Integration-by-parts identities from the viewpoint of differential geometry

Amplitudes 2014, Saclay, June 10th, 2014

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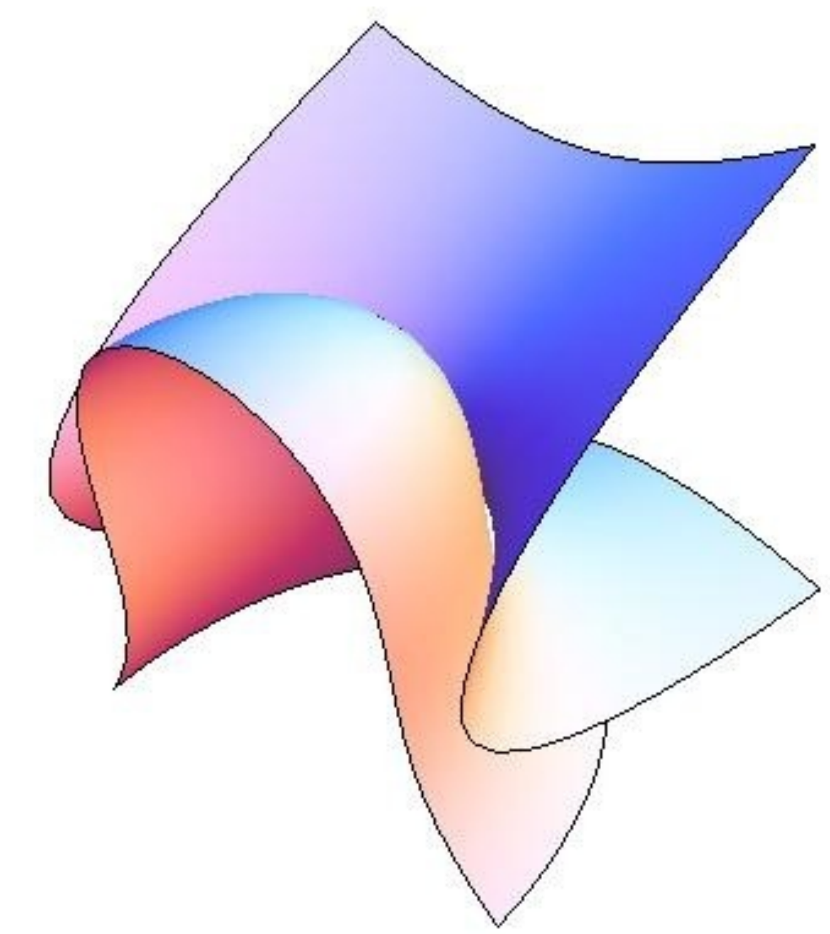
based on
YZ, 1406.xxxx, to appear

Outline

$$\int \frac{d^D l_1}{(i\pi^{D/2})} \cdots \int \frac{d^D l_L}{(i\pi^{D/2})} \frac{N(l_1 \ldots l_L)}{D_1 \ldots D_k}$$

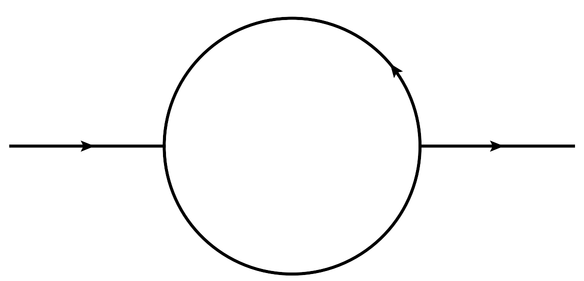
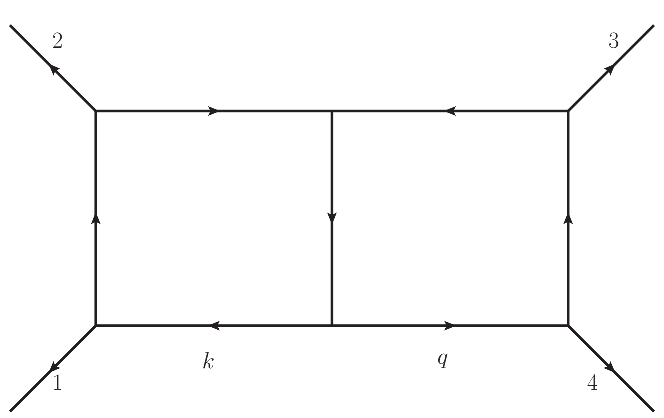
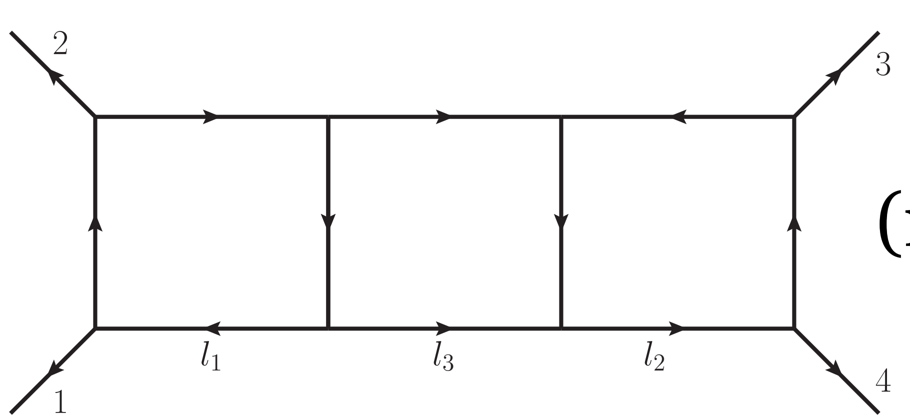
- Review of integration-by-part identities (IBP)
- IBP: geometric meaning
- A new method of constructing IBPs
- 4D Two-loop examples



Integration-by-parts identities

Too many integrals from Feynman diagrams/ integrand reduction…

Just a few master integrals …

		Integrand	MI
		9 terms	1 terms
	(massless)	32 terms	2 terms
	(massless)	398 terms	3 terms

Multiloop Integrand reduction
via Groebner basis and polynomial division

1205.5707, YZ
1205.7087, P. Mastrolia, E. Mirabella, G.Ossola and T. Peraro

Simon Badger
&
Pierpaolo Mastrolia's
talk

Using symmetry of diagram, IBPs to reduce the integrand …

Integration-by-parts identities

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \sum_{i=1}^{L} \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1 \ldots D_k} \right) = 0.$$

DL component

v_i^μ depends on loop momenta and external momenta

Integral reduction programs

FIRE, A.V. Smirnov, V.A. Smirnov
Reduze, A. von Manteuffel, C. Studerus
… …

In most cases, IBP relations contain integrals with doubled propagators

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \sum_{i=1}^{L} \left[\frac{\partial v_i^\mu}{\partial l_i^\mu} \left(\frac{1}{D_1 \ldots D_k} \right) - \sum_{j=1}^{k} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu} \left(\frac{1}{D_1 \ldots D_j^2 \ldots D_k} \right) \right] = 0.$$

Frequently, we only have integrals without doubled propagates…
Feynman diagrams, integrand reduction

suitable v_i^μ to remove doubled propagators?

IBP without doubled propagators

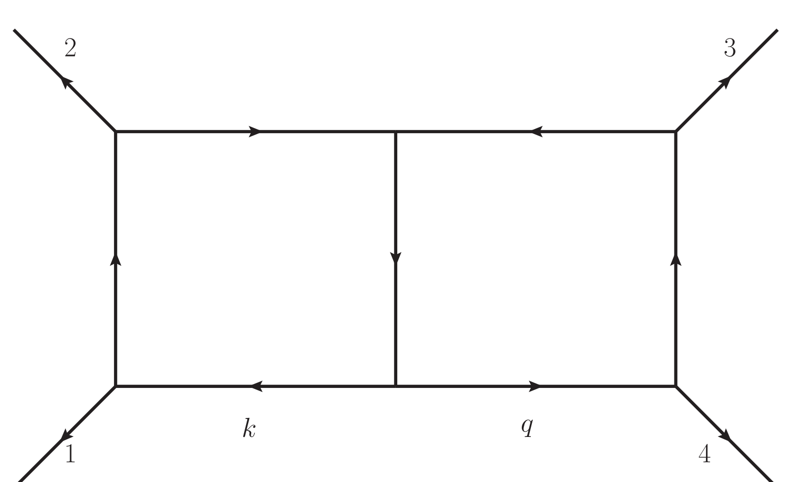
1009.0472 J. Gluza, K. Kajda, D. Kosower (GKK)

1111.4220 R. Schabinger

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \sum_{i=1}^{L} \left[\frac{\partial v_i^\mu}{\partial l_i^\mu} \left(\frac{1}{D_1 \ldots D_k} \right) - \sum_{j=1}^{k} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu} \left(\frac{1}{D_1 \ldots D_j^2 \ldots D_k} \right) \right] = 0.$$

$$\sum_{i=1}^{L} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu} \propto D_j$$

can be solved by algebraic method: Syzygy computation



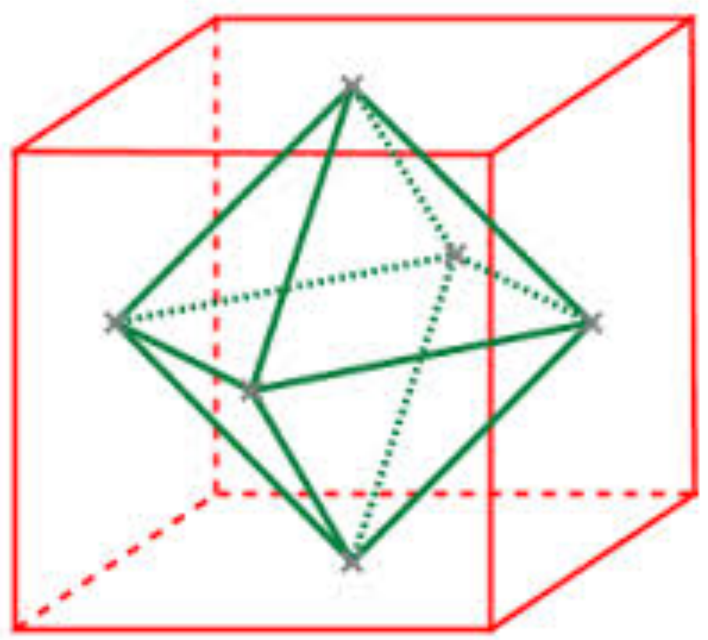
(massless)

3 solutions $v_{1;i}^\mu, \quad v_{2;i}^\mu, \quad v_{3;i}^\mu$

any geometric meaning?

reduced to 2 MIs

Differential forms



Poincare dual: *1*-form $\Longleftrightarrow$ *(DL-1)*-form

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \frac{d^D l_L}{i\pi^{D/2}} \sum_{i=1}^{L} \frac{\partial}{\partial l_i^\mu} \left(\frac{v_i^\mu}{D_1 \ldots D_k} \right) = 0. \quad \Longleftrightarrow \quad \int d \left(\frac{\omega}{D_1 \ldots D_k} \right) = 0.$$

$$\sum_{i=1}^{L} v_i^\mu \frac{\partial D_j}{\partial l_i^\mu} \propto D_j \quad \Longleftrightarrow \quad dD_i \wedge \omega \propto D_i$$

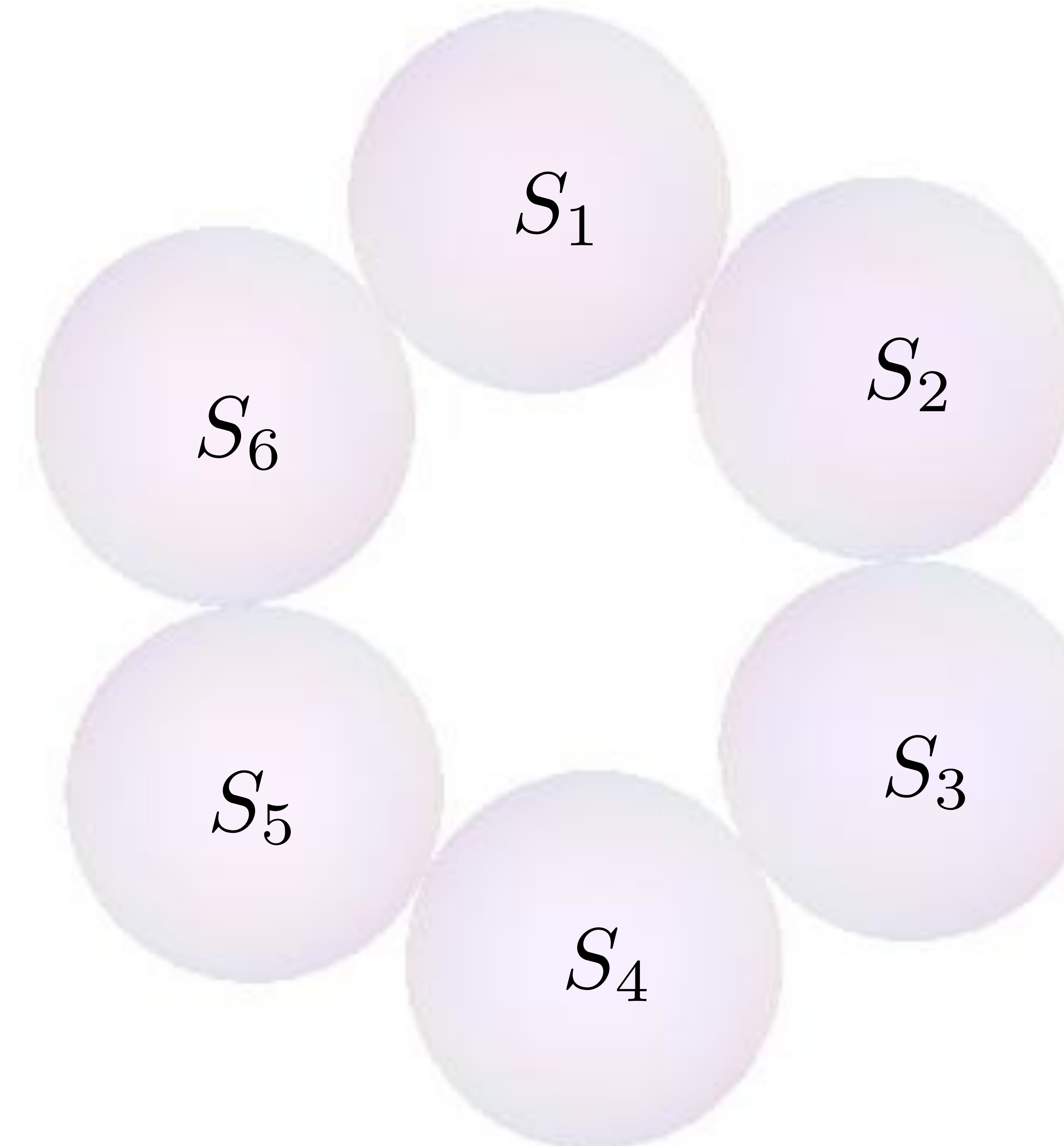
Naive ansatz

polynomial-valued *(DL-k-1)*-form

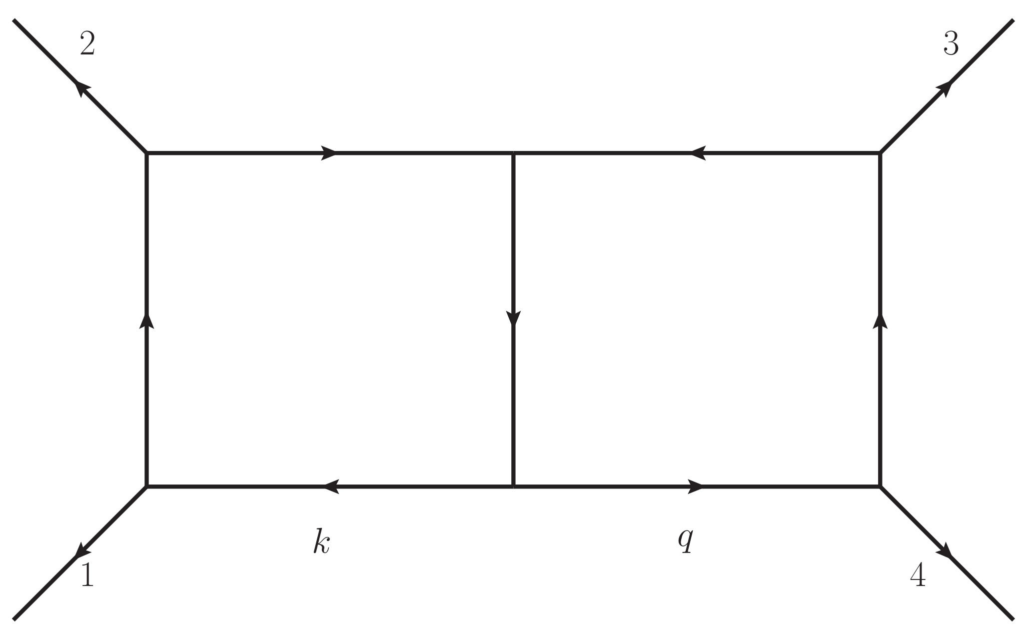
$$\omega = \alpha \wedge dD_1 \wedge \ldots \wedge dD_k$$

The assumption is too strong ...

Local behaviour



massless 4D double box



6 cut solutions

$$S = \{D_1 = \ldots = D_k = 0\}$$

$$I = I_1 \cap I_2 \cap I_3 \cap I_4 \cap I_5 \cap I_6$$

(primary decomposition)

1205.0801, S. Caron-Huot and K. Larsen

$$v^{\mu}_{1;i}, \Longleftrightarrow \omega_1^{\text{GKK}}$$

$$v^{\mu}_{2;i} \Longleftrightarrow \omega_2^{\text{GKK}}$$

$$v^{\mu}_{3;i} \Longleftrightarrow \omega_3^{\text{GKK}}$$

$$\Omega \equiv dD_1 \wedge dD_2 \wedge dD_3 \wedge dD_4 \wedge dD_5 \wedge dD_6 \wedge dD_7$$

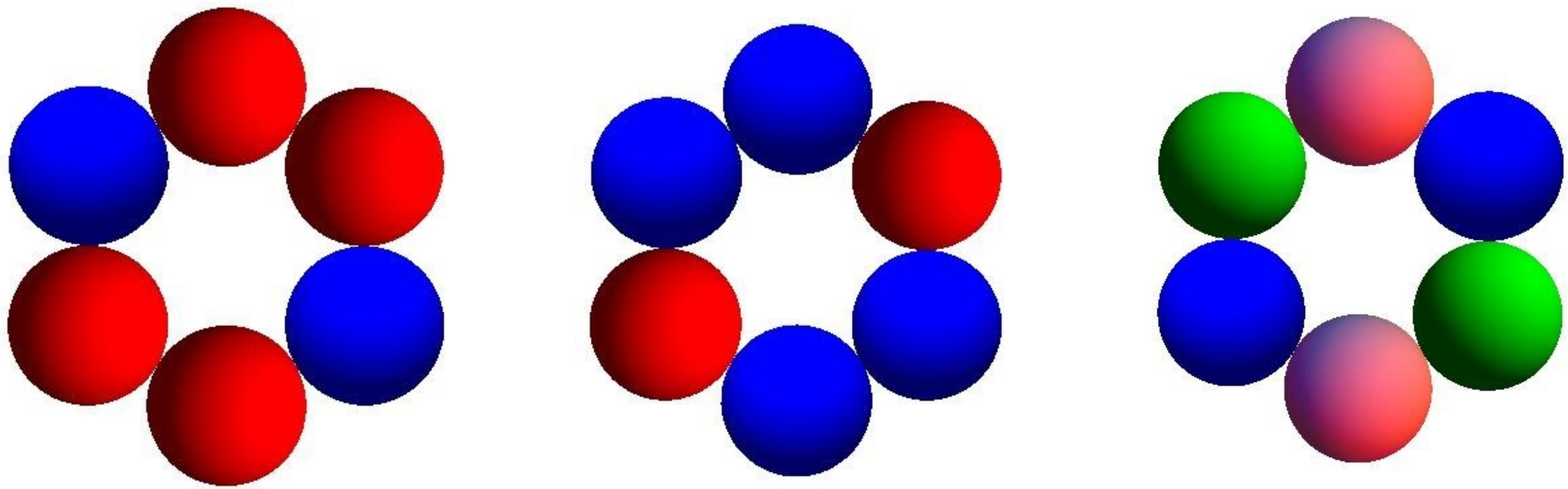
normal form

$$\omega_i^{\text{GKK}} = \alpha_i \Omega \; ?$$

not so simple ...

Local behaviour

On the cut , ω_i^{GKK} is locally proportional to Ω



ω_1^{GKK} ω_2^{GKK} ω_3^{GKK}

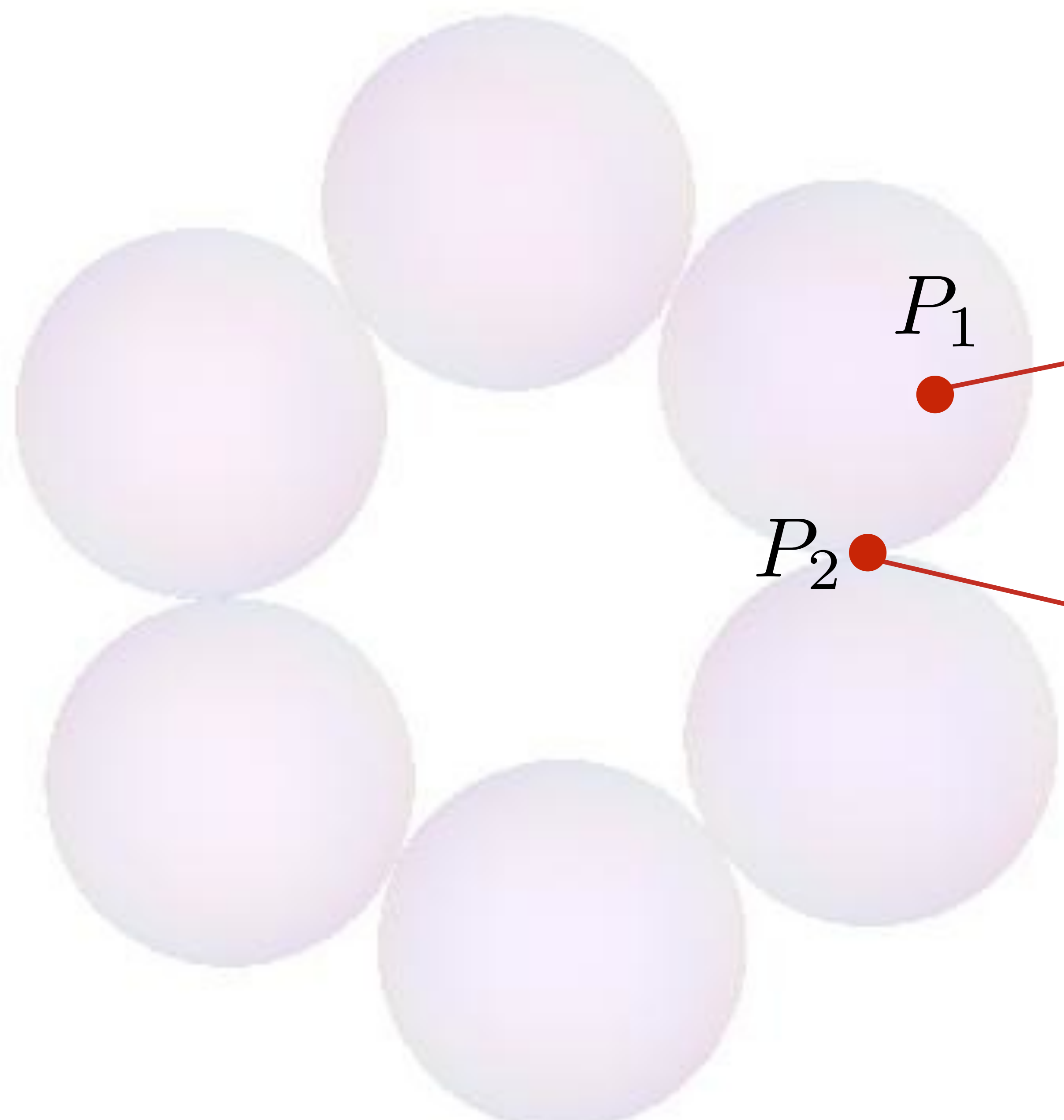
(red) $= +1$ (blue) $= -1$

(pink) $= 1 + \frac{2(l_2 \cdot p_1)}{s}$ (green) $= -1 - \frac{2(l_2 \cdot p_1)}{s}$

not accidental...

Local behaviour

$$\Omega \equiv dD_1 \wedge dD_2 \wedge dD_3 \wedge dD_4 \wedge dD_5 \wedge dD_6 \wedge dD_7$$



Regular point, Jacobian non-degenerate

$$\Omega\Big|_{P_1} \neq 0$$

Singular point, Jacobian degenerate

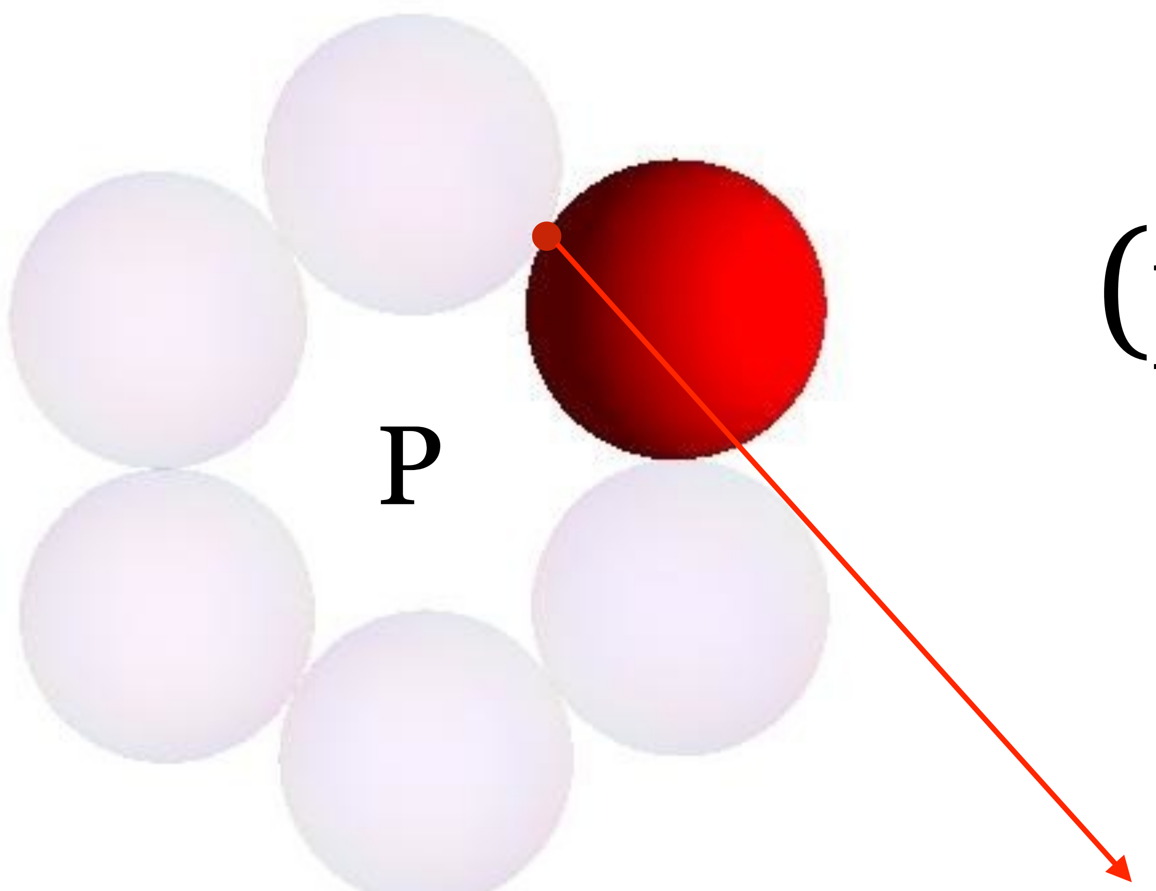
$$\Omega\Big|_{P_2} = 0$$

$$dD_i \wedge \omega \propto D_i \quad \Longrightarrow \quad dD_i \wedge \omega\Big|_S = 0 \quad \Longrightarrow \quad \omega\big|_P = \alpha\big|_P \wedge \Omega\big|_P$$

If P is regular

Almost all points are regular, so by analytic continuation, on the cut , ω is locally proportional to Ω

An ansatz for on-shell IBPs



find 6 fundamental forms: η_i

(partition of Ω)

$$\eta_i\Big|_{S_j} = \delta_{ij}\Omega\Big|_{S_j}$$

Do they exist?

Singular point, $\Omega\Big|_P = 0$

no discontinuity at P !

= +1

= 0

Integrand reduction

↓

IBP reduction from η_i

Congruence equations for differential forms

$$\begin{cases} x = 2 \mod 3 \\ x = 3 \mod 5 \\ x = 2 \mod 7 \end{cases}$$

$$x = 23 + 105n$$

Chinese Remainder Theorem

$$\begin{cases} x = 2 \mod 6 \\ x = 11 \mod 15 \end{cases}$$

not co-prime,
solution may not exist

$$\gcd(6, 15) = 3$$

$$11 - 2 = 9 = 0 \mod 3$$ solution exists!

$$\begin{cases} \eta_2 = 0 \mod I_1 \\ \eta_2 = \Omega \mod I_2 \\ \eta_2 = 0 \mod I_3 \\ \eta_2 = 0 \mod I_4 \\ \eta_2 = 0 \mod I_5 \\ \eta_2 = 0 \mod I_6 \end{cases}$$

not co-prime,
solution may not exist

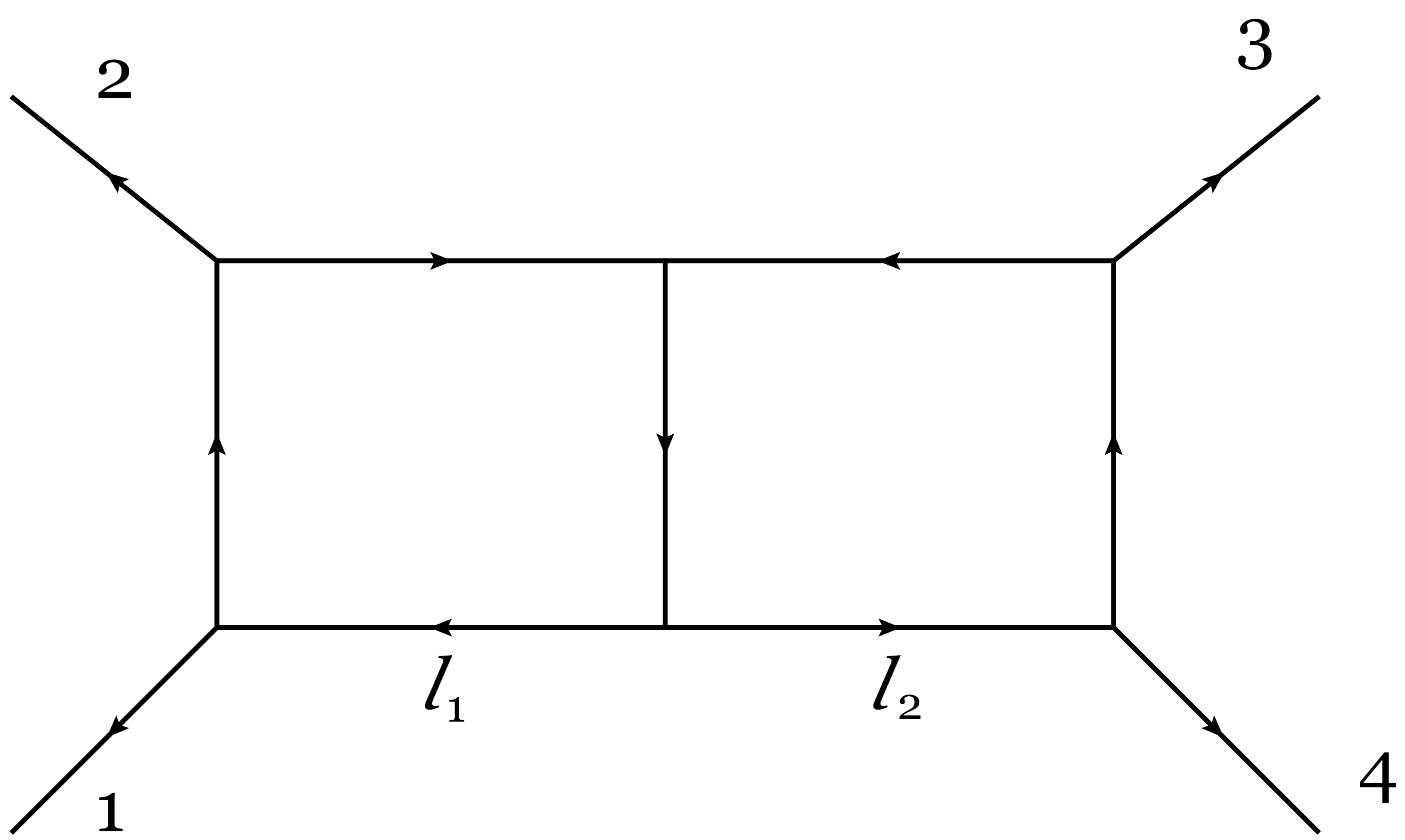
$$\Omega \in I_1 + I_2$$

solution exists!

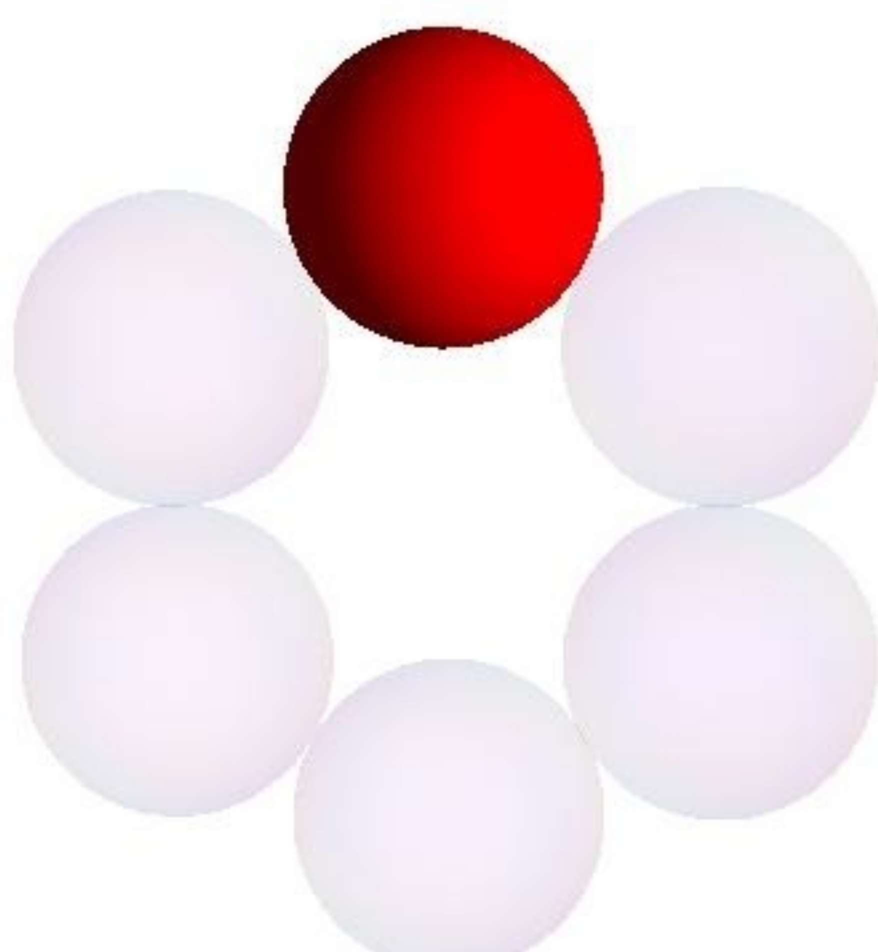
$\Updownarrow$

Ω vanishes on $S_1 \cap S_2$

4D double box



$$x_1 = l_1 \cdot p_1, \quad x_2 = l_1 \cdot p_2, \quad x_3 = l_1 \cdot p_4, \quad x_4 = l_1 \cdot \omega,$$
$$y_1 = l_2 \cdot p_1, \quad y_2 = l_2 \cdot p_2, \quad y_3 = l_2 \cdot p_4, \quad y_4 = l_2 \cdot \omega.$$



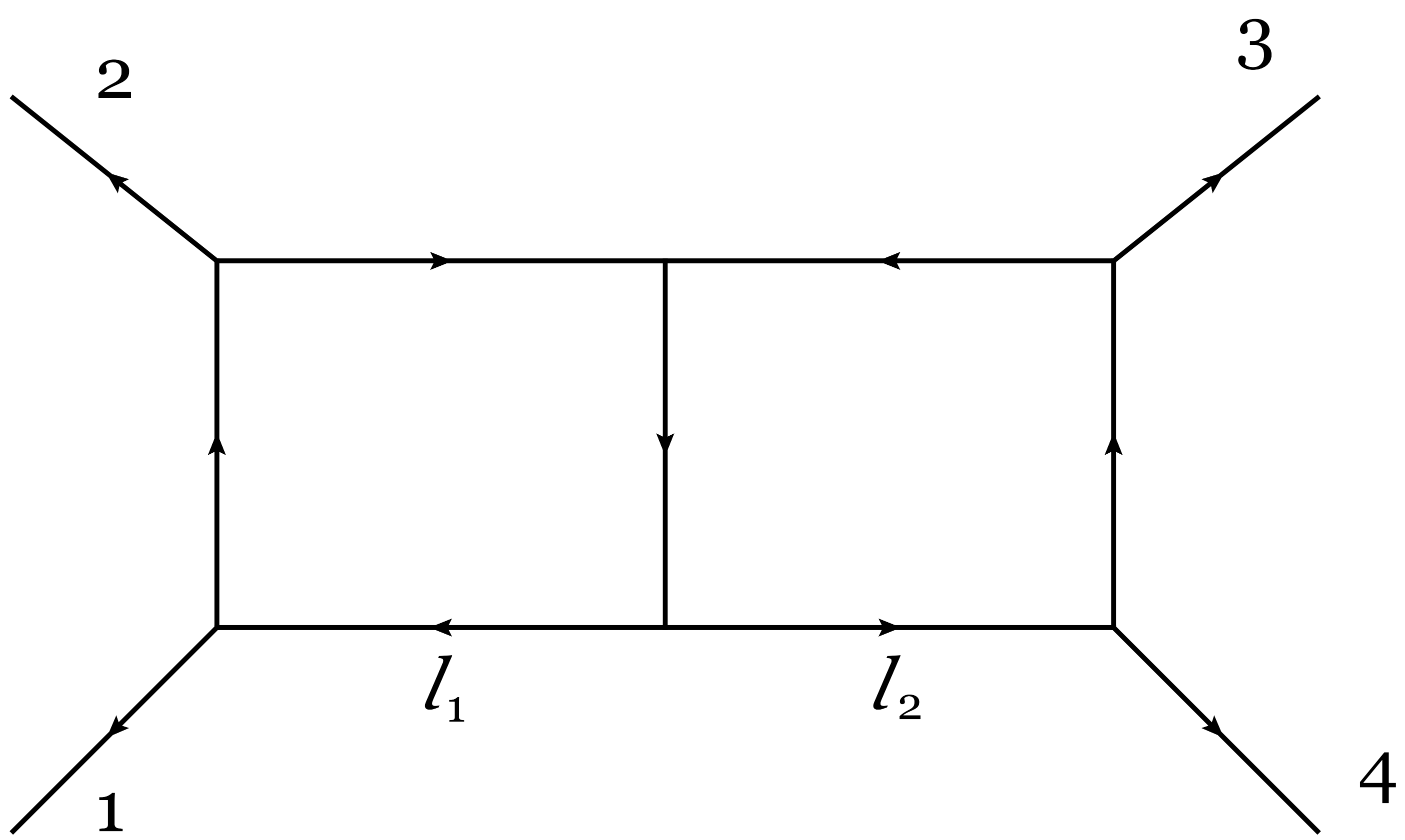
6 fundamental forms

$$\eta_1 = \big(- stx_4 - 2sty_1 - sty_4 + 4sx_3y_1 + 2sx_3y_4 + 2sx_4y_1 + 4sy_1^2 + 4sy_1y_4 + 8x_3y_1^2 + 8x_3y_1y_4 \big) (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 - dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_2 \wedge dy_3 \wedge dy_4)$$

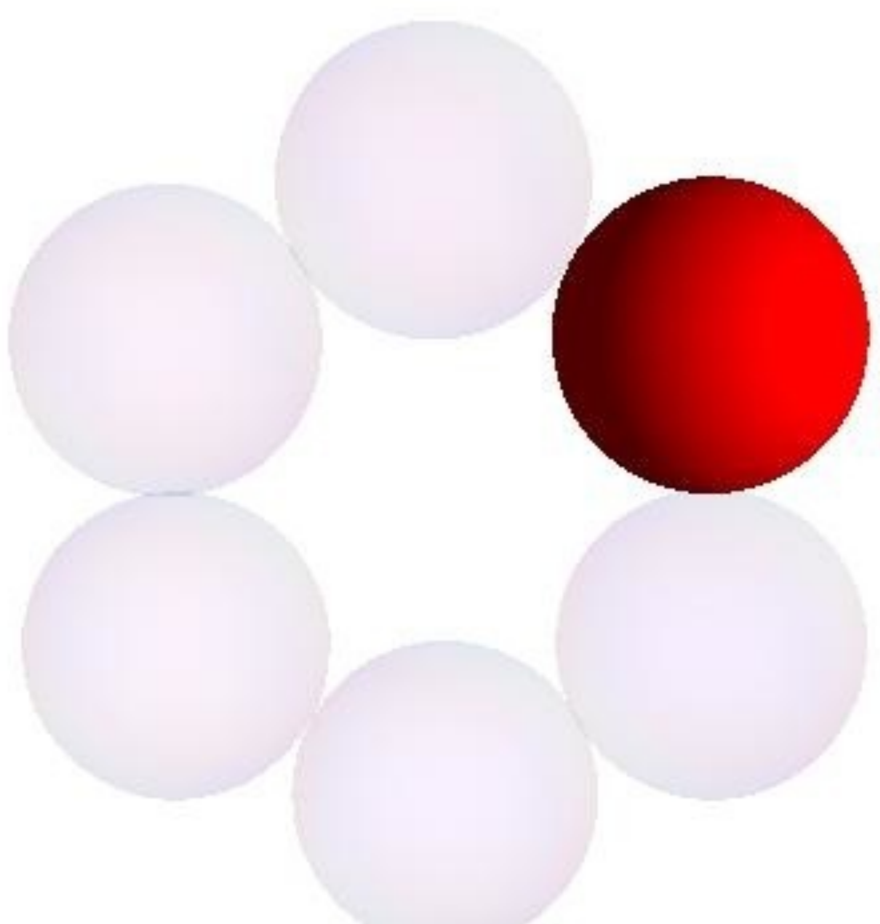
algorithm realized by *MathematicaM2*,
a computational algebraic geometry package, YZ

■ = +1 ■ = 0

4D double box



$$x_1 = l_1 \cdot p_1, \quad x_2 = l_1 \cdot p_2, \quad x_3 = l_1 \cdot p_4, \quad x_4 = l_1 \cdot \omega,$$
$$y_1 = l_2 \cdot p_1, \quad y_2 = l_2 \cdot p_2, \quad y_3 = l_2 \cdot p_4, \quad y_4 = l_2 \cdot \omega.$$



6 fundamental forms

$$\eta_2 = \big(2s^2tx_3 - s^2tx_4 - s^2ty_4 - 4s^2x_3^2 + 4s^2x_3x_4 - 4s^2x_3y_1 + 2s^2x_3y_4 + 2s^2x_4y_1 + 4st^2x_3 - 8stx_3^2 + 4stx_3x_4 - 8stx_3y_1 - 4stx_3y_4 - 8sx_3^2y_1 + 8sx_3^2y_4 - 16tx_3^2y_1\big) (dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 + dx_1 \wedge dx_2 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4)$$

algorithm realized by *MathematicaM2*,
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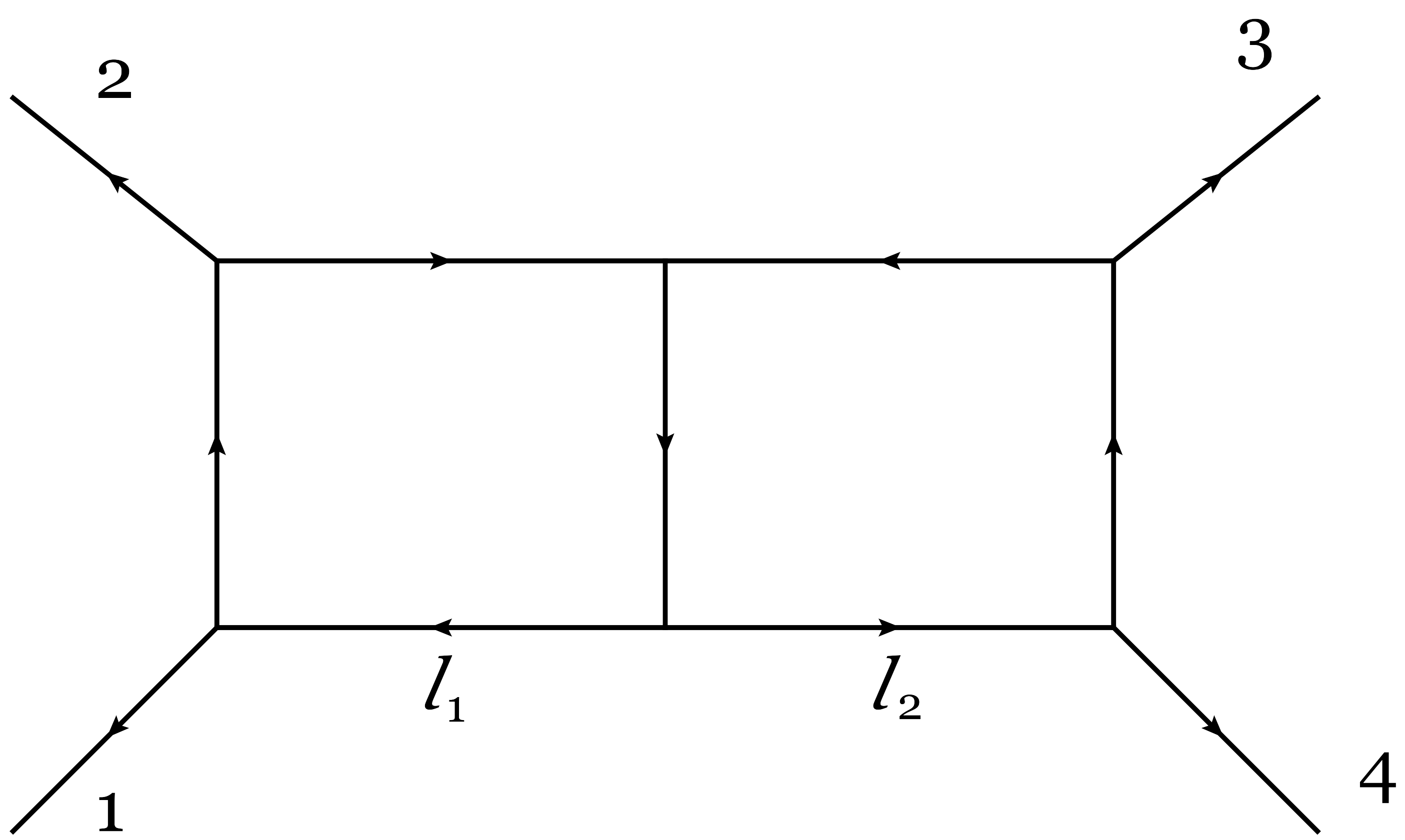
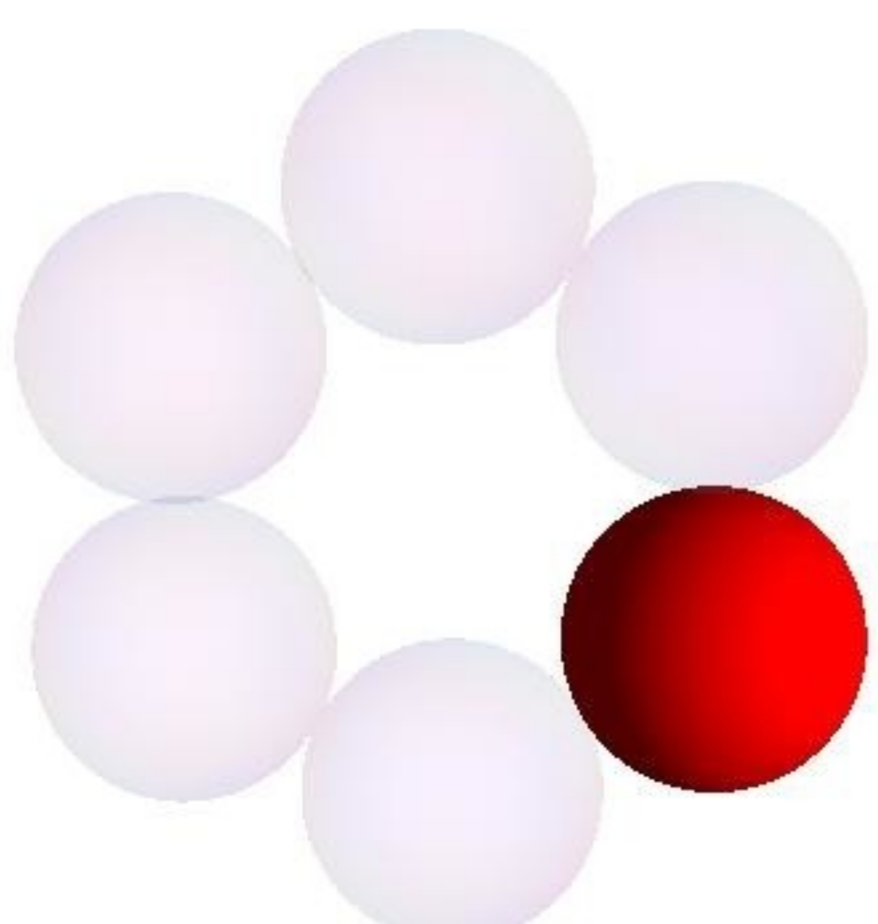
■ = +1 □ = 0

4D double box



$$
\begin{aligned}
x_1 &= l_1 \cdot p_1, \quad x_2 = l_1 \cdot p_2, \quad x_3 = l_1 \cdot p_4, \quad x_4 = l_1 \cdot \omega, \\
y_1 &= l_2 \cdot p_1, \quad y_2 = l_2 \cdot p_2, \quad y_3 = l_2 \cdot p_4, \quad y_4 = l_2 \cdot \omega.
\end{aligned}
$$

6 fundamental forms

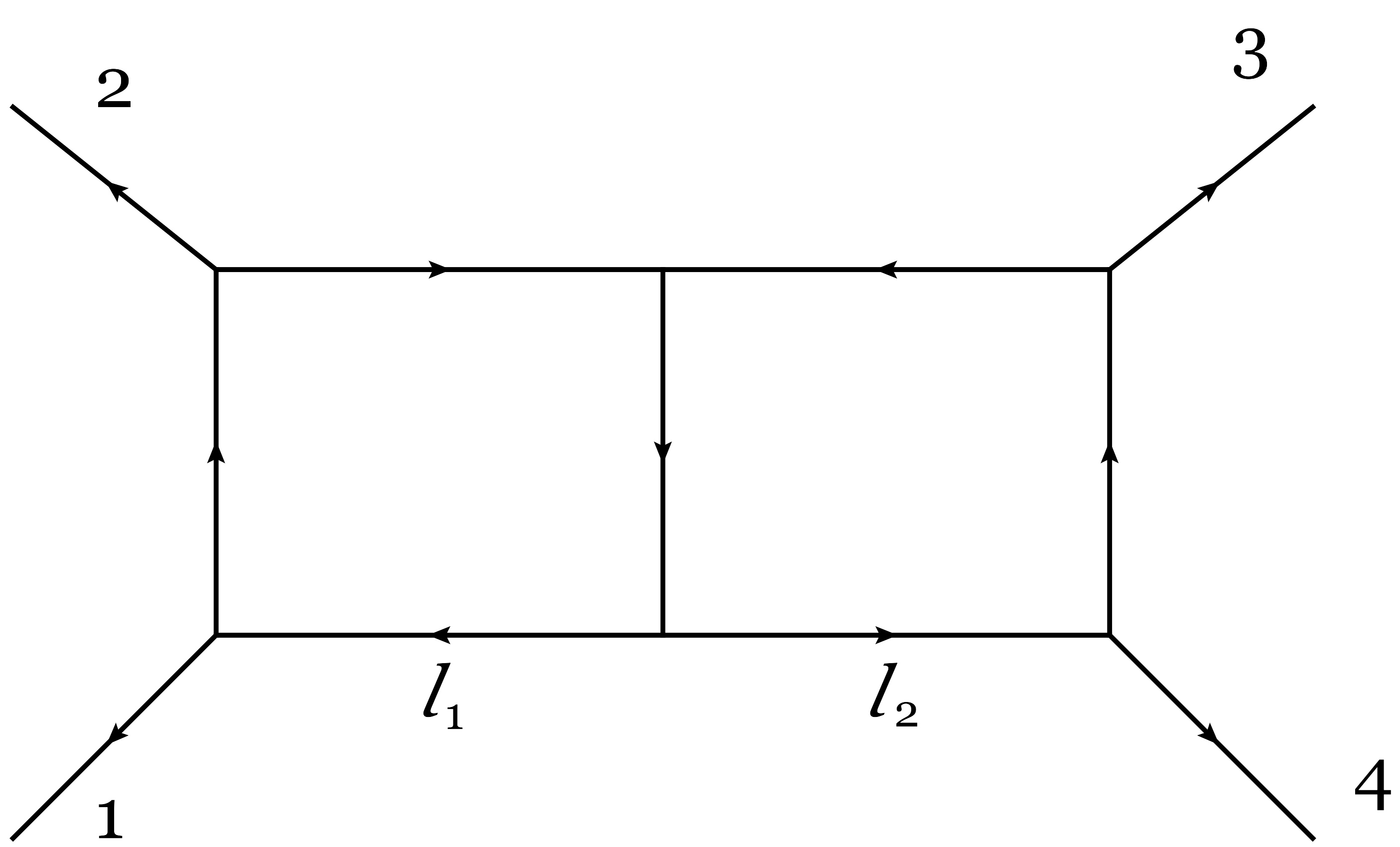


$$
\begin{aligned}
\eta_3 = {} & t x_4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 s^2 \\
& - 4 x_3 y_1 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 s^2 \\
& - 2 x_4 y_1 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 s^2 \\
& + t y_4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 s^2 \\
& - 2 x_3 y_4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 s^2 \\
& - t x_4 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 s^2 \\
& + 4 x_3 y_1 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 s^2 \\
& + 2 x_4 y_1 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dy_1 \wedge dy_3 \wedge dy_4 s^2 + \ldots
\end{aligned}
$$

algorithm realized by *MathematicaM2*,
a computational algebraic geometry package, YZ

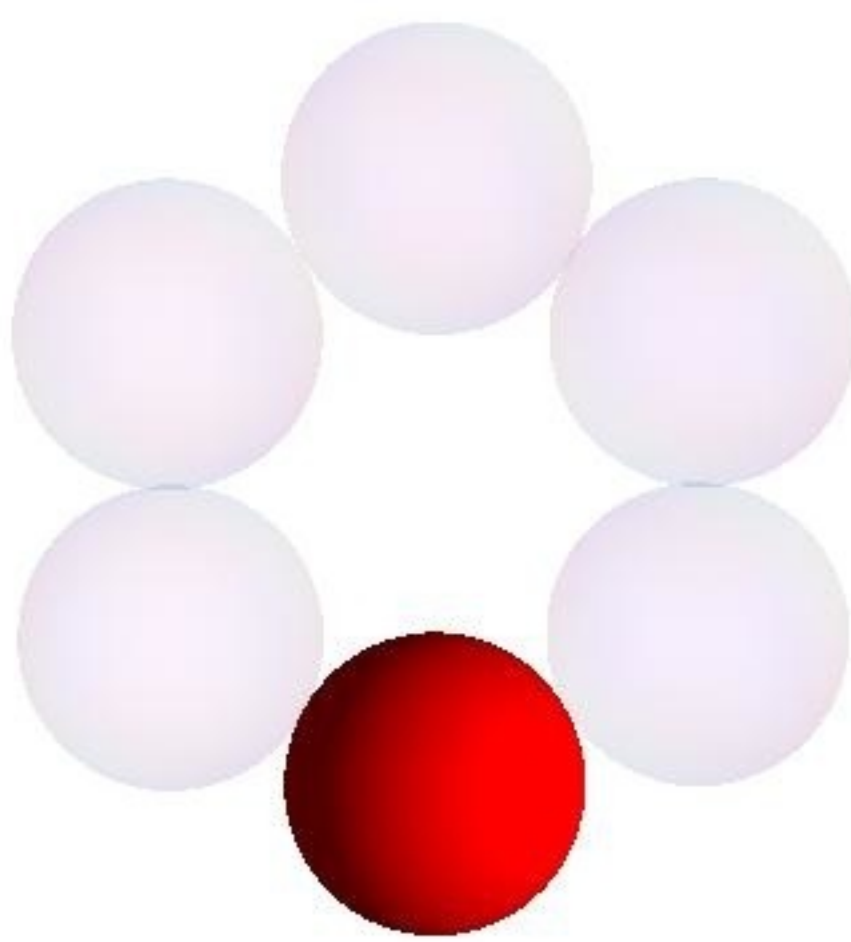
■ = +1 □ = 0

4D double box



$$x_1 = l_1\cdot p_1, \quad x_2 = l_1\cdot p_2, \quad x_3 = l_1\cdot p_4, \quad x_4 = l_1\cdot\omega,$$
$$y_1 = l_2\cdot p_1, \quad y_2 = l_2\cdot p_2, \quad y_3 = l_2\cdot p_4, \quad y_4 = l_2\cdot\omega.$$

6 fundamental forms

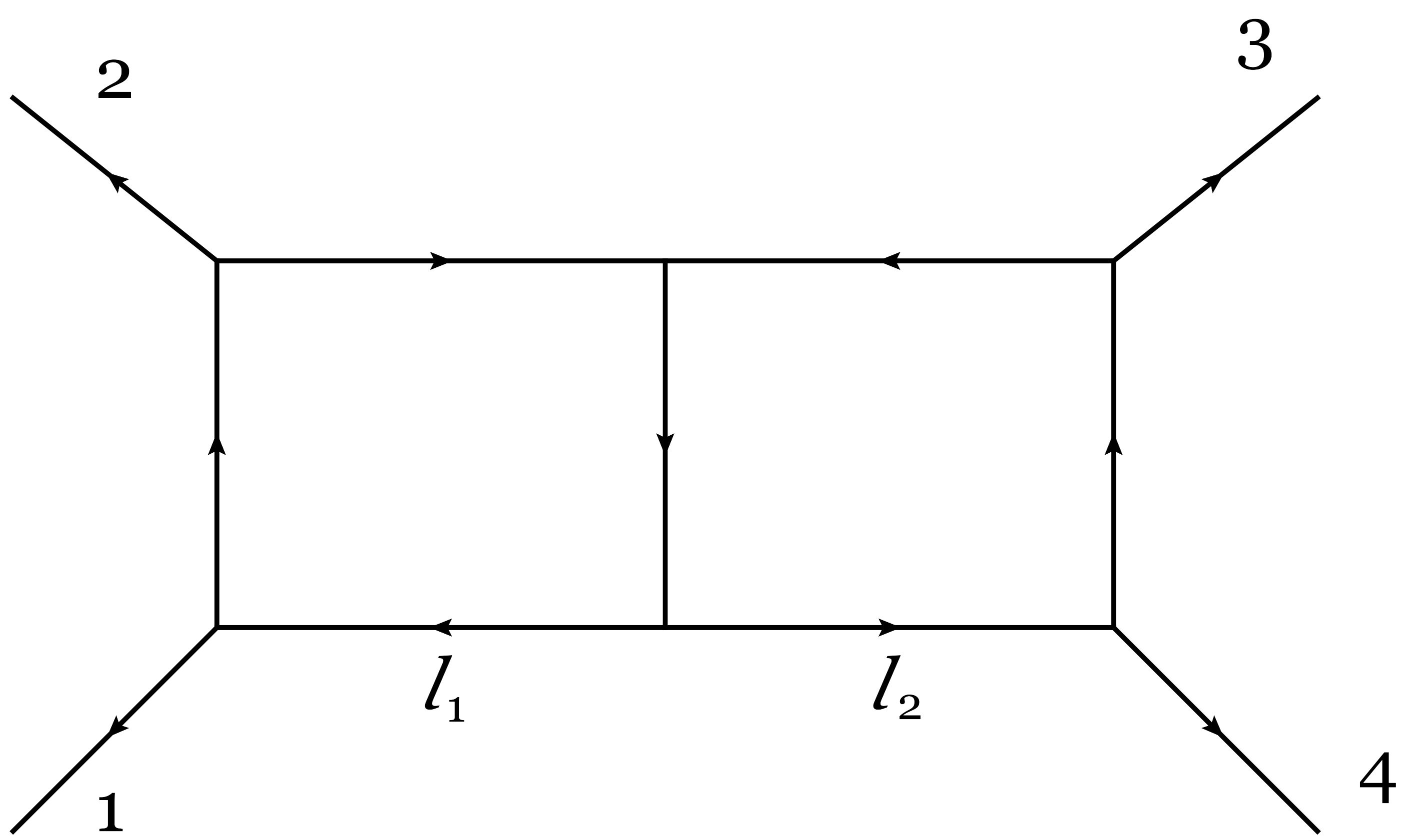


$$\eta_4 = \Big(s(t(x_4-2y_1+y_4)+4x_3y_1-2x_3y_4-2x_4y_1+4y_1^2-4y_1y_4)$$
$$+8x_3y_1(y_1-y_4)\Big)(dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_1\wedge dy_2\wedge dy_3$$
$$+dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_1\wedge dy_3\wedge dy_4+dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_2\wedge dy_3\wedge dy_4)$$

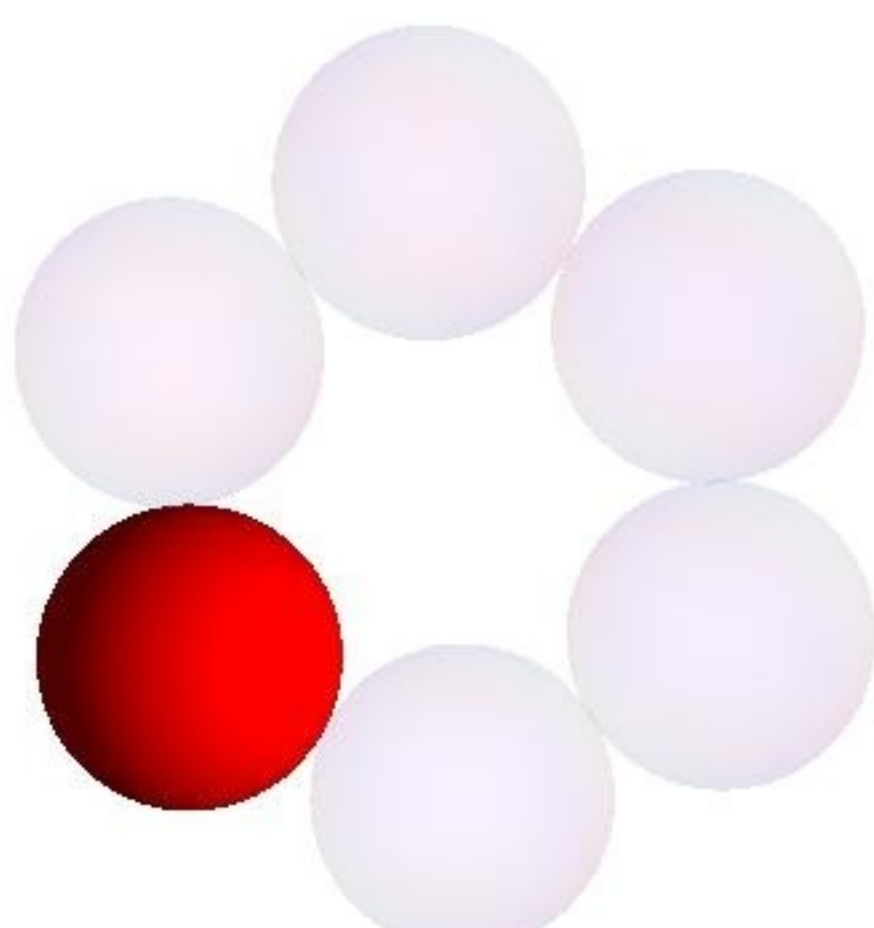
algorithm realized by *MathematicaM2*,
a computational algebraic geometry package, YZ

= +1 = 0

4D double box



$$x_1 = l_1 \cdot p_1, \quad x_2 = l_1 \cdot p_2, \quad x_3 = l_1 \cdot p_4, \quad x_4 = l_1 \cdot \omega,$$
$$y_1 = l_2 \cdot p_1, \quad y_2 = l_2 \cdot p_2, \quad y_3 = l_2 \cdot p_4, \quad y_4 = l_2 \cdot \omega.$$



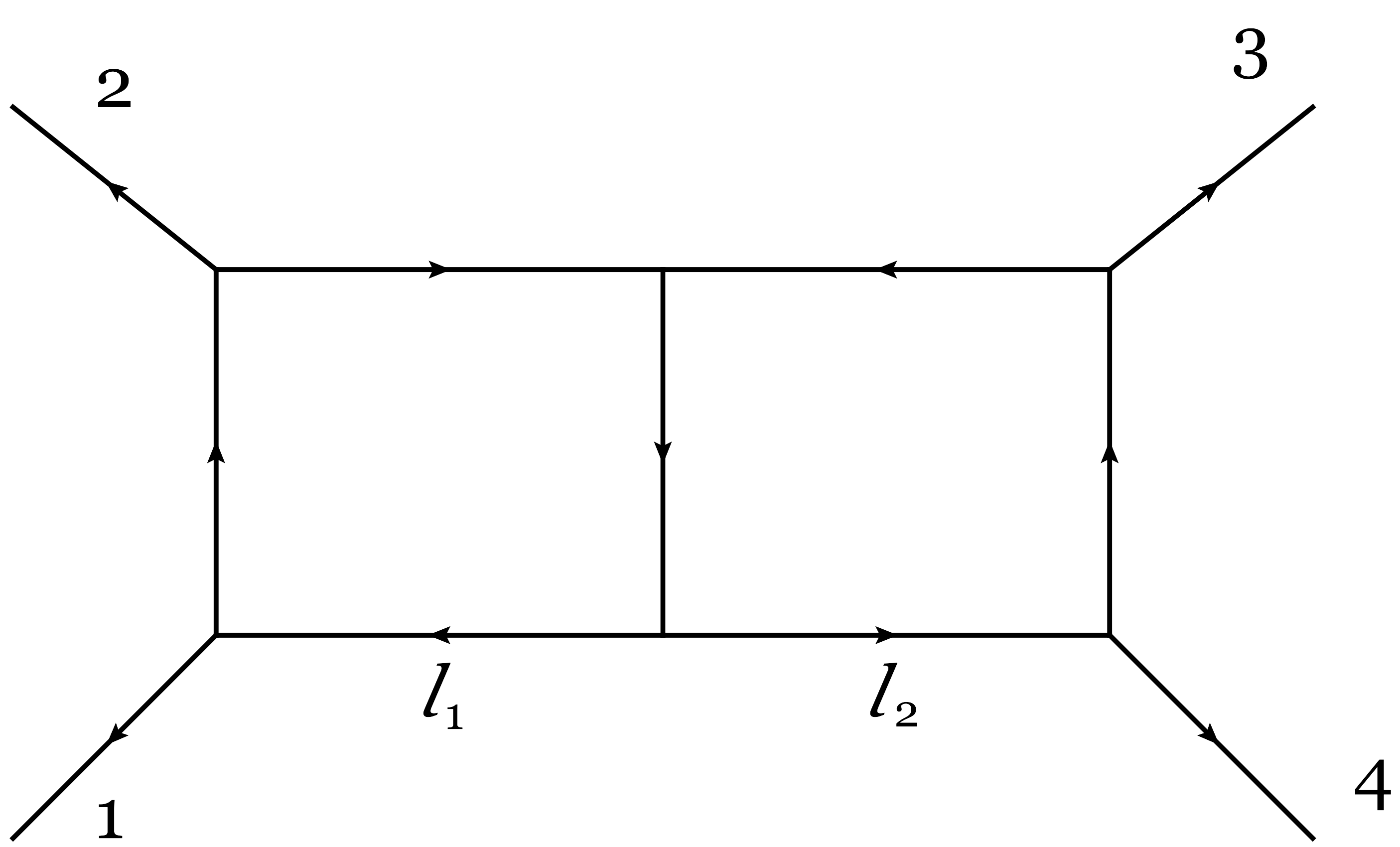
6 fundamental forms

$$\eta_5 = -\Big(s^2(t(2x_3 + x_4 + y_4) - 2(2x_3^2 + x_3(2x_4 + 2y_1 + y_4) + x_4 y_1))$$
$$+ 4sx_3(t^2 - t(2x_3 + x_4 + 2y_1 - y_4) - 2x_3(y_1 + y_4)) - 16tx_3^2 y_1 \Big)$$
$$(dx_1 \wedge dx_2 \wedge dx_3 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 - dx_1 \wedge dx_2 \wedge dx_4 \wedge dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4)$$

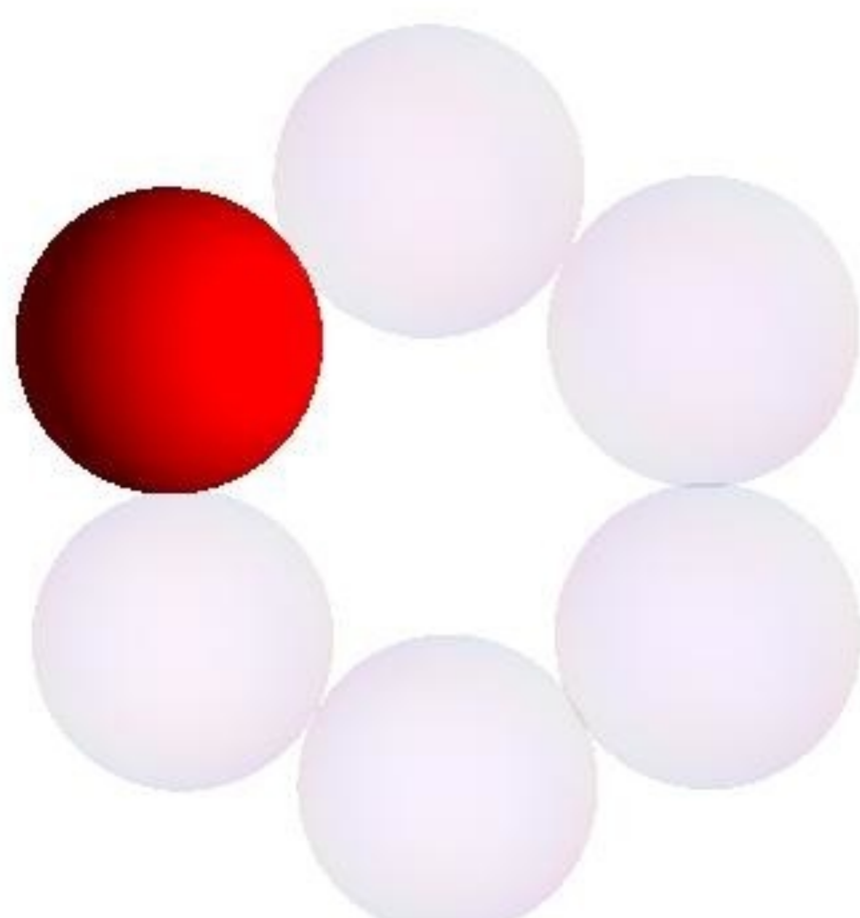
algorithm realized by *MathematicaM2*,
a computational algebraic geometry package,YZ

= +1 = 0

4D double box



$$\begin{aligned} x_1 &= l_1\cdot p_1, \quad x_2 = l_1\cdot p_2, \quad x_3 = l_1\cdot p_4, \quad x_4 = l_1\cdot \omega, \\ y_1 &= l_2\cdot p_1, \quad y_2 = l_2\cdot p_2, \quad y_3 = l_2\cdot p_4, \quad y_4 = l_2\cdot \omega. \end{aligned}$$



6 fundamental forms

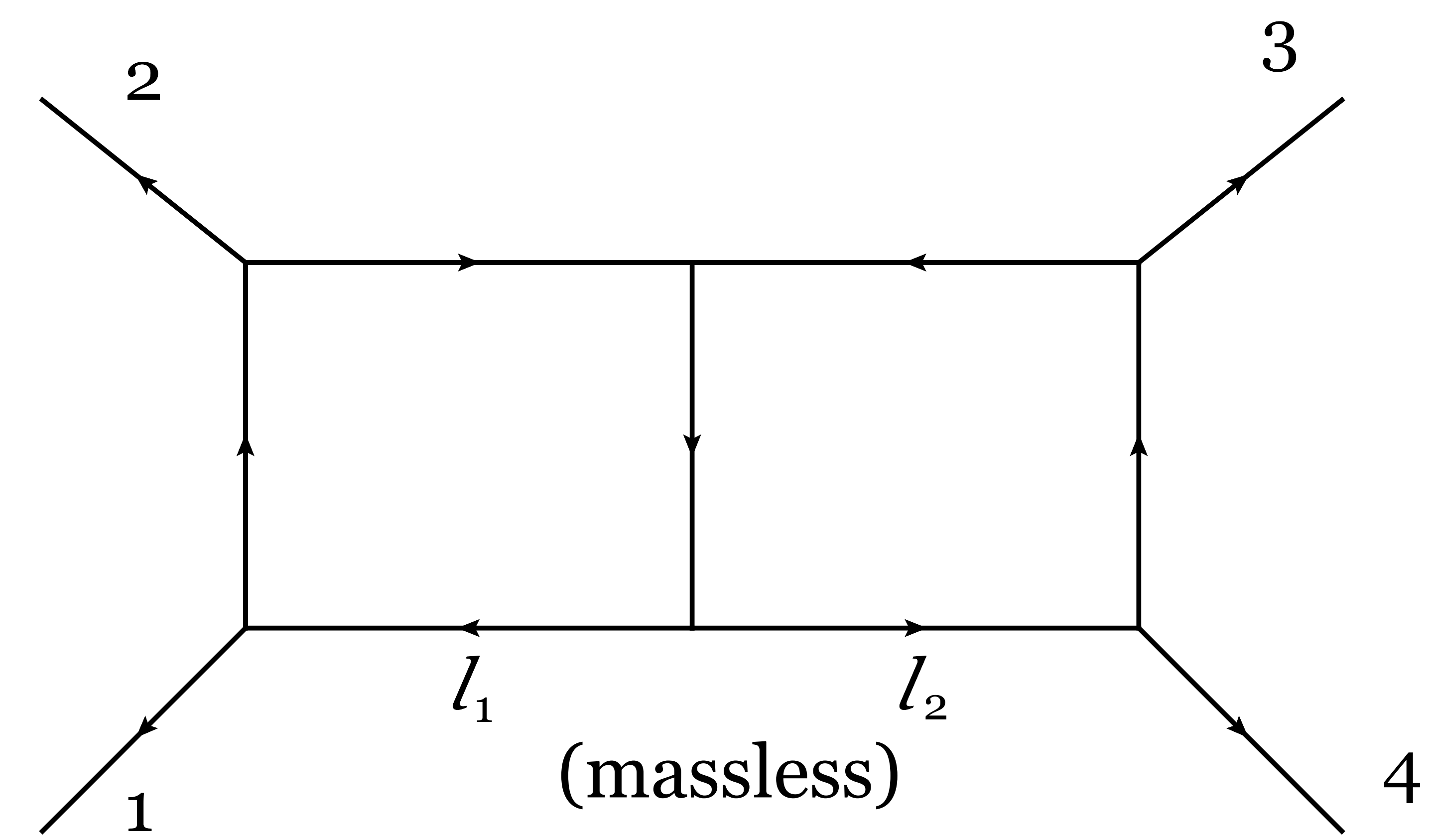
$$\begin{aligned} \eta_6 =\ & t x_4 dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_2\wedge dy_3\wedge dy_4\ s^2 \\ &+ 4x_3y_1 dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_2\wedge dy_3\wedge dy_4 s^2 \\ &- 2x_4y_1 dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_2\wedge dy_3\wedge dy_4 s^2 \\ &+ t y_4 dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_2\wedge dy_3\wedge dy_4 s^2 \\ &- 2x_3y_4 dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dy_2\wedge dy_3\wedge dy_4 s^2 \\ &- t x_4 dx_1\wedge dx_2\wedge dx_3\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4 s^2 \\ &- 4x_3y_1\ dx_1\wedge dx_2\wedge dx_3\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4 s^2 \\ &+ 2\ x_4y_1 dx_1\wedge dx_2\wedge dx_3\wedge dy_1\wedge dy_2\wedge dy_3\wedge dy_4 s^2 + \ldots \end{aligned}$$

algorithm realized by *MathematicaM2*,
a computational algebraic geometry package, YZ

■ = +1 □ = 0

4D double box: result

$$I_{\rm dbox}[(l_1\cdot p_4)^2] = \frac{t}{2} I_{\rm dbox}[l_1\cdot p_4] + \ldots$$
$$I_{\rm dbox}[(l_1\cdot p_4)^3] = \frac{t^2}{4} I_{\rm dbox}[l_1\cdot p_4] + \ldots$$
$$I_{\rm dbox}[(l_1\cdot p_4)^4] = \frac{t^3}{8} I_{\rm dbox}[l_1\cdot p_4] + \ldots$$
$$I_{\rm dbox}[(l_1\cdot p_4)^2(l_2\cdot p_1)] = -\frac{s^2 t}{16} I_{\rm dbox}[1] + \frac{3s^2}{8} I_{\rm dbox}[l_1\cdot p_4] + \ldots$$
$$I_{\rm dbox}[(l_1\cdot p_4)^3(l_2\cdot p_1)] = \frac{s^3 t}{32} I_{\rm dbox}[1] - \frac{3s^3}{16} I_{\rm dbox}[l_1\cdot p_4] + \ldots$$



MIs

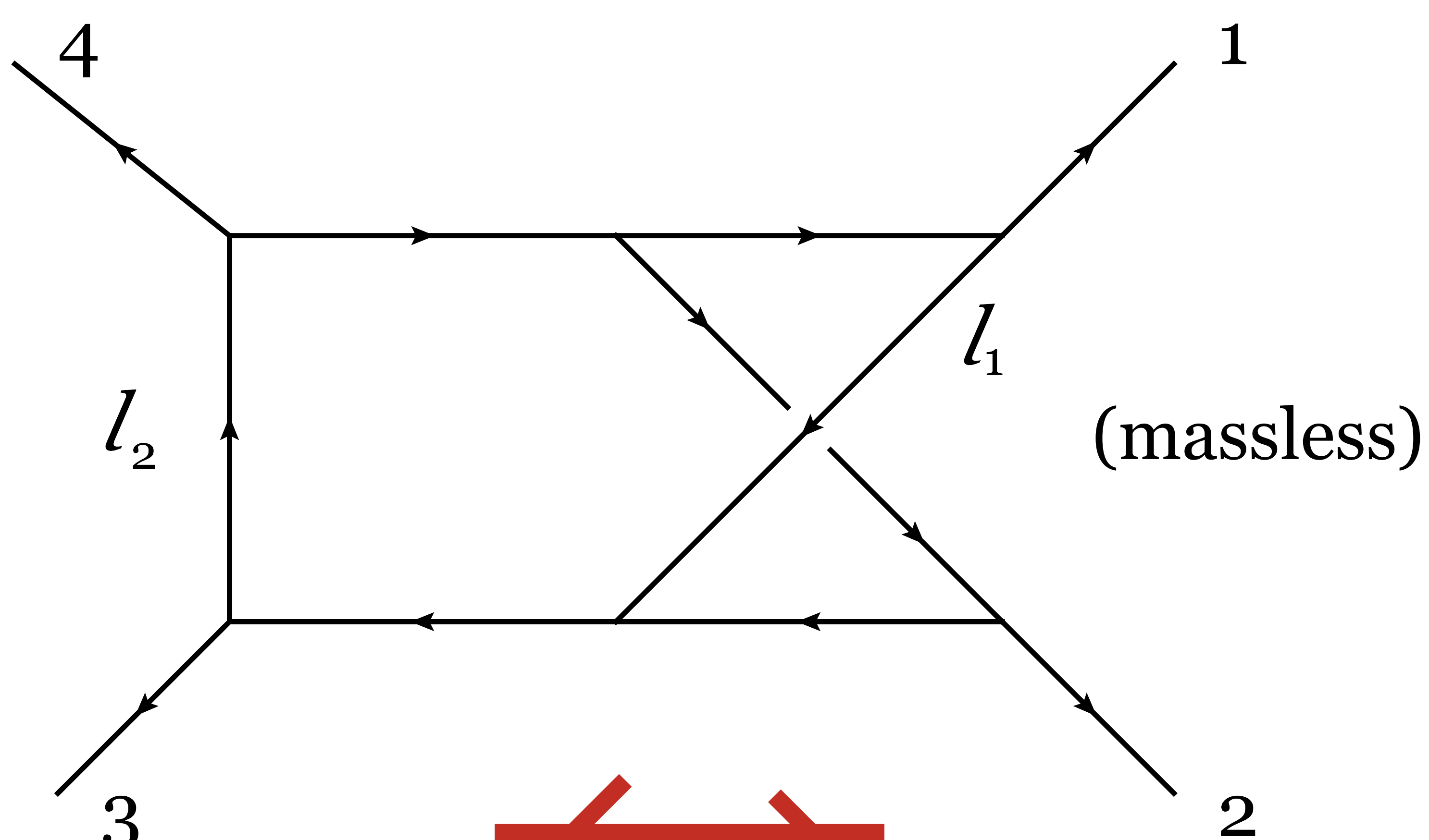
$$I_{\rm dbox}[1], \quad I_{\rm dbox}[l_1\cdot p_4]$$

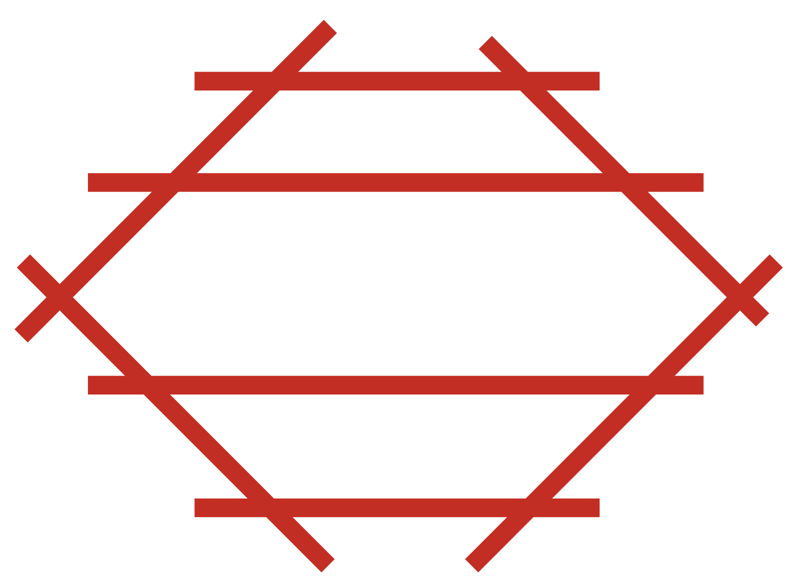
Obtain most on-shell 4D IBPs, except one

$$I_{\rm dbox}[(l_1\cdot p_4)(l_2\cdot p_1)] = \frac{1}{8} st I_{\rm dbox}[1] - \frac{3}{4} s I_{\rm dbox}[l_1\cdot p_4] + \ldots$$

Should be obtained from D-dim formalism

4D non-planar crossed box





8 cut solutions

1302.1023 R. Huang ,YZ

$$\Omega \equiv dD_1 \wedge dD_2 \wedge dD_3 \wedge dD_4 \wedge dD_5 \wedge dD_6 \wedge dD_7$$

8 fundamental forms

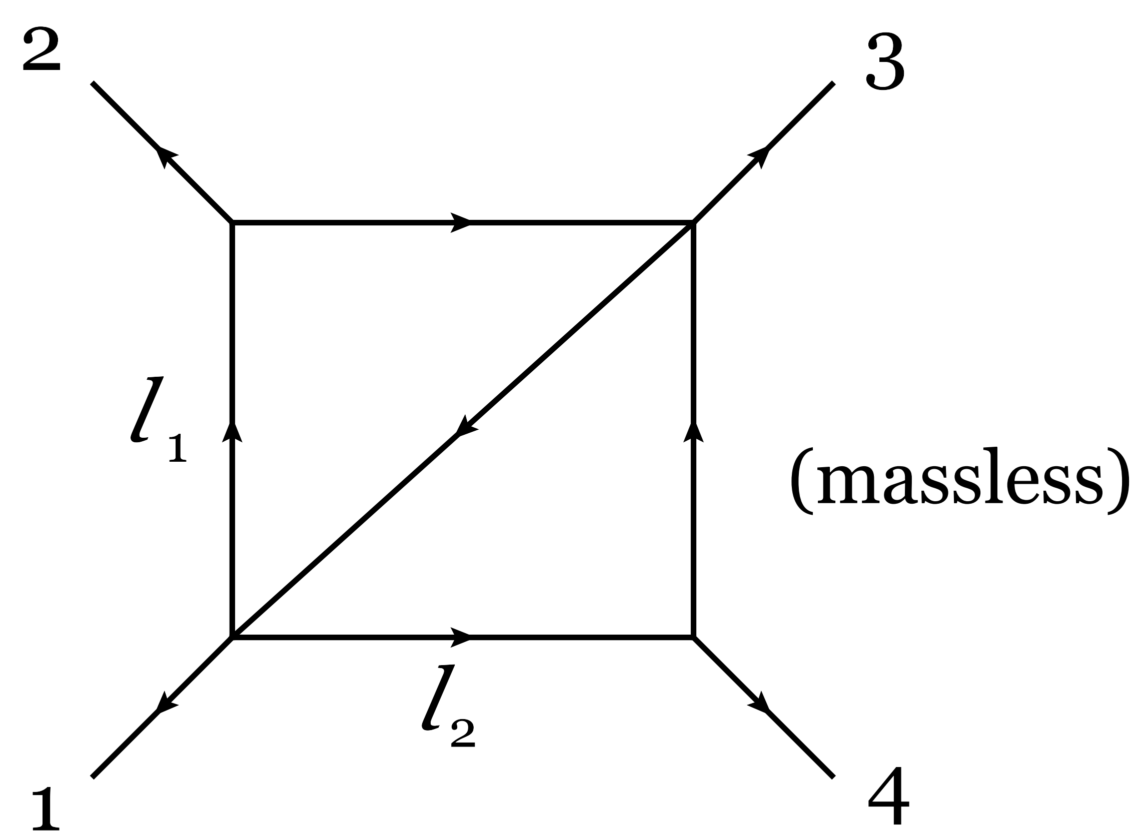
$$I_{\text{xbox}}[(l_2 \cdot p_2)^2] = -\frac{1}{8}t(s+t)I_{\text{xbox}}[1] + \frac{3}{4}(s+2t)I_{\text{xbox}}[l_1 \cdot p_3] + \ldots$$

$$I_{\text{xbox}}[(l_1 \cdot p_3)(l_2 \cdot p_2)^2] = \frac{-t(s^2+3st+2t^2)}{32}I_{\text{xbox}}[1] + \frac{(3s^2+8st+8t^2)}{16}I_{\text{xbox}}[l_1 \cdot p_3] + \ldots$$

$$I_{\text{xbox}}[(l_2 \cdot p_2)^3] = \frac{t(s^2+3st+2t^2)}{16}I_{\text{xbox}}[1] - \frac{(3s^2+8st+8t^2)}{8}I_{\text{xbox}}[l_1 \cdot p_3] + \ldots$$

MIs $\quad I_{\text{xbox}}[1], \quad I_{\text{xbox}}[l_1 \cdot p_3]$

4D slashed box



4 cut solutions,
each is 3-dimensional

not a *(DL-1)*-form

$$\Omega \equiv dD_1 \wedge dD_2 \wedge dD_3 \wedge dD_4 \wedge dD_5$$

$$\alpha^{(1)} = dx_1 \wedge dx_3, \quad \alpha^{(2)} = dx_1 \wedge dy_1, \quad \alpha^{(3)} = dx_1 \wedge dy_2, \quad \alpha^{(4)} = dx_3 \wedge dy_1,$$
$$\alpha^{(5)} = dx_3 \wedge dy_2, \quad \alpha^{(6)} = dy_1 \wedge dy_2, \quad \alpha^{(7)} = dx_4 \wedge dy_4, \quad \alpha^{(8)} = dx_1 \wedge dx_4$$
$$\alpha^{(9)} = dx_3 \wedge dx_4, \quad \alpha^{(10)} = dy_1 \wedge dx_4, \quad \alpha^{(11)} = dy_2 \wedge dx_4, \quad \alpha^{(12)} = dx_1 \wedge dy_4$$
$$\alpha^{(13)} = dx_3 \wedge dy_4, \quad \alpha^{(14)} = dy_1 \wedge dy_4, \quad \alpha^{(15)} = dy_2 \wedge dy_4$$

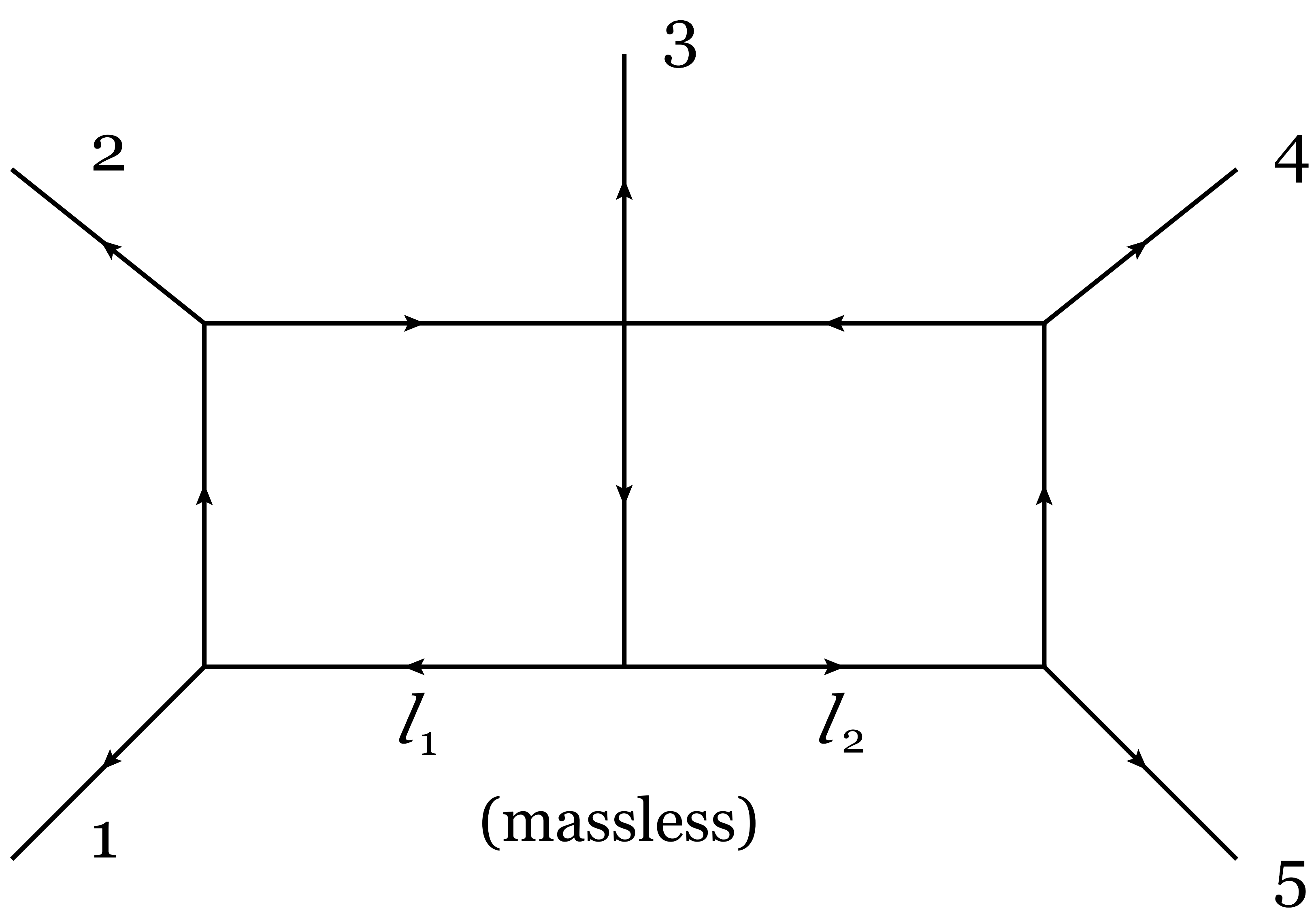
$$\Omega^{(i)} = \alpha^{(i)} \wedge \Omega$$

Each $\Omega^{(i)}$ has 4 fundamental forms

60 fundamental forms

59 integrand terms ⟶ 1 MI

4D "turtle" box: 5pt



6 cut solutions

momentum-twistor parametrization

Simon Badger's talk

$$\tilde{\lambda}_i = \frac{\langle i,\, i+1\rangle \mu_{i-1} + \langle i+1,\, i-1\rangle \mu_i + \langle i-1,\, i\rangle \mu_{i+1}}{\langle i,\, i+1\rangle \langle i-1,\, i\rangle}$$

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{1}{x_1} & \frac{1}{x_1}+\frac{1}{x_2} & \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & x_4 & 1 \\ 0 & 0 & 1 & 1 & \frac{x_5}{x_4} \end{pmatrix}$$

In the final result, it is easy to convert $\{x_1, x_2, x_3, x_4, x_5\}$ to s_{ij}, tr_5...

4 ISP

$$a = l_1 \cdot p_4, \quad b = l_1 \cdot p_5,$$
$$c = l_2 \cdot p_1, \quad d = l_2 \cdot p_2,$$

$$\Omega \equiv dD_1 \wedge dD_2 \wedge dD_3 \wedge dD_4 \wedge dD_5 \wedge dD_6 \wedge dD_7$$

6 fundamental forms $\quad \eta_i\big|_{S_j} = \delta_{ij}\Omega\big|_{S_j}$

4D “turtle” box: result

D-dimensional, 3 MIs $\quad I_{\text{turtle}}[1], \quad I_{\text{turtle}}[l_1 \cdot p_4], \quad I_{\text{turtle}}[l_1 \cdot p_5]$

$\Downarrow$ Gram determinant relation

4-dimensional, 2 MIs $\quad I_{\text{turtle}}[1], \quad I_{\text{turtle}}[l_1 \cdot p_4]$

```
IBPFinal = {Int[b] → -(s15 ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (2 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[a b] → (s15 (s15 - s23 + s45) ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (4 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[b^2] → -(s15^2 ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (4 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[a b^2] → (s15^2 (s15 - s23 + s45) ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (8 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[b^3] → -(s15^3 ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (8 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[a b^3] → (s15^3 (s15 - s23 + s45) ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (16 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[b^4] → -(s15^4 ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (16 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[c] → -(s15 ((s23 s34 + s15 s45 - s34 s45 + s12 (s15 - s23 + 2 s45)) Int[1] + 4 (s12 + s15 - s34) Int[a])) / (2 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[b c] → (s15 (s12 (s15 - s23) + s23 s34 + (s15 - s34) s45) ((s15^2 - s15 (s23 + s34 - 2 s45) + 2 s34 (s23 - s45) + s12 (2 s15 - 2 s23 + 3 s45)) Int[1] + 6 (s12 + s15 - s34) Int[a])) /
    (8 (s15 - s23 - s34) (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[b^2 c] →
   -(s15 (s12^2 (s15 - s23)^2 + 2 s12 s23 s34 (s15 - s23 + s45) + (s23 s34 + (s15 - s34) s45)^2) ((s15^2 - s15 (s23 + s34 - 2 s45) + 2 s34 (s23 - s45) + s12 (2 s15 - 2 s23 + 3 s45)) Int[1] + 6 (s12 + s15 - s34) Int[a])) /
    (16 (-s15 + s23 + s34)^2 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
  Int[b^3 c] →
   (s15 (s12^3 (s15 - s23)^3 + 3 s12^2 (s15 - s23) s23 s34 (s15 - s23 + s45) + 3 s12 s23 s34 (s15 - s23 + s45) (s23 s34 + (s15 - s34) s45) + (s23 s34 + (s15 - s34) s45)^3)
      ((s15^2 - s15 (s23 + s34 - 2 s45) + 2 s34 (s23 - s45) + s12 (2 s15 - 2 s23 + 3 s45)) Int[1] + 6 (s12 + s15 - s34) Int[a])) /
    (32 (s15 - s23 - s34)^3 (2 s15^2 + s12 (s15 - s23) + s34 (s23 - s45) + s15 (-2 s23 - 2 s34 + s45))),
```

numerically verified with *FIRE*

Conclusion

- Geometric meaning of IBPs without doubled propagator
- A simple method to construct the on-shell part of IBPs

Future directions

- Application in generalized unitarity (1108.1180, D. Kosower, K. Larsen)
- D-dimensional formalism
- Off-shell parts, a recursive algorithm
- Combination with integral reduction programs