# Scattering Equations
## Recent Developments and Applications

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# I

# Scattering Equations & Structure of Their Solutions

# Definition & Background

Consider scattering of $n$ massless particles ($\{k_a^\mu, h_a\}, a = 1, \ldots, n$)

$$\sum_{b \neq a} \frac{s_{a,b}}{\sigma_a - \sigma_b} = 0, \quad \forall a.$$

$s_{a,b} = (k_a + k_b)^2$: Mandelstam variables.
$\sigma_a$: inhomogeneous coordinate of $a^{\text{th}}$ marked point on $\mathbb{CP}^1$.

Appearance in previous literature,

[Farlie, Roberts '72], [Roberts '72], [Farlie '08]; [Gross, Mende '88];

[Witten '04]; [Makeenko, Olesen '11]; [Cachazo '12]; [Stieberger, Taylor '13].

Importance in the study of
scattering amplitudes in *any dimensions*
addressed in [Cachazo, He, EYY 2013].

# Some Remarks ($\sum_{b\neq a}\frac{s_{a,b}}{\sigma_a-\sigma_b}=0,\ \forall a$)

(1) Zero set invariant under $\mathrm{SL}(2,\mathbb{C})$ acting on $\{\sigma_a\}$.
$\mathbb{CP}^1$ *with n marked points* as the object of study.

(2) Equivalent statement: *vanishing quadratic differential*

$$\omega^\mu(\sigma)\omega_\mu(\sigma)=0,\qquad \omega^\mu(\sigma)=\underbrace{\sum_{a=1}^{n}\frac{k_a^\mu}{\sigma-\sigma_a}\,d\sigma}_{\text{no pole at }\infty}.$$

(3) $n-3$ independent equations.

(4) Solutions of $\{\sigma_a\}$ for generic $\{k_a\}$ localized on isolated points inside $\mathbb{CP}^n/\mathrm{SL}(2,\mathbb{C})$. $(n-3)!$ distinct solutions in total.

(5) Factorization structure ...

# Bridge Connecting Two Spaces

$$\text{space of kinematics configurations} \xrightarrow{\text{scattering equations}} \text{moduli space of } \mathbb{CP}^1 \text{ with marked points}$$

Consider the limit of a singular kinematics configuration

$$(k_1 + k_2 + \cdots + k_{n_L})^2 \longrightarrow 0.$$

$$(n-3)! \longrightarrow \begin{cases} \underbrace{(n_L + 1 - 3)!}_{\text{left}} \times \underbrace{(n - n_L + 1 - 3)!}_{\text{right}} & \text{singular} \\ (n-3)! - (n_L - 2)! \times (n - n_L - 2)! & \text{regular} \end{cases}$$

Desired *factorization* structure!

(independent of space-time dimensions)

# Amusing Facts in 4d

$$\omega_{\alpha\dot{\alpha}}(\sigma) = \sum_{a=1}^{n} \frac{k_{a,\alpha\dot{\alpha}}}{\sigma - \sigma_a}\, d\sigma = \frac{P_{\alpha\dot{\alpha}}(\sigma)\, d\sigma}{\prod_a (\sigma - \sigma_a)}.$$

$P_{\alpha\dot{\alpha}}(\sigma)$ a polynomial of degree $n-2$, satisfying

$$P_{1\dot{1}}(\sigma) P_{2\dot{2}}(\sigma) = P_{1\dot{2}}(\sigma) P_{2\dot{1}}(\sigma).$$

Have to choose the position for each factor $(\sigma - \sigma_{\text{root}})$.
With any given choice of integers $d, \tilde{d}$ with $d + \tilde{d} = n - 2$

$$P_{\alpha\dot{\alpha}} \propto \underbrace{\lambda_\alpha(\sigma)}_{\text{deg. } d} \underbrace{\tilde{\lambda}_{\dot{\alpha}}(\sigma)}_{\text{deg. } \tilde{d}},$$

The $(n-3)!$ solutions fall into *different sectors*!
(*independent of* any helicity configuration)

# Amusing Facts in 4d, Continued

Polynomial maps $\lambda_\alpha(\sigma), \tilde{\lambda}_{\dot{\alpha}}(\sigma)$ in previous literature

[Witten '03], [Roiban, Spradlin, Volovich '04], [Arkani-Hamed, Cachazo, Cheung, Kaplan '09], [Cachazo, Geyer '12], [Cachazo, Mason, Skinner '12], [Cachazo '13].

For each choice of $d$ ($d = 1, \ldots, n-3$), the number of solutions of $\{\sigma_a\}$ is the Eulerian number $\left\langle {n-3 \atop d-1} \right\rangle$ [Spradlin, Volovich '09].

Summation

$$\underbrace{\left\langle {n-3 \atop 0} \right\rangle}_{\text{"MHV"}} + \underbrace{\left\langle {n-3 \atop 1} \right\rangle}_{\text{"NMHV"}} + \cdots + \underbrace{\left\langle {n-3 \atop n-2} \right\rangle}_{\text{"}\overline{\text{MHV}}\text{"}} = (n-3)!$$

Analytic solutions in 4d further in [Weinzierl '14], [Dolan, Goddard '14].

Similar structure in 3d; deduced from 4d.

... Dimensions higher than 4?

# Polynomial Version [Dolan, Goddard '14]

Solutions of scattering equations form the zero set of the ideal spanned by the following polynomials

$$h_m := \sum_{\substack{S\subset\{1,\ldots,n\}\\ |S|=m}} k_S^2 \sigma_S = 0, \qquad m = 2,\ldots,n-2,$$

with $k_S^\mu = \sum_{b\in S} k_b^\mu$, $\sigma_S = \prod_{b\in S} \sigma_b$.

Partially gauge-fixed version, by setting $\sigma_1 = \infty$ and $\sigma_n = 0$,

$$\tilde{h}_m := \sum_{\substack{S\subset\{2,\ldots,n-1\}\\ |S|=m}} k_{\{1\}\cup S}^2 \sigma_S = 0, \qquad m = 1,\ldots,n-3.$$

An easy derivation is by looking at the coefficients in $\omega^\mu(\sigma)\omega_\mu(\sigma)$.

# Remarks

(1) Manifestly $(n-3)!$ solutions by Bézout's theorem.

(2) Every $h_m$ ($\tilde{h}_m$) is multilinear in each $\sigma_a$.

(3) Easier to obtain solutions (e.g., by elimination theory).

(4) algebra: polynomial rept. of $\mathfrak{sl}(2,\mathbb{C})$ algebra.
Scattering equations $\Longleftrightarrow$ the unique largest irrept.
Other irrepts...

(5) geometry: each irrept. defines a projective variety in $\mathbb{CP}^{n-1}$ (or $\mathbb{CP}^{n-3}$) (*scattering varieties* [He, Matti, Sun '14]).

More systematic way of studying scattering equations and the structure of their solution space!

# II

# Scattering Amplitudes

# CHY Formula [Cachazo, He, EYY '13]

Contour integral in moduli space / summation over solutions

$$\int \frac{d^n\sigma_a}{\text{vol.}} \prod{}' \delta\Big(\sum_{b\neq a} \frac{s_{a,b}}{\sigma_a - \sigma_b}\Big) \mathcal{I}(\{k_a, \epsilon_a\}, \{\sigma_a\}) = \sum_{\substack{(n-3)!\\ \text{solns.}}} \underbrace{\frac{\mathcal{I}(\{k_a, \epsilon_a\}, \{\sigma_a\})}{J(\{k_a\}, \{\sigma_a\})}}_{\text{residue}} .$$

Contour wrapping around all solutions of the equations.

$\mathcal{I} = C\tilde{C}$ ($\phi^3$ with gauge group $U(N) \times U(\tilde{N})$),
or $\mathcal{I} = CE$ (gluon), or $\mathcal{I} = E\tilde{E}$ (graviton).

$$C = \sum_{\alpha \in S_{n-1}} \frac{\text{tr}(T^{a_{\alpha_1}} \cdots T^{a_{\alpha_n}})}{(\sigma_{\alpha_1} - \sigma_{\alpha_2}) \cdots (\sigma_{\alpha_n} - \sigma_{\alpha_1})}. \qquad E = \text{Pf}' \begin{pmatrix} A & -B^T \\ B & D \end{pmatrix}.$$

$$A_{a,b} = \begin{cases} \frac{s_{a,b}}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{a,b} = \begin{cases} \frac{2\,\epsilon_a \cdot k_b}{\sigma_a - \sigma_b} & a \neq b \\ -\sum_{c\neq a} \frac{2\,\epsilon_a \cdot k_c}{\sigma_a - \sigma_c} & a = b \end{cases}, \quad D_{a,b} = \begin{cases} \frac{2\,\epsilon_a \cdot \epsilon_b}{\sigma_a - \sigma_b} & a \neq b \\ 0 & a = b \end{cases}.$$

# Feynman Diagrams from CHY

Consider the limit $k_n^\mu \to 0$. $(\sigma_{a,b} := \sigma_a - \sigma_b)$

$$E_n \longrightarrow \sum_{a=1}^{n-1} \frac{\epsilon_n \cdot k_a}{\sigma_{n,a}} E_{n-1}.$$

Can directly integrate $\{\sigma_1, \dots, \sigma_{n-1}\}$, like those for $M_{m-1}$

$$M_n \longrightarrow \sum_{i=1}^{(n-4)!} \oint d\sigma_n \frac{\left( \frac{\epsilon_n \cdot k_1}{\sigma_{n,1}} + \frac{\epsilon_n \cdot k_2}{\sigma_{n,2}} + \cdots + \frac{\epsilon_n \cdot k_{n-1}}{\sigma_{n,n-1}} \right)^2}{\frac{s_{n,1}}{\sigma_{n,1}} + \frac{s_{n,2}}{\sigma_{n,2}} + \cdots + \frac{s_{n,n-1}}{\sigma_{n,n-1}}} \frac{\left( E_{n-1}^{(i)} \right)^2}{J_{n-1}^{(i)}}.$$

$\sum_a k_a = 0$ & $\epsilon_a \cdot k_a = 0 \implies$ regular at $\sigma_n = \infty$.

Contour deformation leads to Weinberg's soft theorem

$$M_n \longrightarrow \sum_{a=1}^{n-1} (\epsilon_n \cdot k_a)^2 \frac{1}{s_{n,a}} M_{n-1}.$$

# Connecting Different Theories

Define $V^{(i)}[\alpha] := (\sigma_{\alpha(1),\alpha(2)} \cdots \sigma_{\alpha(n),\alpha(1)})^{-1}|_{\text{soln.}[i]}$, $(\sigma_{a,b} := \sigma_a - \sigma_b)$

KLT orthogonality [Cachazo, Geyer '12], [Cachazo, He, Yuan '13]

$$\sum_{\alpha,\beta \in S_{(n-3)!}} V^{(i)}[1,\alpha,n-1,n] \underbrace{\mathcal{S}[\alpha|\beta]}_{\alpha' \to 0 \text{ limit of KLT momentum kernel}} V^{(j)}[1,\beta,n,n-1] = \delta^{ij} J(\sigma)|_{\text{soln.}[i]}$$

[Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove '10]

$$\Updownarrow$$

$$\mathcal{S}[\alpha|\beta]^{-1} = \underbrace{\int \frac{d^n\sigma_a}{\text{vol.}} \prod{}' \delta\Big(\sum_{b \neq a} \frac{s_{a,b}}{\sigma_{a,b}}\Big) V[1,\alpha,n-1,n]\, V[1,\beta,n,n-1]}$$

double-partial amplitude $m[1,\alpha,n-1,n|1,\beta,n,n-1]$
trivalent tree graphs consistent with both orderings

[Broedel, Schlotterer, Stieberger '13], [Cachazo, He, EYY '13]

# Connecting Different Theories, Continued

A natural playground for studying connections between amplitudes in YM and GR!

BCJ relations [Bern, Carrasco, Johansson '08] at the level of residues upon each solution [Cachazo '12], [Cachazo, He, EYY '13], [Litsey, Stankowicz '13].

Lie algebraic structure in kinematics mirroring that of color; kinematic vertices [Monteiro, O'Connell '13] *(Monteiro's talk)*.

Other constructions of explicit BCJ numerators [Cachazo, He, EYY '13], [Naculich '14].

# Loop Extension

Tree level:

Scattering equations $\Longleftrightarrow \omega^\mu(\sigma)\omega_\mu(\sigma) = 0$,
where $\omega^\mu(\sigma)$ is a meromorphic form on $\mathbb{CP}^1$, and has simple poles only at each marked point $\sigma_a$ with residue $k_a^\mu$.

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General:

Scattering equations $\Longleftrightarrow \omega^\mu(\sigma)\omega_\mu(\sigma) = 0$,

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$\Sigma_g$: Riemann surface of genus $g \sim$ Amplitudes at $g$ loops.
$g = 0 \Longrightarrow$ tree level.

$g \geq 1$:
(1) Solution of $\omega^{\mu}(\sigma)$ not fully determined by external data.
(2) For the same external states, amplitudes depend on the contents propagating in the loop.

Guidance: Models on the world-sheet.

# Loop Extension via Ambi-Twistor String

$$\omega^\mu(\sigma)\omega_\mu(\sigma) = 0 \Rightarrow \text{complex null geodesics as target space}$$
$$\Rightarrow \text{ambi-twistor strings}$$

[Mason, Skinner '13], [Adamo, Casali, Skinner '13]; [Berkovits '13], [Gomez, EYY '13].

$$\text{Genus } g \Longrightarrow \underbrace{g \text{ holomorphic 1-forms} \sim g \text{ loop momenta } l_i^\mu}_{\substack{\text{need to be further integrated} \\ \text{potential UV divergence}}}$$

$g$ loops, $n$ particles: $\Sigma_{g,n}$,
$n + 3g - 3$ scattering equations. $\omega^\mu(\sigma)$ determined by $\{k_a^\mu, l_i^\mu\}$.

Evidences: modular invariance of 1-loop partition function in the critical dimension; behavior in the non-separating or separating degeneration of $\Sigma_{g,n}$. [Adamo, Casali, Skinner '13]

# Connection to the Ordinary Strings

$$A^{\text{YM}}(I) = \sum_{\alpha,\beta\in S_{n-3}} m[I|1,\alpha,n,n-1]\mathcal{S}[\alpha|\beta]A^{\text{YM}}(1,\beta,n-1,n),$$

$$A^{\text{String}}(I) = \sum_{\alpha,\beta\in S_{n-3}} \underbrace{Z[I|1,\alpha,n,n-1]}_{\text{disk integral}}\mathcal{S}[\alpha|\beta]A^{\text{SYM}}(1,\beta,n-1,n).$$

[Mafra, Schlotterer, Stieberger '11], [Broedel, Schlotterer, Stieberger '13]

Infinite tension limit v.s. High energy limit?

[Bandos '14] Moving frame formulation of superstring gives rise to ambi-twistor string in both limits.

[Bjerrum-Bohr, Damgaard, Tourkine, Vanhove '14] Method to derive a model from the ordinary string amplitude, matching both CHY at $\alpha' \to 0$ and Gross-Mende at $\alpha' \to \infty$. *(Bjerrum-Bohr's talk)*

# Miscellaneous

CHY formula applied in proving the new soft theorems [Cachazo, Strominger '14] at tree level in any dimensions, for the subleading terms. [Schwab, Volovich '14], [Afkhami-Jeddi '14]

Special Kinematics
(1) analytic solution for generic $n$,
(2) alternative starting point to learn the structure of solutions.

[Cachazo, He, Yuan '13], [Kalousios '14], [Dolan, Goddard '14]

## Outlook (not ordered by relative importance)

- Structure of the equations and solutions, in dimensions higher than 4 (especially 6d).
- Massive extensions for generic spins.
- Formula in terms of on-shell superspace.
- Verification of the loop-level extension.
- Understanding the role of each individual residue.
- Relation between the two limits of strings?
- Further applications to new soft theorems.
- ... ...

# Thank you very much!

comments and questions are welcome