \Sigma\, \mathbf{Q}^\dagger_a{}^\alpha\,,$$

where the braiding factors $\Theta(\mathbf{P})$ and $\overline{\Theta}(\mathbf{P})$ are the diagonal matrices

$$\Theta(\mathbf{P}) = \mathrm{diag}(1, 1, e^{i\mathbf{P}}, e^{i\mathbf{P}})\,, \qquad \overline{\Theta}(\mathbf{P}) = e^{-i\mathbf{P}}\,\Theta(\mathbf{P})\,.$$

Twisting ZF algebra

$$A^\dagger(p) \to A^\dagger(p)\, U(\mathbf{P}, p)\,, \quad A(p) \to U^\dagger(\mathbf{P}, p) A(p)$$

ZF algebra keeps its form but with an *operator-valued S-matrix*

$$\mathcal{S}^U_{12}(p_1, p_2; \mathbf{P}) = U_2(\mathbf{P} + p_1, p_2) U_1(\mathbf{P}, p_1) \mathcal{S}_{12}(p_1, p_2) U_2^\dagger(\mathbf{P}, p_2) U_1^\dagger(\mathbf{P} + p_2, p_1)$$

Taking $U(\mathbf{P}, p) \equiv U(\mathbf{P}) = \text{diag}(e^{\frac{i}{2}\mathbf{P}}, e^{\frac{i}{2}\mathbf{P}}, 1, 1)$ relates gauge and string choices

$$S^{\text{chain}}_{12}(p_1, p_2) = U_2(p_1) S^{\text{string}}_{12}(p_1, p_2) U_1^\dagger(p_2)$$

For $S_{ij} = S^{\text{chain}}_{ij}(p_i, p_j)$ one finds the *twisted YB equation*

$$F_{23}(p_1) S_{23} F_{23}^{-1}(p_1) S_{13} F_{12}(p_3) S_{12} F_{12}^{-1}(p_3) = S_{12} F_{13}(p_2) S_{13} F_{13}^{-1}(p_2) S_{23}$$

Crossing Symmetry

The complete S-matrix

$$\mathcal{S}(p_1, p_2) = \underbrace{S_0(p_1, p_2)}_{\text{prefactor}}^{?} S(p_1, p_2)$$

Compatibility condition of the ZF algebra

$$A_1 A_2 \quad = \quad \mathcal{S}_{12} A_2 A_1\,, \quad A_1^\dagger A_2^\dagger = A_2^\dagger A_1^\dagger \mathcal{S}_{12}\,, \quad A_1 A_2^\dagger = A_2^\dagger \mathcal{S}_{21} A_1 + \delta_{12}$$

Find the conditions on $\mathcal{S}(p_1, p_2)$ under which the ZF algebra admits an automorphism

$$A^\dagger(p) \to B^\dagger(p) = A^t(-p)\mathscr{C}(-p)\,, \qquad A(p) \to B(p) = \mathscr{C}^\dagger(-p)A^{\dagger\, t}(-p)\,,$$

where $\mathscr{C}(p)$ is a “charge-conjugation” matrix.

This automorphism is the “particle-to-antiparticle" transform

Require the transform to be compatible with the $\mathfrak{psu}(2|2)$-symmetry (braiding relations!). This leads to

$$\mathscr{C}(p)\, L_a^{\,b} = -L_b^{\,a}\, \mathscr{C}(p) \qquad \mathscr{C}(p)\, R_\alpha^{\,\beta} = -R_\beta^{\,\alpha}\, \mathscr{C}(p)$$

and

$$e^{-i\frac{p}{2}} \mathscr{C}(p)\, \overline{Q}_a^{\,\alpha}(-p) = -\left(\overline{Q}_a^{\,\alpha}(p)\right)^t \Sigma \mathscr{C}(p)$$
$$\mathscr{C}(p)\; Q_\alpha^{\,a}(-p) = -e^{-i\frac{p}{2}} \left(Q_\alpha^{\,a}(p)\right)^t \Sigma \mathscr{C}(p)$$

Susy generators of anti-particle irrep are anti-hermitian: $(Q_\alpha^{\,a}(-p))^\dagger = -\overline{Q}_a^{\,\alpha}(-p)$

Equations are solved for the charge-conjugation matrix $\mathscr{C}$

$$\mathscr{C}(p) = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -i\,\mathrm{sign}(p)\,\sigma_2 \end{pmatrix},$$

where σ_2 one of the Pauli matrices

Parameters of the anti-particle representation

$$a(-p) = -ie^{-\frac{ip}{2}} b(p)\,\mathrm{sign} p \qquad c(-p) = -ie^{\frac{ip}{2}} d(p)\,\mathrm{sign} p$$
$$b(-p) = -ie^{-\frac{ip}{2}} a(p)\,\mathrm{sign} p \qquad d(-p) = -ie^{\frac{ip}{2}} c(p)\,\mathrm{sign} p$$

Central charges of the anti-particle representation are

$$\mathrm{H}(-p) = -\mathrm{H}(p)\,, \quad C(-p) = -C(p)e^{-ip}\,, \quad C(-p)^\dagger = -C(p)^\dagger e^{ip}\,.$$

If we assume that $p_1 > p_2$ *then the S-matrix must obey*

$$\mathscr{C}_1^{-1}(-p_1)\mathcal{S}_{12}^{t_1}(p_1,p_2)\mathscr{C}_1(-p_1)\mathcal{S}_{12}(-p_1,p_2) = \mathbb{I}$$
$$\mathscr{C}_2^{-1}(-p_2)\mathcal{S}_{21}^{t_2}(p_2,p_1)\mathscr{C}_2(-p_2)\mathcal{S}_{21}(-p_2,p_1) = \mathbb{I}$$

Crossing Equation

Substituting our string S-matrix we find the following equation for the scalar factor

$$S_0(-p_1,p_2)S_0(p_1,p_2)=\frac{\left(\frac{1}{x_1^-}-x_2^-\right)\left(x_1^- - x_2^+\right)}{\left(\frac{1}{x_1^+}-x_2^-\right)\left(x_1^+ - x_2^+\right)}$$

[Janik, hep-th/0603038]

Relation to the “dressing phase" through

$$S_0(p_1,p_2)^2=\frac{x_2^+-x_1^-}{x_1^+-x_2^-}\,\frac{1-\frac{1}{x_1^+x_2^-}}{1-\frac{1}{x_1^-x_2^+}}\,e^{i\theta(p_1,p_2)}$$

[Frolov and G.A., hep-th/0604043]

Dressing v.s. Crossing

$$\theta(p_1,p_2) = \sum_{r,s=2}^{\infty} c_{r,s}(g)\; q_r(p_{[1})q_s(p_{2]})\,, \qquad c_{r,s}(g) = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}$$

where

$$c_{r,s}^{(0)} \quad = \quad \delta_{r+1,s} \qquad\qquad \Leftarrow \qquad\qquad \mathrm{Tree-level}$$

[Frolov, Staudacher and G.A, 0406256]

$$c_{r,s}^{(1)} \quad = \quad -\frac{(1-(-1)^{r+s})}{\pi}\,\frac{(r-1)(s-1)}{(r+s-2)(s-r)} \qquad \Leftarrow \qquad \mathrm{One\ loop}$$

[Hernandez and Lopez, hep-th/0603204]

These two leading orders satisfy the crossing equation up to $\mathcal{O}(1/g^3)$

[Frolov and G.A., hep-th/0604043]

All Loop Proposals for the Dressing Phase

[Beisert, Hernandez and Lopez, hep-th/0609044]

[Beisert, Eden and Staudacher, hep-th/0610251]

String S-matrix

- obeys the standard Yang-Baxter equation

$$\mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12} = \mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23}$$

- obeys the unitarity condition

$$\mathcal{S}_{12}(p_1, p_2)\mathcal{S}_{21}(p_2, p_1) = \mathbb{I}$$

- obeys "hermitian analyticity"

$$\mathcal{S}_{21}^{\dagger}(p_2, p_1) = \mathcal{S}_{12}(p_1, p_2)$$

- obeys the crossing symmetry ($\mathscr{C}$ is the charge conj. matrix)

$$\mathscr{C}_1^{-1}\mathcal{S}_{12}^{t_1}(p_1, p_2)\mathscr{C}_1\mathcal{S}_{12}(-p_1, p_2) = \mathbb{I}$$

Further Properties

Crossing twice gives

$$A^\dagger(p) \to A^\dagger(p)\Sigma\,, \qquad\qquad A \to \Sigma A(p)$$

The ZF algebra implies

$$A_1 A_2 = \mathcal{S}_{12} A_2 A_1 \qquad \Rightarrow \qquad [S_{12}(p_1, p_2), \Sigma \otimes \Sigma] = 0\,,$$

where $\Sigma = \mathrm{diag}(1, 1, -1, -1)$

Under the shift $p \to p + 2\pi$ the S-matrix exhibits the monodromies

$$\begin{aligned} S_{12}(p_1, p_2 + 2\pi) &= -S_{12}(p_1, p_2)\Sigma_1 \\ S_{12}(p_1 + 2\pi, p_2) &= -\Sigma_2 S_{12}(p_1, p_2) \end{aligned}$$

Graded Inverse Scattering Method

Define the fermionic S-operator as

$$\mathbb{S}(p_1,p_2) = (-1)^{\pi_j+\pi_k(\pi_i+\pi_l)}\, \mathcal{S}_{ij}^{kl}(p_1,p_2) \quad \underbrace{E_k{}^i \hat{\otimes} E_l{}^j}_{\text{graded unities}}$$

It obeys the graded YB and the crossing relation

$$\mathscr{C}_1^{-1}(-p_1)\mathbb{S}_{12}^{\mathrm{st}_1}(p_1,p_2)\mathscr{C}_1(-p_1)\mathbb{S}_{12}(-p_1,p_2) = \mathbb{I}$$

The graded YB allows to consistently define the relations between the matrix elements of a “would be" quantum monodromy matrix

$$\mathbb{S}_{12}(p_1,p_2)\mathrm{T}_1(p_1)\mathrm{T}_2(p_2) = \mathrm{T}_2(p_2)\mathrm{T}_1(p_1)\mathbb{S}_{12}(p_1,p_2)$$

Supertransposing we get

$$\mathrm{T}_1^{\mathrm{st}}(p_1)\mathbb{S}_{12}^{\mathrm{st}_1}(p_1,p_2)\mathrm{T}_2(p_2) = \mathrm{T}_2(p_2)\mathbb{S}_{12}^{\mathrm{st}_1}(p_1,p_2)\mathrm{T}_1^{\mathrm{st}}(p_1)\,.$$

Compare to

$$\mathrm{T}_1^{-1}(-p_1)\mathbb{S}_{12}^{-1}(-p_1,p_2)\mathrm{T}_2(p_2) = \mathrm{T}_2(p_2)\mathbb{S}_{12}^{-1}(-p_1,p_2)\mathrm{T}_1^{-1}(-p_1)$$

Monodromy matrix algebra is consistent with the relation

$$\mathrm{T}(-p)^{-1} \sim \mathscr{C}^{-1}(-p)\mathrm{T}^{\mathrm{st}}(p)\mathscr{C}(-p)$$

Leads to description of the center of the $\mathbb{S}\mathrm{TT}$-algebra

[Frolov, Leeuw and G.A., in progress]

Local and Non-local Charges

The S-matrix allows to reconstruct the representation of c.e. extended $\mathfrak{psu}(2|2)$.

A 4-dim rep of $\mathbb{S}\mathrm{TT}$-algebra is provided by the S-matrix itself:

$$\mathrm{T}_1(p_1,\lambda) = S_{13}(p_1,p_3) \qquad \text{with} \qquad p_3 \equiv \underbrace{\lambda}_{\text{spec. par.}}$$

Around $p_3 = 0$ the $\mathbb{S}\mathrm{TT}$-algebra produces *(non-)local charges*

$$S_{12}\Big(\underbrace{S_{23}^{-1}S_{13}^{-1}\partial_3 S_{13}S_{23}}_{\mathrm{B}\otimes J} + \underbrace{S_{23}^{-1}\partial_3 S_{23}}_{\mathbb{I}\otimes J}\Big) = \Big(\underbrace{S_{13}^{-1}\partial_3 S_{13}}_{J\otimes\mathbb{I}} + \underbrace{S_{13}^{-1}S_{23}^{-1}\partial_3 S_{23}S_{13}}_{\mathrm{B}\otimes J}\Big)S_{12}$$

[a là Bernard and LeClair]

Expand YB further and get the higher (non-local) symmetries commuting with the S-matrix. No restriction for the dressing phase!

Summary

- Symmetries of the gauge-fixed string sigma-model are used to find the S-matrix

- The S-matrix fits the axioms of massive integrable systems

- Particle-to-antiparticle transform is an automorphism of the ZF algebra provided the S-matrix obeys crossing symmetry

- In the large tension limit the S-matrix perfectly agrees with the near-plane wave S-matrix

[Klose, McLouglin, Roiban and Zarembo, hep-th/0611169]